THE SPLITTING OF $BO(8) \wedge bo$ AND $MO(8) \wedge bo$

BY

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Abstract. Let $BO(8)$ denote the classifying space for vector bundles trivial on the 7-skeleton, and $MO(8)$ the associated Thom spectrum. It is proved that, localized at 2, $BO(8) \wedge bo$ and $MO(8) \wedge bo$ split as a wedge of familiar spectra closely related to $bo$, where $bo$ is the spectrum for connective $KO$-theory.

1. Introduction. Let $BO(8)$ denote the classifying space for vector bundles trivial on the 7-skeleton, and $MO(8)$ the associated Thom spectrum. Let $bo$ denote the spectrum for connective $ko$-theory, localized at 2, $bo^{(s)}$ the spectrum obtained from $bo$ by killing homotopy classes of Adams filtration less than $s$, and $bo^{(s)}[m]$ the spectrum obtained from $bo^{(s)}$ by killing homotopy classes of degree less then $m$. In this paper we prove

Theorem 1.1. There are equivalences of spectra, localized at 2,

$$BO(8) \wedge bo \simeq MO(8) \wedge bo \simeq K \vee \bigvee_{(U,V)} bo^{(2d+e)}[4d],$$

where $K$ is a wedge of Eilenberg-Mac Lane $K(Z_2)$-spectra, $U$ ranges over all nondecreasing sequences of integers $u \geq 2$ such that $u - 1$ is not an even 2-power, $V$ ranges over all increasing sequences of integers $v$ with $\alpha(v) = 2$, $d = |U| + |V|$ is the sum of the entries of $U$ and $V$, and $e = \sum_{v \in V} (2^{a(v)+1} - 1)$.

Here and throughout the paper $\alpha(v)$ is the number of 1's in the binary expansion of $v$, and $a(v)$ is the exponent of 2 in the prime factorization of $v$.

This splitting is quite similar to that of $bo \wedge bo$ given in [12]. Two important applications of the splitting of $bo \wedge bo$ have been made; analogues of these for $MO(8) \wedge bo$ certainly warrant investigation. The first is to $bo$-resolutions, useful in understanding $\pi_*(S^0)$ [12, 13] and in obstruction theory [6]. The second is in constructing operations $bo \rightarrow bo^{(1)}$ [14] and applying these to deduce restrictions on the $A$-modules which can arise as $H^*X$ [9]. Here and elsewhere $A$ denotes the mod 2 Steenrod algebra. Kane's applications to realizable $A$-modules were for odd primes and utilized the splitting of $bu \wedge bu$, but it seems quite likely that the operations $MO(8) \rightarrow bo^{(1)}$ derived from Theorem 1.1 may be chosen to have nice properties similar to his.

Received by the editors February 25, 1982.

1980 Mathematics Subject Classification. Primary 55P15, 55P42; Secondary 55N20, 55N22.

Key words and phrases. Splittings of spectra, connective $K$-theory, Brown-Gitler spectra, Thom spectra.

1This research was partially supported by a National Science Foundation research grant.
The compositions
\[ BO(8) \to BO(8) \wedge bo \to bo^{(2d+e)}[4d] \]
give rise to \(K\)-theory characteristic classes which may be useful in obstruction theory. In order that these be of much use one will have to show that the orientation \(BO(8) \to \Sigma P_N \wedge MO(8)\) of [5] factors through (1.2).

Another possible application of Theorem 1.1 is as a first step toward calculating the \(\langle 8\rangle\)-cobordism ring, \(\pi_q(MO(8))\) [8]. Since calculation by the classical Adams spectral sequence (ASS) has thus far been too complicated for any sort of general conclusions [8, 4], one might hope for more success from the \(bo\)-resolution, similar to that applied successfully to \(\pi_q(S^0)\) by Mahowald in [11 and 12]. The entire \(E_1\)-term can be written explicitly using Theorem 1.1 and results of [12] which we will also use in our proof of Theorem 1.1.

The proof of Theorem 1.1, which is given in §2, combines the splitting of \(H^*MO(8)\) as an \(A_1\)-module obtained in [4] with the method used for \(bo \wedge bo\) in [12] or [6]. The splitting of \(MO(8) \wedge bo\) encounters an obstruction which was not present for \(bo \wedge bo\); this is circumvented by using a property of Brown-Gitler spectra [2]. \(A_1\) above is a case of \(A_r\), the subalgebra of \(A\) generated by \(\{Sq^j: j \leq 2r\}\).

2. Proof of main theorem. In this section we prove Theorem 1.1 and make some minor corrections in some proofs of [6].

Since we will use many of the results and techniques of [6], we restate Theorem 1.1 in the terminology of that paper.

**Theorem 2.1.**

\[
BO(8) \wedge bo \cong MO(8) \wedge bo \cong K \bigvee_{U, V; d \text{ even}} \Sigma^4 bo^{(e)} \bigvee_{U, V; d \text{ odd}} \Sigma^4 bsp^{(e-1)},
\]

where \(K, U, V, d,\) and \(e\) are as in (1.2).

Here \(bsp\) is the 2-local connective \(\Omega\)-spectrum whose 0th space is \(Z \times BSp\). The equivalence of Theorems 1.1 and 2.1 follows from \(\Sigma^{8k} bo^{(z)} \cong bo^{(4k+z)}[8k]\) and \(\Sigma^4 bsp \cong bo^{(3)}[4]\).

In the splitting of \(bo \wedge bo\) in [12 and 6] certain Thom spectra \(B(n)\) were useful. There the pairings \(B(n) \wedge B(m) \to B(n + m)\) were used, but we shall also require maps \(S^1 \wedge_T (B(2^i) \wedge B(2^i)) \to B(2^{i+1})\). The existence of these maps follows easily from [2 and 3], since \(B(n)\) are closely related to Brown-Gitler spectra (see §3).

