ALMOST CONVERGENT AND WEAKLY ALMOST PERIODIC 
FUNCTIONS ON A SEMIGROUP

BY

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ABSTRACT. Let $S$ be a topological semigroup, $UC(S)$ the set of all bounded 
uniformly continuous functions on $S$, $WAP(S)$ the set of all (bounded) weakly 
almost periodic functions on $S$, $E_0(S) := \{ f \in UC(S) : m(|f|) = 0 \text{ for each left and right invariant mean } m \text{ on } UC(S) \}$ and $W_0(S) := \{ f \in WAP(S) : m(|f|) = 0 \text{ for each left and right invariant mean } m \text{ on } WAP(S) \}$.

Among other results, for a large class of noncompact locally compact topological 
semigroups $S$, we show that the quotient space $E_0(S)/W_0(S)$ contains a linear 
isometric copy of $l^\infty$ and so is nonseparable.

1. Introduction. In this paper we study matters that continued to burn inside our 
mind during and after writing our paper [3]. In particular terms undefined here are 
as defined in [3] and we shall refer to [3] at various stages of the paper.

Let $S$ be a (Hausdorff jointly) continuous topological semigroup, $C(S)$ the set of 
all bounded real-valued continuous functions on $S$, $M(S)$ the set of all bounded 
real-valued Radon measures on $S$ and $M_a(S) := \{ \mu \in M(S) : \text{the maps } x \rightarrow |\mu|(x^{-1}C) \text{ and } x \rightarrow |\mu|(Cx^{-1}) \text{ of } S \text{ into } \mathbb{R} \text{ are continuous, for all compact } C \subseteq S \}$. $S$ is said to be the foundation of $M_a(S)$ if $S$ coincides with the closure of \bigcup \{ \text{supp} (\mu) : \mu \in M_a(S) \}$; where $\text{supp} (\mu)$ stands for the support of $\mu$. For each $f \in C(S)$ and $x \in S$ we define $x f$ and $f_x$ in $C(S)$ by

$$
x f(y) := f(xy) \quad \text{and} \quad f_x(y) := f(yx) \quad (y \in S).
$$

Let $UC(S) := \{ f \in C(S) : \text{the maps } x \rightarrow x f \text{ and } x \rightarrow f_x \text{ of } S \text{ into } C(S) \text{ are norm continuous} \}$ and $WAP(S) := \{ f \in C(S) : \text{the set } \{ x f : x \in S \} \text{ is relatively weakly compact} \}$.

A functional $m \in UC(S)^*$ is called a mean on $UC(S)$ if $\|m\| = m(1) = 1$, where $1$ stands for the constant function one on $S$. A mean $m$ is left (or right) invariant if $m(x f) = m(f)$ (or $m(f_x) = m(f)$, respectively), for all $f \in UC(S)$ and $x \in S$. Similarly one defines left and right invariant means on other subspaces of $C(S)$.

We label our most important definitions as 1.1, 1.2 and 1.3.

DEFINITION 1.1. The following sets consist of special cases of the so-called almost convergent functions: $F_0(S) := \{ f \in UC(S) : m(|f|) = 0 \text{ for every left or right invariant mean } m \text{ on } UC(S) \}$ and $E_0(S) := \{ f \in UC(S) : m(|f|) = 0 \text{ for every left} \}$
and right invariant mean \( m \) on \( \text{UC}(S) \)). We also define \( W_0(S) := \{ f \in \text{WAP}(S): m(|f|) = 0 \text{ for every left and right invariant mean } m \text{ on } \text{WAP}(S) \} \).

The following definition is taken from [3].

**Definition 1.2.** For any subsets \( A_1, \ldots, A_n \) of a semigroup \( S \) we define

\[
A_1 \otimes A_2 := \{ A_1 A_2, A_1^{-1} A_2, A_1 A_2^{-1} \},
\]

\[
A_1 \otimes A_2 \otimes A_3 := \left( \bigcup \{ A_1 \otimes B: B \in A_2 \otimes A_3 \} \right) \cup \left( \bigcup \{ B \otimes A_3: B \in A_2 \otimes A_3 \} \right)
\]

and hence inductively define \( A_1 \otimes \cdots \otimes A_n \).

