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FINITELY GENERIC ABELIAN LATTICE-ORDERED GROUPS

BY

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ABSTRACT. The authors characterize the finitely generic abelian lattice-ordered groups and make application of this characterization to specific examples.

A key goal in Abraham Robinson's development of model-theoretic forcing was to explicate the notion of algebraically closed, even when the appropriate classes may not be first-order axiomatizable. Interesting links sometimes appear between purely algebraic properties and model-theoretic properties such as existentially closed (e.c.) and finitely generic. In this spirit we consider the characterization of finitely generic abelian l-groups, as well as the model-theoretic properties of certain e.c. abelian l-groups.

The model theory of abelian lattice-ordered (l-) groups was developed by Glass and Pierce in **[G-P]** and **[G-P2]**. They showed that every finitely generic structure is hyperarchimedean, and that the group $\bar{C}(X, \mathbf{R})$ is existentially closed. They also stated several problems, including:

(i) Distinguish the finitely generic models among the hyperarchimedean e.c. ones **[G-P**, p. 263].

(ii) Show $\bar{C}(X, \mathbf{R})$ is finitely generic **[G-P2**, p. 146].

(iii) Give an example of a finitely generic e.c. group not representable as a group of real-valued functions with finite range. (Note that an incorrect reading of 14.1.7 of **[B-K-W]** led to an incorrect statement of this on p. 269 of **[G-P]**.)

(iv) Give an example of an archimedean e.c. model which is not hyperarchimedean **[G-P**, p. 263].

In the present paper we solve (i)–(iii); an example for (iv) appears in a separate note **[S-W]**.

For (i) we first show that $\mathcal{C} = \bar{C}(X, \mathbf{Q})$ is the unique prime e.c. model, and hence that $\mathcal{C}$ is finitely generic. We next employ the representation of hyperarchimedean l-groups in **[B-K-W]**, together with a fixed embedding of $\mathcal{C}$ into a suitable hyperarchimedean group, to show that any hyperarchimedean e.c. model is finitely generic. In fact, the finitely generic models are axiomatizable among the e.c. models. As a consequence, we have that $\bar{C}(X, A)$ is finitely generic for any divisible subgroup A of the additive reals; (ii) is a special case of this, for $A = \mathbf{R}$. It follows also that there are $2^{\aleph_0}$ countable nonisomorphic finitely generic abelian l-groups.

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From the axioms for finitely generic models among e.c. models we show that any hyperarchimedean group can be extended to a finitely generic one, which settles (iii). In addition, we give a specific construction of a group for (iii) which is **Q**-generated over $\mathcal{C}$ by two elements.

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0. Preliminaries.

0.1. *l-groups*. The language L for abelian lattice ordered (l-) groups is chosen here with one constant "0" (zero), one unary function "$-$" (minus), and three binary functions "$+$" (addition), "$\wedge$" (meet) and "$\vee$" (join). Note that $x \leqslant y$ is not a formula of L, but is definable, for example, from $\wedge$ via $x \wedge y = x$. (Warning: the symbols $\wedge$ and $\vee$ play dual roles here, as meet/and and join/or, with context the only clue.) An atomic formula of L can be written as $\tau(\vec{x}) = 0$ where τ is a term built up from 0 and variables $\vec{x} = x_1, \ldots, x_n$ using $-$, $+$, $\wedge$ and $\vee$. For an l-group $\mathcal{A}$ we denote the *positive cone* by $\mathcal{A}_+ = \{a \in \mathcal{A} \mid a \geqslant 0\}$. A positive element of $\mathcal{A}$ is thus a nonzero element of $\mathcal{A}_+$.

DEFINITIONS. Let $\mathcal{A}$ be an abelian l-group. We say $\mathcal{A}$ is *existentially closed* (e.c.) provided for any $\vec{a} \in \mathcal{A}$, any finite set of atomic and negated atomic formulae $\tau_1(\vec{x}, \vec{a}) = 0, \ldots, \tau_n(\vec{x}, \vec{a}) = 0, \tau_{n+1}(\vec{x}, \vec{a}) \neq 0, \ldots, \tau_m(\vec{x}, \vec{a}) \neq 0$ has a solution for $\vec{x}$ *in* $\mathcal{A}$ whenever it has a solution in some abelian l-group containing $\mathcal{A}$.

$\mathcal{A}$ is *archimedean* provided for any $a, b \in \mathcal{A}$ with $a > 0$ there is an integer n such that $na \nleqslant b$. Further, $\mathcal{A}$ is *hyperarchimedean* provided all homomorphic images of $\mathcal{A}$ are archimedean; equivalently, provided every proper prime subgroup of $\mathcal{A}$ is maximal.

The *divisible hull* of $\mathcal{A}$ is the unique divisible l-group containing $\mathcal{A}$ and generated as a module over the rationals by $\mathcal{A}$.

REMARK. Recall that multiplication by a positive integer produces an automorphism of any divisible abelian l-group. Thus, in dealing with divisible groups, we often allow rational coefficients in purported atomic L-formulae, since any such "formula" has an honest equivalent in L.

0.2. *Totally ordered groups*. The theory of divisible totally-ordered abelian groups, for convenience taken here in L, admits elimination of quantifiers. Two models of this theory are of special interest here: the reals (**R**) and the rationals (**Q**). In §3 we use the following property of the embedding of **Q** in **R** as L-structures.

RECTANGLE LEMMA. *Let φ be a formula of L, $r_1, \ldots, r_t \in \mathbf{R}_+$ such that $\{r_1, \ldots, r_t\}$ is linearly independent over* **Q**, *and* $\mathbf{R} \models \varphi(r_1, \ldots, r_t)$. *Then there exist rationals $q_{2,1}$, $q_{2,2}, \ldots, q_{t,1}$, $q_{t,2}$ such that $r_1 q_{j,1} < r_j < r_1 q_{j,2}$ for $2 \leqslant j \leqslant t$, and such that for all $s_1, \ldots, s_t \in \mathbf{R}$ with $s_1 > 0$ and with $s_1 q_{j,1} < s_j < s_1 q_{j,2}$ for $2 \leqslant j \leqslant t$, it follows that* $\mathbf{R} \models \varphi(s_1, \ldots, s_t)$.

