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MEROMORPHIC FUNCTIONS THAT SHARE FOUR VALUES

BY

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ABSTRACT. An old theorem of R. Nevanlinna states that if two distinct nonconstant meromorphic functions share four values counting multiplicities, then the functions are Möbius transformations of each other, two of the shared values are Picard values for both functions, and the cross ratio of a particular permutation of the shared values equals -1. In this paper we show that if two nonconstant meromorphic functions share two values counting multiplicities and share two other values ignoring multiplicities, then the functions share all four values counting multiplicities.

1. Introduction. We say that two meromorphic functions f and g share the value c ($c = \infty$ is allowed) provided that $f(z) = c$ if and only if $g(z) = c$. We will state whether a shared value is by **CM** (counting multiplicities), by **IM** (ignoring multiplicities), or by **DM** (by different multiplicities at one point or more). In this paper the term "meromorphic" will mean meromorphic in the whole complex plane.

R. Nevanlinna proved the following two well-known theorems.

THEOREM A [**5**, p. 109]. *If two nonconstant meromorphic functions f and g share five values* **IM**, *then $f \equiv g$.*

THEOREM B [**5**, p. 122]. *If two distinct nonconstant meromorphic functions f and g share four values $\{a_i\}_{i=1}^4$* **CM**, *then f is a Möbius transformation of g, two of the shared values, say a_1 and a_2, must be Picard values, and the cross ratio $(a_1, a_2, a_3, a_4) = -1$. (We mention that it is written incorrectly as $(a_1, a_3, a_2, a_4) = -1$ in* [**5**].)

For example, if h is a nonconstant entire function then e^h and e^{-h} share $0, \infty, \pm 1$ **CM**. Recently the author has shown that the hypothesis of Theorem B can be relaxed somewhat by proving the following result.

THEOREM C [**2**]. *If two nonconstant meromorphic functions f and g share three values* **CM** *and share a fourth value* **IM**, *then f and g share all four values* **CM** (*hence if $f \not\equiv g$ the conclusions of Theorem* B *hold*).

On the other hand the following example [**2**] shows that we cannot simply replace "**CM**" by "**IM**" in Theorem B. If h is a nonconstant entire function and b is a

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nonzero constant then

$$f = \frac{e^h + b}{(e^h - b)^2} \quad \text{and} \quad g = \frac{(e^h + b)^2}{8b^2(e^h - b)} \tag{1}$$

share $0, \infty, 1/b$, and $-1/8b$ by **DM** at *every point*. In contrast to Theorem B, f is not a Möbius transformation of g, none of the shared values are Picard values, and the cross ratio of any permutation of the shared values does not equal -1.

The main purpose of this paper is to further "close the gap" between Theorem C and example (1) by proving the following result.

THEOREM 1. *If two nonconstant meromorphic functions f and g share four values $\{a_i\}_{i=1}^4$, a_1 and a_2 both* **CM** *and a_3 and a_4 both* **IM**, *then f and g share all four values* **CM**.

At the Classical Complex Analysis Conference (Purdue University, March, 1980), Erwin Mues showed the author a proof of the following special case of Theorem 1.

THEOREM D. *If two nonconstant meromorphic functions f and g share* $0, 1$ **CM** *and* $\infty, \frac{1}{2}$ **IM**, *then f and g share all four values* **CM**.

Mues had several proofs of Theorem D; they involved using results from [**2**] and choosing linear combinations of logarithmic derivatives in clever ways so that the Nevanlinna theory could be applied. With these methods he also had a proof of Theorem C that is different from the author's in [**2**]. His proofs of Theorems D and C directly yield all the conclusions of Theorem B (when $f \not\equiv g$) without appealing to Theorem B.

For the proof of Theorem 1 we shall use results from [**2**], some preliminary lemmas and corollaries, the ideas of Mues, plus variations, extensions, and refinements of the ideas of Mues. We mention that some of the methods and ideas used in the proof of Theorem 1 were used to solve another shared value problem in [**3**].

Our proof of Theorem 1 will consist of proving the following two theorems.

THEOREM E. *If two nonconstant meromorphic functions f and g share* $0, \infty$ **CM** *and* $a, -a$ **IM** $(a \neq 0, \infty)$, *then f and g share all four values* **CM**.

THEOREM 2. *If two nonconstant meromorphic functions f and g share* $0, \infty$ **CM** *and* a, b **IM** $(a, b \neq 0, \infty;\ a \neq \pm b)$, *then f and g share all four values* **CM**.

Theorem 1 follows immediately from Theorem E, Theorem 2, and a Möbius transformation. Theorems E and D are equivalent because the Möbius transformation $L(z) = az/(z-1)$ carries the points $0, 1, \infty, \frac{1}{2}$ into the points $0, \infty, a, -a$ respectively. We will not appeal to Theorem C in our proofs of Theorems E and 2.

There is one more "gap" between Theorem 1 and example (1), namely the open question: Does there exist two distinct nonconstant meromorphic functions that share three values by **DM** and a fourth value **CM**?

We also mention the following open question. If two distinct nonconstant meromorphic functions share four values **IM** and the cross ratio of at least one permutation of the shared values is equal to -1, then do the functions necessarily

share all four values **CM**? This open question is suggested by Theorem B, example (1), and Theorem D. Regarding Theorem D (where for example, $(0, 1, \infty, \frac{1}{2}) = -1$) it should be mentioned that when Mues started from the hypothesis of Theorem D with "$\frac{1}{2}$" replaced by a constant "c" ($c \neq 0, 1, \infty$, otherwise arbitrary) and tried to obtain the conclusion of Theorem D, he found that in order for each one of his proofs to work it seemed necessary that $c = \frac{1}{2}$. The reader can see the analogous situation for Theorem E (where for example, $(0, \infty, a, -a) = -1$) in Remark 3 in §4.

Lee A. Rubel originally posed the question to the author of, what can be said when "**CM**" is replaced by "**IM**" in the hypothesis of Theorem B?

This paper is organized as follows.

In §2 we exhibit some general results on distinct nonconstant meromorphic functions that share four values. In §3 we prove Theorem E. In §4 we make some remarks about the proofs of Theorems E and 2. In §5 we prove Theorem 2. Finally, in §6 we use the methods in this paper to give a proof of Theorem B.

These proofs of Theorems E, 2 and B all use the results from §2.

I would like to thank the referee for many comments and suggestions.

2. Some general results. In this section we will exhibit some results on distinct nonconstant meromorphic functions that share four values. These results will be used in the proof of Theorem 1.

We will assume that the reader is familiar with the basic notations and fundamental results of Nevanlinna's theory of meromorphic functions, as found in [**4**]. We make the following two definitions.

DEFINITION 1. If two meromorphic functions f and g share the value c, then we set $\overline{N}(r, c) = \overline{N}(r, f, c) = \overline{N}(r, g, c)$.

DEFINITION 2. We will denote by $S(r, f)$ any quantity satisfying

$$S(r, f) = o(1)T(r, f)$$

as $r \to \infty$, possibly outside of a set of finite linear measure.