If \(\bar{n} = (n_1, \ldots, n_s)\), let \(|\bar{n}| = \Sigma n_i\), \(a(\bar{n}) = \Sigma a(n_i)\), and \(B(\bar{n}) = \wedge B(n_i)\). Then we have

**Lemma 2.2** [6, 3.9].

\[
B(\bar{n}) \wedge bo \cong K \bigvee \begin{cases} bo^{(2|\bar{n}| - a(\bar{n}))} & \text{if } |\bar{n}| \text{ even}, \\ bsp^{(2|\bar{n}| - 1 - a(\bar{n}))} & \text{if } |\bar{n}| \text{ odd}. \end{cases}
\]

2.1 follows from the following result, to be proved below.
Proposition 2.3. There is an equivalence mod $KZ_2$'s

\[
BO \langle 8 \rangle \wedge bo \simeq MO \langle 8 \rangle \wedge bo \simeq \left( \bigwedge_{k \geq 1, \ j \geq 0} S^{8kj} \right) \wedge \left( \bigwedge_{m \text{ odd}, \ a(m-1) \geq 1, \ j \geq 0} \Sigma^{4mj}Z_j \right) \wedge \bigwedge_{a(i) = 2} \left( Z_2 \vee \Sigma^4/(2^{i+1}) \right) \wedge bo,
\]

where $Z$ is a stable complex whose $A_1$-structure is

\[
\begin{array}{ccc}
-2 & -1 & 0 \\
1 & 2
\end{array}
\]

and $Z^j$ is its $j$-fold smash product. Here $(\uparrow \downarrow)$ denotes a stable $n$-cell, $\rightharpoonup$ denotes attaching map $\eta \in \pi_{n+1}(S^n)$, and $\rightharpoonup$ — denotes attaching map $2 \in \pi_n(S^n)$.

Proof that Proposition 2.3 implies Theorem 2.1. Let $\varnothing$, $\wedge$, and $\bar{C}$ be the three $\wedge$-products on the right-hand side of Proposition 2.3, i.e. $RHS = \varnothing \wedge \wedge \wedge \wedge bo$. Using Lemma 2.2, $\varnothing \wedge bo$ becomes

\[
\bigvee \Sigma^4/V . \begin{cases} bo^{(e)}, & |V| \text{ even}, \\ bsp^{(e-1)}, & |V| \text{ odd}, \end{cases}
\]

where $V$ and $e$ are as in Theorem 1.1. Using $Z \wedge Z \wedge bo \simeq K \wedge bo$, $\varnothing \wedge \wedge \wedge \wedge bo$ can be written

\[
K \wedge \bigwedge_{m \text{ not an even } 2\text{-power}} \bigvee_{m \geq 0} \Sigma^{4mj}Z^m \wedge bo = K \wedge \bigvee_{m \text{ not an even } 2\text{-power}} \Sigma^4/U Z^U \wedge bo.
\]

Thus RHS of Proposition 2.3 becomes

\[
\sum_{U, V} \Sigma^4/[U + |V|] . \begin{cases} bo^{(e)}, & |U| \text{ ev, } |V| \text{ ev,} \\ bsp^{(e-1)}, & |U| \text{ ev, } |V| \text{ od,} \\ Z \wedge bo^{(e)}, & |U| \text{ od, } |V| \text{ ev,} \\ Z \wedge bsp^{(e-1)}, & |U| \text{ od, } |V| \text{ od.} \end{cases}
\]

Since $Z \wedge bo^{(e)} = bsp^{(e-1)}$, this implies Theorem 2.1. There are no $KZ_2$'s on the LHS of Theorem 2.1, because the RHS without the $K$ contains none. \(\square\)

We write the $A_1$-splitting of $[4, 2.9]'$ in notation more convenient for this paper. Let $M_n$ denote the $A_1$-module with generators $m_{4i}$ for $0 \leq i \leq n-1$ and relations $Sq^1m_0$, $Sq^1m_{4i} + Sq^2Sq^3m_{4i-4}$ ($1 \leq i \leq n-1$), and $Sq^2Sq^3m_{4n-4}$. Thus $M_n = Q_{1,n-1}$ where $Q_{1,n-1}$ is as in [6, 3.11] and [4, Chapter 2]. $M_n$ is stably isomorphic to $H^\ast B(n)$ if $n$ is a 2-power [6, 3.12], and $M_n$ is stably isomorphic to $(\Sigma^{-1}I)^{2n-1} [4, 2.1]$ if $n$ is even. Let $J = H^\ast Z$ (of our Proposition 2.3) be as in [4, Chapter 2]. If $S$ is a set...
of modules, let \( P(\mathcal{S}) \) and \( E(\mathcal{S}) \) denote the polynomial and exterior algebras on \( \mathcal{S} \), i.e.

\[
P(\mathcal{S}) = \bigotimes_{M \in \mathcal{S}} \bigoplus_{j \geq 0} M^{\otimes j} \quad \text{and} \quad E(\mathcal{S}) = \bigotimes_{M \in \mathcal{S}} (\mathbb{Z}_2 \oplus M).
\]

(This use of \( P(\ ) \) differs from that in [4].) Then [4, 2.9'], modified as in the sentence which follows it, can be restated:

