FINITELY GENERIC ABELIAN LATTICE-ORDERED GROUPS

BY

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ABSTRACT. The authors characterize the finitely generic abelian lattice-ordered groups and make application of this characterization to specific examples.

A key goal in Abraham Robinson's development of model-theoretic forcing was to explicate the notion of algebraically closed, even when the appropriate classes may not be first-order axiomatizable. Interesting links sometimes appear between purely algebraic properties and model-theoretic properties such as existentially closed (e.c.) and finitely generic. In this spirit we consider the characterization of finitely generic abelian l-groups, as well as the model-theoretic properties of certain e.c. abelian l-groups.

The model theory of abelian lattice-ordered (l-) groups was developed by Glass and Pierce in [G-P] and [G-P2]. They showed that every finitely generic structure is hyperarchimedean, and that the group \( \overline{C}(X, R) \) is existentially closed. They also stated several problems, including:

(i) Distinguish the finitely generic models among the hyperarchimedean e.c. ones [G-P, p. 263].
(ii) Show \( \overline{C}(X, R) \) is finitely generic [G-P2, p. 146].
(iii) Give an example of a finitely generic e.c. group not representable as a group of real-valued functions with finite range. (Note that an incorrect reading of 14.1.7 of [B-K-W] led to an incorrect statement of this on p. 269 of [G-P].)
(iv) Give an example of an archimedean e.c. model which is not hyperarchimedean [G-P, p. 263].

In the present paper we solve (i)–(iii); an example for (iv) appears in a separate note [S-W].

For (i) we first show that \( \mathcal{C} = \overline{C}(X, Q) \) is the unique prime e.c. model, and hence that \( \mathcal{C} \) is finitely generic. We next employ the representation of hyperarchimedean l-groups in [B-K-W], together with a fixed embedding of \( \mathcal{C} \) into a suitable hyperarchimedean group, to show that any hyperarchimedean e.c. model is finitely generic. In fact, the finitely generic models are axiomatizable among the e.c. models. As a consequence, we have that \( \overline{C}(X, A) \) is finitely generic for any divisible subgroup \( A \) of the additive reals; (ii) is a special case of this, for \( A = R \). It follows also that there are \( 2^{2^{\aleph_0}} \) countable nonisomorphic finitely generic abelian l-groups.

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From the axioms for finitely generic models among e.c. models we show that any hyperarchimedean group can be extended to a finitely generic one, which settles (iii). In addition, we give a specific construction of a group for (iii) which is $\mathbb{Q}$-generated over $\mathfrak{C}$ by two elements.

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0. Preliminaries.

0.1. $l$-groups. The language $L$ for abelian lattice ordered ($l$-) groups is chosen here with one constant “0” (zero), one unary function “$-$” (minus), and three binary functions “$+$” (addition), “$\wedge$” (meet) and “$\vee$” (join). Note that $x \leq y$ is not a formula of $L$, but is definable, for example, from $\wedge$ via $x \wedge y = x$. (Warning: the symbols $\wedge$ and $\vee$ play dual roles here, as meet/and and join/or, with context the only clue.) An atomic formula of $L$ can be written as $\tau(\bar{x}) = 0$ where $\tau$ is a term built up from 0 and variables $\bar{x} = x_1, \ldots, x_n$ using $-, +, \wedge$ and $\vee$. For an $l$-group $\mathfrak{A}$ we denote the positive cone by $\mathfrak{A}_+ = \{a \in \mathfrak{A} \mid a > 0\}$. A positive element of $\mathfrak{A}$ is thus a nonzero element of $\mathfrak{A}_+$.

Definitions. Let $\mathfrak{A}$ be an abelian $l$-group. We say $\mathfrak{A}$ is existentially closed (e.c.) provided for any $\bar{a} \in \mathfrak{A}$, any finite set of atomic and negated atomic formulae $\tau_1(\bar{x}, \bar{a}) = 0, \ldots, \tau_m(\bar{x}, \bar{a}) = 0, \tau_{m+1}(\bar{x}, \bar{a}) \neq 0, \ldots, \tau_n(\bar{x}, \bar{a}) \neq 0$ has a solution for $\bar{x}$ in $\mathfrak{A}$ whenever it has a solution in some abelian $l$-group containing $\mathfrak{A}$.

$\mathfrak{A}$ is archimedean provided for any $a, b \in \mathfrak{A}$ with $a > 0$ there is an integer $n$ such that $na \leq b$. Further, $\mathfrak{A}$ is hyperarchimedean provided all homomorphic images of $\mathfrak{A}$ are archimedean; equivalently, provided every proper prime subgroup of $\mathfrak{A}$ is maximal.

The divisible hull of $\mathfrak{A}$ is the unique divisible $l$-group containing $\mathfrak{A}$ and generated as a module over the rationals by $\mathfrak{A}_+$.

Remark. Recall that multiplication by a positive integer produces an automorphism of any divisible abelian $l$-group. Thus, in dealing with divisible groups, we often allow rational coefficients in purported atomic $F$-formulae, since any such “formula” has an honest equivalent in $L$.

0.2. Totally ordered groups. The theory of divisible totally-ordered abelian groups, for convenience taken here in $L$, admits elimination of quantifiers. Two models of this theory are of special interest here: the reals ($\mathbb{R}$) and the rationals ($\mathbb{Q}$). In §3 we use the following property of the embedding of $\mathbb{Q}$ in $\mathbb{R}$ as $L$-structures.

Rectangle Lemma. Let $\varphi$ be a formula of $L$, $r_1, \ldots, r_t \in \mathbb{R}_+$ such that $\{r_1, \ldots, r_t\}$ is linearly independent over $\mathbb{Q}$, and $\mathbb{R} \models \varphi(r_1, \ldots, r_t)$. Then there exist rationals $q_{1,1}, \ldots, q_{1,t}$, $q_{2,1}, \ldots, q_{2,t}$ such that $r_1q_{j,1} < r_j < r_1q_{j,2}$ for $2 \leq j \leq t$, and such that for all $s_1, \ldots, s_t \in \mathbb{R}$ with $s_1 > 0$ and with $s_1q_{j,1} < s_j < s_1q_{j,2}$ for $2 \leq j \leq t$, it follows that $\mathbb{R} \models \varphi(s_1, \ldots, s_t)$.

