NONFACTORIZATION THEOREMS 
IN WEIGHTED BERGMAN AND HARDY SPACES 
ON THE UNIT BALL OF C^n (n > 1) 

BY 
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ABSTRACT. Let \( A^{p,a}(B) \), \( A^{q,a}(B) \) and \( A^{1,a}(B) \) be weighted Bergman spaces on the unit ball of \( C^n \) \((n > 1)\). We prove:

**Theorem 1.** If \( 1/l = 1/p + 1/q \) then \( A^{p,a}(B) \cdot A^{q,a}(B) \) is of first category in \( A^{1,a}(B) \).

**Theorem 2.** Theorem 1 holds for Hardy spaces in place of weighted Bergman spaces.

We also show that Theorems 1 and 2 hold for the polydisc \( U^n \) in place of \( B \).

1. Introduction. Let \( U \) be the unit disc in \( C \). For \( 0 < t < \infty \) and \( -1 < \alpha < \infty \), let \( H^t(U) \) be the Hardy space of all holomorphic functions \( f \) on \( U \) satisfying

\[
\sup_{0<r<1} \int_{-\pi}^{\pi} |f(re^{i\theta})|^t \, d\theta < \infty,
\]

and let \( A^{t,a}(U) \) be the weighted Bergman space of all holomorphic functions \( f \) on \( U \) satisfying

\[
\int_{U} |f(z)|^t \left(1 - |z|^2\right)^a \, dm(z) < \infty,
\]

where \( dm(z) \) denotes the Lebesgue measure on \( U \). If \( 0 < p, q, l < \infty \) and \( 1/p + 1/q = 1/l \), then it is well known that \( H^p(U) \cdot H^q(U) = H^l(U) \), where the left-hand side consists of all products of the form \( f \cdot g \) with \( f \in H^p(U) \) and \( g \in H^q(U) \). Horowitz [3] proved that \( A^{p,a}(U) \cdot A^{q,a}(U) = A^{1,a}(U) \) whenever \( \alpha > 0 \) and \( 1/p + 1/q = 1/l \).

In \( C^n \) \((n > 1)\), the above results are no longer valid. Rudin [6] and Miles [4] showed that \( H^2(U^n) \cdot H^2(U^n) \) is a proper subset of \( H^1(U^n) \) for \( n \geq 3 \). (Here \( U^n \) denotes the unit polydisc in \( C^n \).) Rosay [5] showed that \( H^2(U^n) \cdot H^2(U^n) \) is of first category in \( H^1(U^n) \) for \( n \geq 2 \), thereby completely solving the Factorization Problem (see [6, 4.2]) in Hardy spaces of the polydisc. In [7, Problem 19.3.1], Rudin asked whether \( H^2(B) \cdot H^2(B) \) is properly contained in \( H^1(B) \), where \( B \) denotes the unit ball of \( C^n \) \((n > 1)\). In this paper we show that \( H^p(B) \cdot H^q(B) \) is of first category in \( H^1(B) \) whenever \( 0 < p, q, l < \infty \) and \( 1/p + 1/q = 1/l \). We prove a similar result.__

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(Theorem 1) for the weighted Bergman spaces on the unit ball $B$ (see §2 for notations and terminology). Essential ideas required to prove these results come from Rosay [5].

Coifman, Rochberg and Weiss [1] proved that any function in $H^1(B)$ is an infinite sum of the form $\sum_{i=1}^{\infty} f_i g_i$ where $f_i$ and $g_i$ belong to $H^2(B)$ for all $i$. We do not know if the infinite sum can be replaced by a finite sum (see Remark 4).

2. Preliminaries. Notations are as in [7]. For $z = (z_1, z_2, \ldots, z_n)$ and $w = (w_1, w_2, \ldots, w_n)$ in $\mathbb{C}^n$, let $\langle z, w \rangle = \sum_{i=1}^{n} z_i \overline{w}_i$ and $|z|^2 = \sum_{i=1}^{n} |z_i|^2$; let $B = B_n = \{z \in \mathbb{C}^n: |z| < 1\}$ and $S = \{z \in \mathbb{C}^n: |z| = 1\}$. For $z \in \mathbb{C}^n$ we sometimes write $z = (z_1, z')$ where $z' = (z_2, z_3, \ldots, z_n)$, $e_1 = (1, 0, 0, \ldots, 0)$.

Let $a, z \in B$ and $a \neq 0$, let

$$\phi(z) = \frac{\frac{a - P_0 z}{1 - \langle z, a \rangle} - (1 - |a|^2)^{1/2} Q_0 z}{1 - \langle z, a \rangle},$$

where $P_0 z = \langle z, a \rangle a / \langle a, a \rangle$ and $Q_0 z = z - P_0 z$. $\phi(z)$ is a holomorphic automorphism of $B$ satisfying $\phi(\phi(z)) = z$.