**Proposition 2.4.** \( H^*BO(\langle 8 \rangle) \) is stably \( A_1 \)-isomorphic to

\[
P\left( \left\{ \Sigma^2\mathbb{Z}_2 : i \geq 3 \right\} \right) \otimes P\left( \left\{ \Sigma^8\mathbb{Z}_2 : \alpha(i) > 1 \right\} \right)
\]

\[
\otimes P\left( \left\{ \Sigma^{8i}\mathbb{Z}_2 : \alpha(i - 1) > 1 \right\} \right)
\]

\[
\otimes E\left( \left\{ \Sigma^4\mathbb{Z}_2 : \alpha(i - 1) > 1 \right\} \right)
\]

\[
\otimes E\left( \{\Sigma^4\mathbb{Z}_2^{\otimes 0} : \alpha(1) = 2\} \right).
\]

If \( g_i \in H^i(BO(\langle 8 \rangle)) \) is the generator of [4, 2.4], then the initial classes of these summands are, respectively, \( g_2^1, g_4^1 + g_4^1, g_4^1 + 18g_4^1, g_2^1, g_2^1, \) and \( g_2^1. \)

Here we use that the Thom isomorphism \( H^*BO(\langle 8 \rangle) \to H^*MO(\langle 8 \rangle) \) is an \( A_1 \)-equivalence (in fact an \( A_2 \)-equivalence) to identify the two. The only properties of \( MO(\langle 8 \rangle) \) that we use in the remainder of the paper are its \( A_1 \)-cohomology and its multiplication, both of which are true of \( BO(\langle 8 \rangle) \), so that everything said for \( MO(\langle 8 \rangle) \) from now on can be said of \( BO(\langle 8 \rangle) \). Note that if the first two factors of Proposition 2.4 are combined and the third and fourth factors combined, the decomposition of Proposition 2.4 corresponds directly to Proposition 2.3.

From now on, we shall delete \( \mathbb{Z}_2 \times 0 \) from the second component of \( Ext(\ , \ ) \), and denote \( MO(\langle 8 \rangle) \wedge bo \) by \( Mb \). From Proposition 2.4 (or perhaps [4, 2.3] is slightly more convenient for this purpose) and [4, 2.2] one can easily write the chart for \( Ext_{A_1}(H^*MO(\langle 8 \rangle)) \approx Ext_{A}(H^*Mb) \). The chart begins as in the figure and continues in this fashion.

Here, and elsewhere, we use the ASS charts introduced in [1] and used extensively in papers such as [6 and 7]. A dot in \((x, y)\)-position \((t - s, s)\) corresponds to an element in a basis of \( Ext^{t-s}(H^*X) \), and if it survives the spectral sequence contributes to \( \pi_{t-s}(X) \). Vertical lines correspond to multiplication by 2 in \( \pi_{t-s}(X) \). Often, as in the chart above, we just indicate the vertical lines without indicating the dots which they connect.
There are no possible nonzero differentials in the ASS converging to \( \pi_\ast(Mb) \), and no exotic multiplications by 2. To see the latter, in degree \( d \equiv 0 \pmod{2 \eta} \) (4) an exotic multiplication can be avoided by rechoosing the generator, while in degree \( d \equiv 1 \pmod{2 \eta} \) (4) \( \pi_\ast(Mb) = 0 \). Filtration zero \( Z_2 \)'s due to free \( A_1 \)'s in \( H^\ast MO(\langle 8 \rangle) \), which are not pictured in the above chart, cannot extend into elements of filtration \( > 0 \) because the latter is acted on freely by \( P \in \pi_8 bo \). Finally, a filtration zero \( Z_2 \) such as the one in the chart in degree 26 satisfies \( P \cdot 2g_{4i+2} = 2 \eta y_{4i+9} = 0 \), so that \( 2g_{4i+2} = 0 \).

The first four factors of Proposition 2.4 multiply out to give a stable sum of \( \Sigma^8kZ_2 \)'s and \( \Sigma^{8i+4}J \)'s (since \( J \wedge J \approx Z_2 \), stably).

**Proposition 2.5.** For every \( \Sigma^8kZ_2 \) (resp. \( \Sigma^{8i+4}J \)) in the \( A_1 \)-splitting of \( H^\ast BO(\langle 8 \rangle) \) given by Proposition 2.4 (after multiplying out) there is a map \( S^8k \to Mb \) (resp. \( \Sigma^{8i+4}Z \to Mb \)) such that under the isomorphisms

\[
\text{Hom}_{A_1}^8(H^\ast Mb, Z_2) \cong \text{Hom}_{A_1}^8(H^\ast MO(\langle 8 \rangle), Z_2) \\
\cong \text{Hom}_{A_1}^8(\Sigma^8kZ_2, Z_2) \oplus \text{Hom}_{A_1}^8(C, Z_2),
\]

(resp. \( \text{Hom}_{A_1}^{8i+2}(H^\ast Mb, Z_2) \cong \text{Hom}_{A_1}^{8i+2}(\Sigma^{8i+4}Z, Z_2) \oplus \text{Hom}_{A_1}^{8i+2}(C', Z_2) \)), where \( C \) (resp. \( C' \)) is the complementary summand in the \( A_1 \)-splitting, \( [f] \) (resp. \( [f \circ i] \) where \( i : S^{8i+2} \to \Sigma^{8i+4}Z \) corresponds to \( \hat{g} \oplus 0 \), where \( \hat{g} \) is the nonzero element.

**Proof.** The element of \( \text{Ext}_{A_1}^{0,8k}(H^\ast Mb, Z_2) \cong \text{Hom}_{A_1}^8(H^\ast Mb, Z_2) \) corresponds under the ASS to the desired map \( S^8k \to Mb \). For \( \Sigma^{8i+4}J \), the element of \( \text{Ext}_{A_1}^{0,8i+2}(H^\ast Mb, Z_2) \) gives the map on the bottom cell of \( \Sigma^{8i+4}Z \). Since \( 2[f_{8i+2}] = 0 \), \( \pi_{8i+3}(Mb) = 0 \), and \( [\Sigma^{8i+4} \cup_2 e^{8i+5}, Mb] = 0 \) above filtration zero, there are no obstructions to extending this map over \( \Sigma^{8i+4}Z \).