We will use the following result throughout the paper.

THEOREM F. *If two distinct nonconstant meromorphic functions f and g share four values $\{a_i\}_{i=1}^4$* **IM**, *then the following conditions hold*:

$$T(r, f) = T(r, g) + S(r, g); \tag{2}$$

$$\sum_{i=1}^{4} \overline{N}(r, a_i) = 2T(r, f) + S(r, f); \tag{3}$$

$$\overline{N}(r, f, c) + S(r, f) = T(r, f) \quad \textit{and} \quad \overline{N}(r, g, c) + S(r, g) = T(r, g) \tag{4}$$

if $c \neq a_i$ $(i = 1, 2, 3, 4)$.

Equations (2) and (3) are proven in [**2**], while (4) is an immediate consequence of (2), (3), and Nevanlinna's second fundamental theorem.

We will now prove some results.

LEMMA 1. *Let f and g be two distinct nonconstant meromorphic functions that share four values $\{a_i\}_{i=1}^4$* **IM**. *Then the following properties hold*:

$$\sum_{i=1}^{4} \overline{N}(r, a_i) = \overline{N}(r, f-g, 0) + S(r, f) \tag{5}$$

if $a_i \neq \infty$ for $i = 1, 2, 3, 4$;

$$\sum_{i=2}^{4} \overline{N}(r, a_i) = \overline{N}(r, f-g, 0) + S(r, f) \quad \text{if } a_1 = \infty; \tag{6}$$

$$N(r, f-g, 0) = \overline{N}(r, f-g, 0) + S(r, f); \tag{7}$$

$$N_5(r, f-g, 0) = S(r, f) \tag{8}$$

where $N_5(r, f-g, 0)$ "counts" only those points z such that $f(z) = g(z) \neq a_i$ for $i = 1, 2, 3, 4$.

REMARK. We mention that (8) was proven in [**2**].

PROOF OF LEMMA 1. First suppose that $a_i \neq \infty$ for $i = 1, 2, 3, 4$. From (2), (3), and Jensen's theorem we obtain the following inequalities:

$$\begin{aligned} 2T(r, f) + S(r, f) = \sum_{i=1}^{4} \overline{N}(r, a_i) &\leq \overline{N}(r, f-g, 0) \\ &\leq \sum_{i=1}^{4} \overline{N}(r, a_i) + N_5(r, f-g, 0) \\ &\leq N(r, f-g, 0) \leq T(r, f-g, 0) \\ &= T(r, f-g) + O(1) \\ &\leq T(r, f) + T(r, g) + O(1) \\ &= 2T(r, f) + S(r, f). \end{aligned} \tag{9}$$

It follows that all the expressions in (9) are equal to $2T(r, f) + S(r, f)$. Thus we see that (5), (7), and (8) hold.

Now suppose that $a_1 = \infty$. Let c be a constant that is not one of the shared values and set

$$F = \frac{1}{f-c} \quad \text{and} \quad G = \frac{1}{g-c}. \tag{10}$$

Then F and G share four finite values **IM**. Since

$$F - G = \frac{g-f}{(f-c)(g-c)},$$

it follows that

$$N(r, f-g, 0) - N_{cc}(r, f-g, 0) + \overline{N}(r, \infty) \leq N(r, F-G, 0) \tag{11}$$

where $N_{cc}(r, f-g, 0)$ "counts" only those points z such that $f(z) = g(z) = c$. From (2), (3), (10), (11), and Nevanlinna's first fundamental theorem we obtain the following inequalities:

$$
\begin{aligned}
(12)\quad 2T(r,f) + S(r,f) &= \sum_{i=1}^{4} \overline{N}(r, a_i) \\
&\leqslant \overline{N}(r, f-g, 0) - \overline{N}_{cc}(r, f-g, 0) + \overline{N}(r, \infty) \\
&\leqslant N_5(r, f-g, 0) - N_{cc}(r, f-g, 0) + \sum_{i=1}^{4} \overline{N}(r, a_i) \\
&\leqslant N(r, f-g, 0) - N_{cc}(r, f-g, 0) + \overline{N}(r, \infty) \\
&\leqslant N(r, F-G, 0) \\
&\leqslant T(r, F) + T(r, G) + O(1) \\
&= 2T(r, f) + S(r, f).
\end{aligned}
$$

Thus all the expressions in (12) are equal to $2T(r, f) + S(r, f)$. It follows that

$$(13)\qquad N_5(r, f-g, 0) - N_{cc}(r, f-g, 0) = S(r, f).$$

Hence if b is not a shared value and $b \neq c$ then from (13) we obtain

$$(14)\qquad N_{bb}(r, f-g, 0) = S(r, f).$$

Since c was an arbitrarily chosen nonshared value, it is clear from (14) that

$$(15)\qquad N_{cc}(r, f-g, 0) = S(r, f).$$

If we substitute (15) back into (13) and (12) we will obtain (6), (7), and (8). The proof of Lemma 1 is now complete.

COROLLARY 1. *If two distinct nonconstant meromorphic functions f and g share four values $\{a_i\}_{i=1}^4$* **IM** *then we have the following two properties*:

I. *If a_j is shared* **CM** *then*

$$(16)\qquad N(r, f, a_j) = N(r, g, a_j) = \overline{N}(r, a_j) + S(r, f).$$

II. *If $N_4(r, a_i)$ refers only to those a_i-points that are multiple for both f and g and "counts" each point the number of times of the smaller of the two multiplicities, then*

$$(17)\qquad \sum_{i=1}^{4} N_4(r, a_i) = S(r, f).$$

PROOF. We can assume that all the shared values are finite because if $a_1 = \infty$ we can consider F and G in (10). From (5) and (7) we obtain

$$(18)\qquad \sum_{i=1}^{4} \overline{N}(r, a_i) + S(r, f) = N(r, f-g, 0).$$

Suppose that a_j is shared **CM**. If f and g both have an a_j-point of multiplicity k, then $f-g$ has a zero of multiplicity at least k. Hence

$$N(r, f, a_j) - \overline{N}(r, a_j) + \sum_{i=1}^{4} \overline{N}(r, a_i) \leqslant N(r, f-g, 0). \tag{19}$$

From (18) and (19) we obtain (16).

Now suppose that z_1 is an a_i-point of multiplicity $k \geqslant 2$ for f and of multiplicity $m \geqslant 2$ for g, and set $n = \min(k, m)$. Then z_1 will be a zero of multiplicity at least n for $f-g$. Since z_1 is "counted" exactly n times in $N_4(r, a_i)$ we see that

$$\sum_{i=1}^{4} \left(N_4(r, a_i) - \overline{N}_4(r, a_i)\right) + \sum_{i=1}^{4} \overline{N}(r, a_i) \leqslant N(r, f-g, 0). \tag{20}$$

From (20) and (18) we obtain

$$\sum_{i=1}^{4} \left(N_4(r, a_i) - \overline{N}_4(r, a_i)\right) = S(r, f). \tag{21}$$

Since $N_4(r, a_i) \geqslant 2\overline{N}_4(r, a_i)$ for each i, we see that (17) follows from (21).