Proof. Let $r_1, \ldots, r_t$, $\varphi$ be as in the hypothesis with $\mathbb{R} \models \varphi(r_1, \ldots, r_t)$. Without loss of generality we consider only the situation when $r_1 = 1$ and, moreover, find conditions only for $s_1 = 1$, since multiplication by a positive real determines an
automorphism of \( \mathbb{R} \). By quantifier-elimination we have that \( \varphi \) is equivalent to a formula in disjunctive normal form, some disjunct—say \( \theta \)—of which holds at \( 1, r_2, \ldots, r_t \). (Here \( \theta \) is a conjunction of atomic and negated atomic formulae.) It is now convenient to rewrite \( \theta \) without meets and joins, and so we choose \( q_{j,1}', q_{j,2}' \) so that \( q_{j,1}' < r_j < q_{j,2}' \), \( 2 \leq j \leq t \), and so that for all \( s_2, \ldots, s_t \) satisfying \( q_{j,1}' < s_j < q_{j,2}' \), \( 2 \leq j \leq t \), the ordering of all terms occurring in \( \theta \) is the same for \( s_2, \ldots, s_t \) as it is for \( r_2, \ldots, r_t \). Next we replace terms of the form \( \tau_1 \land \tau_2 \) and \( \tau_1 \lor \tau_2 \) by the appropriate term \( \tau_1 \) according to the ordering at \( 1, r_2, \ldots, r_t \). The resulting formula \( \theta' \) is a conjunction of inequalities of the form \( \tau(1, y_2, \ldots, y_t) \neq 0 \), since all equations must be trivial by the independence of \( 1, r_2, \ldots, r_t \). Now these inequalities hold on an open subset of \( \mathbb{R}^{t-1} \), hence there exist \( q_{j,1}, q_{j,2} \) with \( q_{j,1}' \leq q_{j,1} < q_{j,2} \leq q_{j,2}' \) such that \( \tau(1, s_2, \ldots, s_t) \neq 0 \) for \( s_2, \ldots, s_t \) satisfying \( q_{j,1}' < s_j < q_{j,2}' \), \( 2 \leq j \leq t \). Thus \( \mathbb{R} \models \theta'(1, s_2, \ldots, s_t) \) for such \( s_2, \ldots, s_t \), and so \( \mathbb{R} \models \varphi(1, s_2, \ldots, s_t) \) as well.

1. The smallest finitely generic structure. Throughout this paper let \( X = \bigcup_{n \in \omega} X_n \) be a disjoint union of countably many Cantor sets, let \( Q \) be the additive rationals with discrete topology, and let \( \mathcal{C} = \mathcal{C}(X, Q) \) be the \( l \)-group of all continuous functions from \( X \) to \( Q \) with compact support. (Here the group operation is pointwise addition, and sups and infs are pointwise maxima and minima.) Note that \( \mathcal{C} \) is countable, abelian, and that each element of \( \mathcal{C} \) has finite range: see [G-P] for further details. For \( c \in \mathcal{C} \) we denote the support of \( c \) by \( \text{supp}(c) = \{ x \in X | c(x) \neq 0 \} \). The goal of this section is to show that \( \mathcal{C} \) is the unique prime c.c. abelian \( l \)-group, hence is finitely generic. To do this we make use of the very pleasant nature of the structure of \( \mathcal{C} \).

**Lemma 1.1.** Given \( X = Y_1 \cup \cdots \cup Y_n \cup Y = Y'_1 \cup \cdots \cup Y'_n \cup Y' \), where each \( Y_i \), \( Y'_i \) is nonempty compact open, then there is a homeomorphism from \( X \) onto itself sending \( Y_i \) onto \( Y'_i \).

**Proof.** Each \( Y_i \) and \( Y'_i \) is a Cantor set, and \( Y \) and \( Y' \) are countable unions of Cantor sets.

**Lemma 1.2.** Let \( s_1, \ldots, s_n, s'_1, \ldots, s'_n \in \mathcal{C}^+ \), all nonzero, such that \( s_i \land s_j = s'_i \land s'_j = 0 \) for all \( i \neq j \). Then there is an automorphism \( \sigma \) of \( \mathcal{C} \) such that \( \sigma(s_i) = s'_i \), \( i = 1, \ldots, n \).

**Proof.** Let \( Y_i = \text{supp}(s_i), Y'_i = \text{supp}(s'_i) \). Then each \( Y_i, Y'_i \) is nonempty compact open, and \( Y_i \cap Y_j = Y'_i \cap Y'_j = \emptyset \) for \( i \neq j \). This gives by 1.1 a homeomorphism \( h \) of \( X \) mapping \( Y_i \) onto \( Y'_i \). For \( c \in \mathcal{C} \) define \( \sigma(c) \) as follows:

\[
\sigma(c)(x) = \begin{cases} 
  c(h^{-1}(x)) \cdot s'_i(x)/s_i(h^{-1}(x)), & x \in \text{supp} s'_i, \\
  c(h^{-1}(x)), & x \notin \bigcup_{i=1}^n \text{supp} s'_i.
\end{cases}
\]

Then \( \sigma \) is the desired automorphism of \( \mathcal{C} \).
Lemma 1.3. For any $c_1, \ldots, c_k \in \mathcal{C}$ there exist $s_1, \ldots, s_n \in \mathcal{C}^+$, the $s_i$'s pairwise disjoint and nonzero, and there exist rationals $q_{ij}$, $i = 1, \ldots, k$, $j = 1, \ldots, n$, such that $c_i = \sum_{j=1}^{n} q_{ij} s_j$, $i = 1, \ldots, k$. Moreover, if $c_1', \ldots, c_k' \in \mathcal{C}$ and $s_1', \ldots, s_n' \in \mathcal{C}^+$, the $s_i$'s pairwise disjoint and nonzero, such that $c_i' = \sum_{j=1}^{n} q_{ij} s_j'$, $i = 1, \ldots, k$, then there is an automorphism of $\mathcal{C}$ mapping $c_i$ to $c_i'$, $i = 1, \ldots, k$.