$d\sigma$ denotes the rotation invariant probability measure on $S$. $dv(z) = dv_n(z) = 2n \pi^{2n-1} dr \, d\sigma(\xi)$ is the normalized Lebesgue measure on $B$. Here $z = r \xi, r = |z|$ and $\xi \in S$.

$H(B)$ denotes the space of all holomorphic functions on $B$.

$C(B)$ denotes the space of all continuous functions on $\overline{B}$.

$A(B) = H(B) \cap C(B)$ is the ball algebra.

For $0 < t < \infty$, $H^t(B)$ is the Hardy space of all $f \in H(B)$ satisfying

$$\|f\|_{t,a} = \left( \sup_{0 < r < 1} \int_S |f(r \xi)| t \, d\sigma(\xi) \right)^{1/t} < \infty.$$ 

Let

$$d\mu(z) = (1 - |z|^2)^{\alpha} \, dv(z) / nB(n, \alpha + 1)$$

where $-1 < \alpha < \infty$ and $B(n, \alpha + 1)$ denotes the Beta function. For $-1 < \alpha < \infty$ and $0 < t < \infty$, we write $A^{t,a}(B)$ to denote the space of all $f \in H(B)$ satisfying

$$\|f\|_{t,a} = \left( \int_B |f|^t \, d\mu_\alpha \right)^{1/t} < \infty.$$ 

We note that $d\mu_\alpha$ is a probability measure on $B$ and

$$\lim_{\alpha \to 1} \int_B f(z) \, d\mu_\alpha(z) = \int_S f(\xi) \, d\sigma(\xi)$$

for all $f \in C(\overline{B})$. (The above relation holds for monomials $z_1^{\beta_1} z_2^{\beta_2} \ldots z_n^{\beta_n} \overline{z}_1^{\gamma_1} \ldots \overline{z}_n^{\gamma_n}$ and hence for linear combinations of monomials. The Stone-Weierstrass theorem proves (1) for any $f \in C(\overline{B})$.)
Because of (1) we can think of $H^p(B)$ as a “limiting” case of $A^{p,\alpha}(B)$ for $\alpha = -1$. Let $f \in H(B)$. Then from [7, Theorem 7.2.5],
\[
|f(z)|^p \left(1 - \frac{|z|}{r}\right)^n \leq 2^n \int_{S} |f(r\xi)|^p \, d\sigma(\xi)
\]
for $|z| < r < 1$ and $0 < p < \infty$.

Let $K$ be a compact subset of $B$. If we multiply the above inequality by $(1 - r^2)\alpha r^{2n-1} \, dr$ and integrate over the interval $(1 + |z|)/2 < r < 1$, we get
\[
|f(z)| \leq C_{n,\alpha,p,K} \|f\|_{p,\alpha}(1 - |z|)^{-n/p} \quad (\forall z \in K)
\]
where $C_{n,\alpha,p,K}$ is a constant depending only on its subscripts.

The above two inequalities, together with a normality argument, give

**Fact 1.** Every bounded sequence in $H^p(B)$ (or in $A^{p,\alpha}(B)$) has a subsequence which converges uniformly on compact subsets of $B$.

From this it follows that $A^{p,\alpha}(B)$ and $H^p(B)$ are $F$-spaces.

Let $f \in H^p(B)$ ($A^{p,\alpha}(B)$) and $f_r(z) = f(rz)$ for $0 < r < 1$. Then $f_r \to f$ in $H^p(B)$ (in $A^{p,\alpha}(B)$) as $r \to 1$. For a suitable $r$ and $\delta$ ($0 < \delta < 1$), $(1 - z_1)f_r(z)/(1 - \delta z_1)$ is close to $f$ in $H^p(B)$ (in $A^{p,\alpha}(B)$) and vanishes at $e_1$. Hence we have

**Fact 2.** The set of all $f \in A(B), f(e_1) = 0$, is dense in $H^p(B)$ ($A^{p,\alpha}(B)$).

We need the following identities [7, Proposition 1.4.7]:

\[
(2) \int_S f(\xi) \, d\sigma(\xi) = \int_S \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}\xi) \, d\theta\right) \, d\sigma(\xi),
\]

\[
(3) \int_S f(\xi_1, \xi') \, d\sigma(\xi) = \int_{B_{e_1}} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}\xi_1, \xi') \, d\theta\right) \, d\nu_{n-1}(\xi').
\]