A subset \( B \) of \( S \) is said to be relatively neo-compact if \( B \) is contained in a (finite) union of sets in \( A_1 \otimes \cdots \otimes A_n \) for some compact subsets \( A_1, \ldots, A_n \) of \( S \).

We note in particular that if \( S \) is such that \( C^{-1} D \) and \( DC^{-1} \) are compact sets for all compact subsets \( C \) and \( D \) of \( S \), then \( B \subset S \) is relatively neo-compact if and only if \( B \) is relatively compact.

In the following definition we urge the reader to note that in the case of a topological group all the sets mentioned in (b) are compact.

**Definition 1.3.** A locally compact topological semigroup \( S \) with an identity element 1 is said to have property (E) if \( S \) contains a nonrelatively neo-compact subset (also denoted by the letter) \( E \) such that if \( U \) is a neighbourhood of 1, then for all \( x \in E \) we have that

(a) there is a neighbourhood \( V \) of 1 such that

\[
xV \subseteq Ux \quad \text{and} \quad Vx \subseteq Ux;
\]

(b) if \( C \) and \( D \) are compact subsets and \( p, t \in S \), then

\[
\ell[C, xD; t] := \bigcup \left\{ (yCt^{-1})(yxD): y \in S \right\},
\]

\[
\ell_1[xD; t, p] := \bigcup \left\{ y^{-1}((yxD)t^{-1})p: y \in S \right\},
\]

\[
\ell[C, Dx; t] := \bigcup \left\{ (Dxy)(t^{-1}Cy)^{-1}: y \in S \right\},
\]

\[
\ell_1[Dx; t, p] := \bigcup \left\{ (p(t^{-1}(Dxy)))y^{-1}: y \in S \right\}
\]

are relatively neo-compact subsets of \( S \).

For convenience of notation we shall write

\[
L[xD] := \ell[\{1\}, xD; 1], \quad L[C, FD; T] := \bigcup \{ L[C, xD, t]: x \in F \text{ and } t \in T \}
\]

and similarly define \( R[DX], \ell[C, DF; T], \ell_1[DF; T, P] \) and \( \ell_1[DF; F, F] \) for finite subsets \( F, T \) and \( P \) of \( S \).

When \( S \) is a noncompact locally compact topological group then our definition of property (E) coincides with that of property (E) as defined in [2]. Further, one can prove that if \( S \) is an infinite cancellative discrete topological semigroup with identity element, then indeed \( S \) has property (E). Also if \( S \) is a noncompact locally compact topological semigroup that is cancellative, commutative and has an identity element, then \( S \) has property (E).

Burckel [1, Theorem 3.19] proved that \( W_0(\mathbb{R}) \) is a proper subset of \( E_0(\mathbb{R}) \) where \( \mathbb{R} \) is the additive group of real numbers with the usual topology. Generalising this result, Ching Chou [2, Theorem 5.1] proved that if \( G \) is a locally compact topological group with property (E) and such that \( \{ xax^{-1}: x \in E \cup E^{-1} \} \) is relatively compact.
for each \( a \in G \), then the quotient space \( F_0(G)/W_0(G) \) contains a linear isometric copy of \( l^\infty \). Also Ching Chou [2, p. 177] asked whether the conclusion of his Theorem 5.1 can be extended to every discrete topological group.

In this paper our main result says that if \( S \) is a locally compact topological semigroup such that \( S \) is the foundation of \( M_d(S) \) and \( S \) has property \( (E) \) then \( E_0(S)/W_0(S) \) contains an isometric copy of \( l^\infty \).

So as to maintain clarity we collect together some preliminary results in §2. Our main result is proved in §3. Finally we examine property \( (E) \) in §4.

We are indebted to the referee for helpful criticisms.

2. Preliminaries. We need the following result which is a special case of [4, Corollary 1.4].

**Proposition 2.1.** Let \( S \) be a locally compact topological semigroup with identity element and \( WUC(S) := \{ f \in C(S): \text{the maps } x \rightarrow f_x \text{ and } x \rightarrow xf \text{ of } S \text{ into } C(S) \text{ are weakly continuous} \} \). Then if \( S \) is the foundation of \( M_d(S) \), we have that \( WUC(S) = UC(S) \).