For the last factor of Propositions 2.3 and 2.4 we have the following result, which sounds similar to Proposition 2.5, but is much more difficult to prove.

**Proposition 2.6.** For every \( l \) with \( \alpha(l) = 2 \) there is a map \( \Sigma^4lB(2^l) \to Mb \) such that \( f_l^\ast(g_4 \otimes 1) \neq 0 \) and \( f_l^\ast(g_l \otimes 1) = 0 \) if \( g_l \) is a product of two or more of the generators of \( H^\ast(BO(\langle 8 \rangle)) \) of \([4, 2.4]\).

Before proving Proposition 2.6, we use it and Proposition 2.5 to prove Proposition 2.3 (and hence Theorem 1.1).

**Proof of Proposition 2.3.** For any \( \Sigma^{8m+4}J \) (or similarly for \( \Sigma^{8k}Z_2 \)) occurring in the expansion of the first four factors of Proposition 2.4 and any finite set \( \{l_1, \ldots, l_r\} \) of integers with \( \alpha(l_j) = 2 \), we use the ring structure of \( Mb \) and the maps of Propositions 2.5 and 2.6 to form the map

\[
\Sigma^{8m+4}Z \wedge \Sigma^{4l_1}B(2^{l_1}) \wedge \cdots \wedge \Sigma^{4l_r}B(2^{l_r}) \to Mb \wedge Mb \wedge \cdots \wedge Mb \to Mb.
\]
We form the wedge of all such maps, apply $\wedge bo$, and follow by the map $Mb \wedge bo \to Mb$ to obtain a map from the RHS of Proposition 2.3 without the $K$ into $Mb$.

We thus have a map $F$ between two spectra whose homotopy charts are isomorphic except for filtration above zero $\mathbb{Z}_2$'s. We will show that the induced map of homotopy groups is surjective above filtration zero, which implies Proposition 2.3. We will use the action of $\pi_* bo$ on the homotopy groups of the spectra, which is valid because of the following lemma.

**Lemma 2.7.** If $X \wedge b \to Y \wedge b$ satisfies $F = (Y \wedge m_b)(f \wedge b)$ for some map $f$: $X \to Y \wedge b$, then $F_*(rx) = rF_*(x)$ if $r \in \pi_* b$ and $x \in \pi_*(X \wedge b)$.

**Proof.**

Thus for example, a map of this type between two charts of the form

\[
\begin{array}{ccc}
X \wedge b & \xrightarrow{f \wedge b} & Y \wedge b \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y \\
\end{array}
\]

which sends the bottom class across induces an isomorphism, by considering the action of the generators of $\pi_* bo$ on the generator of the first tower.

In considering the cohomology map of $F$, we must be somewhat careful because the product structure used in splitting $H^{*}MO(8)$ in Proposition 2.4 was the product in $H^{*}BO(8)$, while the product used in forming the maps was $MO(8) \wedge MO(8) \to MO(8)$, corresponding to the diagonal $\Delta$ in $H^{*}MO(8)$. There is some compatibility here, since the Thom isomorphism $\Phi$ is a coalgebra morphism, and the diagonal of $\mathbb{Z}^{*}BO(8)$ is an algebra morphism.

Referring to the calculation of $\pi_* Mb$ from $\text{Ext}_{A_1}(H^{*}MO(8))$ using Proposition 2.4, the summands of $\pi_* Mb$ due to $\mathbb{Z}_2$'s and $J$'s are in $\text{im}(F_*)$ by construction (Proposition 2.5), as are those due to $\Sigma^{4i} M_{2^{m}(l)}$ (by Proposition 2.6 and Lemma 2.7). For

\[
\Sigma^{4i} B(2^{l_i}) \wedge \cdots \wedge \Sigma^{4i} B(2^{l_i}) \xrightarrow{p} Mb \wedge \cdots \wedge Mb \xrightarrow{m} Mb
\]

\[
(mp)^*(\Phi(g_i) \otimes 1) = p^*(\Phi\Delta(g_i) \otimes 1) = \text{bottom class iff } g_{4i_1} \otimes \cdots \otimes g_{4i_t} \in \Delta_{g_{4i}},
\]

which is true if $g_{4i} = g_{4i_1} \cdots g_{4i_t}$ and perhaps for some shorter products, but not for any longer products. Since the homotopy charts due to the shorter products will...
already be accounted for in im $F_*$, we deduce that the first tower in the summand of $\pi_\ast M_b$ corresponding to

$$\Sigma^{4l_i}M_{2^{x_1(l)}} \otimes \cdots \otimes \Sigma^{4l_i}M_{2^{x_2(l)}}$$

is in im $F_*$, and utilizing Lemma 2.7 that this entire sequence of towers is in im $F_*$.

Finally we consider a summand

$$\Sigma^{8l+4}J \otimes \Sigma^{4l_i}M_{2^{x_1(l)}} \otimes \cdots \otimes \Sigma^{4l_i}M_{2^{x_2(l)}}$$

in $H^* M(O)_8$ from Proposition 2.4. (If $\Sigma^{8l+4}J$ is replaced by $\Sigma^{8k}Z_2$, a similar but easier argument applies.) The map $\Sigma^{8l+4}Z \to M_b$ sends $\Phi(g_7^2) \otimes 1$ to the class in $H^{8l+4}(\Sigma^{8l+4}Z)$, where $g_7$ is a product of an odd number of distinct $g_i$'s with $l \equiv 2 \mod 4$. (To see this for a basic $J$ in Proposition 2.4, $\Phi(g_7^2 + g_{2i-1}g_{2i+1}) \otimes 1 \mapsto \text{Sq}^2$ (bottom class), but $\Phi(g_{2i-1}g_{2i+1}) \otimes 1$ cannot map to it, because it is in $\text{im}(\text{Sq}^1)$.)