3. Lemmas.

**Lemma 1.** Let $0 < t < \infty$ and $\alpha > -1$. Then $\|z_n^2\|_{t,\alpha} \sim N^{-(n+\alpha)}$ as $N \to \infty$.

**Proof.** We have
\[
\|z_n^2\|_{t,\alpha}^t = \frac{2}{B(n, \alpha + 1)} \int_B \left((z_n^2)^t (1 - r^2) \alpha r^{2n-1} \, dr \, d\sigma(\xi)\right) = \left(\int_S z_n^2 \, d\sigma(\xi)\right) \left(\frac{2}{B(n, \alpha + 1)} \int_0^1 r^{Nt + 2n - 1} (1 - r^2) \alpha \, dr\right).
\]

The second integral, on putting $u = r^2$ becomes $\frac{1}{2} B(Nt/2 + n, \alpha + 1)$. By Stirling’s formula this behaves like $1/N^{\alpha+1}$ as $N \to \infty$. For the first integral, we use the identity [7, 1.4.5, p. 15]
\[
\int_S f(\xi, \eta) \, d\sigma(\xi) = \frac{n-1}{\pi} \int_U (1 - r^2)^{n-2} f(re^{i\theta}) r \, dr \, d\theta.
\]

We get
\[
\int_S z_n^2 \, d\sigma(\xi) = \frac{n-1}{\pi} \int_0^1 (1 - r^2)^{n-2} r^{Nt+1} \, dr = \frac{n-1}{\pi} B \left(\frac{Nt + 1}{2}, n - 1\right) \sim 1/N^{n-1}
\]
by Stirling’s formula.
Hence
\[ \| z_2^N \|_{t, \alpha} \sim \frac{1}{N^{n-1}} \cdot \frac{1}{N^{n+1}} = N^{-(n+1)}. \]

**Remark 1.** \[ \| z_2^N \|_{t, \alpha} \sim N^{-(n-1)}. \]

**Lemma 2.** Let \( K(z) = \sum_{i=N}^\infty K_i(z) \) be holomorphic in \( B \), where \( K_i(z) \) is a homogeneous polynomial of degree \( i \) and \( N \) is a positive integer. Then for \( 0 < t < \infty \), there exists a constant \( M \) (depending only on \( t \)) such that
\[ \| K_N \|_{t, \alpha} \leq M \cdot \| K \|_{t, \alpha}, \]
\[ \| K_N \|_{t, \alpha} \leq M \cdot \| K \|_{t, \alpha}. \]

**Proof.** For \( 0 < t < \infty \), there exists an \( M \) such that if \( G(\lambda) = a_0 + a_1 \lambda + a_2 \lambda^2 + \cdots \) is in the disc algebra \( A(U) \) then
\[ |a_i| \leq M \cdot \frac{1}{2\pi} \int_{-\pi}^\pi |G(e^{i\theta})|^t d\theta. \]
(For \( t \geq 1 \) we can take \( M = 1 \). For \( 0 < t < 1 \), see [2, Theorem 6.4, p. 98]. In fact, \( M = 2^{1/t} \) works for any \( t \).) Now for a fixed \( z \), let
\[ G(\lambda) = K(\lambda z)/\lambda^{N-1} = K_{N-1}(z) + \lambda K_N(z) + \cdots. \]
We get
\[ |K_N(z)|^t \leq M \cdot \frac{1}{2\pi} \int_{-\pi}^\pi |K(e^{i\theta}z)|^t d\theta. \]
We let \( z = r \zeta \), integrate both sides with respect to \( d\sigma(\zeta) \) and use (2) to get
\[ \int_S |K_N(r \zeta)|^t d\sigma(\zeta) \leq M \int_S |K(r \zeta)|^t d\sigma(\zeta). \]
Taking the supremum over \( r \) in the interval \( 0 < r < 1 \) and \( t \) th roots, we get (4). To get (4'), we multiply both sides of (5) by \( (2/B(n, \alpha + 1)) r^{2n-1}(1 - r^2)^{\alpha} dr \), integrate over \( 0 < r < 1 \) and take \( t \) th roots.

**Lemma 3.** Let \( 0 < p, q, l < \infty, 1/l = 1/p + 1/q, -1 < \alpha < \infty \) and \( n > 1 \). Then the product map \((h, k) \to h \cdot k \) from \( A^{p, \alpha}(B) \times A^{q, \alpha}(B) \) to \( A^{l, \alpha}(B) \) is not open at the origin, i.e., for any constant \( C > 0 \), there exists \( f \in A^{l, \alpha}(B) \) such that \( \| f \|_{l, \alpha} \leq C \) and if \( f = h \cdot k \) with \( h \in A^{p, \alpha}(B), k \in A^{q, \alpha}(B) \) then at least one of \( \| h \|_{p, \alpha}, \| k \|_{q, \alpha} \) is larger than \( C \).