PROOF. Let $r_1, \ldots, r_t$, φ be as in the hypothesis with $\mathbf{R} \models \phi(r_1, \ldots, r_t)$. Without loss of generality we consider only the situation when $r_1 = 1$ and, moreover, find conditions only for $s_1 = 1$, since multiplication by a positive real determines an

automorphism of $\mathbf{R}$. By quantifier-elimination we have that φ is equivalent to a formula in disjunctive normal form, some disjunct—say θ—of which holds at $1, r_2, \ldots, r_t$. (Here θ is a conjunction of atomic and negated atomic formulae.) It is now convenient to rewrite θ without meets and joins, and so we choose $q'_{j,1}$, $q'_{j,2}$ so that $q'_{j,1} < r_j < q'_{j,2}$, $2 \leqslant j \leqslant t$, and so that for all $s_2, \ldots, s_t$ satisfying $q'_{j,1} < s_j < q'_{j,2}$, $2 \leqslant j \leqslant t$, the ordering of all terms occurring in θ is the same for $s_2, \ldots, s_t$ as it is for $r_2, \ldots, r_t$. Next we replace terms of the form $\tau_1 \wedge \tau_2$ and $\tau_1 \vee \tau_2$ by the appropriate term τ_i according to the ordering at $1, r_2, \ldots, r_t$. The resulting formula θ' is a conjunction of inequalities of the form $\tau(1, y_2, \ldots, y_t) \neq 0$, since all equations must be trivial by the independence of $1, r_2, \ldots, r_t$. Now these inequalities hold on an open subset of $\mathbf{R}^{t-1}$, hence there exist $q_{j,1}$, $q_{j,2}$ with $q'_{j,1} \leqslant q_{j,1} < q_{j,2} \leqslant q'_{j,2}$ such that $\tau(1, s_2, \ldots, s_t) \neq 0$ for $s_2, \ldots, s_t$ satisfying $q_{j,1} < s_2 < q_{j,2}$, $2 \leqslant j \leqslant t$. Thus $\mathbf{R} \models \theta(1, s_2, \ldots, s_t)$ for such $s_2, \ldots, s_t$, and so $\mathbf{R} \models \varphi(1, s_2, \ldots, s_t)$ as well.

1. The smallest finitely generic structure. Throughout this paper let $X = \dot{\bigcup}_{n \in \omega} X_n$ be a disjoint union of countably many Cantor sets, let $\mathbf{Q}$ be the additive rationals with discrete topology, and let $\mathcal{C} = \overline{C}(X, \mathbf{Q})$ be the l-group of all continuous functions from X to $\mathbf{Q}$ with compact support. (Here the group operation is pointwise addition, and sups and infs are pointwise maxima and minima.) Note that $\mathcal{C}$ is countable, abelian, and that each element of $\mathcal{C}$ has finite range: see **[G-P]** for further details. For $c \in \mathcal{C}$ we denote the support of c by $\operatorname{supp}(c) = \{x \in X \mid c(x) \neq 0\}$. The goal of this section is to show that $\mathcal{C}$ is the unique prime e.c. abelian l-group, hence is finitely generic. To do this we make use of the very pleasant nature of the structure of $\mathcal{C}$.

LEMMA 1.1. *Given $X = Y_1 \dot{\cup} \cdots \dot{\cup} Y_n \dot{\cup} Y = Y'_1 \dot{\cup} \cdots \dot{\cup} Y'_n \dot{\cup} Y'$, where each Y_i, Y'_i is nonempty compact open, then there is a homeomorphism from X into itself sending Y_i onto Y'_i.*

PROOF. Each Y_i and Y'_i is a Cantor set, and Y and Y' are countable unions of Cantor sets.

LEMMA 1.2. *Let $s_1, \ldots, s_n$, $s'_1, \ldots, s'_n \in \mathcal{C}_+$, all nonzero, such that $s_i \wedge s_j = s'_i \wedge s'_j = 0$ for all $i \neq j$. Then there is an automorphism σ of $\mathcal{C}$ such that $\sigma(s_i) = s'_i$, $i = 1, \ldots, n$.*

PROOF. Let $Y_i = \operatorname{supp}(s_i)$, $Y'_i = \operatorname{supp}(s'_i)$. Then each Y_i, Y'_i is nonempty compact open, and $Y_i \cap Y_j = Y'_i \cap Y'_j = \varnothing$ for $i \neq j$. This gives by 1.1 a homeomorphism h of X mapping Y_i onto Y'_i. For $c \in \mathcal{C}$ define $\sigma(c)$ as follows:

$$\sigma(c)(x) = \begin{cases} c\big(h^{-1}(x)\big) \cdot s'_i(x)/s_i\big(h^{-1}(x)\big), & x \in \operatorname{supp} s'_i, \\ c\big(h^{-1}(x)\big), & x \notin \bigcup_{i=1}^{n} \operatorname{supp} s'_i. \end{cases}$$

Then σ is the desired automorphism of $\mathcal{C}$.

LEMMA 1.3. *For any $c_1,\dots,c_k \in \mathcal{C}$ there exist $s_1,\dots,s_n \in \mathcal{C}_+$, the s_i's pairwise disjoint and nonzero, and there exist rationals q_{ij}, $i = 1,\dots,k$, $j = 1,\dots,n$, such that $c_i = \sum_{j=1}^n q_{ij}s_j$, $i = 1,\dots,k$. Moreover, if $c_1',\dots,c_k' \in \mathcal{C}$ and $s_1',\dots,s_n' \in \mathcal{C}_+$, the s_i''s pairwise disjoint and nonzero, such that $c_i' = \sum_{j=1}^n q_{ij}s_j'$, $i = 1,\dots,k$, then there is an automorphism of $\mathcal{C}$ mapping c_i to c_i', $i = 1,\dots,k$.*

PROOF. Given $c_i,\dots,c_k$, we indicate one reasonable choice of s_i's and q_{ij}'s. For $c_1 = \cdots = c_k = 0$ take $n = 1$, s_1 *any* nonzero element of $\mathcal{C}_+$, and $q_{i1} = 0$, $i = 1,\dots,k$. Otherwise write $\bigcup_{i=1}^k \operatorname{supp} c_i = Y_1 \dot\cup \cdots \dot\cup Y_n$ where each Y_j is compact open nonempty, and where $c_i \restriction Y_j$ is constant for all i and j. Then let

$$s_j(x) = \begin{cases} 1, & x \in Y_j, \\ 0, & \text{otherwise}, \end{cases}$$

and let $q_{ij} = c_i(x)$ for any (all) $x \in Y_j$.