The next result was stated in [**2**].

LEMMA 2. *If two distinct nonconstant meromorphic functions f and g share four values $\{a_i\}_{i=1}^4$* **IM**, *then*

$$N_0(r, f', 0) = S(r, f) \quad \textit{and} \quad N_0(r, g', 0) = S(r, g), \tag{22}$$

where $N_0(r, f', 0)$ refers only to those roots of $f'(z) = 0$ such that $f(z) \neq a_i$ for $i = 1, 2, 3, 4$ ($N_0(r, g', 0)$ is similarly defined).

PROOF. From the second fundamental theorem we obtain

$$2T(r, f) \leqslant \sum_{i=1}^{4} \overline{N}(r, a_i) - N_0(r, f', 0) + S(r, f).$$

Thus $N_0(r, f', 0) = S(r, f)$ from (3). The proof that $N_0(r, g', 0) = S(r, g)$ is similar.

We next make some observations. First note that example (1) satisfies the following properties:

$$\sum_{i=1}^{4} N(r, f, a_i) + \sum_{i=1}^{4} N(r, g, a_i) = 6T(r, f) + S(r, f);$$

$$\frac{N(r, f', 0)}{T(r, f')} \to \frac{1}{3}, \quad \frac{N(r, g', 0)}{T(r, g')} \to \frac{2}{3}$$

as $r \to \infty$, perhaps outside a set of finite linear measure. On the other hand, let f and g be functions as in Theorem B. From the second fundamental theorem, Theorem B, and (2) we obtain

$$\sum_{i=1}^{4} N(r, f, a_i) + \sum_{i=1}^{4} N(r, g, a_i) = 4T(r, f) + S(r, f).$$

Since f and g both have two Picard values it is easy to deduce that

$$N(r, f', 0) = S(r, f') \quad \text{and} \quad N(r, g', 0) = S(r, g'). \tag{23}$$

Regarding (23) we mention the book of Wittich [**6**, II, 3].

This leads to the following two open questions. Let f and g be two distinct nonconstant meromorphic functions that share four values $\{a_i\}_{i=1}^4$ **IM**.

1. Is $\Sigma_{i=1}^4 N(r, f, a_i) + \Sigma_{i=1}^4 N(r, g, a_i) \leqslant 6T(r, f) + S(r, f)$?

2. Is $N(r, f', 0)/T(r, f') + N(r, g', 0)/T(r, g') \leqslant 1 + o(1)$ as $r \to \infty$ outside a possible exceptional set of finite linear measure?

I would like to thank Thomas P. Czubiak and the referee for the same remark that concerned question 1.

We conclude this section by noting the following result.

THEOREM G. *If two distinct nonconstant meromorphic functions share four values* **IM** *then both functions are transcendental.*

PROOF. Since the two functions share four values it follows from Picard's theorem that either both functions are rational or both functions are transcendental. Adams and Straus [**1**] have shown that two nonconstant rational functions that share four values **IM** are identical. Theorem G follows.

REMARK. It has not been necessary to distinguish between rational and transcendental functions in our proofs here or in [**2**].

3. Proof of Theorem E. There will be a recurring theme in the proof. The purpose of Remark 1 in §4 is to illustrate this theme; hence the reader may want to read this remark before reading the proof here.

The next result will play an important role in the proofs of Theorems E and 2.

LEMMA 3. *If h is a meromorphic function that has a zero of order k at z_0, then in the Laurent expansion of h'/h about z_0,*

$$\frac{h'(z)}{h(z)} = \frac{k}{z - z_0} + \sum_{n=0}^{\infty} A_n (z - z_0)^n,$$

we have

$$A_0 = \frac{1}{k+1} \cdot \frac{h^{(k+1)}(z_0)}{h^{(k)}(z_0)}. \tag{24}$$

If $k = 1$ then

$$A_1 = \frac{h'''(z_0)}{3h'(z_0)} - \left(\frac{h''(z_0)}{2h'(z_0)}\right)^2. \tag{25}$$

The proof of Lemma 3 is elementary.

The first part of the proof of Theorem E (through (40)) is a rearrangement of the proof of Theorem D that was shown to the author by Mues. The author does not know what similarities (if any) exist between the remaining part of the proof of Theorem E and the proofs of Theorem D by Mues. An idea of Mues that will be utilized in this remaining part is stated in Remark 1 after the proof.

We now begin the proof of Theorem E. We will assume that $f \not\equiv g$ because there is nothing to prove if $f \equiv g$. Consider the following function:

$$\gamma_1 = \frac{f''}{f'} - \frac{f'}{f-a} - \frac{f'}{f+a} - 2\frac{f'}{f} - \frac{g''}{g'} + \frac{g'}{g-a} + \frac{g'}{g+a} + 2\frac{g'}{g}. \tag{26}$$

We will show that $T(r, \gamma_1) = S(r, f)$. It is well-known [**5**, pp. 103–104] that Nevanlinna's fundamental estimate of the logarithmic derivative can be used to show that $T(r, h') \leqslant 2T(r, h) + S(r, h)$ holds for any meromorphic function h. Hence from (26), Nevanlinna's fundamental estimate, and (2) it follows that

$$m(r, \gamma_1) \leqslant S(r, f') + S(r, f) + S(r, g') + S(r, g) = S(r, f). \tag{27}$$

Now we note that since γ_1 is the logarithmic derivative of

$$H = \frac{f'(g-a)(g+a)g^2}{g'(f-a)(f+a)f^2}, \tag{28}$$

this means that

$$N(r, \gamma_1) = \overline{N}(r, H, 0) + \overline{N}(r, H). \tag{29}$$

Since f and g share 0, ∞ **CM** and a, $-a$ **IM**, it can be seen from (28) and (22) that

$$N(r, H, 0) \leqslant N_0(r, f', 0) = S(r, f), \tag{30}$$

and

$$N(r, H) \leqslant N_0(r, g', 0) = S(r, g). \tag{31}$$

Consequently, from (31), (30), (29), and (2) we obtain

$$N(r, \gamma_1) = S(r, f). \tag{32}$$

From (27) and (32) we have

$$T(r, \gamma_1) = S(r, f). \tag{33}$$

Now suppose that z_0 is a simple zero of f and g. Since z_0 is neither a zero or a pole of H in (28), γ_1 is analytic at z_0. Furthermore, if equation (24) is applied to (26), it will be found that

$$\gamma_1(z_0) = 0. \tag{34}$$

Now suppose that $\gamma_1 \not\equiv 0$. Then because of (34) we can say that

$$\overline{N}(r, 0) \leqslant N(r, \gamma_1, 0) + N(r, f, 0) - \overline{N}(r, 0). \tag{35}$$

From Jensen's theorem and (33) we obtain

$$N(r, \gamma_1, 0) \leqslant T(r, \gamma_1, 0) = T(r, \gamma_1) + O(1) = S(r, f). \tag{36}$$