Proof. Given $c_1, \ldots, c_k$, we indicate one reasonable choice of $s_i$'s and $q_{ij}$'s. For $c_1 = \cdots = c_k = 0$ take $n = 1$, $s_1$ any nonzero element of $\mathcal{C}^+$, and $q_{i1} = 0$, $i = 1, \ldots, k$. Otherwise write $\bigcup_{i=1}^{k} \text{supp } c_i = \bigcup_{j=1}^{n} Y_j$ where each $Y_j$ is compact open nonempty, and where $c_i \upharpoonright Y_j$ is constant for all $i$ and $j$. Then let

$$s_j(x) = \begin{cases} 1, & x \in Y_j, \\ 0, & \text{otherwise}, \end{cases}$$

and let $q_{ij} = c_i(x)$ for any (all) $x \in Y_j$.

The second part of the lemma is immediate from 1.2.

Remark. Given $c_1, \ldots, c_k \in \mathcal{C}$ we can in fact define some choice of $s_1, \ldots, s_n$ in a quantifier-free manner and thus argue that the $k$-type of $c_1, \ldots, c_k$ in $\mathcal{C}$ is principal and is given by a quantifier-free formula, although we never make full use of this. Also, we note that 1.3 tells us that $\mathcal{C}$ is $\omega$-homogeneous.

Lemma 1.4. $\mathcal{C}$ is an existentially closed abelian $l$-group.

Proof. The proof is exactly that given by Glass and Pierce (Theorem 8, p. 261 of [G-P]) for the existential closedness of $\overline{C}(X, \mathbb{R})$, since $\mathbb{R}$ can be replaced there by any divisible totally-ordered abelian group—in the present case by $\mathbb{Q}$.

We collect here some easy facts about e.c. abelian $l$-groups.

Lemma 1.5. Let $\mathcal{D}$ be an e.c. abelian $l$-group. Then:

(i) For $u \in \mathcal{D}$ with $u > 0$ and $n > 0$ there exist $u_1, \ldots, u_n$ pairwise disjoint positive elements of $\mathcal{D}$ with $u = u_1 + \cdots + u_n$.

(ii) Any maximal set of pairwise disjoint positive elements of $\mathcal{D}$ is infinite.

(iii) $\mathcal{D}$ is divisible.

Proof. This is easily checked directly, but we can also use the expressibility of all the assertions by $\forall \exists$ sentences true in $\mathcal{C}$, hence true in any e.c. $\mathcal{D}$. For example, for (ii) we note that

$$\mathcal{C} \models \forall x_1 \cdots \forall x_n \exists y \left( y > 0 \land \bigwedge_{i=1}^{n} (x_i \land y) = 0 \right)$$

which gives (ii).

Theorem 1.6. The group $\mathcal{C}$ is the unique prime e.c. abelian $l$-group.

Proof. Let $\mathcal{D}$ be e.c. and let $(f_n)_{n \in \omega}$ be any infinite set of pairwise disjoint positive elements of $\mathcal{D}$ (which exists by 1.5(ii)), and let $(e_n)_{n \in \omega} \subseteq \mathcal{C}$ be the set of elements defined by

$$e_n(x) = \begin{cases} 1, & x \in X_n, \\ 0, & \text{otherwise}. \end{cases}$$
Map $e_n$ to $f_n$, $n \in \omega$, to form a partial isomorphism of $\mathcal{C}$ into $\mathcal{D}$. Extend this map to the domain of all characteristic functions of compact open sets by listing all such elements of $\mathcal{C}$ and choosing images, one-by-one, in $\mathcal{D}$ via 1.5(i). The resulting map extends uniquely to the divisible hull, using 1.5(iii), which is all of $\mathcal{C}$. Thus $\mathcal{C}$ is prime.

In case $\mathcal{D}$ is a substructure of $\mathcal{C}$ we can first choose $\{f_n\}_{n \in \omega}$ maximal, and then, by working back and forth as above, we get an isomorphism between $\mathcal{C}$ and $\mathcal{D}$. Thus $\mathcal{C}$ is unique.

Note that $\mathcal{C}$ is not minimal. If we write $X = Y \cup Z$, where $Y$ and $Z$ are each homeomorphic to $X$, then $\mathcal{C}(Y, \mathbb{Q}) \subseteq \mathcal{C}$ but $\mathcal{C}(Y, \mathbb{Q}) \not\approx \mathcal{C}$.

**Corollary 1.7.** $\mathcal{C}$ is a finitely generic abelian $l$-group.

**Proof.** Certainly some $e.c.$ structure is finitely generic, and since by 1.6 $\mathcal{C}$ embeds in any $e.c.$ structure, we get that $\mathcal{C}$ is an $e.c.$ substructure of a finitely generic model. This implies $\mathcal{C}$ is itself finitely generic (see 5.15 of [H-W]).

**Remark.** In [L] Lacava states without proof that a certain structure $\mathcal{Q}_\omega$ is a prime $e.c.$ model, where $\mathcal{Q}_\omega$ is constructed using the unique countable $e.c.$ distributive lattice. From this claim and 1.6 it follows that $\mathcal{Q}_\omega$ is isomorphic to $\mathcal{C}$, answering affirmatively a conjecture of Glass [G]. It is likely that Lacava had in mind an embedding similar to ours; such an argument does prove his claim.

2. Hyperarchimedean $e.c.$ structures. Throughout this section we assume $\mathcal{H}$ is a countable hyperarchimedean (necessarily abelian) $l$-group satisfying (i)–(iii) of Lemma 1.5. Our goal is to show that all such groups are finitely generic. This converse to Theorem 1.11 of [G-P2] then characterizes the finitely generic structures among those $e.c.$ as exactly the hyperarchimedean ones. Also, the $\forall \exists$-sentences expressing 1.5(i)–(iii) axiomatize the finitely generic (or equivalently, the $e.c.$) structures among the hyperarchimedean models. Turning this around, it shows that the finitely generic models are axiomatizable among the $e.c.$ models of the theory of abelian $l$-groups, since hyperarchimedean is axiomatizable there.