The second part of the lemma is immediate from 1.2.

REMARK. Given $c_1,\dots,c_k \in \mathcal{C}$ we can in fact define some choice of $s_1,\dots,s_n$ in a quantifier-free manner and thus argue that the k-type of $c_1,\dots,c_k$ in $\mathcal{C}$ is principal and is given by a quantifier-free formula, although we never make full use of this. Also, we note that 1.3 tells us that $\mathcal{C}$ is ω-homogeneous.

LEMMA 1.4. *$\mathcal{C}$ is an existentially closed abelian l-group.*

PROOF. The proof is exactly that given by Glass and Pierce (Theorem 8, p. 261 of [**G-P**]) for the existential closedness of $\overline{C}(X, \mathbf{R})$, since $\mathbf{R}$ can be replaced there by any divisible totally-ordered abelian group—in the present case by $\mathbf{Q}$.

We collect here some easy facts about e.c. abelian l-groups.

LEMMA 1.5. *Let $\mathfrak{D}$ be an e.c. abelian l-group. Then*:

(i) *For $u \in \mathfrak{D}$ with $u > 0$ and $n > 0$ there exist $u_1,\dots,u_n$ pairwise disjoint positive elements of $\mathfrak{D}$ with $u = u_1 + \cdots + u_n$.*

(ii) *Any maximal set of pairwise disjoint positive elements of $\mathfrak{D}$ is infinite.*

(iii) *$\mathfrak{D}$ is divisible.*

PROOF. This is easily checked directly, but we can also use the expressibility of all the assertions by $\forall\exists$ sentences true in $\mathcal{C}$, hence true in any e.c. $\mathfrak{D}$. For example, for (ii) we note that

$$\mathcal{C} \models \forall x_1 \cdots \forall x_n \exists y \left(y > 0 \wedge \bigwedge_{i=1}^n (x_i \wedge y) = 0 \right)$$

which gives (ii).

THEOREM 1.6. *The group $\mathcal{C}$ is the unique prime e.c. abelian l-group.*

PROOF. Let $\mathfrak{D}$ be e.c. and let $\{f_n\}_{n\in\omega}$ be any infinite set of pairwise disjoint positive elements of $\mathfrak{D}$ (which exists by 1.5(ii)), and let $\{e_n\}_{n\in\omega} \subseteq \mathcal{C}$ be the set of elements defined by

$$e_n(x) = \begin{cases} 1, & x \in X_n, \\ 0, & \text{otherwise}. \end{cases}$$

Map e_n to f_n, $n \in \omega$, to form a partial isomorphism of $\mathcal{C}$ into $\mathfrak{D}$. Extend this map to the domain of all characteristic functions of compact open sets by listing all such elements of $\mathcal{C}$ and choosing images, one-by-one, in $\mathfrak{D}$ via 1.5(i). The resulting map extends uniquely to the divisible hull, using 1.5(iii), which is all of $\mathcal{C}$. Thus $\mathcal{C}$ is prime.

In case $\mathfrak{D}$ is a substructure of $\mathcal{C}$ we can first choose $\{f_n\}_{n\in\omega}$ maximal, and then, by working back and forth as above, we get an isomorphism between $\mathcal{C}$ and $\mathfrak{D}$. Thus $\mathcal{C}$ is unique.

Note that $\mathcal{C}$ is not minimal. If we write $X = Y \dot{\cup} Z$, where Y and Z are each homeomorphic to X, then $\overline{C}(Y, \mathbf{Q}) \subsetneq \mathcal{C}$ but $\overline{C}(Y, \mathbf{Q}) \approx \mathcal{C}$.

COROLLARY 1.7. *$\mathcal{C}$ is a finitely generic abelian l-group.*

PROOF. Certainly some e.c. structure is finitely generic, and since by 1.6 $\mathcal{C}$ embeds in any e.c. structure, we get that $\mathcal{C}$ is an e.c. substructure of a finitely generic model. This implies $\mathcal{C}$ is itself finitely generic (see 5.15 of **[H-W]**).

REMARK. In **[L]** Lacava states without proof that a certain structure Q_ω is a prime e.c. model, where Q_ω is constructed using the unique countable e.c. distributive lattice. From this claim and 1.6 it follows that Q_ω is isomorphic to $\mathcal{C}$, answering affirmatively a conjecture of Glass **[G]**. It is likely that Lacava had in mind an embedding similar to ours; such an argument does prove his claim.

2. Hyperarchimedean e.c. structures. Throughout this section we assume $\mathcal{H}$ is a countable hyperarchimedean (necessarily abelian) l-group satisfying (i)–(iii) of Lemma 1.5. Our goal is to show that all such groups are finitely generic. This converse to Theorem 1.11 of **[G-P2]** then characterizes the finitely generic structures among those e.c. as exactly the hyperarchimedean ones. Also, the $\forall\exists$-sentences expressing 1.5(i)–(iii) axiomatize the finitely generic (or equivalently, the e.c.) structures among the hyperarchimedean models. Turning this around, it shows that the finitely generic models are axiomatizable among the e.c. models of the theory of abelian l-groups, since hyperarchimedean is axiomatizable there.