Since f and g share 0 **CM**, we have

$$N(r, f, 0) - \overline{N}(r, 0) = S(r, f) \tag{37}$$

from (16). Substituting (37) and (36) into (35) gives $\overline{N}(r, 0) = S(r, f)$. We have shown that

$$\text{if } \gamma_1 \not\equiv 0 \quad \text{then } \overline{N}(r, 0) = S(r, f). \tag{38}$$

Now set $F = 1/f$, $G = 1/g$, $A = 1/a$, and

$$\gamma_2 = \frac{F''}{F'} - \frac{F'}{F-A} - \frac{F'}{F+A} - 2\frac{F'}{F} - \frac{G''}{G'} + \frac{G'}{G-A} + \frac{G'}{G+A} + 2\frac{G'}{G}. \tag{39}$$

Then F and G satisfy the hypothesis of Theorem E, and by inspection of (26), (38), and (39) we can deduce that

$$\text{if } \gamma_2 \not\equiv 0 \quad \text{then } \overline{N}(r,\infty) = S(r,f). \tag{40}$$

Four cases result from (38) and (40).

Case 1. $\gamma_1 \equiv \gamma_2 \equiv 0$. From integration of $\gamma_1 \equiv 0$ we obtain

$$\frac{f'(g^2-a^2)g^2}{g'(f^2-a^2)f^2} \equiv C \tag{41}$$

where C is some nonzero constant, while from integration of $\gamma_2 \equiv 0$ we will obtain

$$\frac{f'(g^2-a^2)f^2}{g'(f^2-a^2)g^2} \equiv K \tag{42}$$

where K is some nonzero constant. Combining (41) and (42) we get $Cf^4 \equiv Kg^4$. Hence f and g share a and $-a$ **CM**.

Case 2. $\gamma_1 \not\equiv 0$ and $\gamma_2 \not\equiv 0$. Since $\overline{N}(r,0) = S(r,f)$ and $\overline{N}(r,\infty) = S(r,f)$ from (38) and (40), it follows from the second fundamental theorem that

$$\overline{N}(r,a) + S(r,f) = T(r,f) \quad \text{and} \quad \overline{N}(r,-a) + S(r,f) = T(r,f). \tag{43}$$

That f and g share a and $-a$ **CM** will follow from

LEMMA 4. *If, in addition to the hypothesis of Theorem* E, (43) *holds, then*

$$(g-a)^2(f+a)^2 \equiv (f-a)^2(g+a)^2. \tag{44}$$

PROOF OF LEMMA 4. From (43), the first fundamental theorem, and (2) we obtain

$$N(r,f,a) - \overline{N}(r,a) = S(r,f), \quad N(r,g,a) - \overline{N}(r,a) = S(r,f), \tag{45}$$

and

$$N(r,f,-a) - \overline{N}(r,-a) = S(r,f), \quad N(r,g,-a) - \overline{N}(r,-a) = S(r,f). \tag{46}$$

We shall now consider the following two functions:

$$\gamma_3 = \frac{f'(f+a)}{f(f-a)} - \frac{g'(g+a)}{g(g-a)}, \tag{47}$$

$$\gamma_4 = \frac{f'(f-a)}{f(f+a)} - \frac{g'(g-a)}{g(g+a)}. \tag{48}$$

Since

$$\frac{f'(f+a)}{f(f-a)} = 2\frac{f'}{f-a} - \frac{f'}{f},$$

we see from (47), (48), the fundamental estimate of the logarithmic derivative, and (2) that

$$m(r,\gamma_i) = S(r,f) \quad \text{for } i = 3,4. \tag{49}$$

If z_0 is a zero of order k of f and g, then from (47) and (48), the principal parts of γ_3 and γ_4 at z_0 will both equal

$$-\frac{k}{z-z_0}+\frac{k}{z-z_0}=0;$$

hence γ_3 and γ_4 are analytic at the zeros of f and g. γ_3 and γ_4 are also analytic at the poles of f and g. Furthermore, γ_3 (γ_4) is analytic at a-points ($(-a)$-points) that are simple for both f and g. Therefore, any pole of γ_3 (γ_4) must occur at an a-point (a $(-a)$-point) of f and g that is multiple for either f or g or both. Since any pole of γ_3 (γ_4) will be simple, it follows from (45) and (46) that

$$N(r,\gamma_i)=S(r,f)\quad \text{for } i=3,4. \tag{50}$$

From (49) and (50) we have

$$T(r,\gamma_i)=S(r,f)\quad \text{for } i=3,4. \tag{51}$$

If $\gamma_3 \not\equiv 0$ then from (47) and (51) we obtain

$$\begin{aligned}\overline{N}(r,-a)&\leqslant \overline{N}(r,\gamma_3,0)\leqslant T(r,\gamma_3,0)\\&=T(r,\gamma_3)+O(1)=S(r,f),\end{aligned}$$

which contradicts (43). Hence $\gamma_3\equiv 0$. We can deduce similarly that $\gamma_4\equiv 0$. Combining $\gamma_3\equiv 0$ and $\gamma_4\equiv 0$ gives (44). This proves Lemma 4.

Case 2 is now complete.

Case 3. $\gamma_1\equiv 0$ and $\gamma_2\not\equiv 0$. If z_0 is an a-point (a $(-a)$-point) of order k for f and an a-point (a $(-a)$-point) of order m for g, then from (41) we find that

$$k=Cm. \tag{52}$$

We will now consider the function

$$\gamma_5=\frac{f'f}{f^2-a^2}-C\frac{g'g}{g^2-a^2}. \tag{53}$$

An analysis similar to that used to obtain (49) from (47) and (48) will show that

$$m(r,\gamma_5)=S(r,f). \tag{54}$$

Because of the condition (52), an analysis of (53) similar to that used on (47) and (48) (e.g. see the sentence that follows (49)) will show that γ_5 is analytic at a-points and $(-a)$-points of f and g. Since

$$\overline{N}(r,\infty)=S(r,f) \tag{55}$$

because of (40), it follows from inspection of (53) that

$$N(r,\gamma_5)=S(r,f). \tag{56}$$

From (54) and (56) we have

$$T(r,\gamma_5)=S(r,f). \tag{57}$$

If $\gamma_5\equiv 0$ then from (53) and (41) we obtain $f^3\equiv g^3$. Hence f and g share a and $-a$ **CM**.

Now suppose that $\gamma_5 \not\equiv 0$. Then from (53) and (57) we obtain

$$\begin{aligned}\bar{N}(r,0) \leqslant \bar{N}(r,\gamma_5,0) &\leqslant T(r,\gamma_5,0) \\ &= T(r,\gamma_5) + O(1) = S(r,f).\end{aligned} \tag{58}$$

Combining (58) and (55) with the second fundamental theorem implies that (43) holds. Then f and g share a and $-a$ **CM** from (44).

Case 4. $\gamma_1 \not\equiv 0$ and $\gamma_2 \equiv 0$. We can deduce that f and g share a and $-a$ **CM** from Case 3.