From [6, Theorems 2.2 and 2.4] we have

**Proposition 2.2.** Let \( S \) be a locally compact topological semigroup with identity element 1 and such that \( S \) is the foundation of \( M_d(S) \). Then every compact neighbourhood \( U \) contains an element \( u \) such that \( U^{-1}u \cap uU^{-1} \) is a neighbourhood of 1.

3. The main result. First we prove a lemma. Item (i) of the following lemma is a special form of [3, Key Lemma 3] and is only mentioned here for completeness without reproducing the proof. Thus the proof of our next lemma is essentially a proof for item (ii) of the lemma.

**Lemma 3.1.** Let \( S \) be a locally compact topological semigroup with identity element 1, such that \( S \) is the foundation of \( M_d(S) \). Then if \( S \) has property \( (E) \) we can find a compact neighbourhood \( V \) of 1, sequences \( \{x_1, x_2, \ldots \}, \{y_1, y_2, \ldots \}, \{w_1, w_2, \ldots \} \) and \( \{z_1, z_2, \ldots \} \) in \( E \), with \( w_1 = z_1 = 1 \); such that

\[
V^{-1}(Vx_1y)V^{-1} \cap V^{-1}(Vx_ny)V^{-1} = \emptyset
\]

if any one of the following conditions holds:

(a) \( n \leq m \) and \( i > j \);

(b) \( n > m \), \( i > j \) and \( n \neq i \);

(c) \( n \leq m \), \( i < j \) and \( m \neq j \).

(ii) If \( n \neq m \) then

(a) \( i > j \) and \( k > l \) imply \( V^{-1}(Vx_1y)V^{-1} \cap V^{-1}(Vx_ky)V^{-1} = \emptyset \);

(b) \( i \leq j \) and \( k \leq l \) imply \( (Vz_n)^{-1}(Vx_1y)V^{-1} \cap (Vz_m)^{-1}(Vx_ky)V^{-1} = \emptyset \),

for all \( i, j, k, l, n, m \) in \( \mathbb{N} \).

**Proof.** Without losing generality we assume that \( E \) is a subsemigroup of \( S \). By Proposition 2.2 we fix compact neighbourhoods \( C, D \) of 1 and \( c \in C, d \in D \) such that \( C \subset D^{-1}d \cap dD^{-1} \) and \( C^{-1}c \cap cC^{-1} \) is a neighbourhood of 1. Next, by property \( (E) \) and Proposition 2.2, we choose compact neighbourhoods \( U_1, U_2, U_3, U_4 \) and \( V \) of 1 and \( u \in U_1 \) meeting the following inclusion relations.
\[ \begin{align*}
U_2^2 & \subseteq U_1 \subseteq C^{-1}c, \\
U_3 & \subseteq U_2, \ xU_3 \subseteq U_2x \text{ and } U_4x \subseteq xU_2 \quad (x \in E), \\
U_4' & \subseteq U_3, \\
V & \subseteq U_4' \cup \cup U_4'^{-1} \cap U_4.
\end{align*} \]

For convenience let \( X_p := \{x_1, \ldots, x_p\}, \ Y_p := \{y_1, \ldots, y_p\}, \ W_p := \{w_1, \ldots, w_p\}, \ Z_p := \{z_1, \ldots, z_p\} \) and \( N_p := \{1, 2, \ldots, p\} \) for \( p \in \mathbb{N} \).

Now suppose, by the inductive hypothesis, we have the lemma (i.e. items (i) and (ii)) and item (iii) that:

(iii) \( n < m \) implies that

(a) \( w_n \cup C \subseteq L[W_n \cup C] = \emptyset \),

(b) \( C \subseteq L[R(C \cup Z^n) = \emptyset \),

(c) \( w_n \cup C \cap \mathcal{E}_1 \subseteq \subsetneq L[W_n \cup C; Y_n, Y_n^c] = \emptyset \),

(d) \( C \subseteq R \mathcal{E}_1 \subseteq \subsetneq L[C \cup Z^n; cX_n, cX_n] = \emptyset \),

(e) \( y_n \subseteq \subsetneq L[W_n \cup C; W_n \cup C; Y_n, Y_n^c] \),

(f) \( cx_n \subseteq \subsetneq \mathcal{E}_1[L[C \cup Z^n; C \cup Z^n; cX_n; cX_n] \),

for all \( i, j, k, \) in \( N_p \).