Our starting point is the representation afforded any hyperarchimedean group as a group of real-valued functions $\mathcal{H} \subseteq \mathbf{R}^Y$, where the compact open sets of Y are of the form $\{\operatorname{supp}(h) \mid h \in \mathcal{H}\}$ and provide a base for a Hausdorff topology on Y. (See §14.1 of **[B-K-W]**.) Because $\mathcal{H}$ is countable we have that Y is second countable and regular, hence metrizable. The space Y is not compact, since by 1.5(ii) any maximal set of pairwise disjoint positive elements of $\mathcal{H}$ is infinite: fix one such, $\{f_n \mid n \in \omega\}$, with $\operatorname{supp} f_n = Y_n$ compact open, $Y = \dot{\bigcup}_{n\in\omega} Y_n$. By 1.5(i), any compact open set in Y can be divided into a disjoint union of compact open sets, hence Y is totally disconnected and perfect. Each Y_n is thus totally disconnected, perfect, compact metric, and so is homeomorphic to the Cantor set. By scaling the values of elements of $\mathcal{H}$ at each point of Y via positive real multiplication, we may assume that

$$f_n(y) = \begin{cases} 1, & y \in Y_n, \\ 0, & \text{otherwise.} \end{cases}$$

Under this adjustment we have in $\mathcal{H}$ the characteristic function of any compact open set in Y. It follows that the restriction of any $f \in \mathcal{H}$ to a compact open subset of Y must also lie in $\mathcal{H}$.

We now fix an embedding of $\mathcal{C}$ into $\mathcal{H}$. Since this embedding could be given by homeomorphisms from X_n onto Y_n, we regard $\mathcal{C} \subseteq \mathcal{H} \subseteq \mathbf{R}^X$, identifying e_n's and f_n's, X_n's and Y_n's. Our task is to obtain information about $\mathcal{H}$, largely from local behavior under this representation and from properties of the embedding of $\mathcal{C}$ into $\mathcal{H}$.

Notation. Given any finite subset $F = \{h_1,\ldots,h_n\} \subseteq \mathcal{H}$ we write $\operatorname{supp} F = \operatorname{supp} \vec{h} = \bigcup \{\operatorname{supp}(h_i) \mid i = 1,\ldots,n\}$.

LEMMA 2.1. *Let S be compact open in X, $t(y_1,\ldots,y_n)$ a term of L, $\theta(y_1,\ldots,y_n)$ a formula of L, and $h_1,\ldots,h_n \in H$. Then the sets $A_1 = \{x \in X \mid t(\vec{h}(x)) \neq 0\}$, $A_2 = \{x \in X \mid t(\vec{h}(x)) = 0\}$, and $A_3 = \{x \in X \mid \mathbf{R} \models \theta(\vec{h}(x))\}$ each intersect S in a compact open set.*

PROOF. Since $t(\vec{h}) \in \mathcal{H}$ we have $\operatorname{supp} t(\vec{h})$ is compact open, and so $A_1 \cap S = \operatorname{supp} t(\vec{h}) \cap S$ is compact open. Therefore $A_2 \cap S = (X - \operatorname{supp} t(\vec{h})) \cap S$ is also compact open. Now θ has an equivalent in $\operatorname{Th}(\mathbf{R})$ which is quantifier-free (see 0.2) and in disjunctive normal form. Thus A_3 is a finite union of finite intersections of sets defined by atomic or negated atomic formulae, each of which is of the form A_1 or A_2, giving that $A_3 \cap S$ is compact open.

LEMMA 2.2. *Let $a_1,\ldots,a_n \in \mathcal{H}$, $\theta(x_1,\ldots,x_k, y_1,\ldots,y_n)$ a formula in L and $w_0 \in X$ such that*

$$\mathbf{R} \models \exists x_1 \cdots \exists x_k\, \theta(x_1,\ldots,x_k, a_1(w_0),\ldots,a_n(w_0)).$$

Then given W a compact open set containing w_0, there exists W_0 compact open, $w_0 \in W_0 \subseteq W$ and $b_1,\ldots,b_k \in \mathcal{H}$ with $\operatorname{supp} \vec{b} \subseteq W_0$ such that for all $w \in W_0$,

$$\mathbf{R} \models \theta(b_1(w),\ldots,b_k(w), a_1(w),\ldots,a_n(w)).$$

PROOF. Consider $A = \{h(w_0) \mid h \in \mathcal{H}\}$. Since $\mathcal{H}$ is divisible, A is a divisible abelian subgroup of $\mathbf{R}$, hence $A \prec \mathbf{R}$. In particular,

$$A \models \exists x_1 \cdots \exists x_k\, \theta(x_1,\ldots,x_k, a_1(w_0),\ldots,a_n(w_0)),$$

so that $A \models \theta(c_1,\ldots,c_k, a_1(w_0),\ldots,a_n(w_0))$ for some $c_1,\ldots,c_k \in A$. Given W as above, choose b_i', $i = 1,\ldots,k$, such that $c_i = b_i'(w_0)$ and let $W_0 = \{w \in W \mid \mathbf{R} \models \theta(\vec{b'}(w), \vec{a}(w))\}$, $b_i = b_i' \restriction W_0$, $i = 1,\ldots,k$. By 2.1, W_0 is compact open; by choice of b_i' and b_i all the above conditions are met.

LEMMA 2.3 (PIECING TOGETHER). *Let $a_1,\ldots,a_n \in \mathcal{H}$, $\theta(x_1,\ldots,x_k, y_1,\ldots,y_n)$ in L, and W nonempty compact open in X, such that for all $w \in W$,*

$$\mathbf{R} \models \exists \vec{x}\, \theta(\vec{x}, \vec{a}(w)).$$

Then there exist $b_1,\ldots,b_k \in \mathcal{H}$ with $\operatorname{supp} \vec{b} \subseteq W$ such that for all $w \in W$,

$$\mathbf{R} \models \theta(\vec{b}(w), \vec{a}(w)).$$

PROOF. By the previous lemma we can go from any point $x \in W$ to solutions $b_1^x, \ldots, b_k^x$ good on some compact open W_x with $x \in W_x \subseteq W$. Since $W = \bigcup_{x \in W} W_x$ is an open cover of W, there are $x_1, \ldots, x_r$ such that $W = W_{x_1} \cup \cdots \cup W_{x_r}$. By restricting certain of the $b_j^{x_i}$'s to avoid overlapping supports we can find b_j which work on all of W.