**Proof.** Let \( F(z) = z_1^{N-1} + z_2^N, N > 1 \). Suppose \( F(z) = H(z) \cdot K(z) \) with \( H \) and \( K \) holomorphic in \( B \). We expand \( H(z) \) and \( K(z) \) in terms of homogeneous polynomials: \( H = H_1 + H_{i+1} + \cdots, K = K_{N-1-i} + K_{N-1-i} + \cdots \). Here, as usual, subscript refers to the degree, \( H_i \equiv 0 \) and \( K_{N-1-i} \equiv 0 \). From \( F = H \cdot K \) we get, by comparing degrees,
\[ H_i \cdot K_{N-1-i} = z_1^{N-1} \]
and
\[ H_i K_{N-1-i} + H_{i+1} K_{N-1-i} = z_2^N. \]
From (6) and (7) we get \( i = 0 \) or \( N - 1 \). We assume for a moment that \( i = 0 \). Then

\[
AK_N(z) = z_i^N - \left( H_i(z) \cdot z_i^{N-1} \right)/A.
\]

Letting \( z = r(e^{i\theta}, \xi') \) we get

\[
AK_N\left( e^{i\theta}, \xi' \right) = z_i^N - A^{-1} H_i(e^{i\theta}, \xi') e^{i(N-1)\theta} z_i^{N-1}.
\]

Therefore, \( z_i^N \) is the constant term in the polynomial \( AK_N(\lambda^i, \xi') \) in \( \lambda \). By subharmonicity,

\[
\left| z_i^N \right| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| AK_N\left( e^{i\theta}, \xi' \right) \right| d\theta \quad \text{for } 0 < t < \infty.
\]

Now we multiply both sides of (8) by \( d\nu_{n-1}(\xi') \) and integrate over \( B_{n-1} \). Using (3), we get

\[
\int_S \left| z_i^N \right| d\sigma(\xi) \leq \int_S \left| AK_N(\xi) \right| d\sigma(\xi)
\]

and

\[
\int_S \left| (r z_i^N) \right| d\sigma(\xi) \leq \int_S \left| AK_N(r \xi) \right| d\sigma(\xi).
\]

We multiply both sides of (9) by \((2/B(n, a + 1)) r^{2n-1}(1 - r^2)^a dr\), integrate over \( 0 < r < 1 \) and take \( t \)th roots to get

\[
\| z_i^N \|_{t, a} \leq \| A \| \| K_N \|_{t, a}.
\]

Since \( |H(z)|^t \) is subharmonic and \( A = H(0) \), we have \( |A|^t \leq \int_S |H(\xi)|^t d\sigma(\xi) \). From this we get \( |A|^t \leq \| H \|_{t, a} \). Hence

\[
\| z_i^N \|_{t, a} \leq \| A \| \| K_N \|_{t, a} \leq \| H \|_{t, a} \| K_N \|_{t, a}.
\]

Using Lemma 2, we get \( \| z_i^N \|_{t, a} \leq M \| H \|_{t, a} \| K \|_{t, a} \). By symmetry, this inequality holds when \( i = N - 1 \). Now let \( f = F/\| F \|_{t, a} \). Then \( \| f \|_{t, a} = 1 \). Suppose \( f = h \cdot k \) where \( h \in A^{p, a}(B) \) and \( k \in A^{q, a}(B) \). Then \( F = H \cdot K \) where \( H = \| F \|_{t, a} h \) and \( K = k \). Therefore

\[
\| z_i^N \|_{t, a} \leq M \| H \|_{t, a} \| K \|_{t, a} \leq M \| F \|_{t, a} \| h \|_{t, a} \| k \|_{t, a}.
\]

Now we take \( t = \min(p, q) \). Then \( \| h \|_{t, a} \leq \| h \|_{p, a} \) and \( \| k \|_{t, a} \leq \| k \|_{q, a} \). We have

\[
\| F \|_{t, a} = \int_B \left| z_i^{N-1} + z_i^N \right|^t d\mu_a \leq 2^t \left[ \| z_i^{N-1} \|_{t, a}^t + \| z_i^N \|_{t, a}^t \right].
\]

By Lemma 1, the right side of the above inequality is like \( N^{-(n+1)} \) for large \( N \). We see that

\[
\| h \|_{p, a} \cdot \| k \|_{q, a} \geq \| z_i^N \|_{t, a} / M \| F \|_{t, a}.
\]

Hence \( \| h \|_{p, a} \| k \|_{q, a} \) is bigger than a constant times \( N^{-(n+1)}(1/t^t + 1/i^i) \) which goes to \( \infty \) as \( N \to \infty \) (recall \( t = \min(p, q) > 1 \)). Therefore, for any constant \( C \), we can find a large \( N \) so that \( \| h \|_{p, a} \cdot \| k \|_{q, a} > C^2 \). This completes the proof.