By the definition of relative neo-compactness we can choose \( w_{p+1} \) and \( z_{p+1} \) in \( E \) such that

(1a) \( w_{p+1} \subseteq (V^{-1}(VX_p \cup Y_p)(W_p \cup Y_p)^{-1} - (V^{-1}(VX_p \cup Y_p)(V^{-1})), \)

(1b) \( w_{p+1} \subseteq (L[W_p \cup C])D(ud)^{-1}, \)

(1c) \( w_{p+1} \subseteq L[W_p \cup C; Y_n, Y_n^c](u \cup C)^{-1}, \)

(1a') \( z_{p+1} \subseteq (V^{-1}(VX_p \cup Y_p)^{-1}((VZ_p)^{-1}(VX_p \cup Y_p)^{-1}), \)

(1b') \( z_{p+1} \subseteq (du)^{-1}(DR(C \cup Z_p)), \)

(1c') \( z_{p+1} \subseteq (C \cup Z_p \subseteq \mathcal{E}_1[C \cup Z_p; cX_p, cX_p] \).

Next we choose \( x_{p+1} \) and \( y_{p+1} \) in \( E \) such that item (i) is met for \( i, j, m, n \) in \( N_{p+1} \)

which is possible by our proof of [3, Lemma 3] (and by our definition of a relatively

neo-compact set),

(2a) \( x_{p+1} \subseteq (V^{-1}(VX_p \cup Y_p)(W_{p+1} \cup V)^{-1} - (V^{-1}(VX_p \cup Y_p)(V^{-1}), \)

(2b) \( x_{p+1} \subseteq c^{-1}(C \cup Z_{p+1} \cup Z_{p+1}; cX_p), \)

(3a) \( y_{p+1} \subseteq (VX_{p+1})^{-1}((VZ_{p+1})^{-1}(VX_{p+1} \cup Y_p)^{-1}), \)

(3b) \( y_{p+1} \subseteq \mathcal{E}_1[L[W_{p+1} \cup C, W_{p+1} \cup C; Y_p, c] \).

Now for \( i, j, k, l, m, n \) in \( N_{p+1} \) we have from (*)

\[ V^{-1}(VX_i \cup Y_i) \cup V^{-1}(VX_k \cup Y_k)^{-1} \cap V^{-1}(VX_n \cup Y_n)^{-1} \cap u^{-1} \cap V^{-1}(VX_i \cup Y_i)^{-1} \cap V^{-1}(VX_k \cup Y_k)^{-1} \cap V^{-1}(VX_n \cup Y_n)^{-1} \cap u^{-1} \]

\[ \subseteq u^{-1}(VX_i \cup Y_i)^{-1} \cap u^{-1}(VX_k \cup Y_k)^{-1} \cap u^{-1}(VX_n \cup Y_n)^{-1} \cap u^{-1} \]

\[ \subseteq u^{-1}((VX_i \cup Y_i)^{-1} \cap (VX_k \cup Y_k)^{-1} \cap (VX_n \cup Y_n)^{-1}) \]

\[ \subseteq u^{-1}((VX_i \cup Y_i)^{-1} \cap (VX_k \cup Y_k)^{-1} \cap (VX_n \cup Y_n)^{-1}) \]

\[ \subseteq u^{-1}((VX_i \cup Y_i)^{-1} \cap (VX_k \cup Y_k)^{-1} \cap (VX_n \cup Y_n)^{-1}) \]