Our starting point is the representation afforded any hyperarchimedean group as a group of real-valued functions $\mathcal{K} \subseteq \mathbb{R}^Y$, where the compact open sets of $Y$ are of the form $\{\text{supp}(h) \mid h \in \mathcal{K}\}$ and provide a base for a Hausdorff topology on $Y$. (See §14.1 of [B-K-W].) Because $\mathcal{K}$ is countable we have that $Y$ is second countable and regular, hence metrizable. The space $Y$ is not compact, since by 1.5(ii) any maximal set of pairwise disjoint positive elements of $\mathcal{K}$ is infinite: fix one such, $\{f_n \mid n \in \omega\}$, with $\text{supp } f_n = Y_n$ compact open, $Y = \bigcup_{n \in \omega} Y_n$. By 1.5(i), any compact open set in $Y$ can be divided into a disjoint union of compact open sets, hence $Y$ is totally disconnected and perfect. Each $Y_n$ is thus totally disconnected, perfect, compact metric, and so is homeomorphic to the Cantor set. By scaling the values of elements of $\mathcal{K}$ at each point of $Y$ via positive real multiplication, we may assume that

$$f_n(y) = \begin{cases} 1, & y \in Y_n, \\ 0, & \text{otherwise.} \end{cases}$$
Under this adjustment we have in \( \mathcal{H} \) the characteristic function of any compact open set in \( Y \). It follows that the restriction of any \( f \in \mathcal{H} \) to a compact open subset of \( Y \) must also lie in \( \mathcal{H} \).

We now fix an embedding of \( \mathcal{C} \) into \( \mathcal{H} \). Since this embedding could be given by homeomorphisms from \( X_n \) onto \( Y_n \), we regard \( \mathcal{C} \subseteq \mathcal{H} \subseteq \mathbb{R}^X \), identifying \( e_n \)'s and \( f_n \)'s, \( X_n \)'s and \( Y_n \)'s. Our task is to obtain information about \( \mathcal{H} \), largely from local behavior under this representation and from properties of the embedding of \( \mathcal{C} \) into \( \mathcal{H} \).

**Notation.** Given any finite subset \( F = \{ h_1, \ldots, h_n \} \subseteq \mathcal{H} \) we write \( \text{supp} F = \text{supp} \ h = \bigcup \{ \text{supp}(h_i) \mid i = 1, \ldots, n \} \).

**Lemma 2.1.** Let \( S \) be compact open in \( X \), \( t(y_1, \ldots, y_n) \) a term of \( L \), \( \theta(y_1, \ldots, y_n) \) a formula of \( L \), and \( h_1, \ldots, h_n \in H \). Then the sets \( A_1 = \{ x \in X \mid t(h(x)) \neq 0 \} \), \( A_2 = \{ x \in X \mid t(h(x)) = 0 \} \), and \( A_3 = \{ x \in X \mid R \vdash \theta(h(x)) \} \) each intersect \( S \) in a compact open set.

**Proof.** Since \( t(h) \in \mathcal{H} \) we have \( \text{supp} t(h) \) is compact open, and so \( A_1 \cap S = \text{supp} t(h) \cap S \) is compact open. Therefore \( A_2 \cap S = (X - \text{supp} t(h)) \cap S \) is also compact open. Now \( \theta \) has an equivalent in \( \text{Th}(\mathbb{R}) \) which is quantifier-free (see 0.2) and in disjunctive normal form. Thus \( A_3 \) is a finite union of finite intersections of sets defined by atomic or negated atomic formulae, each of which is of the form \( A_1 \) or \( A_2 \), giving that \( A_3 \cap S \) is compact open.

**Lemma 2.2.** Let \( a_1, \ldots, a_n \in \mathcal{H} \), \( \theta(x_1, \ldots, x_k, y_1, \ldots, y_n) \) a formula in \( L \) and \( w_0 \in X \) such that
\[
R \vdash \exists x_1 \cdots \exists x_k \ \theta(x_1, \ldots, x_k, a_1(w_0), \ldots, a_n(w_0)).
\]
Then given \( W \) a compact open set containing \( w_0 \), there exists \( W_0 \) compact open, \( w_0 \in W_0 \subseteq W \) and \( b_1, \ldots, b_k \in \mathcal{H} \) with \( \text{supp} b \subseteq W_0 \) such that for all \( w \in W_0 \),
\[
R \vdash \theta(b_1(w), \ldots, b_k(w), a_1(w), \ldots, a_n(w)).
\]

**Proof.** Consider \( A = \{ h(w_0) \mid h \in \mathcal{H} \} \). Since \( \mathcal{H} \) is divisible, \( A \) is a divisible abelian subgroup of \( \mathbb{R} \), hence \( A < \mathbb{R} \). In particular,
\[
A \vdash \exists x_1 \cdots \exists x_k \ \theta(x_1, \ldots, x_k, a_1(w_0), \ldots, a_n(w_0)),
\]
so that \( A \vdash \theta(c_1, \ldots, c_k, a_1(w_0), \ldots, a_n(w_0)) \) for some \( c_1, \ldots, c_k \in A \). Given \( W \) as above, choose \( b'_i, i = 1, \ldots, k \), such that \( c_i = b'_i(w_0) \) and let \( W_0 = \{ w \in W \mid R \vdash \theta(b'_i(w), a(w)) \} \), \( b_i = b'_i \cap W_0, i = 1, \ldots, k \). By 2.1, \( W_0 \) is compact open; by choice of \( b'_i \) and \( b_i \) all the above conditions are met.

**Lemma 2.3 (Piecing Together).** Let \( a_1, \ldots, a_n \in \mathcal{H} \), \( \theta(x_1, \ldots, x_k, y_1, \ldots, y_n) \) in \( L \), and \( W \) nonempty compact open in \( X \), such that for all \( w \in W \),
\[
R \vdash \exists \tilde{x} \ \theta(\tilde{x}, a(w)).
\]
Then there exist \( b_1, \ldots, b_k \in \mathcal{H} \) with \( \text{supp} \ b \subseteq W \) such that for all \( w \in W \),
\[
R \vdash \theta(b(w), a(w)).
\]
Proof. By the previous lemma we can go from any point \( x \in W \) to solutions \( b_1, \ldots, b_k \) good on some compact open \( W_x \) with \( x \in W_x \subseteq W \). Since \( W = \bigcup_{x \in W} W_x \) is an open cover of \( W \), there are \( x_1, \ldots, x_r \) such that \( W = W_{x_1} \cup \cdots \cup W_{x_r} \). By restricting certain of the \( b_j \)'s to avoid overlapping supports we can find \( b_j \) which work on all of \( W \).