**Remark 2.** By considering \( H^p \)-norms instead of \( A^{p, a} \)-norms, one can get the nonopenness of the product map (at the origin) for \( H^p \)-spaces.
Lemma 4. For $a \in B$ and $z \in \overline{B}$, let

$$K(a, z) = \left(1 - |a|^2 / |1 - \langle z, a \rangle|^2\right)^{(n+1)+\alpha)}.$$

Then:

(i) $K(a, \phi_a(z)) \cdot K(a, z) = 1$.

(ii) $\int_B f(\omega) \, d\mu_a(\omega) = \int_B f(\phi_a(z)) K(a, z) \, d\mu_a(z)$ for all $f \in C(\overline{B})$.

(iii) $\int_B f(\phi_a(z)) \, d\mu_a(z) \to f(e_1)$ as $a \to e_1$ for all $f \in C(\overline{B})$.

Proof. From [7, Theorem 2.2.5], we have

$$1 - \langle \phi_a(z), a \rangle = (1 - |a|^2) / (1 - \langle z, a \rangle).$$

Taking absolute values and using the definition of $K$, we get (i). From [7, Theorem 2.2.6] we have

$$\int_B f(\omega) (1 - |\omega|^2)^{\alpha} \, d\nu(\omega) = \int_B f(\phi_a(z)) (1 - |\phi_a(z)|^2)^{\alpha} \left(1 - |a|^2 / \left|1 - \langle z, a \rangle\right|^2\right)^{(n+1)+\alpha} \, d\nu(z).$$

Using

$$1 - |\phi_a(z)|^2 = (1 - |a|^2)(1 - |z|^2) / |1 - \langle z, a \rangle|^2 \quad \text{(see [7, Theorem 2.2.5])},$$

$$\int_B f(\omega) (1 - |\omega|^2)^{\alpha} \, d\nu(\omega) = \int_B f(\phi_a(z)) \left(1 - |a|^2\right)^{(n+1)+\alpha} \left(1 - |z|^2\right)^{\alpha} / \left|1 - \langle z, a \rangle\right|^{2(n+1)+\alpha} \, d\nu(z).$$

Hence

$$\int_B f(\omega) \, d\mu_a(\omega) = \int_B f(\phi_a(z)) K(a, z) \, d\mu_a(z).$$

This is (ii). Since $\lim_{a \to e_1} \phi_a(z) = e_1$, an application of the Bounded Convergence Theorem gives (iii).

Remark 3. In the above lemma we assumed that $\alpha > -1$. The following statements hold when $\alpha = -1$.

(i) $K(a, \phi_a(z)) \cdot K(a, z) = 1$.

(ii) $\int_S f(\eta) \, d\sigma(\eta) = \int_S f(\phi_a(\xi)) K(a, \xi) \, d\sigma(\xi)$ for all $f \in C(S)$.

(iii) $\int_S f(\phi_a(\xi)) \, d\sigma(\xi) \to f(e_1)$ as $a \to e_1$ for all $f \in C(S)$.

We observe that when $\alpha = -1$, $K(a, z)$ is the Poisson kernel and statements (i) and (ii) are well known. Since $\int_S f(\phi_a(\xi)) \, d\sigma(\xi)$ is the Poisson integral of $f$, (iii) follows (see, e.g., [7, Theorem 3.3.4(a)]).

Lemma 5. Let

$$\psi_a(z) = \left[1 + \sqrt{1 - |a|^2} / (1 - \langle z, a \rangle)\right]^{2(n+1)+\alpha)}.$$

Then

$$\max\{1, K(a, z)\} \leq |\psi_a(z)| \leq 2^{(n+\alpha)+1} \{1 + K(a, z)\}$$

for all $z \in \overline{B}$, $a \in B$ and $\alpha > -1$. 
Proof. For $\lambda \in \mathbb{C}$ and $\text{Re} \lambda \geq 0$, we have $\max\{1, |\lambda|\} \leq |1 + \lambda|$. This can be seen by plotting $\lambda$ and $1 + \lambda$ in the complex plane. Also, $|1 + \lambda|^m \leq (1 + |\lambda|)^m \leq 2^{m-1}(1 + |\lambda|^m)$ for $m \geq 1$. Taking $\lambda = \sqrt{1 - |a|^2/(1 - \langle z, a \rangle)}$ and $m = 2(n + 1 + \alpha)$, we get the lemma.