\[ \subseteq u^{-1}((VX_i \cup Y_i)^{-1} \cap (VX_k \cup Y_k)^{-1} \cap (VX_n \cup Y_n)^{-1}) \]
Hence setting \( A(i, j, n; k, l, m) := (x_i y_j c)(w_n u C)^{-1} \) \( \cap \) \( (x_k y_l c)(w_m u C)^{-1} \) we have
\[
(4) \quad V^{-1}(V x_i y_j V) V^{-1} w_n^{-1} \cap V^{-1}(V x_k y_l V) V^{-1} w_m^{-1} \subseteq u^{-1} A(i, j, n; k, l, m).
\]
Similarly setting \( B(i, j, n; k, l, m) := (C u z_n)^{-1}(c x_i y_j) \cap (C u z_m)^{-1}(c x_k y_l) \) we obtain
\[
(5) \quad z_n^{-1} V^{-1}(V x_i y_j V) V^{-1} \cap z_m^{-1} V^{-1}(V x_k y_l V) V^{-1} \subseteq B(i, j, n; k, l, m) u^{-1}.
\]
We are now in a position to verify the inductive step. Already item (i) is done as remarked before. Items (iii)(a) and (b), for \( n, m \) in \( \mathbb{N}_{p+1} \), should be clear in view of (1b), (1'b) and the inclusion relationships
\[
L[W_p u C] (u C)^{-1} \subseteq L[W_p u C] D (u u d)^{-1},
\]
\[
(C u)^{-1} R[C u Z_p] \subseteq (du)^{-1} (C R[C u Z_p]).
\]
Also with \( \mathbb{N}_{p+1} \) in place of \( \mathbb{N}_p \) items (iii)(c), (d), (e) and (f) follow from items (1c), (1'c), (3b) and (2b) (respectively) and the inductive hypothesis.

Next we prove (ii)(a) for \( i, j, k, l, m, n \) in \( \mathbb{N}_{p+1} \). To this end we assume that \( n < m \) and consider the following cases:

Case (a): \( i, k < p \).
If \( m < p \), the result follows by the inductive hypothesis, so we assume \( m = p + 1 \).
Now item (1a) is equivalent to
\[
v_p x_p y_p v_p \{ w_p y_p x_p \} x_p w_p x_p = 0,
\]
which in turn implies (ii)(a) (under the present case).

Case (b): \( i = p + 1 \) or \( k = p + 1 \) and \( i \neq k \).
The reader can easily deduce (ii)(a), for this case, from item (2a).

Case (c): \( i = k = p + 1 \) and \( j, l < m \).
From item (4) it is sufficient to show that \( A = A(p+1, j, n; p+1, l, m) = \emptyset \).
Suppose on the contrary there exists \( a \in A \). Then from the definition of \( A \) we have
\[
aw_n u c_1 = x_{p+1} y_1 c \quad \text{and} \quad aw_m u c_2 = x_{p+1} y_1 c
\]
for some \( c_1 \) and \( c_2 \) in \( C \).

Hence
\[
w_m \in a^{-1}\left( (a W_n u C)(Y_c)^{-1}) Y_1 c \right)(u C)^{-1} \subseteq C_1 [W_n u C; Y_c, Y_1 c] (u C)^{-1}
\]
where \( t := \max\{j, l\} \). This contradicts (iii)(c) (with \( \mathbb{N}_{p+1} \) in place of \( \mathbb{N}_p \)). By this conflict we have (ii)(a) under the present case.

Case (d): \( i = k = p + 1 \), \( j \neq l \) and \( m \leq \max\{j, l\} \).
We assume that \( l < j \) and suppose there exists \( a \in A(p+1, j, n; p+1, l, m) \).
Then by the definition of the latter set we have
\[
y_1 c \in \left( (a W_n u C)(Y_c)^{-1}) (a W_n u C) \subseteq C [W_n u C, W_m u C; Y_1 c]
\]
which contradicts item (iii)(e) (with \( \mathbb{N}_{p+1} \) in place of \( \mathbb{N}_p \)). By this conflict \( A = \emptyset \) and item (4) gives the result.

Case (e): \( (i, j) = (k, l) \).
Suppose there exists \( a \in A(i, j, n; k, l, m) \). Then
\[
aw_n u c_1 = x_i y_j c = x_k y_l c = aw_m u c_2 \quad \text{for some} \ c_1, c_2 \in C.
\]
Hence
\[ w_m uC_2 \subseteq a^{-1}(aw_n uC) \cap w_m uC \subseteq L(W_n uC) \cap w_m uC, \]
which contradicts item (iii)(a) (with \( N_{p+1} \) in place of \( N_p \)). By this conflict \( A = \emptyset \) and item (4) implies the result.

This completes our proof for the inductive step for (ii)(a).
Similarly from items (iii)(b), (d) and (f); (l’a), (l’c); (2b), (3a) and (5), we obtain the inductive step for (ii)(b).