Definitions. Let \( \varphi = \varphi(x_1, \ldots, x_k, y_1, \ldots, y_n) \) be a quantifier-free formula of \( L \) in disjunctive normal form, \( \varphi = \bigvee_{i=1}^l \varphi_i \), where \( \varphi_i = \varphi_i^+ \land \rho_{i,1} \land \cdots \land \rho_{i,n} \), with \( \varphi_i^+ \) a conjunction of equations and each \( \rho_{i,j} \) a single inequation. Given \( 1 \leq k_1 < \cdots < k_r = k \) and \( -1 \leq j \leq r \) we define \( S_j = S_j(\varphi, k_1, \ldots, k_r) \) as follows:

\[
S_{-1} = \{ \theta | \theta \text{ is either } \varphi_i^+ \text{ or } \varphi_i^+ \land \rho_{i,j} \text{ for some } i, j, 1 \leq i \leq t, 1 \leq j \leq n_i \},
\]

\[
S_0 = \left\{ \bigwedge_{i=1}^m \theta_i | \theta_i \in S_{-1}, m > 0 \right\},
\]

and, for \( 0 \leq j < r \),

\[
S_{j+1} = \left\{ \exists x_{k_{j-r}}, \ldots, \exists x_{k_{j-r+j+1}} \cdots \exists x_{k_r} \bigwedge_{i=1}^m \theta_i = \theta_i' \text{ or } \theta_i = -\theta_i', \theta_i' \in S_j, m > 0 \right\},
\]

where all the above conjunctions are irredundant, so that each \( S_j \) is finite. We also define

\[
\psi_1 = \psi_1(\varphi, k) = \exists x_1 \cdots \exists x_k, \forall x_{k_1+1} \cdots \forall x_{k_2} \cdots \varphi,
\]

the formula obtained by \( r-1 \) alternations of quantifiers, according to \( k \), with \( \exists \) first, and \( \psi_2 \) as the corresponding formula beginning with universal quantifiers,

\[
\psi_2 = \psi_2(\varphi, k) = \forall x_1 \cdots \forall x_k, \exists x_{k_1+1} \cdots \exists x_{k_2} \cdots \varphi.
\]

For \( h_1, \ldots, h_n \in \tilde{C} \) and \( w \in X \), define the \( S_r \)-local type of \( \tilde{h} \) as

\[
S_{r, \tilde{h}(w)} = \{ \theta \in S_r | R \models \theta(\tilde{h}(w)) \}.
\]

In the following we see the link between “local” and “global” behavior for \( C \) and \( \tilde{C} \).

Theorem 2.4. Given \( \varphi, \tilde{k}, S_r, S_j \) and \( \psi_j \) as above, and given \( h_1, \ldots, h_n \in \tilde{C} \), \( c_1, \ldots, c_n \in C \) such that for all \( w \in X \), \( S_{r, \tilde{h}(w)} = S_{r, \tilde{c}(w)} \), then for \( j = 1 \) and \( 2 \), \( \tilde{C} \models \psi_j(h) \) if and only if \( C \models \psi_j(c) \).

Proof. We proceed by induction on \( r \), the number of quantifier blocks in \( \psi_1 \) or \( \psi_2 \).

If \( r = 0 \) then \( \psi_1 = \psi_2 = \varphi \) is quantifier-free, and the result follows readily. [One direction: If \( \tilde{C} \models \varphi(\tilde{h}) \) then \( \tilde{C} \models \varphi(h) \) for some \( h \), and so \( R \models \varphi_i^+ (\tilde{h}(w)) \) for all \( w \in X \), while \( R \models \varphi_i^+ \land \rho_{i,j} (\tilde{h}(w)) \) for some \( w_1, \ldots, w_n \in X \). This implies \( \varphi_i^+ \in S_{0, \tilde{c}(w)} \) for all \( w \in X \) and \( \varphi_i^+ \land \rho_{i,j} \in S_{0, \tilde{c}(w)}, j = 1, \ldots, n_i \), by hypothesis. Thus \( \tilde{C} \models \varphi(\tilde{c}) \) and so \( C \models \varphi(c) \) as desired.]

Now suppose \( r > 0 \) and \( C \models \psi_j(c) \), where \( \psi_j(c) = \exists x_1 \cdots \exists x_k, \psi_j(x_1, \ldots, x_k, c) \).

Then there exist \( d_1, \ldots, d_{k_1} \in \tilde{C} \) such that \( \tilde{C} \models \psi_j(d, \tilde{c}) \). Let

\[
W = \text{supp}\{d_1, \ldots, d_{k_1}, c_1, \ldots, c_n, h_1, \ldots, h_n\}.
\]
For each \( w \in W \) define
\[
\sigma_w = \sigma_\vec{d}(w) = \bigwedge \{ \sigma(x_1, \ldots, x_k, \vec{y}) \mid \mathcal{R} \models \sigma(\vec{d}(w), \vec{c}(w)) \},
\]
where \( \sigma \) or \( -\sigma \in S_{r-1} \).

Note that \( \exists x_1 \cdots \exists x_k \sigma_w \in S_{r-1}(w) \) and that \( \sigma_\vec{d}(w) \) is the conjunction of the set of formulae \( S_{r-1}(\vec{d}(w)) \) associated with the formula
\[
\varphi \left( y_{n+1}, \ldots, y_{n+k_1}, x_{k_1+1}, \ldots, x_{k_2}, y_1, \ldots, y_n \right),
\]
the tuple \( k_2, \ldots, k_r \), the tuple \( \vec{d} \), and the point \( w \).