Thus in all four cases of (38) and (40) we have found that f and g share a and $-a$ **CM**. The proof of Theorem E is complete.

REMARK 1. The idea of using expressions like (47) and (48) for problems on meromorphic functions that share values is due to Mues. The use of (53) is a variation of this idea to fit the particular situation of Case 3.

REMARK 2. When $f \not\equiv g$ the conclusions in Theorem B can be obtained directly from this proof of Theorem E. We now indicate how this can be done but will leave out most of the details.

In the proof of Theorem E it can be found that:

(i) Case 1 led to $Cf^4 \equiv Kg^4$;

(ii) Case 2 led to $(g-a)^2(f+a)^2 \equiv (f-a)^2(g+a)^2$;

(iii) Case 3 led to $f^3 \equiv g^3$ and $(g-a)^2(f+a)^2 \equiv (f-a)^2(g+a)^2$;

(iv) Case 4 will lead to $(1/f)^3 \equiv (1/g)^3$ and $(1/g - 1/a)^2(1/f + 1/a)^2 \equiv (1/f - 1/a)^2(1/g + 1/a)^2$.

There are only three distinct identities in (i)–(iv).

Regarding (i), suppose that $Cf^4 \equiv Kg^4$. From this identity it follows that $f \equiv Dg$ where D is a nonzero constant. Since $D \neq 1$, this implies that a and $-a$ must be Picard values of f and g. Since f and g can have no other Picard values from Picard's theorem, it follows that $a = -Da$, or $D = -1$. Thus $f \equiv -g$. Since $(a, -a, 0, \infty) = -1$ we have obtained the conclusions of Theorem B.

Elementary analysis can also be used on the remaining identities $(g-a)^2(f+a)^2 \equiv (f-a)^2(g+a)^2$ and $f^3 \equiv g^3$.

4. Some remarks on the proofs of Theorems E and 2. First we make the following comments about the preceding proof of Theorem E.

1. Five different linear combinations $\{\gamma_i\}_{i=1}^5$ of logarithmic derivatives were constructed from f, g, f', and g', so that each γ_i satisfied the following three conditions:

(a) $m(r, \gamma_i) = S(r, f)$;

(b) $N(r, \gamma_i) = S(r, f)$;

(c) the zeros of γ_i contained all or "essentially all" of the roots of $f(z) = g(z) = d_i$ where d_i was one of the shared values.

Condition (a) was always an easy consequence of Nevanlinna's fundamental estimate of the logarithmic derivative and (2). Conditions (b) and (c) were always accomplished by using the shared value properties of f and g. If $\gamma_i \equiv 0$ then this condition was utilized. If $\gamma_i \not\equiv 0$ then $\bar{N}(r, d_i) = S(r, f)$ followed from (a), (b), and (c). The second fundamental theorem was then utilized.

2. The second fundamental theorem (when used) was always applied to f or g (which were assumed to be nonconstant) and never to any of the γ_i's. Thus it was valid to consider only whether $\gamma_i \equiv 0$ or $\gamma_i \not\equiv 0$; i.e. we did not have to consider separately the possibility of $\gamma_i \equiv C$ where C is a nonzero constant.

3. A natural question to ask is whether some generalization of the proof of Theorem E can be used to prove Theorem 1? The author has been unable to obtain such a generalization; Mues was earlier unable to obtain such a generalization of any of his proofs of Theorem D. The following observation illustrates the problem that we both have had with finding such a generalization. This observation is a rearrangement of the observation that Mues made about his proofs of Theorem D referred to in the Introduction.

Suppose that we start with the hypothesis that f and g are two nonconstant meromorphic functions that share $0, \infty$ **CM** and a, b **IM** ($a, b \neq 0, \infty$; $a \neq b$). Consider the function

$$\gamma = \frac{f''}{f'} - \frac{f'}{f-a} - \frac{f'}{f-b} - \lambda\frac{f'}{f} - \frac{g''}{g'} + \frac{g'}{g-a} + \frac{g'}{g-b} + \lambda\frac{g'}{g}, \tag{59}$$

where λ is an arbitrarily fixed complex number. Note that when $\lambda = 2$ and $a = -b$ then $\gamma \equiv \gamma_1$ in (26). From the analysis used on (26) we see that $m(r, \gamma) = S(r, f)$. Since

$$\frac{f''}{f'} - \frac{f'}{f-a} \quad \text{and} \quad \frac{g''}{g'} - \frac{g'}{g-a}$$

both have a simple pole with residue -1 at a-points of f and g, we see that γ is analytic at such points. γ is also analytic at b-points of f and g. Since f and g share 0 and ∞ **CM** we can determine from (59) that γ is analytic at the zeros and poles of f and g. Thus from (59), (22), and (2) it follows that $N(r, \gamma) = S(r, f)$. Hence $T(r, \gamma) = S(r, f)$.

Now suppose that z_0 is a simple zero of both f and g. Formula (24), when applied to (59), shows that

$$\gamma(z_0) = (1 - \lambda/2)\left(\frac{f''(z_0)}{f'(z_0)} - \frac{g''(z_0)}{g'(z_0)}\right) + \left(\frac{1}{a} + \frac{1}{b}\right)(f'(z_0) - g'(z_0)).$$

Hence if $\lambda = 2$ and $a = -b$ we obtain $\gamma(z_0) = 0$; in this case $\gamma \equiv \gamma_1$ in (26) and $\gamma(z_0) = 0$ is condition (34). On the other hand, if $\lambda \neq 2$ or $a \neq -b$, it was not clear to Mues and it is not clear to the author whether $\gamma(z_0) = 0$ can be concluded; this was the problem that we both have had in finding the stated generalization.

Now we will remark on the upcoming proof of Theorem 2 in §5. Due to the length of this proof it may be worthwhile to give the following brief outline.

We will start with the assumption that either a or b is shared by **DM**. It will be shown that

$$\overline{N}(r, a) + S(r, f) \geqslant \tfrac{1}{2}T(r, f), \tag{60}$$

and

$$\overline{N}(r, b) + S(r, f) \geqslant \tfrac{1}{2}T(r, f). \tag{61}$$

We will partition the set of a-points of f and g into six disjoint subsets $\{E_i\}_{i=1}^6$. It will be shown that E_1 and E_2 contribute $S(r, f)$ to $\overline{N}(r, a)$. For the sets E_3 and E_4 we will find corresponding functions u_3 and u_4 respectively such that the following conditions are satisfied (for $i = 3, 4$):

(A) $T(r, u_i) = S(r, f)$;

(B) if $z_0 \in E_i$ then $u_i(z_0) = 0$;

(C) $u_i \not\equiv 0$.

From (A), (B), and (C) it follows that E_3 and E_4 contribute $S(r, f)$ to $\overline{N}(r, a)$. This will immediately imply that E_5 and E_6 must also contribute $S(r, f)$ to $\overline{N}(r, a)$. But then it will follow that $\overline{N}(r, a) = S(r, f)$, which contradicts (60). This contradiction will mean that our original assumption is false, and hence will prove Theorem 2.