Repeating the argument countably many times we obtain our lemma.

We now give our main result.

**Theorem 3.2.** Let \( S \) be a locally compact topological semigroup with identity element 1 and such that \( S \) is the foundation of \( M_a(S) \). Then if \( S \) has property (E) we have that the quotient space \( E_0(S)/W_0(S) \) contains a linear isometric copy of \( l^\infty \) and so is nonseparable.

**Proof.** We now indicate how the proof of [3, Theorem 2.1] can be extended to yield our result.
Let \( \rho \) be a positive measure in \( M_a(S) \) with \( \| \rho \| = 1 \) and \( \text{supp}(\rho) \subseteq V \); where \( V \) is as stated in Lemma 3.1. We assume the notation of Lemma 3.1 throughout this proof. Take the measures \( \nu \) and \( \mu \) used in the proof of [3, Theorem 2.1] to be equal to \( \rho \) and define the sequence of functions \( \{ f_k \} \) as done there (i.e. in the proof of [3, Theorem 2.1]). Now Proposition 2.1 says that \( \text{WUC}(S) = \text{UC}(S) \), consequently \( \text{WAP}(S) \subseteq \text{UC}(S) \).

So the proof of [3, Theorem 2.1] would show that the mapping
\[ \{ c_k \} \rightarrow \sum_{k=1}^{\infty} c_k f_k + W_0(S) \]
is a linear isometric map of \( l^\infty \) into \( E_0(S)/W_0(S) \) if we can show that the function \( f := \sum_{k=1}^{\infty} c_k f_k \) is in \( E_0(S) \). Already \( f \in \text{WUC}(S) = \text{UC}(S) \), so it remains to show that \( m(|f|) = 0 \) for every left and right invariant mean \( m \) on \( \text{UC}(S) \).

From the definition of the \( f_k \)'s we can write \( f = h - g \) for some \( h, g \in \text{UC}(S) \) such that

(1) \( \text{supp}(z_1 | h |) \subseteq \bigcup_{i=1}^{\infty} \bigcup_{i<j} (Vz_n)^{-1}(Vx_i y_j V)V^{-1}, \)

(2) \( \text{supp}(|g| w_n) \subseteq \bigcup_{j=1}^{\infty} \bigcup_{j>i} V^{-1}(Vx_i y_j V)(Vw_n)^{-1}, \)

for all \( n \in \mathbb{N} \). From (1) and Lemma 3.1(ii)(b) we have that any two members of the sequence \( \{ z_n | h | \} \) have disjoint supports. So if \( m \) is any left and right invariant mean on \( \text{UC}(S) \), then for any \( n \in \mathbb{N} \) we have that

\[ nm(|h|) = m(z_1 | h |) + m(z_2 | h |) + \cdots + m(z_n | h |) \]
\[ = m(z_1 | h | + z_2 | h | + \cdots + z_n | h |) \leq \| h \| m(1) = \| h \|. \]
Hence $m(|h|) = 0$. Similarly by using (2) and Lemma 3.1(ii)(a) we obtain that $m(|g|) = 0$ and hence $m(|f|) = 0$ and $f \in E_0(S)$. This completes our proof.

As an immediate consequence we have

**Corollary 3.3.** If $G$ is a locally compact topological group with property $(E)$ then $E_0(G)/W_0(G)$ contains an isometric linear copy of $l^\infty$.

### 4. Some remarks on property $(E)$.  

#### 4.1.  
Every infinite discrete cancellative semigroup with identity element has property $(E)$. Also if $T$ is a (noncompact) cancellative commutative locally compact topological semigroup with identity element and $H$ any compact semigroup with identity, then clearly the product semigroup $T \times H$ has property $(E)$. In this way one can construct various examples of noncancellative and noncommutative topological semigroups with property $(E)$.

