A CORRECTION AND SOME ADDITIONS TO
"FUNDAMENTAL SOLUTIONS FOR DIFFERENTIAL EQUATIONS
ASSOCIATED WITH THE NUMBER OPERATOR"

BY

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ABSTRACT. Let \((H, B)\) be an abstract Wiener pair and \(\mathcal{R}\) the operator defined by \(\mathcal{R}u(x) = -\text{trace}_B D^2 u(x) + (x, Du(x))\), where \(x \in B\) and \((\cdot, \cdot)\) denotes the \(B-B^*\) pairing. In this paper, we point out a mistake in the previous paper concerning the existence of fundamental solutions of \(\mathcal{R}^k\) and intend to make a correction. For this purpose, we study the fundamental solution of the operator \((\mathcal{R} + \lambda I)^k (\lambda > 0)\) and investigate its behavior as \(\lambda \to 0\). We show that there exists a family \(\{Q_\lambda(x, dy)\}\) of measures which serves as the fundamental solution of \((\mathcal{R} + \lambda I)^k\) and, for a suitable function \(f\), we prove that the solution of \(\mathcal{R}^k u = f\) can be represented by \(u(x) = \lim_{\lambda \to 0} \int_B f(y)Q_\lambda(x, dy) + C\), where \(C\) is a constant.

In our previous paper [2, §3], we have shown that the solution of the equation \(\mathcal{R}^k u(x)f(x) (f \in L_0)\) is of the form \(G_k f(x) + \text{a constant}\), where \(G f(x) = \int_0^\infty [\int_B f(y)Q_\lambda(x, dy)] dt\) and \(G^k f = G(G^{k-1} f)\) with \(G^1 f = f\). Viewing the representation of \(G_k f\), we then intuitively claimed that the family \(\{Q_\lambda(x, dy)\}\) of \(k\)-fold convolution of \(G(x, dy) = \int_0^\infty Q_\lambda(x, dy) dt\) forms rigorously the "fundamental solution" of \(\mathcal{R}^k\). Unfortunately, the "fundamental solution" is only formal. The mistake is caused by the fact that \(G_k f(x)\) may not equal \(\int_B f(y)Q_\lambda(x, dy)\) when \(f \in L_0\) (though \(G_k f(x) = \int_B f(y)Q(x, dy)\) for all \(f \geq 0\)). In order to obtain a correct representation of \(G_k f(x)\) by an integral with respect to certain measure, we study the fundamental solution of the differential operator \((\mathcal{R} + \lambda I)^k\), where \(\lambda > 0\), and then investigate its behavior as \(\lambda\) goes to zero. We show that the fundamental solution of \((\mathcal{R} + \lambda I)^k\) exists in the sense of measure, which means that there exists a family of measures, say \(\{Q_\lambda(x, dy)\}\), so that, for any member \(f\) of a certain reasonable large class of functions, the integral \(Q_\lambda f(x) = \int_B f(y)Q_\lambda(x, dy)\) exists and \((\mathcal{R} + \lambda I)^k(Q_\lambda f)(x) = f(x)\). As \(\lambda\) goes to zero, we show that \(\lim_{\lambda \to 0} \int_B f(y)Q_\lambda(x, dy) = G_k f(x)\) for any \(f \in L_0\).

DEFINITIONS AND NOTATION. We give in the following some new definitions and notations which did not appear in the previous paper. For the others, we refer the reader to [2].

For each \(x\) in \(B\) and for each Borel set \(A\) in \(B\), we define
\[
G_\lambda(x, A) = \int_0^\infty e^{-\lambda t}Q_\lambda(x, A) dt \quad (\lambda > 0),
\]
\[ R_\lambda(x, A) = \int_0^\infty e^{-\lambda t}[q_t(x, A) - p_1(A)] \, dt; \]

and let
\[ G_\lambda f(x) = \int_0^\infty \int_B e^{-\lambda t}f(y)o_t(x, dy) \, dt \quad \text{(if it exists)}, \]
\[ R_\lambda f(x) = \int_0^\infty \int_B e^{-\lambda t}f(y)[o_t(x, dy) - p_1(dy)] \, dt. \]

Evidently, \( G_\lambda f \) and \( R_\lambda f \) exist when \( f \) is bounded and continuous. Furthermore, we have

**Lemma 1.** (a) \( G_\lambda(x, \cdot) \) and \( R_\lambda(x, \cdot) \) are Borel measures with total variation \( \lambda^{-1} \) and \( 2\lambda^{-1} \), respectively.

(b) If \( f \in \mathcal{L} \), then \( R_\lambda f \in \mathcal{L} \) and \( G_\lambda f \in \mathcal{L} \); and, if \( f \in \mathcal{L}_0 \), then \( R_\lambda f(x) = G_\lambda f(x) \) and \( G_\lambda f \in \mathcal{L}_0 \).

(c) If \( f \in \mathcal{L} \), \( f \) is integrable with respect to \( R_\lambda(x, \cdot) \) and \( G_\lambda(x, \cdot) \). Moreover, we have:

\[ \mathbf{Proof.} \] (a) follows from the fact that \( o_t(x, \cdot) \) and \( p_1(\cdot) \) are mutually singular probability measures.

(b) follows by arguments similar to [2, Proposition 3.1].

It remains to prove (c). First of all, we observe that \( R_\lambda f(x) = G_\lambda f(x) - \lambda^{-1}\int_B f(y)p_1(dy) \) and \( R_\lambda(x, \cdot) = G_\lambda(x, \cdot) - \lambda^{-1}p_1(\cdot) \), so it suffices to verify (2).