DEFINITIONS. Let $\varphi = \varphi(x_1, \ldots, x_k, y_1, \ldots, y_n)$ be a quantifier-free formula of L in disjunctive normal form, $\varphi = \bigvee_{i=1}^{t} \varphi_i$, where $\varphi_i = \varphi_i^+ \wedge \rho_{i,1} \wedge \cdots \wedge \rho_{i,n_i}$, with φ_i^+ a conjunction of equations and each $\rho_{i,j}$ a single inequation. Given $1 \leqslant k_1 < \cdots < k_r = k$ and $-1 \leqslant j \leqslant r$ we define $S_j = S_j(\varphi, k_1, \ldots, k_r)$ as follows:

$$S_{-1} = \{\theta \mid \theta \text{ is either } \varphi_i^+ \text{ or } \varphi_i^+ \wedge \rho_{i,j} \text{ for some } i, j, 1 \leqslant i \leqslant t, 1 \leqslant j \leqslant n_i\},$$

$$S_0 = \left\{ \bigwedge_{l=1}^{m} \theta_l \mid \theta_l \in S_{-1}, m > 0 \right\},$$

and, for $0 \leqslant j < r$,

$$S_{j+1} = \left\{ \exists x_{k_{r-j-1}+1} \cdots \exists x_{k_{r-j}} \bigwedge_{l=1}^{m} \theta_l \mid \theta_l = \theta_l' \text{ or } \theta_l = \neg\theta_l', \theta_l' \in S_j, m > 0 \right\},$$

where all the above conjunctions are irredundant, so that each S_j is finite. We also define

$$\psi_1 = \psi_1(\varphi, \vec{k}) = \exists x_1 \cdots \exists x_{k_1} \forall x_{k_1+1} \cdots \forall x_{k_2} \cdots \varphi,$$

the formula obtained by $r-1$ alternations of quantifiers, according to $\vec{k}$, with $\exists$ first, and ψ_2 as the corresponding formula beginning with universal quantifiers,

$$\psi_2 = \psi_2(\varphi, \vec{k}) = \forall x_1 \cdots \forall x_{k_1} \exists x_{k_1+1} \cdots \exists x_{k_2} \cdots \varphi.$$

For $h_1, \ldots, h_n \in \mathcal{H}$ and $w \in X$, define the S_r-w-local type of $\vec{h}$ as

$$S_{r,\vec{h}(w)} = \{\theta \in S_r \mid R \models \theta(\vec{h}(w))\}.$$

In the following we see the link between "local" and "global" behavior for $\mathcal{C}$ and $\mathcal{H}$.

THEOREM 2.4. *Given φ, $\vec{k}$, S_r and ψ_j as above, and given $h_1, \ldots, h_n \in \mathcal{H}$, $c_1, \ldots, c_n \in \mathcal{C}$ such that for all $w \in X$, $S_{r,\vec{h}(w)} = S_{r,\vec{c}(w)}$, then for $j = 1$ and 2, $\mathcal{H} \models \psi_j(\vec{h})$ if and only if $\mathcal{C} \models \psi_j(\vec{c})$.*

PROOF. We proceed by induction on r, the number of quantifier blocks in ψ_1 or ψ_2.

If $r = 0$ then $\psi_1 = \psi_2 = \varphi$ is quantifier-free, and the result follows readily. [One direction: If $\mathcal{H} \models \varphi(\vec{h})$ then $\mathcal{H} \models \varphi_i(\vec{h})$ for some i, and so $\mathbf{R} \models \varphi_i^+(\vec{h}(w))$ for all $w \in X$, while $\mathbf{R} \models \varphi_i^+ \wedge \rho_{i,j}(\vec{h}(w_j))$ for some $w_1, \ldots, w_{n_i} \in X$. This implies $\varphi_i^+ \in S_{0,\vec{c}(w)}$ for all $w \in X$ and $\varphi_1^+ \wedge \rho_{i,j} \in S_{0,\vec{c}(w_j)}$, $j = 1, \ldots, n_i$, by hypothesis. Thus $\mathcal{C} \models \varphi_i(\vec{c})$ and so $\mathcal{C} \models \varphi(\vec{c})$ as desired.]

Now suppose $r > 0$ and $\mathcal{C} \models \psi_1(\vec{c})$, where $\psi_1(\vec{c}) = \exists x_1 \cdots \exists x_{k_1} \psi_1'(x_1, \ldots, x_{k_1}, \vec{c})$. Then there exist $d_1, \ldots, d_{k_1} \in \mathcal{C}$ such that $\mathcal{C} \models \psi_1'(\vec{d}, \vec{c})$. Let

$$W = \operatorname{supp}\{d_1, \ldots, d_{k_1}, c_1, \ldots, c_n, h_1, \ldots, h_n\}.$$

For each $w \in W$ define

$$\sigma_w = \sigma_{\vec{d},w} = \bigwedge \left\{\sigma(x_1,\ldots,x_{k_1}, \vec{y}) \mid \mathbf{R} \models \sigma(\vec{d}(w), \vec{c}(w)), \text{ where } \sigma \text{ or } \neg\sigma \in S_{r-1}\right\}.$$

Note that $\exists x_1 \cdots \exists x_{k_1}\sigma_w \in S_{r,\vec{c}(w)}$ and that $\sigma_{\vec{d},w}$ is the conjunction of the set of formulae $S_{r-1,\vec{d}\frown\vec{c}(w)}$ associated with the formula

$$\varphi(y_{n+1},\ldots,y_{n+k_1}, x_{k_1+1},\ldots,x_{k_2}, y_1,\ldots,y_n),$$

the tuple $k_2,\ldots,k_r$, the tuple $\vec{d} \frown \vec{c}$, and the point w.