Using 2.1 we know that for each \( w \in W \), \( \{ u \in W \mid \mathcal{R} \models \sigma_w(d(u), c(u)) \} \) is compact open and contains \( w \). We choose \( \sigma_1, \ldots, \sigma_m \) and \( W_1, \ldots, W_m \) such that \( W = W_1 \cup \cdots \cup W_m \). \( \sigma_i = \sigma_w \) for each \( w \in W_i \), \( W_i \) compact open. Also for each \( w \in W_i \), \( \mathcal{R} \models \exists x_1 \cdots \exists x_k \sigma_i(x_1, \ldots, x_k, \vec{c}(w)) \), hence \( \exists x_1 \cdots \exists x_k \sigma_i \in S_{r-1}(w) \), and so by hypothesis \( \exists x_1 \cdots \exists x_k \sigma_i \in S_{r-1}(w) \), hence \( \mathcal{R} \models \exists x_1 \cdots \exists x_k \sigma_i(x_1, \ldots, x_k, \vec{c}(w)) \).

Applying 2.3 we get \( I_1^{(i)}, \ldots, I_k^{(i)} \) with \( \text{supp}(I_1^{(i)}, \ldots, I_k^{(i)}) \subseteq W_i \) such that \( \mathcal{R} \models \sigma_i(I_1^{(i)}(w), \ldots, I_k^{(i)}(w), \vec{c}(w)) \) for all \( w \in W_i \). Now let \( l_j = I_1^{(j)} + \cdots + I_k^{(j)}, \) \( j = 1, \ldots, k_1 \). Then \( S_{r-1}(\vec{d}(w)) = S_{r-1}(\vec{d} \cap \vec{c}(w)) \) for all \( w \in W \), hence for all \( w \in X \). By induction, then, \( c \models \psi_1(d, \vec{c}) \) implies \( \mathcal{K} \models \psi_1(l, \vec{h}) \), and so \( \mathcal{K} \models \psi_1(\vec{h}) \).

Now suppose \( \mathcal{C} \models \psi_2(\vec{c}) \), where \( \psi_2 = \forall x_1 \cdots \forall x_k \psi_2(x_1, \ldots, x_k, \vec{c}) \). We must verify \( \mathcal{K} \models \psi_2(\vec{h}) \); i.e., for any \( l_1, \ldots, l_k \in \mathcal{K} \) that \( \mathcal{K} \models \psi_2(l_1, \ldots, l_k, \vec{h}) \). To do this we proceed as above, using the fact that \( Q \prec \mathcal{R} \), to produce a tuple \( d_1, \ldots, d_k \in C \) bearing the same relationship to \( \vec{c} \) as \( l \) does to \( \vec{h} \). (Since functions in \( C \) are locally constant, we do not use 2.1 and 2.3 in this case, and it is less complex.)

The other direction is analogous, with "\( \mathcal{K} \models \psi_2(\vec{h}) \) implies \( \mathcal{C} \models \psi_2(\vec{c}) \)" being the more complicated of the two cases.

**Corollary 2.5.** \( \text{Th}(\mathcal{C}) = \text{Th}(\mathcal{K}) = T' \), the finite forcing companion of the theory of abelian \( l \)-groups.

**Proof.** Immediate.

Now we are prepared to prove the main result of this section:

**Theorem 2.6.** Any hyperarchimedean \( l \)-group satisfying 1.5(i)-(iii) is finitely generic.

**Proof.** It suffices to prove that any countable hyperarchimedean \( l \)-group \( \mathcal{C} \) satisfying 1.5(i)-(iii) is finitely generic, since this implies that all countable elementary substructures of an arbitrary \( \mathcal{C}' \) as above are finitely generic, hence \( \mathcal{C}' \) is itself finitely generic.

We verify the following criterion for finite genericity of \( \mathcal{K} \) (see [S] for this formulation):

\( \star \) Given \( h_1, \ldots, h_n \in \mathcal{K} \) and a formula \( \psi(y_1, \ldots, y_n) \) of \( L \) such that \( \mathcal{K} \models \psi(\vec{h}) \), there is an existential formula \( \theta(y_1, \ldots, y_n) \) of \( L \) such that \( \mathcal{K} \models \psi(\vec{h}) \),

(i) \( \mathcal{K} \models \theta(\vec{h}) \) and

(ii) \( T' \models \forall y(\theta(y) \rightarrow \psi(y)) \).

Let \( h_1, \ldots, h_n \in \mathcal{K} \). Without loss of generality we assume \( h_1, \ldots, h_n \in \mathcal{K}_+ \) and that \( \psi(y_1, \ldots, y_n) \) is in prenex normal form, beginning with \( \exists \):

\[
\psi(y_1, \ldots, y_n) \equiv \exists x_1 \cdots \exists x_{k_1} \forall x_{k_1+1} \cdots \forall x_{k_2} \cdots \varphi(x_1, \ldots, x_k, y_1, \ldots, y_n).
\]
Consider, as before, the set of local types \( \{ S_{r, h(w)} \mid w \in X \} \). For each \( w \in W = \text{supp} \, h \), define \( \theta_w = \bigwedge S_{r, h(w)} \). Now choose \( t_w \) such that \( h_1(w), \ldots, h_t(w) \) are linearly independent over \( \mathbb{Q} \), and \( h_{t_w+1}(w), \ldots, h_n(w) \) are dependent on \( h_1(w), \ldots, h_t(w) \) over \( \mathbb{Q} \). Denote by \( G_j = G_j(z_1, \ldots, z_n) \in \mathbb{Q}[z] \) the polynomials such that \( h_j(w) = G_j(h_1(w), \ldots, h_t(w)) \), \( j = t_w + 1, \ldots, n \). (Here we choose to ignore the necessary rearrangements of \( h_i \)'s, to avoid notational chaos.) By the Rectangle Lemma applied to \( h_1(w), \ldots, h_t(w) \) and

\[
\theta_w' = \theta_w(y_1, \ldots, y_n, G_{t_w+1}(y), \ldots, G_n(y))
\]