Next, noting that if \( f \) is in \( \mathcal{L} \) then \( f^+, f^- \) and \( |f| \) are also in \( \mathcal{L} \); it suffices to prove that any nonnegative member \( f \) in \( \mathcal{L} \) is integrable with respect to \( G_\lambda(x, \cdot) \) and (2) holds. But, by the definition of \( G_\lambda(x, \cdot) \), it is easy to see that (2) holds when \( f \) is a simple function and so, by the monotone convergence theorem, (2) holds if \( f \) is a nonnegative function. Now the integrability of a nonnegative member in \( \mathcal{L} \) follows immediately from (b). □

**Proposition 1.** For each \( x \) in \( B \) and each Borel set \( E \) in \( B \), define
\[ Q_\lambda(x, E) = \int_B \ldots \int_B G_\lambda(y_{k-1}, E)G_\lambda(y_{k-2}, dy_{k-1}) \ldots G_\lambda(y_1, dy_2)G_\lambda(x, dy_1). \]

We have:

(a) The total variation of \( Q_\lambda(x, \cdot) \) is \( \lambda^{-k} \).

(b) \( \mathcal{L} \subset L^1(Q_\lambda(x, \cdot)) \) for each \( x \) in \( B \) and \( \lambda > 0 \) and
\[ G^k_\lambda f(x) = \int_B f(y)Q_\lambda(x, dy). \]

(c) If \( f \) is a function in \( \mathcal{L} \), then \( u(x) = G^k_\lambda f(x) \) satisfies the equation \( (\mathcal{K} + \lambda I)^k u = f \) (cf. [1]).
PROOF. (a) follows from Lemma 1(a).

(b) Using the same idea as in the proof of Lemma 1(c), we see that $f^+$ and $f^-$ are integrable with respect to $Q_k(x, dy)$ and $G^k f^+(x) = \int_B f^+(y) Q_k(x, dy)$, $G^k f^-(x) = \int_B f^-(y) Q_k(x, dy)$, which yield the identity (4).

Finally, imitating the proof of [2, Theorem 3.5], (c) follows immediately. □

REMARK. Proposition 1 shows that the fundamental solution of $(\mathcal{R} + \lambda I)^k$ exists in the sense of measure which is given by the family $\{Q_k(x, \cdot)\}$. □

PROPOSITION 2. Let $\{f_\lambda: \lambda \in R^+\}$ be a net of functions in $C$ satisfying the following conditions:

(C-1) There exist constants $c, c'$ such that

\[ |f_\lambda(x) - f_\lambda(y)| \leq c \cdot e^{c'\|x\|} e^{c'\|y\|} \|x - y\| \]

for all $x, y \in B$ and $\lambda \in R^+$.

(C-2) $\lim_{\lambda \to 0} f_\lambda(x) = f(x)$.

Then we have

(5) $\lim_{\lambda \to 0} R_\lambda f_\lambda(x) = \int_0^\infty \left[ o_1 f(x) - p_1 f(0) \right] dt$.

In particular, if $f \in C$, then $\lim_{\lambda \to 0} R_\lambda f(x) = Rf(x)$, where $Rf(x)$ is defined by the limit function of (5).

Proof. Write out the expression of $R_\lambda f_\lambda(x)$ and use Lebesgue’s dominated convergence theorem. □

COROLLARY 1. Assume $f \in C_0$. Then

\[ G^k f(x) = \lim_{\lambda \to 0} \int_B f(y) Q_k(x, dy). \]

Proof. Noting that the net $\{G_k f\}$ satisfies (C-1) and (C-2) of Proposition 2, the Corollary follows immediately. □

REMARK. To correct the previous paper, we should change properly all the statements concerning the fundamental solution of $\mathcal{R}^k$ according to the above results. In view of Corollary 1. Theorem 3.5(b) of [2] should read:

Assume $f$ is a function in $C_0$ and $Q_k(x, \cdot)$ is defined as in (3). Then $G^k f(x) = \lim_{\lambda \to 0} \int_B f(y) Q_k(x, dy)$ exists, $G^k f \in C(k)$, and $\mathcal{R}^k(G^k f)(x) = f(x)$. □

REMARK. It is not known so far if the fundamental solution of $\mathcal{R}^k$ exists in the sense of measure. When $k = 1$ and $f \in C_0$, we see that $p_1 f(0) = 0$ and

\[ Gf(x) = Rf(x) = \int_0^\infty \left[ o_1 f(x) - o_1 f(0) \right] dt = \int_0^\infty \int_B f(y) \left[ o_1(x, dy) - p_1(dy) \right] dt. \]

Since the last integral exists for all $f \in C$, one might conjecture that the the set function $R(x, A) = \int_0^\infty \left[ o_1(x, A) - p_1(A) \right] dt$ could define a measure and the family $\{R(x, A)\}$ might form the fundamental solution of $\mathcal{R}$. Unfortunately, if one takes $A = \text{the concentrated set of } p_1$, then $R(x, A) = -\infty$ and $R(x, A^n) = +\infty$, thus
$R(x, \cdot)$ fails to be a measure. From this observation, we conjecture that the fundamental solution of $\mathcal{A}$ does not exist in the sense of measure and neither does that of $\mathcal{A}^k$. However, a proof is lacking. □

REFERENCES


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