4. Main theorem.

**Theorem 1.** Let $n > 1$, $-1 < \alpha < \infty$, $0 < p, q, l < \infty$ and $1/l = 1/p + 1/q$. Then $A^p, \alpha(\mathcal{B}) \cdot A^q, \alpha(\mathcal{B})$ is of first category in $A^l, \alpha(\mathcal{B})$.

Proof. Let $V$ and $W$ be the closed unit balls in $A^p, \alpha(\mathcal{B})$ and $A^q, \alpha(\mathcal{B})$, respectively. We claim that $V \cdot W$ is closed in $A^l, \alpha(\mathcal{B})$. Let $g_m \in V$, $h_m \in W$ such that $g_m \cdot h_m \to f$ in $A^l, \alpha(\mathcal{B})$. By Fact 1 (of §2) we may assume, without loss of generality, that $g_m \to g$ and $h_m \to h$ uniformly on compact subsets of $\mathcal{B}$. By Fatou’s Lemma, $g \in V$ and $h \in W$. Since $g_m \cdot h_m \to f$ uniformly on compact subsets of $\mathcal{B}$, $f = g \cdot h \in V \cdot W$. Hence the claim. We have $A^p, \alpha(\mathcal{B}) \cdot A^q, \alpha(\mathcal{B}) = \bigcup_{m=1}^{\infty} (mV \cdot W)$. We show that $mV \cdot W$ has empty interior in $A^l, \alpha(\mathcal{B})$ for each $m \geq 1$. Assume the contrary. Then some $mV \cdot W$ will have an interior point in $A^l, \alpha(\mathcal{B})$. There exist an $R \in A^l, \alpha(\mathcal{B})$ and a constant $C$ such that

\begin{equation}
\int_{\mathcal{B}} |R - F|^l d\mu_{\alpha} \leq 2^{l+1+\alpha} A^l, \alpha(\mathcal{B}), \text{ implies } \\
F = g \cdot h \text{ with } \|g\|_{p, \alpha} \leq C \text{ and } \|h\|_{q, \alpha} \leq C.
\end{equation}

By Fact 2 (of §2), we may assume that $R$ is a function in $A(\mathcal{B})$ vanishing at $e_1$. Now by Lemma 3, for the constant $C$ there is an $f \in A^l, \alpha(\mathcal{B})$ such that $\|f\|_{l, \alpha} \leq 1$ and $f = g \cdot h$, $g \in A^p, \alpha(\mathcal{B})$, $h \in A^q, \alpha(\mathcal{B})$ imply that at least one of $\|g\|_{p, \alpha}$, $\|h\|_{q, \alpha}$ is larger than $C$. There is an $\epsilon > 0$ such that

\begin{equation}
\|f - f_1\|_{l, \alpha} \leq \epsilon \text{ and } f_1 = g_1 \cdot h_1 \in A^p, \alpha(\mathcal{B}) \cdot A^q, \alpha(\mathcal{B}) \text{ implies either } \|g_1\|_{p, \alpha} > C \text{ or } \|h_1\|_{q, \alpha} > C.
\end{equation}

We may assume, after Fact 2, that $f$ is a function in $A(\mathcal{B})$ vanishing at $e_1$.

We now come to perhaps the most important single step in the proof (see [5]). Let $F(z) = f(\phi_a(z))\psi_\alpha^{1/l}(z) + R(z)$. ($\phi_a(z)$ is defined in §2 and $\psi_\alpha(z)$ is defined in Lemma 5.) Now

\begin{align*}
\int_{\mathcal{B}} |F - R|^l d\mu_{\alpha} &= \int |f(\phi_a(z))|^{1/l} \psi_\alpha(z)\ d\mu_{\alpha} \\
&\leq 2^{2(n+\alpha)+1} \left[ \int_{\mathcal{B}} |f(\phi_a)|^{1/l} d\mu_{\alpha} + \int_{\mathcal{B}} |f(\phi_a(z))|^{1/l} K(a, z)\ d\mu_{\alpha}(z) \right]
\end{align*}