Using 2.1 we know that for each $w \in W$, $\{u \in W \mid \mathbf{R} \models \sigma_w(\vec{d}(u), \vec{c}(u))\}$ is compact open and contains w. We choose $\sigma_1,\ldots,\sigma_m$ and $W_1,\ldots,W_m$ such that $W = W_1 \dot\cup \cdots \dot\cup W_m$, $\sigma_i = \sigma_w$ for each $w \in W_i$, W_i compact open. Also for each $w \in W_i$, $\mathbf{R} \models \exists x_1 \cdots \exists x_{k_1}\sigma_i(x_1,\ldots,x_{k_1}, \vec{c}(w))$, hence $\exists x_1 \cdots \exists x_{k_1}\sigma_i \in S_{r,\vec{c}(w)}$, and so by hypothesis $\exists x_1 \cdots \exists x_{k_1}\sigma_i \in S_{r,\vec{h}(w)}$, hence $\mathbf{R} \models \exists x_1 \cdots \exists x_{k_1}\sigma_i(x_1,\ldots,x_k, \vec{h}(w))$.

Applying 2.3 we get $l_1^{(i)},\ldots,l_{k_1}^{(i)}$ with $\operatorname{supp}\{l_1^{(i)},\ldots,l_{k_1}^{(i)}\} \subseteq W_i$ such that $\mathbf{R} \models \sigma_i(l_1^{(i)}(w),\ldots,l_{k_1}^{(i)}(w), \vec{h}(w))$ for all $w \in W_i$. Now let $l_j = l_j^{(1)} + \cdots + l_j^{(m)}$, $j = 1,\ldots,k_1$. Then $S_{r-1,\vec{l}\frown\vec{h}(w)} = S_{r-1,\vec{d}\frown\vec{c}(w)}$ for all $w \in W$, hence for all $w \in X$. By induction, then, $\mathcal{C} \models \psi_1'(\vec{d}, \vec{c})$ implies $\mathcal{H} \models \psi_1'(\vec{l}, \vec{h})$, and so $\mathcal{H} \models \psi_1(\vec{h})$.

Now suppose $\mathcal{C} \models \psi_2(\vec{c})$, where $\psi_2 = \forall x_1 \cdots \forall x_{k_1}\psi_2'(x_1,\ldots,x_{k_1}, \vec{c})$. We must verify $\mathcal{H} \models \psi_2(\vec{h})$; i.e., for any $l_1,\ldots,l_{k_1} \in \mathcal{H}$ that $\mathcal{H} \models \psi_2'(l_1,\ldots,l_{k_1}, \vec{h})$. To do this we proceed as above, using the fact that $\mathbf{Q} \prec \mathbf{R}$, to produce a tuple $d_1,\ldots,d_{k_1}$ in $\mathcal{C}$ bearing the same relationship to $\vec{c}$ as $\vec{l}$ does to $\vec{h}$. (Since functions in $\mathcal{C}$ are locally constant, we do not use 2.1 and 2.3 in this case, and it is less complex.)

The other direction is analogous, with "$\mathcal{H} \models \psi_2(\vec{h})$ implies $\mathcal{C} \models \psi_2(\vec{c})$" being the more complicated of the two cases.

COROLLARY 2.5. $\operatorname{Th}(\mathcal{C}) = \operatorname{Th}(\mathcal{H}) = T^f$, *the finite forcing companion of the theory of abelian l-groups.*

PROOF. Immediate.

Now we are prepared to prove the main result of this section:

THEOREM 2.6. *Any hyperarchimedean l-group satisfying* 1.5(i)–(iii) *is finitely generic.*

PROOF. It suffices to prove that any countable hyperarchimedean l-group $\mathcal{H}$ satisfying 1.5(i)–(iii) is finitely generic, since this implies that all countable elementary substructures of an arbitrary $\mathcal{H}'$ as above are finitely generic, hence $\mathcal{H}'$ is itself finitely generic.

We verify the following criterion for finite genericity of $\mathcal{H}$ (see **[S]** for this formulation):

(*) Given $h_1,\ldots,h_n \in \mathcal{H}$ and a formula $\psi(y_1,\ldots,y_n)$ of L such that $\mathcal{H} \models \psi(\vec{h})$, there is an *existential* formula $\theta(y_1,\ldots,y_n)$ such that

(i) $\mathcal{H} \models \theta(\vec{h})$ and

(ii) $T^f \models \forall\vec{y}(\theta(\vec{y}) \to \psi(\vec{y}))$.

Let $h_1,\ldots,h_n \in \mathcal{H}$. Without loss of generality we assume $h_1,\ldots,h_n \in \mathcal{H}_+$ and that $\psi(y_1,\ldots,y_n)$ is in prenex normal form, beginning with $\exists$:

$$\psi(y_1,\ldots,y_n) \equiv \exists x_1 \cdots \exists x_{k_1} \forall x_{k_1+1} \cdots \forall x_{k_2} \cdots \varphi(x_1,\ldots,x_k, y_1,\ldots,y_n).$$

Consider, as before, the set of local types $\{S_{r,\vec{h}(w)} \mid w \in X\}$. For each $w \in W = \operatorname{supp} \vec{h}$, define $\theta_w = \bigwedge S_{r,\vec{h}(w)}$. Now choose t_w such that $h_1(w),\ldots,h_{t_w}(w)$ are linearly independent over $\mathbf{Q}$, and $h_{t_w+1}(w),\ldots,h_n(w)$ are dependent on $h_1(w),\ldots,h_{t_w}(w)$ over $\mathbf{Q}$. Denote by $G_j = G_j(z_1,\ldots,z_{t_w}) \in \mathbf{Q}[\vec{z}]$ the polynomials such that $h_j(w) = G_j(h_1(w),\ldots,h_{t_w}(w))$, $j = t_w + 1,\ldots,n$. (Here we choose to ignore the necessary rearrangements of h_i's, to avoid notational chaos.) By the Rectangle Lemma applied to $h_1(w),\ldots,h_{t_w}(w)$ and

$$\theta'_w = \theta_w\big(y_1,\ldots,y_{t_w}, G_{t_w+1}(\vec{y}),\ldots,G_n(\vec{y})\big)$$