we find \( q_{2,1}, q_{2,2}, \ldots, q_{t_w,1}, q_{t_w,2} \in \mathbb{Q} \) such that \( h_i(w)q_{j,1} < h_i(w) < h_i(w)q_{j,2} \), \( 2 \leq j \leq t_w \), and such that for any \( s_1, \ldots, s_{t_w} \in \mathbb{R} \) with \( s_i > 0 \) and \( s_1q_{j,1} < s_j < s_1q_{j,2} \) for \( 2 \leq j \leq t_w \), we have \( \mathbb{R} \ni \theta_{w'}(s_1, \ldots, s_{t_w}) \). Now let

\[
\psi_w(y_1, \ldots, y_n) = \left( \bigwedge_{j=2}^{t_w} y_1q_{j,1} < y_j < y_1q_{j,2} \right) \land (y_1 > 0) \land \bigwedge_{j=t_w+1}^n \left( y_j = G_j(y_1, \ldots, y_n) \right)
\]

and let \( W_w = \{ x \in W \mid \mathbb{R} \ni \psi_w(h_1(x), \ldots, h_n(x)) \} \). Then \( W_w \) is compact open, by 2.1, and is nonempty since \( w \in W_w \). Thus we write \( W = \bigcup_{w \in W} W_w = W_1 \cup \cdots \cup W_m \) as usual, further cutting back \( W_w \)'s as necessary for disjointness, to get each \( W_j \) a nonempty compact open subset of some \( W_w \), with \( \psi_j = \psi_{w_j} \). Let

\[
\theta(y) = \exists y_{ij} \left( \bigwedge_{1 \leq i, i' \leq n} \bigwedge_{1 \leq j, j' \leq m} (y_{ij} \land y_{i'j'} = 0) \land \bigwedge_{i=1}^n \left( y_i = \sum_{j=1}^m y_{ij} \right) \land \bigwedge_{j=1}^m \psi_j(y_1, \ldots, y_n) \right).
\]

Certainly \( \mathcal{C} \models \theta(h) \), with \( h_j \upharpoonright W_j \) taken for \( y_{ij} \). To see that \( T' \models \forall y \theta(y) \rightarrow \psi(y) \), hence verifying (\( * \)), we check that \( \mathcal{C} \models \forall y (\theta(y) \rightarrow \psi(y)) \) and use the fact that \( T' = \text{Th}(\mathcal{C}) \).

Let \( c_1, \ldots, c_n \in \mathcal{C} \) such that \( \mathcal{C} \models \theta(\bar{c}) \), and let \( c_{ij} \in \mathcal{C} \) be elements which work as \( y_{ij} \) in \( \theta \). Let \( Y_j = \text{supp} \{ c_1, \ldots, c_n \} \), \( j = 1, \ldots, m \), and note that \( Y_j \cap Y_{j'} = \emptyset \) for \( j \neq j' \). Let \( g : X \to X \) be a homeomorphism sending \( Y_j \) onto \( W_j \), \( j = 1, \ldots, m \), and let \( c'_1, \ldots, c'_n \) be the images of \( c_1, \ldots, c_n \) under the automorphism of \( \mathcal{C} \) induced by \( g \). Then by choice of the \( \theta_{w} \)'s and \( \psi_{w} \)'s we have—for each \( w \in W \), hence for each \( w \in X \)—that

\[
S_{r, h(w)} = S_{r, \bar{c}(w)}.
\]

Therefore by 2.4 we have \( \mathcal{C} \models \psi(\bar{c}) \), hence \( \mathcal{C} \models \psi(\bar{c}) \), as desired.

**Corollary 2.7.** Any hyperarchimedean e.c. l-group is finitely generic.

**Proof.** Immediate from 1.5 and 2.6.

As an immediate corollary we have a generalization of a conjecture of Glass and Pierce [G-P2] concerning the finite genericity of \( \overline{C}(X, \mathbb{R}) \). Recall that for \( A \) any
divisible totally ordered abelian subgroup of \( \mathbb{R} \) (with discrete topology) we denote by \( \mathcal{C}(X, A) \) the \( l \)-group of all continuous functions with compact support from \( X \) to \( A \).

**Corollary 2.8.** All groups of the form \( \mathcal{C}(X, A) \) are finitely generic.

**Proof.** Again, the proof of Theorem 8 of [G-P] applies to \( A \) as well as \( \mathbb{R} \), giving that \( \mathcal{C}(X, A) \) is existentially closed. (Alternately, it is an easy check that 1.5(i)-(iii) hold for \( \mathcal{C}(X, A) \).) Since \( \mathcal{C}(X, A) \) is hyperarchimedean (all proper prime subgroups are zero sets, hence maximal), the corollary follows from 2.7.

Our description of finitely generic models also enables us to prove the following embedding result:

**Corollary 2.9.** Any hyperarchimedean \( l \)-group can be embedded in a finitely generic abelian \( l \)-group.

**Proof.** Let \( \mathcal{G} \) be hyperarchimedean. By 2.6 it suffices to find \( \mathcal{G} \supseteq \mathcal{A} \), \( \mathcal{A} \) hyperarchimedean and satisfying 1.5(ii)-(iii). Let \( \mathcal{A}_{-1} \) be the divisible hull of \( \mathcal{A} \) and let \( \mathcal{A}_0 = \mathcal{A}_{-1} \oplus \mathbb{C} \). For each \( j \geq 0 \) choose \( \mathcal{A}_{j+1} \supseteq \mathcal{A}_j \) such that \( \mathcal{A}_{j+1} \approx \mathcal{A}_j \oplus \mathcal{A}_j \), with each \( a \in \mathcal{A}_j \) corresponding to \( a \oplus a \) in \( \mathcal{A}_j \oplus \mathcal{A}_j \). Then let \( \mathcal{G} = \bigcup_{j \in \omega} \mathcal{A}_j \). Each element of \( \mathcal{A}_j \) is a sum of \( n \) disjoint elements in \( \mathcal{A}_{j+n} \), and so 1.5(i) holds, while 1.5(i)-(iii) remain in force from \( \mathcal{A}_0 \) on. Each \( \mathcal{A}_j \) is hyperarchimedean, hence \( \mathcal{G} \) is, and we have the desired extension of \( \mathcal{A} \).