We have two further remarks about the proof of Theorem 2.

(i) We will create several linear combinations $\{\mu\}$ of logarithmic derivatives from f, g, f', and g', and use variations, extensions, and refinements of the ideas in the proof of Theorem E.

(ii) In the proof there exists a nonconstant entire function w such that $f = e^w g$ because f and g share 0 and ∞ **CM** and either a or b is assumed to be shared by **DM**. We mention that the following two properties (which will be proved) will play key roles in obtaining the contradiction:

$$T(r, w') = S(r, f); \quad \overline{N}(r, e^w, 1) = \overline{N}(r, a) + \overline{N}(r, b) + S(r, f).$$

CONCLUDING REMARK. Each linear combination η of logarithmic derivatives in this paper was constructed with the following motivations:

(a) to obtain $m(r, \eta) = S(r, f)$ as an easy consequence of the fundamental estimate and (2);

(b) to use the shared value properties of f and g to accomplish two things: (i) to make $N(r, \eta)$ as "small" as possible, and (ii) to have the zeros of η (or the function created from η) contain as "many" as possible of the roots of $f(z) = g(z) = a_j$ where a_j is one of the shared values.

5. Proof of Theorem 2. Before we begin the proof, we have two comments.

(i) The proof of Theorem 2 will be by contradiction. The proof will consist of proving the statements referred to in the outline of the proof that was given in §4.

(ii) The proofs of Theorems E and 2 can be combined in a natural way; specifically, the proof of Theorem E can be naturally incorporated into the proof of Theorem 2 (see the remark at the end of this section).

We now begin the proof of Theorem 2. We make the assumption that either a or b is shared by **DM**. Then $f \not\equiv g$. First we prove

LEMMA 5. *The hypothesis of Theorem* 2 *and our additional assumption imply that the following properties hold*:

$$\overline{N}(r, a) + S(r, f) \geqslant \tfrac{1}{2}T(r, f); \tag{60}$$

$$\overline{N}(r, b) + S(r, f) \geqslant \tfrac{1}{2}T(r, f); \tag{61}$$

$$\overline{N}_1(r, a) + \overline{N}_1(r, b) = S(r, f) \tag{62}$$

where $\overline{N}_1(r, a)$ *"counts" only those a-points that are simple for both f and g (and similarly for* $\overline{N}_1(r, b)$*).*

PROOF OF LEMMA 5. We will first consider the function

$$\psi = \frac{f'(f-a)}{f(f-b)} - \frac{g'(g-a)}{g(g-b)}. \tag{63}$$

From the analysis previously used on (47) and (48) we find that

$$m(r, \psi) = S(r, f), \tag{64}$$

and also that ψ is analytic at (i) the zeros and poles of f and g (since f and g share 0 and ∞ **CM**) and (ii) the b-points that are simple for both f and g. For a meromorphic function h and a constant $c \neq \infty$ we define $\overline{N}_2(r, h, c)$ to "count" only multiple zeros of $h(z) - c$ and "count" them once each. It then follows from (63) that

$$N(r, \psi) \leqslant \overline{N}_2(r, f, b) + \overline{N}_2(r, g, b). \tag{65}$$

From (64) and (65) we have

$$T(r, \psi) \leqslant \overline{N}_2(r, f, b) + \overline{N}_2(r, g, b) + S(r, f). \tag{66}$$

If $\psi \equiv 0$, then from (63) it can be seen that f and g would share a and b **CM**, which contradicts our assumption. Thus $\psi \not\equiv 0$. Then from (63), (17), and (66), it follows that

$$\begin{aligned} \overline{N}(r, a) &\leqslant \overline{N}_1(r, a) + \overline{N}_2(r, f, a) + \overline{N}_2(r, g, a) \\ &= \overline{N}(r, a) + \overline{N}_4(r, a) \leqslant N(r, \psi, 0) + S(r, f) \\ &\leqslant T(r, \psi) + S(r, f) \\ &\leqslant \overline{N}_2(r, f, b) + \overline{N}_2(r, g, b) + S(r, f) \\ &\leqslant \overline{N}(r, b) + \overline{N}_4(r, b) + S(r, f) \\ &= \overline{N}(r, b) + S(r, f). \end{aligned} \tag{67}$$

If we interchange a and b in (63) and repeat this argument we will obtain

$$\begin{aligned} \overline{N}(r, b) &\leqslant \overline{N}_1(r, b) + \overline{N}_2(r, f, b) + \overline{N}_2(r, g, b) \\ &\leqslant \overline{N}_2(r, f, a) + \overline{N}_2(r, g, a) + S(r, f) \\ &\leqslant \overline{N}(r, a) + S(r, f). \end{aligned} \tag{68}$$

We can deduce from (67) and (68) the following three properties:

$$\overline{N}(r, a) = \overline{N}(r, b) + S(r, f); \tag{69}$$

$$\overline{N}_2(r, f, a) + \overline{N}_2(r, g, a) = \overline{N}(r, a) + S(r, f); \tag{70}$$

$$\overline{N}_2(r, f, b) + \overline{N}_2(r, g, b) = \overline{N}(r, a) + S(r, f). \tag{71}$$

If we substitute (71), (70), and (69) back into (67) and (68) it can be seen that (62) holds.

Now consider the following function:

$$\phi_1 = \frac{f''}{f'} - \frac{f'}{f-a} + \frac{bf'}{a(f-b)} - 2\frac{f'}{f} - \frac{g''}{g'} + \frac{g'}{g-a} - \frac{bg'}{a(g-b)} + 2\frac{g'}{g}. \tag{72}$$

From the analysis that was used on (26) we obtain

$$m(r, \phi_1) = S(r, f). \tag{73}$$

By using the analysis that was applied to (59) it can be seen that ϕ_1 is analytic at a-points, zeros, and poles of f and g. Thus it follows from (72), (22) and (2) that

$$\begin{aligned} N(r, \phi_1) &\leqslant \overline{N}(r, b) + N_0(r, f', 0) + N_0(r, g', 0) \\ &= \overline{N}(r, b) + S(r, f). \end{aligned} \tag{74}$$

From (73) and (74) we have

$$T(r, \phi_1) \leqslant \overline{N}(r, b) + S(r, f). \tag{75}$$

Suppose that z_0 is a simple zero of f and g. Formula (24), when applied to (72), shows that

$$\phi_1(z_0) = 0. \tag{76}$$

Suppose that $\phi_1 \equiv 0$. If z_1 is a b-point of order k for f and of order m for g, then by equating the principal part of ϕ_1 at z_1 to zero we will obtain from (72) that $(k - m)(b/a + 1) = 0$. Since $a + b \neq 0$ from the hypothesis, this means that $k = m$. Then b is shared **CM** by f and g. But if we let η_1 be ϕ_1 in (72) with "a" and "b" interchanged, repeat this argument, and assume that $\eta_1 \equiv 0$, then we will obtain that a is also shared **CM**. Since this would contradict our original assumption we must have either $\phi_1 \not\equiv 0$ or $\eta_1 \not\equiv 0$. If $\phi_1 \not\equiv 0$, then since

$$N(r, f, 0) - \overline{N}(r, 0) = S(r, f) \tag{77}$$

from (16), it will follow from (75), (76), and (77) that

$$\begin{aligned} \overline{N}(r, 0) &\leqslant \overline{N}(r, \phi_1, 0) + S(r, f) \leqslant T(r, \phi_1) + S(r, f) \\ &\leqslant \overline{N}(r, b) + S(r, f). \end{aligned} \tag{78}$$