The element of $H^*(\Sigma^{8l+4}Z \wedge \Sigma^{4l_i}B(2^{x_1(l)}) \wedge \cdots \wedge \Sigma^{4l_i}B(2^{x_2(l)}))$

from which the first homotopy tower arises is

(middle class of $H^*(Z)) \otimes (\text{bottom classes of $B$'s}$).

Using $\Delta$ as in the preceding paragraph, the map

$mp: \Sigma^{8l+4}Z \wedge \Sigma^{4l_i}B(2^{x_1(l)}) \wedge \cdots \wedge \Sigma^{4l_i}B(2^{x_2(l)}) \to M_b$

sends $\Phi(g_7^2g_{4l_1} \cdots g_{4l_i}) \otimes 1$, but no longer products, to this “bottom class”. The argument of the last sentence of the preceding paragraph completes the argument.

It remains to prove Proposition 2.6. Throughout the rest of this section let $n = 2^{x'(l)}$. Most of our work will be to prove the last statement of the following result.

**Proposition 2.8.** If $f_i: \Sigma^{4l_i}B(n) \to M_b$ is as in Proposition 2.6, and $\psi$ is the composite

$$\Sigma^{4l_i}B(n) \wedge \Sigma^{4l_i}B(n) \wedge bo \xrightarrow{\psi} M_b \wedge bo \xrightarrow{M(O)_8 \wedge M_b} M_b,$$

then $\pi_{8l+8n-4}(\Sigma^{4l_i}B(n) \wedge \Sigma^{4l_i}B(n) \wedge bo) \approx Z,(2)$ with generator $\alpha_{8l+8n-4}$, and $\psi_\ast(\alpha_{8l+8n-4})$ is divisible by 2.

**Proof that Proposition 2.8 implies Proposition 2.6.** The proof is by induction on $x(l)$. The case $x(l) = 0$ of Proposition 2.6 is easily handled as in Proposition 2.5 since $B(1) = S^0 \cup e^2 \cup_2 e^3$ and $\pi_{8l+6}(M_b) = 0 = \pi_{8l+7}(M_b)$. We will show below that Proposition 2.8 implies that in the diagram

$$\begin{array}{c}
\text{fiber (} \phi \wedge bo \text{)} \\
\downarrow i \\
(2.9) \\
\Sigma^{4l_i}B(n) \wedge \Sigma^{4l_i}B(n) \wedge bo \xrightarrow{\psi} M_b \\
\downarrow \psi \wedge bo \\
\Sigma^{8l_i}B(2n) \wedge bo \\
\end{array}$$

fiber (} \phi \wedge bo \text{)

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If \( m' \) denotes the multiplication of \( \text{MO}(8) \), \( \Phi \) is the Thom isomorphism, and \( g_i \) is as in [4, 2.4], then \( m'((\Phi g_i)) = \Phi g_{4i} \otimes \Phi g_{4i} + \text{others if } I = 8l \).

**Proof.** As in the proof of Proposition 2.3, this cannot be true for a decomposable \( g_i \) since \( \Delta_{H^*BO(8)} \) preserves products. To see that this is true for \( g_{8l} \), note that

\[
g_{2^{a+2b}} = (1)\text{Sq}(2^a - 2^{b+1}, 2^b),
\]

so that it is equivalent to

\[
\Delta(\chi\text{Sq}(2^{a+1} - 2^{b+2}, 2^{b+1}))U = \chi\text{Sq}(2^a - 2^{b+1}, 2^b)U \otimes \chi\text{Sq}(2^a - 2^{b+1}, 2^b)U + \text{others}
\]

in \( H^*\text{MO}(8) \), and hence to a similar statement (without the \( U \)'s) in \( A//A_2 \). This is immediate from

\[
\Delta\chi\text{Sq}(R) = \sum_{S_1 + S_2 = R} \chi\text{Sq}(S_1) \otimes \chi\text{Sq}(S_2).
\]

The proof that \( \psi \circ i \neq 0 \) in (2.9) is similar to the proof of the case \( k = 2^i \) of [6, 3.15]. There are some minor errors in [6, 3.16 and 3.17]; we wrote a chart for \( \pi_*(B_{2^i} \wedge B_{2^i} \wedge bo) \) but called it a chart for \( \pi_*(B_{2^i} \wedge B_{2^i}) \). By making minor changes in the proof of [6, 3.16], mostly adding \( \wedge bo \), we obtain the following result (\( n = 2^{i(l)} \) here, \( n = 2^i \) for [6]).

**Lemma 2.11.** If \( n \) is a 2-power, let \( F_n = \Sigma^{8n-5} M_2 \wedge B(1) \). There is a map \( F_n \to B(n) \wedge B(n) \wedge bo \) such that the cofibre of the composite

\[
F_n \wedge bo \to B(n) \wedge B(n) \wedge bo \wedge bo \to B(n) \wedge B(n) \wedge bo
\]

is equivalent mod \( K(Z_2) \)'s to \( B(2n) \wedge bo \).