by Lemma 5. The second integral in the above inequality is $\int_{\mathcal{B}} |f|^l d\mu_{\alpha}$ by (ii) of Lemma 4 and the first integral goes to zero as $a \to e_1$, by (iii) of Lemma 4 (recall...
that \( f(e_1) = 0 \). Hence when \( a \) is close to \( e_1 \),
\[
\int_B |F - R|' \, d\mu_a \leq 2^{2(n+a)+1} \left[ 1 + \int_B |f|' \, d\mu_a \right]
\leq 2^{2(n+a)+1} [1 + 1] \quad \text{(since } \|f\|_{\ell, a} \leq 1) \]
\[
= 4^{n+a+1}.
\]
By (10), \( F = g \cdot h \) with \( \|g\|_{p,a} \leq C \) and \( \|h\|_{q,a} \leq C \). Therefore \( f(\phi_a(z)) \cdot \psi_a^{1/q}(z) + R(z) = g(z) \cdot h(z) \). Replacing \( z \) by \( \phi_a(z) \) and using \( \phi_a(\phi_a(z)) = z \), we get
\[
f(z) + \frac{R(\phi_a(z))}{\psi_a^{1/q}(\phi_a(z))} = \frac{g(\phi_a(z)) \cdot h(\phi_a(z))}{\psi_a^{1/q}(\phi_a(z))} = \frac{g(\phi_a(z))}{\psi_a^{1/p}(\phi_a(z))} \cdot \frac{h(\phi_a(z))}{\psi_a^{1/q}(\phi_a(z))}.
\]
We have
\[
\int_B \left| \frac{R(\phi_a)}{\psi_a^{1/p}(\phi_a)} \right|^' \, d\mu_a \leq \int_B |R(\phi_a)|' \, d\mu_a
\]
by Lemma 5. Since \( R(e_1) = 0 \), the right side integral in the above inequality goes to zero as \( a \to e_1 \) by (iii) of Lemma 4. Hence if \( a \) is close to \( e_1 \), (11) holds with \( f_1 = g_1 \cdot h_1 \) where
\[
g_1 = g(\phi_a) / \psi_a^{1/p}(\phi_a) \quad \text{and} \quad h_1 = h(\phi_a) / \psi_a^{1/q}(\phi_a).
\]
Therefore either \( \|g_1\|_{p,a} > C \) or \( \|h_1\|_{q,a} > C \). Suppose \( \|g_1\|_{p,a} > C \). Then
\[
C^p < \int_B \left| \frac{g(\phi_a)}{\psi_a^{1/p}(\phi_a)} \right|^p \, d\mu_a
\leq \int_B \left| \frac{|g|}{K(a, \phi_a(z))} \right|^p \, d\mu_a(z) \quad \text{(by Lemma 5)}
\leq \int_B |g| K(a, z) \, d\mu_a(z) \quad \text{(by (i) of Lemma 4)}
\leq C^p \quad \text{(since } \|g\|_{p,a} \leq C).
\]
We reach a contradiction. Similarly \( \|h_1\|_{q,a} > C \) gives a contradiction. Hence all \( m(V \cdot W) \) have empty interiors. So
\[
A^{p,a}(B) \cdot A^{q,a}(B) = \bigcup_{m=1}^{\infty} m(V \cdot W)
\]
is of first category in \( A^{l,a}(B) \).

5. Other results. Here is a nonfactorization theorem for Hardy spaces.

**Theorem 2.** Let \( n > 1 \) and \( 0 < p, q, l < \infty \). If \( 1/l = 1/p + 1/q \) then \( H^p(B) \cdot H^q(B) \) is of first category in \( H^l(B) \).
The proof of this theorem is very similar to that of Theorem 1. One has to integrate functions in the Hardy class $H'(B)$ (for $t = p, q$ and $l$) with respect to $d\sigma$ over $S$. $\alpha$ should be replaced by $-1$ (relation (1) can also be used at appropriate places). We omit the details. Theorem 2 can also be proved, for $n > 2$, using Theorem 1 (with $\alpha = 0$) and Theorem 7.2.4 in [7].

Remark 4. Let $T$ be the mapping $(f_1, g_1, f_2, g_2, \ldots, f_k, g_k) \to \Sigma_{i=1}^k f_i g_i$. The proof of Theorem 1 shows that

$$T: A^{p,\alpha}(B) \times A^{q,\alpha}(B) \times \cdots \times A^{p,\alpha}(B) \times A^{q,\alpha}(B) \to A^{l,\alpha}(B)$$

$(1/l = 1/p + 1/q)$ is onto if and only if it is open at the origin. Nonopenness of $T$ at the origin would imply the existence of a function in $A^{l,\alpha}(B)$ which is not of the form $\Sigma_{i=1}^k f_i g_i$, with $f_i \in A^{p,\alpha}(B)$ and $g_i \in A^{q,\alpha}(B)$. However, any function $F$ in $A^{l,\alpha}(B)$ (for $\alpha = 0, 1, 2, \ldots$) can be written as $F = \Sigma_{i=1}^\infty G_i H_i$ where $G_i$ and $H_i$ belong to $A^{p,\alpha}(B)$ (see [1, Theorem IV]). Similar statements can be made for Hardy spaces.