If $\eta_1 \not\equiv 0$ then we obtain

$$\overline{N}(r, 0) \leqslant \overline{N}(r, a) + S(r, f). \tag{79}$$

Since either (78) or (79) must hold, it follows from (69) that

$$\overline{N}(r, 0) \leqslant \overline{N}(r, a) + S(r, f). \tag{80}$$

If we let ϕ_2 be ϕ_1 in (72) with f, g, a, b, replaced by $F = 1/f$, $G = 1/g$, $A = 1/a$, $B = 1/b$, respectively, and repeat this argument, we will obtain

$$\overline{N}(r, \infty) \leqslant \overline{N}(r, a) + S(r, f). \tag{81}$$

Inequalities (60) and (61) are now a consequence of (81), (80), (69), and (3) (or we could use the second fundamental theorem instead of (3)). Since we have already shown that (62) holds, the proof of Lemma 5 is now complete.

Proceeding with the proof of Theorem 2 we next note that since f and g share $0, \infty$ **CM** and either a or b is assumed to be shared by **DM**, there exists a nonconstant entire function w such that

$$f = e^w g. \tag{82}$$

From (82) and (2) we obtain

$$T(r, e^w) \leq T(r, f) + T(r, g) + O(1) = 2T(r, f) + S(r, f). \tag{83}$$

Since w' is the logarithmic derivative of e^w, a consequence of the fundamental estimate and (83) is that

$$T(r, w') = S(r, f). \tag{84}$$

Now (82) can be rewritten as

$$e^w - 1 = \frac{f-g}{g}. \tag{85}$$

Of course $f \not\equiv g$ and $e^w \not\equiv 1$. If z_0 is a zero of f and g such that $e^{w(z_0)} = 1$, then from (85) we see that z_0 will be a multiple zero of $f - g$. Since from (82), $1/g = e^w \cdot 1/f$, we can deduce that if z_1 is a pole of f and g such that $e^{w(z_1)} = 1$, then z_1 will be a multiple zero of $1/g - 1/f$. Therefore, from (85) we can say that

$$\begin{aligned} \overline{N}(r, e^w, 1) \leq{}& \overline{N}(r, a) + \overline{N}(r, b) + N(r, f-g, 0) - \overline{N}(r, f-g, 0) \\ &+ N\left(r, \frac{1}{g} - \frac{1}{f}, 0\right) - \overline{N}\left(r, \frac{1}{g} - \frac{1}{f}, 0\right) + \overline{N}_5(r, f-g, 0) \end{aligned} \tag{86}$$

where $\overline{N}_5(r, f-g, 0)$ "counts" only those points z such that $f(z) = g(z) \neq 0, a, b, \infty$. Noting that $1/g$ and $1/f$ share four values, an application of (7) and (8) to (86) yields

$$\overline{N}(r, e^w, 1) \leq \overline{N}(r, a) + \overline{N}(r, b) + S(r, f).$$

Since $\overline{N}(r, a) + \overline{N}(r, b) \leq \overline{N}(r, e^w, 1)$ is obvious from (82), we obtain that

$$\overline{N}(r, e^w, 1) = \overline{N}(r, a) + \overline{N}(r, b) + S(r, f). \tag{87}$$

The properties (84) and (87) will play an important role in this proof. We will now consider the following two functions:

$$\alpha_1 = \frac{f''}{f'} - \frac{f'}{f-a} - \frac{f'}{f-b} + \frac{w'e^w}{e^w - 1}; \tag{88}$$

$$\alpha_2 = \frac{g''}{g'} - \frac{g'}{g-a} - \frac{g'}{g-b} + \frac{w'e^w}{e^w - 1}. \tag{89}$$

From (83) we obtain

$$m(r, \alpha_1) \leq S(r, f) + S(r, e^w - 1) \leq S(r, f). \tag{90}$$

Since α_1 is the logarithmic derivative of

$$H_1 = \frac{f'(e^w - 1)}{(f-a)(f-b)}, \tag{91}$$

we have that $N(r, \alpha_1) = \overline{N}(r, H_1, 0) + \overline{N}(r, H_1)$. Since a-points and b-points of f are zeros of $e^w - 1$ from (82), we see that H_1 has no poles. On the other hand, if z_0 is a

zero of H_1, then by inspection of (91), it can be seen that z_0 will have to satisfy one of the following four conditions:

(i) z_0 is a multiple pole of f;

(ii) $f'(z_0) = 0$ and $f(z_0) \neq a, b$;

(iii) $e^{w(z_0)} = 1$ and $f(z_0) \neq a, b$;

(iv) z_0 is a multiple zero of $e^w - 1$ and is also either an a-point or a b-point of f.

Note that $w' \not\equiv 0$ from (82). Therefore, since $N(r, \alpha_1) = \overline{N}(r, H_1, 0)$, we can say that

$$(92)\quad N(r,\alpha_1) \leqslant N(r,f) - \overline{N}(r,\infty) + N(r,f,0) - \overline{N}(r,0) + N_0(r,f',0) + \overline{N}(r,e^w,1) - \overline{N}(r,a) - \overline{N}(r,b) + N(r,w',0),$$

where $N_0(r, f', 0)$ refers to those roots of $f'(z) = 0$ such that $f(z) \neq 0, a, b$. Hence from (92), (16) with $a_1 = 0$ and $a_2 = \infty$, (22), (87), and (84) we deduce that

$$(93)\qquad N(r,\alpha_1) = S(r,f).$$

From (90) and (93) we have

$$(94)\qquad T(r,\alpha_1) = S(r,f).$$

From inspection of (88) and (89) we can use (2) and deduce that

$$(95)\qquad T(r,\alpha_2) = S(r,f).$$

Next we note that, in view of (60), (61), (62), and (17), "essentially all" of the a-points and b-points of f and g are simple for one of the functions and multiple for the other function. If z_0 is such an a-point or b-point then from (85) we obtain

$$(96)\qquad e^{w(z_0)} = 1 \quad \text{and} \quad w'(z_0) \neq 0.$$

If we can show that either $\overline{N}(r, a) = S(r, f)$ or $\overline{N}(r, b) = S(r, f)$, then because of (60) and (61) we will have a contradiction. Our goal will be to show that $\overline{N}(r, a) = S(r, f)$. To this end, we shall now prove the following two statements.