The corrected version of [6, 3.17] should read (in the notation of that paper):

[6, Lemma 3.17] If \( \theta \in [B_{2^i} \wedge B_{2^i} \wedge bo, bo \wedge bo] \) satisfies \( \theta*(\xi^2 \otimes 1) \wedge bo \), there exists \( q \) of filtration \( \geq i + 4 \) such that the following composite is inessential:

\[
F_i^1 \to F_i^1 \wedge bo \to \theta + q
\]

[6, 3.15] follows from this once we note that the cofibre of the composite \( c \),

\[
F_i \wedge bo \to B_{2^i} \wedge bo \to B_{2^i} \wedge bo \wedge bo \to B_{2^i} \wedge B_{2^i} \wedge bo
\]

is also equivalent mod \( K(Z_2) \)'s to \( B_{2^{i+1}} \wedge bo \), and \( [(\theta + q)c] = 0 \) so that \( \theta + q \) factors through \( B_{2^{i+1}} \wedge bo \).

The analogue of the corrected [6, 3.17] for \( M_b \) is not true. [6, 3.17] was proved by calculating an ASS in which there were no possible nonzero differentials; but the analogue for \( M_b \) has possible nonzero differentials.
Instead we use Proposition 2.8 to see that
\[ \Sigma^8 F_n^j \to \Sigma^4 B(n) \wedge \Sigma^4 B(n) \wedge \text{bo} \to Mb \]
is trivial on the bottom two cells, and hence on all of \( \Sigma^8 F_n \) since \( \pi_{8i+8n-\varepsilon}(Mb) = 0 \)
for \( 1 \leq \varepsilon \leq 3 \). If \( \wedge \text{bo} \) is applied to this composite, and we follow by \( Mb \wedge \text{bo} \to Mb \),
we obtain a trivial map homotopic to \( \Sigma^8 \) (the composite of Lemma 2.11) followed
by \( \psi \). Thus by Lemma 2.11 \( \psi \) factors as claimed in (2.9). \( \square \)

**Proof of Proposition 2.8.** The calculation of \( \pi_{8i+8n-4}(\Sigma^4 B(n) \wedge \Sigma^4 B(n) \wedge \text{bo}) \)
is easily performed as in [6, Chapter 3]. The possible difficulty in showing its image
under \( \psi \) divisible by 2 is exemplified below for \( l = 6 \).

\[
\begin{array}{c}
\pi_*(\Sigma^2 B(2) \wedge \Sigma^2 B(2) \wedge \text{bo}) \\
\pi_*\text{Mb}
\end{array}
\]

It is conceivable that \( \psi_*(g_{60}) \) might contain among its terms the filtration 3
generator and hence not be divisible by 2. When this possibility arose for \( \text{bo} \wedge \text{bo} \),
it was possible to vary \( f_i \wedge f_i \) to cancel this behavior, but here it is not possible,
essentially because the sequence of towers in \( \pi_*(\Sigma^2 B(2) \wedge \Sigma^2 B(2) \wedge \text{bo}) \)
begins earlier (in 48) than the sequence of towers in \( \pi_*(Mb) \) beginning in 56 which contains
the possible obstruction.

In order to prove the divisibility by 2, we consider the diagram
\[
\begin{array}{c}
\Sigma^4 B(n) \wedge \Sigma^4 B(n) \\
\downarrow \\
S^1 \wedge_T \Sigma^4 B(n) \wedge \Sigma^4 B(n)
\end{array}
\]
and show that \( (j \wedge \text{bo})_*(\alpha_{8l+8n-4}) \) is divisible by 2, plus perhaps an element of
order 2 of positive filtration. But an element of the latter type must map to 0 in
\( \pi_{8l+8n-4}(Mb) \), since it is torsion free above filtration 0. The factorizability of
\( B(n) \wedge B(n) \to B(2n) \) through \( j \) is in §3. The existence of \( G \) follows from (i) \( j \) is the
cofibre of \( 1 - T: \Sigma^4 B(n) \wedge \Sigma^4 B(n) \to \Sigma^4 B(n) \wedge \Sigma^4 B(n) \)
and (ii) \( m'(f_i \wedge f_i) \approx m'(f_i \wedge f_i)T \).

We begin to calculate \( \pi_*(S^1 \wedge_T \Sigma^4 B(n) \wedge \Sigma^4 B(n) \wedge \text{bo}) \). There is a short
exact sequence of \( \mathcal{A} \)-modules
\[
0 \to \Sigma \text{coker}(1 + T) \to H^*(S^1 \wedge_T \Sigma^4 B(n) \wedge \Sigma^4 B(n)) \to \text{ker}(1 + T) \to 0
\]
where \( 1 + T: H^*(\Sigma^4 B(n) \wedge \Sigma^4 B(n)) \to \).
By \([6, 3.12]\) \( H^*B(n) \approx Q_{1,n-1} \oplus F \) as \( \mathcal{A}_1 \)-modules, where \( Q_{1,n-1} \) is as in \([6, 3.11]\)
and \( F \) is a free \( \mathcal{A}_1 \)-module. When there is a splitting of \( \mathcal{A}_1 \)-modules \( H \approx Q \oplus F \), the morphism \( 1 + T: H \otimes H \to \)
spits into components on \( Q \otimes Q, F \otimes F \), and \( (Q \otimes F) \oplus (F \otimes Q) \). The following calculation
will be presented later.
Lemma 2.12. There are isomorphisms of stable $A_1$-modules

(i) $\ker(1 + T: Q_{1,m} \otimes Q_{1,m} \leftrightarrow) \approx Q_{1,2m+1}^U \oplus \bigoplus_{i=1}^m \Sigma^{8i+2}Z_2 \oplus \bigoplus_{i=1}^m \Sigma^{8i-2}J$,

$\coker(1 + T: Q_{1,m} \otimes Q_{1,m} \leftrightarrow) \approx \bigoplus_{i=0}^m \Sigma^{8i}Z_2 \oplus \bigoplus_{i=0}^m \Sigma^{8i+4}J$;

(ii) $\ker(1 + T: A_1 \otimes A_1 \leftrightarrow) \approx (\Sigma^2 \oplus \Sigma^6)Q_1^U$,

$\coker(1 + T: A_1 \otimes A_1 \leftrightarrow) \approx (\Sigma^6 \oplus \Sigma^{10})DQ_1^U$

where $Q_1^U$ is $Q_{1,n}$ without the top $(4n + 3)$ dimensional class, and $DQ_1^U$ is the module dual to $Q_1^U$ ($DQ_1^U$ begins in degree $-6$).