**3. Examples.** Not all hyperarchimedean \( l \)-groups are representable as finite range real-valued functions. It therefore follows from 2.9 that not all finitely generic groups can be thus represented. We choose nonetheless to construct a specific group, one which is \( \mathbb{Q} \)-generated over \( \mathcal{C} \) by two elements. Example 1.13 in [G-P2], of a hyperarchimedean e.c. not thus representable is unfortunately incorrect (their group as stated is representable as finite range functions).

We start with \( \mathcal{C} = \mathcal{C}(X, \mathbb{Q}) \) as above and divide each \( X_n \) into disjoint Cantor sets, \( X_n = Y_n \cup Z_n \), with \( X = \bigcup_{n \in \omega} X_n \). We define \( \mathbf{1}, \mathbf{b} \in \mathbb{Q}^X \) as follows:

\[
\mathbf{1}(x) = \begin{cases} 
1, & x \in Y_n \text{ for some } n, \\
0, & \text{otherwise},
\end{cases}
\]

and

\[
\mathbf{b}(x) = \begin{cases} 
\pi_n, & x \in Y_n, \\
0, & \text{otherwise},
\end{cases}
\]

where \( \pi_n \in \mathbb{Q} \) is chosen so that \( |\pi_n - \pi| < 10^{-n} \) for the real \( \pi, \pi_1 < \pi_2 < \cdots < \pi \).

Now take \( \mathfrak{B} \) to be the divisible hull of the group generated by \( \mathcal{C} \cup \{ \mathbf{1}, \mathbf{b} \} \). Any element of \( \mathfrak{B} \) can be written uniquely as \( c + q_1 \mathbf{1} + q_2 \mathbf{b} \), where \( c \in \mathcal{C} \), \( q_1, q_2 \in \mathbb{Q} \).

**Example 3.1.** The group \( \mathfrak{B} \) is finitely generic and is not representable as a group of real-valued functions with finite range.

**Proof.** First we describe the proper prime subgroups of \( \mathfrak{B} \): those of \( \mathcal{C} \) are the zero sets \( \mathcal{P}_x = \{ f \in \mathcal{C} \mid f(x) = 0 \} \), where \( x \in X \) (see [G-P]). Let \( \mathfrak{L} \) be a proper prime in \( \mathfrak{B} \). Then \( \mathfrak{L} \cap \mathcal{C} \) is prime, hence \( \mathfrak{L} \cap \mathcal{C} \) is either \( \mathcal{C} \) itself or some \( \mathcal{P}_x \). We claim that \( \mathcal{C} \) is maximal in \( \mathfrak{B} \), so that \( \mathfrak{L} \cap \mathcal{C} = \mathcal{C} \) implies \( \mathfrak{L} = \mathcal{C} \). For let \( d \in \mathfrak{B} - \mathcal{C} \), \( d = c + q_1 \mathbf{1} + q_2 \mathbf{b} \), where \( q_1 \) and \( q_2 \) are rationals, not both zero, and \( c \in \mathcal{C} \). Then the convex
subgroup generated by \( \{q_1 + q_2 b\} \cup C \) is \( \mathbb{B} \), because \( q_1 + q_2 b \) is bounded away from zero on all but at most one \( Y_m \). Thus \( \mathbb{L} = C \).

Now suppose \( \mathbb{L} \cap C = \mathbb{P}_x \). We claim \( \mathbb{L} = \mathbb{L}_x = \{f \in \mathbb{B} | f(x) = 0\} \). Let \( f \in \mathbb{L}_x \).

Choose \( f_0 \in \mathbb{P}_x \subseteq \mathbb{L} \) such that either \( f - f_0 \in \mathbb{B}_+ \) or \( f_0 - f \in \mathbb{B}_+ \), say the former.

Let \( S \) be a compact set containing \( x \), and define \( g \in C_+ \) such that

\[
g(w) = \begin{cases} 1, & w \in S, \\ 0, & \text{otherwise}. \end{cases}
\]

Then \( (f - f_0) \wedge g \in C_+ \) and \( (((f - f_0) \wedge g)(x) = 0 \), hence \( (f - f_0) \wedge g \in \mathbb{P}_x \subseteq \mathbb{L} \).

But \( g \notin \mathbb{P}_x = \mathbb{L} \cap C \), hence \( g \in \mathbb{L} \), so \( f - f_0 \notin \mathbb{L} \) and thus \( f \in \mathbb{L} \). This shows \( \mathbb{L}_x \subseteq \mathbb{L} \); it is easy to see that \( \mathbb{L}_x \) is maximal in \( \mathbb{B} \), hence \( \mathbb{L} = \mathbb{L}_x \).

Thus the proper prime subgroups of \( \mathbb{B} \) are \( C \) and the \( \mathbb{L}_x \)'s. In particular, all proper primes are maximal, and \( \mathbb{B} \) is hyperarchimedean.

It is a routine check that \( \mathbb{B} \) satisfies 1.5(i)-(iii), hence is finitely generic by 2.6.

To see that \( \mathbb{B} \) cannot be represented as a group of functions with finite range, note that for each \( n \),

\[
1 = \left( \sum_{i=1}^{n} e_i \wedge 1 \right) + d_n, \quad b = \sum_{i=1}^{n} (e_i \wedge b) + d'_n
\]

where \( d_n \wedge e_i = d'_n \wedge e_i = 0 \) for \( i \leq n \), and where \( e_i \wedge 1, e_i \wedge b, d_n, d'_n \in \mathbb{B} \). Since \( e_i \wedge b = \pi(e_i \wedge 1) \) and there are infinitely many such \( \pi \), at least one of \( b \) and \( 1 \) must take on infinitely many values in any representation.

**Example 3.2.** There exist \( 2^\aleph_0 \) countable finite generic abelian \( l \)-groups.

**Proof.** The groups \( C(X, \mathbb{Q} \oplus \mathbb{Q}a) \) for \( a \in A, A \) a set of algebraically independent irrationals, are all nonisomorphic, as indicated in [G-P]. By 2.8, each such group is finitely generic.

We note that the number of nonelementarily equivalent e.c. abelian \( l \)-groups is not known to us.

**Added in proof.** Françoise Point has given an alternate treatment of the results of §2, via quantifier-elimination in an expanded language (to appear).

**Bibliography**


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