#### 4.2.  
In this item we sharpen [3, Remark 7.5]. Let $G$ be a locally compact topological group throughout this item and $C_00(G)$ be the set of all functions in $C(G)$ with compact support. Following [2], a set $X \subseteq G$ is called an $E$-set if given a neighbourhood $U$ of 1 the set $\{x^{-1}Ux: x \in X\}$ is again a neighbourhood of $G$. It is shown in [2] that an $E$-set $X \subseteq G$ has the property that, for a compact neighbourhood $U$ of 1 such that $xU \cap yU = \emptyset$ for distinct $x, y \in X$ and function $f \in C_00(G)$ with $\text{supp}(f) \subseteq U$, we have that $g := \sum_{x \in X} x f$ belongs to $UC(G)$. Such sums of translates of a function with compact support are useful in finding functions in $UC(G) \setminus WAP(G)$. Our next proposition teaches us that, conversely, if $X \subseteq G$ is a set such that “such functions” $g := \sum_{x \in X} x f$ are in $UC(G)$, then the set $X$ must be an $E$-set.

**Proposition A.** Let $V$ and $U$ be a compact neighbourhoods of the identity of $G$, let $f \in C_00(G)$ be such that $f(1) = 1$ and $\text{supp}(f) \subseteq V$, $V^2 \subseteq U$ and let $X \subseteq G$ be such that for all $x, y \in X$ with $x \neq y$ we have that $xU \cap yU = \emptyset$. Then if $g := \sum_{x \in X} x f$ is in $UC(G)$, we have that $\cap \{x^{-1}Ux: x \in X\}$ is a neighbourhood of 1.

**Proof.** Assuming $g \in UC(G)$, we can find a neighbourhood $W$ of 1 such that $W \subseteq V$ and

$$|g(ay) - g(y)| \leq \|a g - g\| < 1 \quad (a \in W \text{ and } y \in G).$$

Thus $|\sum_{x \in X} (f(xay) - f(xy))| < 1$. Taking $y = x_0^{-1}$ (for any fixed $x_0$ in $X$) and recalling that $xU \cap yU = \emptyset$ for distinct $x, y$ in $X$ we get $|f(x_0 ax_0^{-1}) - f(1)| < 1$. Hence $x_0 ax_0^{-1} \in V$, for all $a \in W$, or $W \subseteq x_0^{-1}Vx_0$. Thus $W \subseteq \cap \{x^{-1}Vx: x \in X\}$ and our proposition follows.

Note that by choosing $V$’s contracting to 1, the preceding proposition says that, for every neighbourhood $V$ of 1 we have that $\cap \{x^{-1}Vx: x \in X\}$ is a neighbourhood of 1 and thus $X$ is an $E$-set.

**Definitions.** A set $B \subseteq G$ is said to be right uniformly discrete if there is a neighbourhood $U$ of 1 such that $Ux \cap Uy = \emptyset$ for distinct $x, y \in B$.

$G$ is said to be $\alpha$-compact if $G$ can be written as a union of $\alpha$ compact sets and cannot be written as a union of $\beta$ compact sets if $\beta < \alpha$. 

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Proposition B (cf. [5, Theorem 2.3]). If $G$ is $\alpha$-compact, then $G$ has equivalent left and right uniformities if and only if for each right-uniformly discrete set $B \subseteq G$ satisfying $\text{card}(B) < \alpha$ and each neighbourhood $U$ of 1, we have $\cap \{x^{-1}Ux : x \in B\}$ a neighbourhood of 1.

Hence from Propositions A and B we have

Proposition C. If $G$ is $\alpha$-compact then every right-uniformly discrete $B \subseteq G$ yields $g := \Sigma_{x \in B} f$ in UC($G$) if and only if $G$ has equivalent left and right uniformities; where $f$ is chosen such that $(\text{supp}(f))x \cap (\text{supp}(f))y = \emptyset$ for distinct $x, y$ in $B$.

Equivalently (to Proposition C) we have that every right uniformly discrete set in $G$ is an $E$-set if and only if $G$ has equivalent left and right uniformities.

4.3. In view of [2, Theorem 4.6], the following conjecture seems reasonable.

Conjecture. Let $S$ be a locally compact semigroup with identity element such that $x^{-1}K$ and $Kx^{-1}$ are compact sets for all compact $K \subseteq S$ and $x \in S$ and with $S$ the foundation of $M_\sigma(S)$. Then if $S$ has property $(E)$ we have that $W_0(S)/C_0(S)$ is nonseparable. (Here $C_0(S)$ is the set of functions in $C(S)$ vanishing at infinity.)

References


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