Because 2.12(ii) implies that the $A_1$'s in $H^*B(n)$ give rise to significant homotopy of $S^1 \times_T B(n) \wedge B(n) \wedge b_0$, we must be careful to show that they cannot affect us. Let $B = \Sigma^4B(n)$ and $b = b_0$. There is a map $X \vee F \to B \wedge b$ such that $(B \wedge m_b)(g \wedge b)$ is an equivalence $(X \vee F) \wedge b \to B \wedge b$ where $H^*X \approx \Sigma^i Q_{1,n-1}$ and $H^*F$ is a free $A_1$-module. ($X$ can be taken to be $\Sigma^4(\ast \cup \chi CP_{n+2}^1)$ and the map constructed as in [7, 2.1].) The composite $\theta$,

$$(X \vee F) \wedge (X \vee F) \wedge b \to (B \wedge b) \wedge (B \wedge b) \wedge b \to B \wedge B \wedge b,$$

is an equivalence and the following diagram commutes:

$$
\begin{array}{ccc}
(X \vee F) \wedge (X \vee F) \wedge b & \to & B \wedge B \wedge b \\
\downarrow (1 - T) \wedge b & & \downarrow (1 - T) \wedge b \\
(X \vee F) \wedge (X \vee F) \wedge b & \to & B \wedge B \wedge b
\end{array}
$$

Thus so does the map of cofibres:

$$
\begin{array}{ccc}
(X \wedge X \wedge b) \vee (F \wedge F \wedge b) \vee ((X \wedge F \vee F \wedge X) \wedge b) & \to & B \wedge B \wedge b \\
\downarrow j^* \wedge b & & \downarrow j \wedge b \\
(S^1 \times_T X \wedge X \wedge b) \vee (S^1 \times_T F \wedge F \wedge b) \vee W \wedge b & \vee & S^1 \times_T (B \wedge B) \wedge b
\end{array}
$$

where $W = \text{cof}(1 - T: X \wedge F \vee F \wedge X \leftrightarrow)$. Since the generator $\alpha$ of $\pi_{8l+8n-4}(B \wedge B \wedge b)$ comes from that of $X \wedge X \wedge b$, the divisibility by 2 of $(j \wedge b)_*\alpha$ follows from that of $(j_x^* \wedge b)_*\alpha$ in $\pi_{8l}^*(S^1 \times_T X \wedge X \wedge b)$.

There is a long exact sequence

$$
\rightarrow \text{Ext}_{A_1}^{i}(K) \rightarrow \text{Ext}_{A_1}^{i}(H^*(S^1 \times_T X \wedge X)) \rightarrow \text{Ext}_{A_1}^{i}(\Sigma C) \rightarrow \text{Ext}_{A_1}^{i+1}(K) \rightarrow \nabla
$$

where by Lemma 2.12 there are stable $A_1$-isomorphisms

$$
K \approx \Sigma^{8l} \left( Q_{2n-1}^U \oplus \bigoplus_{i=1}^{n-1} \Sigma^{8i+2}Z_2 \oplus \bigoplus_{i=1}^{n-1} \Sigma^{8i-2}J \right),
$$

$$
C \approx \Sigma^{8l} \left( \bigoplus_{i=0}^{n-1} \Sigma^{8i}Z_2 \oplus \bigoplus_{i=0}^{n-1} \Sigma^{8i+4}J \right).$$
Thus $\text{Ext}_A(H^*(S^1 \times_T X \wedge X))$ above filtration zero is a sum of the following three types of charts with possible $d_i$-differentials from the third type to either of the first two. These charts also depict $\pi_*(S^1 \times_T X \wedge X \wedge bo)$, with the possibility of $d_r$-differentials for $r \geq 1$ from the third type.

\[ \begin{array}{c}
0 & 4 & 8n-4 & 8n+4 & 8n+8 \vspace{1cm} \\
\text{\vdots} & \text{\vdots} & \text{\vdots} & \text{\vdots} & \text{\vdots} \\
\end{array} \]

In particular $\pi_{8l+8n-4}(S^1 \times_T X \wedge X \wedge bo)$ is $\mathbb{Z}_2(2)$ or $\mathbb{Z}/2^i$ for some $i \geq 2$ plus high filtration $\mathbb{Z}_2$'s in $\text{im}(\pi^2)$. It is easily verified that

$\text{Ext}_A(H^*(X \wedge X)) \to \text{Ext}_A(H^*(S^1 \times_T X \wedge X))$

maps onto towers in $t - s \equiv 0 \pmod{4}$, $t - s < 8l + 8n$. Thus $\pi_{8l+8n-4}(X \wedge X \wedge b) \to \pi_{8l+8n-4}(S^1 \times_T X \wedge X \wedge b)$ sends generator to 2-generator plus perhaps $\mathbb{Z}_2$'s in positive filtration. As noted earlier in the proof, this implies $\pi_{8l+8n-4}(X \wedge X \wedge b) \to \pi_{8l+8n-4}(Mb)$ has image divisible by 2, completing the proof of 2.8 (modulo Lemma 2.12). \[ \square \]