we find $q_{2,1}, q_{2,2},\ldots,q_{t_w,1}, q_{t_w,2} \in \mathbf{Q}$ such that $h_1(w)q_{j,1} < h_j(w) < h_1(w)q_{j,2}$, $2 \leqslant j \leqslant t_w$, and such that for any $s_1,\ldots,s_{t_w} \in \mathbf{R}$ with $s_1 > 0$ and $s_1 q_{j,1} < s_j < s_1 q_{j,2}$ for $2 \leqslant j \leqslant t_w$, we have $\mathbf{R} \vDash \theta'_w(s_1,\ldots,s_{t_w})$. Now let

$$\psi_w(y_1,\ldots,y_n) = \left(\bigwedge_{j=2}^{t_w} y_1 q_{j,1} < y_j < y_1 q_{j,2}\right) \wedge (y_1 > 0) \wedge$$
$$\wedge \bigwedge_{j=t_w+1}^{n} \big(y_j = G_j(y_1,\ldots,y_{t_w})\big)$$

and let $W_w = \{x \in W \mid \mathbf{R} \vDash \psi_w(h_1(x),\ldots,h_n(x))\}$. Then W_w is compact open, by 2.1, and is nonempty since $w \in W_w$. Thus we write $W = \bigcup_{w \in W} W_w = W_1 \dot\cup \cdots \dot\cup W_m$ as usual, further cutting back W_w's as necessary for disjointness, to get each W_j a nonempty compact open subset of some W_{w_j}, with $\psi_j = \psi_{w_j}$. Let

$$\theta(\vec{y}) = \exists y_{ij}\left(\bigwedge_{1 \leqslant i,i' \leqslant n} \bigwedge_{1 \leqslant j < j' \leqslant m} (y_{ij} \wedge y_{i'j'} = 0) \wedge \right.$$
$$\left. \wedge \bigwedge_{i=1}^{n} \left(y_i = \sum_{j=1}^{m} y_{ij}\right) \wedge \bigwedge_{j=1}^{m} \psi_j(y_{1j},\ldots,y_{nj})\right).$$

Certainly $\mathcal{H} \vDash \theta(\vec{h})$, with $h_i \restriction W_j$ taken for y_{ij}. To see that $T^f \vdash \forall \vec{y}(\theta(\vec{y}) \to \psi(\vec{y}))$, hence verifying $(*)$, we check that $\mathcal{C} \vDash \forall_y(\theta(y) \to \psi(\vec{y}))$ and use the fact that $T^f = \operatorname{Th}(\mathcal{C})$.

Let $c_1,\ldots,c_n \in \mathcal{C}$ such that $\mathcal{C} \vDash \theta(\vec{c})$, and let $c_{ij} \in \mathcal{C}$ be elements which work as y_{ij} in θ. Let $Y_j = \operatorname{supp}\{c_{1j},\ldots,c_{nj}\}$, $j = 1,\ldots,m$, and note that $Y_j \cap Y_{j'} = \varnothing$ for $j \neq j'$. Let $g\colon X \to X$ be a homeomorphism sending Y_j onto W_j, $j = 1,\ldots,m$, and let $c'_1,\ldots,c'_n$ be the images of $c_1,\ldots,c_n$ under the automorphism of $\mathcal{C}$ induced by g. Then by choice of the θ_w's and ψ_w's we have—for each $w \in W$, hence for each $w \in X$—that

$$S_{r,\vec{h}(w)} = S_{r,\vec{c}'(w)}.$$

Therefore by 2.4 we have $\mathcal{C} \vDash \psi(\vec{c}')$, hence $\mathcal{C} \vDash \psi(\vec{c})$, as desired.

COROLLARY 2.7. *Any hyperarchimedean e.c. l-group is finitely generic.*

PROOF. Immediate from 1.5 and 2.6.

As an immediate corollary we have a generalization of a conjecture of Glass and Pierce [**G-P2**] concerning the finite genericity of $\overline{C}(X,\mathbf{R})$. Recall that for A any

divisible totally ordered abelian subgroup of $\mathbf{R}$ (with discrete topology) we denote by $\bar{C}(X, A)$ the l-group of all continuous functions with compact support from X to A.

COROLLARY 2.8. *All groups of the form $\bar{C}(X, A)$ are finitely generic.*

PROOF. Again, the proof of Theorem 8 of [**G-P**] applies to A as well as R, giving that $\bar{C}(X, A)$ is existentially closed. (Alternately, it is an easy check that 1.5(i)–(iii) hold for $\bar{C}(X, A)$.) Since $\bar{C}(X, A)$ is hyperarchimedean (all proper prime subgroups are zero sets, hence maximal), the corollary follows from 2.7.

Our description of finitely generic models also enables us to prove the following embedding result:

COROLLARY 2.9. *Any hyperarchimedean l-group can be embedded in a finitely generic abelian l-group.*

PROOF. Let $\mathcal{A}$ be hyperarchimedean. By 2.6 it suffices to find $\tilde{\mathcal{A}} \supseteq \mathcal{A}$, $\tilde{\mathcal{A}}$ hyperarchimedean and satisfying 1.5(ii)–(iii). Let $\mathcal{A}_{-1}$ be the divisible hull of $\mathcal{A}$ and let $\mathcal{A}_0 = \mathcal{A}_{-1} \oplus \mathcal{C}$. For each $j \geqslant 0$ choose $\mathcal{A}_{j+1} \supseteq \mathcal{A}_j$ such that $\mathcal{A}_{j+1} \approx \mathcal{A}_j \oplus \mathcal{A}_j$, with each $a \in \mathcal{A}_j$ corresponding to $a \oplus a$ in $\mathcal{A}_j \oplus \mathcal{A}_j$. Then let $\tilde{\mathcal{A}} = \bigcup_{j \in \omega} \mathcal{A}_j$. Each element of $\mathcal{A}_j$ is a sum of n disjoint elements in $\mathcal{A}_{j+n}$, and so 1.5(i) holds, while 1.5(i)–(iii) remain in force from $\mathcal{A}_0$ on. Each $\mathcal{A}_j$ is hyperarchimedean, hence $\tilde{\mathcal{A}}$ is, and we have the desired extension of $\mathcal{A}$.