Remark 5. Let $0 < t < \infty$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_i > -1$. Let $A^{t,\alpha}(U^n)$ be the space of all holomorphic functions satisfying $\|f\|_{t,\alpha} = \left( \int_{U^n} |f|^t \mu_\alpha \right)^{1/t} < \infty$ where $d\mu_\alpha(z) = \prod_{i=1}^n (1 - |z_i|^2)^{\alpha_i} dm_i(z_i)$ being the Lebesgue measure on $U$ for all $i = 1, 2, \ldots, n$. Then Theorem 1 holds for $U_n$ in place of $B$. We sketch a proof of this statement. If $K(z) = \Sigma_{i=1}^{\infty - 1} K_i(z)$ is as in Lemma 2 then

$$\int_{-\pi}^{\pi} |K_N(r e^{i\theta}, e^{i\theta} z')|^t \ d\theta \leq C_t \int_{-\pi}^{\pi} |K(r e^{i\theta}, e^{i\theta} z')|^t \ d\theta$$

and hence $\|K_N\|_{t,\alpha} \leq M_t \|K\|_{t,\alpha}$ where $C_t$ and $M_t$ are constants depending only on $t$. Without loss of generality let $\alpha_1 \geq \alpha_2$. We have

$$\|z_i^N\|_{t,\alpha} \sim N^{-(1+\alpha_i)} \quad (i = 1, 2),$$

by Lemma 1 and using $F = z_2^{N-1} + z_2^N$ we get (imitating the proof of Lemma 3) the nonopenness of the product map from $A^{p,\alpha}(U^n) \times A^{q,\alpha}(U^n)$ to $A^{l,\alpha}(U^n)$ where $1/p + 1/q = 1/l$. (If

$$AK_N(z) = z_2^N - (H_1(z) \cdot z_1^{N-1})/A$$

then

$$AK_N(r e^{i\theta}, z') = z_2^N - \frac{H_1(r e^{i\theta}, z')}{A} e^{i(N-1)\theta} r_1^{N-1}$$

and

$$\int_{-\pi}^{\pi} |z_2^N|^t \ d\theta \leq C_t \|z_2^N\|^t \int_{-\pi}^{\pi} |K_N(r e^{i\theta}, z')|^t \ d\theta \quad \text{etc.}$$

For $0 < r < 1$, let

$$a = (r, 0, 0, \ldots, 0), \quad \phi_a(z) = (r - z)/(1 - rz), \quad z_2, z_3, \ldots, z_n, \quad$$

$$K(a, z) = \left( (1 - r^2)/|1 - rz|^2 \right)^{2+\alpha_1}$$

and

$$\phi_a(z) = \left( 1 + \sqrt{1 - r^2}/(1 - rz) \right)^{2(2+\alpha_1)}.$$
We note that as $r \to 1$, $a \to (1, 0, \ldots, 0)$ and $\phi_0(z) \to (1, z')$. Observe that functions $f$ in $A(U^n)$ with $f(1, z') \equiv 0$ form a dense subset of $A^{p,0}(U^n)$. With minor changes, one can get results similar to Lemmas 4 and 5. By imitating the proof of Theorem 1, we get the polydisc version of Theorem 1.

**Remark 6.** Let $H'(U^n)$ be the Hardy space of all holomorphic functions $f$ in $U^n$ satisfying

$$
\|f\|_{t,a} = \left( \sup_{0<r<1} \int_{T^n} |f(r\xi)|^t \, d\sigma(\xi) \right)^{1/t} < \infty
$$

where $T^n$ is the torus in $\mathbb{C}^n$ and $d\sigma$ is the normalized Haar measure on $T^n$.


To sketch a proof, let $P = (z_1 + z_2)^N - z_1^N - Nz_1^{N-1}z_2$. Then $\|P\|_{t,a} / \|P\|_{t,a} \to \infty$ as $N \to \infty$ whenever $t > l$. There exists a constant $C_t$ such that if $AK_N = P(z) + z_1^{N-1}Q(z)$, where $Q(z)$ is any linear polynomial in $z$, then $\|P\|_{t,a} \leq C_t \|A\| \|K_N\|_{t,a}$ (use subharmonicity in $z_2$). The function $f = (P + z_1^{N-1})/\|P + z_1^{N-1}\|_{t,a}$ gives the nonopenness of the product map. Changing $a_1$ to $-1$ and making other minor changes in the proof of Remark 5, we get Theorem 2 for $U^n$.

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**References**


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