**Proof of Lemma 2.12.** (ii) is omitted since it is a single calculation, and it is not used in this paper. Let $K = \ker(1 + T)$. Then $K$ has as basis elements $(i, j)$ for $i < j$, $\{i, j\} \subseteq \{k: 0 \leq k \leq 4m + 3, k \neq 1\}$, where $(i, i) = x_i \otimes x_i$, and if $i < j$, $(i, j) = x_i \otimes x_j + x_j \otimes x_i$. Let $S_d$ denote the sum of all elements $(i, j)$ of degree $d$. These elements $S_d$ form an $A_1$-submodule of $K$ isomorphic to $\mathbb{Q}_0$. Sending $\Sigma^{8i-2} \mathbb{Z}_2$ to $(4i + 1, 4i + 1)$ and $\Sigma^{8i-2} J$ to the $A_1$-submodule generated by $(4i - 2, 4i - 2)$ gives a monomorphism

$Q_{2m+1} \oplus \bigoplus_{i=1}^{m} \Sigma^{8i+2} \mathbb{Z}_2 \oplus \bigoplus_{i=1}^{m} \Sigma^{8i-2} J \to K$

which induces an isomorphism in $Q_0$- and $Q_1$-homology and hence is a stable $A_1$-isomorphism by [15]. The isomorphism in $Q_r$-homology follows from $H_*(K; Q_0) = \langle (i, i): i = 0 \text{ or } i \text{ odd} \rangle$ and $H_*(K; Q_1) = \langle (i, i): i = 4m + 2 \text{ or } i \text{ odd} \rangle$. This is
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seen by splitting $K$ as a $Q_0$-module and then as a $Q_1$-module. For example as a $Q_1$-module it is a free module on

$$\{(i, j): i \text{ even}, i \leq 4n, j \geq i, j \notin \{i + 1, i + 3\}\}$$

plus a trivial module on $\{(i, i): i = 4n + 2 \text{ or } i \text{ odd}\}$. A similar argument works for coker, with elements $\langle a, b \rangle, a \leq b$, representing the equivalence class $x_a \otimes x_b \sim x_b \otimes x_a$. The $Z_2$'s are $\langle 4i, 4i \rangle + \langle 4i - 1, 4i + 1 \rangle$ and the $J$'s are generated by $\langle 4i, 4i + 2 \rangle$. □

It seemed reasonable to try to deduce Theorem 2.1 from Proposition 2.4 without using any specific knowledge of the cohomology classes involved in the splitting, i.e. to avoid the need for the maps $B(n) \wedge B(m) \to B(n + m)$. The author attempted to mimic the cellular method of [7, Chapter 2] to prove that if $H^*X$ is stably $A_1$-isomorphic to

$$\bigoplus \Sigma^{8n}(\Sigma^{-1})^n \otimes \bigoplus \Sigma^{8m} + 4(\Sigma^{-1})^I J,$$

then $X \wedge bo \cong K \vee \Sigma^{8n}bo(e) \vee \Sigma^{8m} + 4bo(e)$. An obstruction was encountered, which led to the following example.

**Example 2.13.** There is a spectrum $Y$ with $H^*Y \approx H^*((S^0 \cup \wedge CP^{14}) \vee S^8)$ as $A$-modules, but $Y \wedge bo \neq (S^0 \cup \wedge CP^{14}) \wedge bo \vee \Sigma^3bo (= bo^7) \vee \Sigma^3bo$.

**Proof.** Let $W = D(S^0 \cup \wedge CP^{14})$ denote the Spanier-Whitehead dual of the mapping cone. Then $\pi_1W \approx \pi_1F_{-15}$ for $i \leq -2$. This is given in [10, p. 54] by

Let $f: S^{-9} \to W$ be a non trivial map of filtration 3. Then $Y = D(W \cup f e^{-8})$ is the desired counterexample, for if $Y \wedge bo$ splits, there is a degree 1 map $Y \to S^8 \wedge bo$, and hence a filtration zero element in $\pi_{-6}(DY \wedge bo)$. But a chart for $DY \wedge bo$ in this range is

3. **Properties of $B(n)$**. We recall from [6 or 12] the definition of $B(n)$. Let $W = \text{fibre}(\Omega^2S^3 \to S^1)$ and $F_n(W)$ be the filtration induced by the May filtration on $\Omega^2S^3$. Then $B(n)$ is the Thom spectrum of

$$F_{2n}(W) \cong F_{2n}(\Omega^2S^3) \to \Omega^2S^3 \Omega^2g \to BO.$$

The multiplication $\Omega^2S^3 \times \Omega^2S^3 \to \Omega^2S^3$ sends $F_{2n} \times F_{2m} \to F_{2n+2m}$ and $W \times W \to W$ (since $m$ covers $S^1 \times S^1 \to S^1$). Thomifying, one obtains $B(n) \wedge B(m) \to B(n + m)$. 

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In [2 and 3] it was noted that May's operad structure gives maps $S^1 \ltimes_T F_{2n}(\Omega^2 S^3) \times F_{2n}(\Omega^2 S^3) \to F_{4n}(\Omega^2 S^3)$. This induces maps $S^1 \ltimes_T F_{2n}(W) \times F_{2n}(W) \to F_{4n}(W)$ because the composite

$$S^1 \ltimes_T F_{2n}(W)^2 \to S^1 \ltimes_T F_{2n}(\Omega^2 S^3)^2 \to \Omega^2 S^3 \to S^1 = K(\mathbb{Z}, 1)$$

is trivial. Results of [3] imply that the induced map of Thom spaces is $S^1 \ltimes_T B(n) \wedge B(n) \to B(2n)$ which has the desired effect in cohomology if $n$ is a 2-power.

**References**


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