3. Examples. Not all hyperarchimedean l-groups are representable as finite range real-valued functions. It therefore follows from 2.9 that not all finitely generic groups can be thus represented. We choose nonetheless to construct a specific group, one which is $\mathbf{Q}$-generated over $\mathcal{C}$ by two elements. Example 1.13 in [**G-P2**], of a hyperarchimedean e.c. not thus representable is unfortunately incorrect (their group as stated is representable as finite range functions).

We start with $\mathcal{C} = \bar{C}(X, \mathbf{Q})$ as above and divide each X_n into disjoint Cantor sets, $X_n = Y_n \dot{\cup} Z_n$, with $X = \dot{\bigcup}_{n \in \omega} X_n$. We define $\mathbf{1}, \mathbf{b} \in \mathbf{Q}^X$ as follows:

$$\mathbf{1}(x) = \begin{cases} 1, & x \in Y_n \text{ for some } n, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\mathbf{b}(x) = \begin{cases} \pi_n, & x \in Y_n, \\ 0, & \text{otherwise,} \end{cases}$$

where $\pi_n \in \mathbf{Q}$ is chosen so that $|\pi_n - \pi| < 10^{-n}$ for the real π, $\pi_1 < \pi_2 < \cdots < \pi$. Now take $\mathcal{B}$ to be the divisible hull of the group generated by $\mathcal{C} \cup \{\mathbf{1}, \mathbf{b}\}$. Any element of $\mathcal{B}$ can be written uniquely as $c + q_1\mathbf{1} + q_2\mathbf{b}$, where $c \in \mathcal{C}$, $q_1, q_2 \in \mathbf{Q}$.

EXAMPLE 3.1. The group $\mathcal{B}$ is finitely generic and is not representable as a group of real-valued functions with finite range.

PROOF. First we describe the proper prime subgroups of $\mathcal{B}$: those of $\mathcal{C}$ are the zero sets $\mathcal{P}_x = \{f \in \mathcal{C} \mid f(x) = 0\}$, where $x \in X$ (see [**G-P**]). Let $\mathcal{Q}$ be a proper prime in $\mathcal{B}$. Then $\mathcal{Q} \cap \mathcal{C}$ is prime, hence $\mathcal{Q} \cap \mathcal{C}$ is either $\mathcal{C}$ itself or some $\mathcal{P}_x$. We claim that $\mathcal{C}$ is maximal in $\mathcal{B}$, so that $\mathcal{Q} \cap \mathcal{C} = \mathcal{C}$ implies $\mathcal{Q} = \mathcal{C}$. For let $d \in \mathcal{B} - \mathcal{C}$, $d = c + q_1\mathbf{1} + q_2\mathbf{b}$, where q_1 and q_2 are rationals, not both zero, and $c \in \mathcal{C}$. Then the convex

subgroup generated by $\{q_1\mathbf{1} + q_2\mathbf{b}\} \cup \mathcal{C}$ is $\mathfrak{B}$, because $q_1\mathbf{1} + q_2\mathbf{b}$ is bounded away from zero on all but at most one Y_m. Thus $\mathcal{Q} = \mathcal{C}$.

Now suppose $\mathcal{Q} \cap \mathcal{C} = \mathcal{P}_x$. We claim $\mathcal{Q} = \mathcal{Q}_x = \{f \in \mathfrak{B} \mid f(x) = 0\}$. Let $f \in \mathcal{Q}_x$. Choose $f_0 \in \mathcal{P}_x \subseteq \mathcal{Q}$ such that either $f - f_0 \in \mathfrak{B}_+$ or $f_0 - f \in \mathfrak{B}_+$, say the former. Let S be a compact set containing x, and define $g \in C_+$ such that

$$g(w) = \begin{cases} 1, & w \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(f - f_0) \wedge g \in \mathcal{C}_+$ and $((f - f_0) \wedge g)(x) = 0$, hence $(f - f_0) \wedge g \in \mathcal{P}_x \subseteq \mathcal{Q}$. But $g \notin \mathcal{P}_x = \mathcal{Q} \cap \mathcal{C}$, hence $g \in \mathcal{Q}$, so $f - f_0 \in \mathcal{Q}$ and thus $f \in \mathcal{Q}$. This shows $\mathcal{Q}_x \subseteq \mathcal{Q}$; it is easy to see that $\mathcal{Q}_x$ is maximal in $\mathfrak{B}$, hence $\mathcal{Q} = \mathcal{Q}_x$.

Thus the proper prime subgroups of $\mathfrak{B}$ are $\mathcal{C}$ and the $\mathcal{Q}_x$'s. In particular, all proper primes are maximal, and $\mathfrak{B}$ is hyperarchimedean.

It is a routine check that $\mathfrak{B}$ satisfies 1.5(i)–(iii), hence is finitely generic by 2.6.

To see that $\mathfrak{B}$ cannot be represented as a group of functions with finite range, note that for each n,

$$\mathbf{1} = \left(\sum_{i=1}^{n} e_i \wedge \mathbf{1} \right) + d_n, \qquad \mathbf{b} = \sum_{i=1}^{n} (e_i \wedge \mathbf{b}) + d_n'$$

where $d_n \wedge e_i = d_n' \wedge e_i = 0$ for $i \leqslant n$, and where $e_i \wedge \mathbf{1}$, $e_i \wedge \mathbf{b}$, d_n, $d_n' \in \mathfrak{B}$. Since $e_i \wedge \mathbf{b} = \pi_i(e_i \wedge \mathbf{1})$ and there are infinitely many such π_1, at least one of $\mathbf{b}$ and $\mathbf{1}$ must take on infinitely many values in any representation.

EXAMPLE 3.2. There exist $2^{\aleph_0}$ countable finite generic abelian l-groups.

PROOF. The groups $\bar{C}(X, \mathbf{Q} \oplus \mathbf{Q}a)$ for $a \in A$, A a set of algebraically independent irrationals, are all nonisomorphic, as indicated in [**G-P**]. By 2.8, each such group is finitely generic.

We note that the number of nonelementarily equivalent e.c. abelian l-groups is not known to us.

ADDED IN PROOF. Françoise Point has given an alternate treatment of the results of §2, via quantifier-elimination in an expanded language (to appear).

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