I. If z_0 is a simple a-point of f and a double a-point of g then α_1 is analytic at z_0 and

$$(97)\qquad \alpha_1'(z_0) = \beta_1(z_0)$$

where

$$(98)\quad \beta_1 = \frac{w'''}{w'} + 4h_1 + 8\frac{h_1h_2}{w'} + \left(\frac{5}{2} + \frac{a}{a-b}\right)w'' + \left(\frac{3}{4} - \frac{ab}{(a-b)^2}\right)(w')^2 - \frac{3}{4}\left(2\alpha_1 + \frac{a+b}{a-b}w' - \frac{w''}{w'}\right)^2 - \frac{2aw'\alpha_1}{a-b} - \frac{1}{4}\left(\frac{w''}{w'}\right)^2,$$

for

$$(99)\qquad h_1 = w'\alpha_1 + \frac{b(w')^2}{a-b} - w'' \quad \text{and} \quad h_2 = \alpha_2 - \frac{1}{2}\frac{w''}{w'} - \frac{1}{2}w'.$$

II. If z_0 is an a-point that is simple for f and is of multiplicity at least three for g, then α_1 is analytic at z_0 and

$$(100)\qquad \alpha_1(z_0) = \frac{w''(z_0)}{w'(z_0)} + \frac{b}{b-a}w'(z_0).$$

The proof of assertion I will involve some tedious calculations. Suppose that z_0 is a simple a-point of f and a double a-point of g. From (96) and (88) it follows that α_1 is analytic at z_0. By using (25) with $h = f - a$ and $h = e^w - 1$ we can make a calculation and verify from (88) that the coefficient of $(z - z_0)$ in the Taylor series of α_1 about z_0 will be equal to the following:

$$(101)\quad \alpha_1'(z_0) = \frac{2}{3}\frac{f'''(z_0)}{f'(z_0)} - \frac{3}{4}\left(\frac{f''(z_0)}{f'(z_0)}\right)^2 - \frac{f''(z_0)}{a-b} + \frac{(f'(z_0))^2}{(a-b)^2}$$
$$+ \frac{w'''(z_0)}{3w'(z_0)} + \frac{1}{2}w''(z_0) + \frac{1}{12}(w'(z_0))^2 - \frac{1}{4}\left(\frac{w''(z_0)}{w'(z_0)}\right)^2.$$

We will now derive expressions for $f'(z_0)$, $f''(z_0)$, and $f'''(z_0)$ in order to substitute back into (101). First we differentiate (82) three times. If we substitute $z = z_0$ into the three equations and note that $e^{w(z_0)} = 1$ and $g'(z_0) = 0$, then we will obtain the following three equations:

$$(102)\qquad f'(z_0) = aw'(z_0),$$

$$(103)\qquad f''(z_0) = aw''(z_0) + g''(z_0) + a(w'(z_0))^2,$$

$$(104)\qquad f'''(z_0) = aw'''(z_0) + 3w'(z_0)g''(z_0) + g'''(z_0)$$
$$+3aw'(z_0)w''(z_0) + a(w'(z_0))^3.$$

Now formula (24), when applied to (88) and (89), will yield the following two equations:

$$(105)\qquad \alpha_1(z_0) = \frac{f''(z_0)}{2f'(z_0)} - \frac{f'(z_0)}{a-b} + \frac{w''(z_0) + (w'(z_0))^2}{2w'(z_0)},$$

$$(106)\qquad \alpha_2(z_0) = \frac{g'''(z_0)}{6g''(z_0)} + \frac{w''(z_0) + (w'(z_0))^2}{2w'(z_0)}.$$

Substitution of (102) into (105) gives

$$(107)\quad f''(z_0) = 2aw'(z_0)\alpha_1(z_0) + \frac{2a^2(w'(z_0))^2}{a-b} - aw''(z_0) - a(w'(z_0))^2.$$

Then substitution of (107) into (103) will yield

$$(108)\qquad g''(z_0) = 2ah_1(z_0)$$

where h_1 is given in (99). From (108) and (106) we obtain

$$(109)\qquad g'''(z_0) = 12ah_1(z_0)h_2(z_0)$$

where h_2 is given in (99). Now we substitute (109) and (108) back into (104) and obtain

$$(110)\qquad f'''(z_0) = aw'''(z_0) + 6aw'(z_0)h_1(z_0) + 12ah_1(z_0)h_2(z_0)$$
$$+3aw'(z_0)w''(z_0) + a(w'(z_0))^3.$$

Finally, we substitute (110), (107), and (102) back into (101). After simplification this will reduce to the equation $\alpha_1'(z_0) = \beta_1(z_0)$ where β_1 is given by (98). This proves (97).

To prove (100), suppose that z_0 is an a-point that is simple for f and is of multiplicity at least three for g. Then α_1 is analytic at z_0 from (96) and (88). Formula (24), when applied to (88), shows that

$$\alpha_1(z_0) = \frac{1}{2}\frac{f''(z_0)}{f'(z_0)} - \frac{f'(z_0)}{a-b} + \frac{w''(z_0) + (w'(z_0))^2}{2w'(z_0)}. \tag{111}$$

From (82) we obtain

$$f'(z_0) = aw'(z_0) \quad \text{and} \quad f''(z_0) = aw''(z_0) + a(w'(z_0))^2. \tag{112}$$

Substitution of (112) into (111) will give (100).

For our proof we shall also need to show that

$$\alpha_1 \not\equiv \alpha_2. \tag{113}$$

To prove (113) we will make the assumption that $\alpha_1 \equiv \alpha_2$. From integration of $\alpha_1 \equiv \alpha_2$ we obtain from (88) and (89) that

$$\frac{f'(g-a)(g-b)}{g'(f-a)(f-b)} \equiv C \tag{114}$$

where C is some nonzero constant. If z_0 is an a-point (a b-point) of order k for f and an a-point (a b-point) of order m for g, then from (114) it follows that

$$k = Cm. \tag{115}$$

Now either a or b is shared by **DM**, and neither a or b is a Picard value for f and g from (60) and (61). Hence from (115), C is a positive rational number such that $C \neq 1$. Suppose that $C > 2$. Then from (115), all the a-points (and b-points) of f are of multiplicity at least three. But then

$$3\overline{N}(r, a) \leqslant N(r, f, a) \leqslant T(r, f) + O(1),$$

which contradicts (60). Hence $C \leqslant 2$. From (115) we can deduce that $C < \frac{1}{2}$ is also impossible. On the other hand, if $1 < C < 2$ or $\frac{1}{2} < C < 1$, it can be seen from (115) that all the a-points (and b-points) of f and g will be multiple for both f and g. But then $\overline{N}(r, a) = S(r, f)$ from (17), which contradicts (60). Hence we have shown that either $C = 2$ or $C = \frac{1}{2}$.

Suppose that $C = \frac{1}{2}$, and consider the following function:

$$\alpha = \frac{2f'}{f-a} + \frac{2f'}{f-b} - \frac{3g'}{g-a} - \frac{3g'}{g-b} + \frac{2w'e^w}{e^w - 1} - \frac{f''}{f'} + \frac{2g''}{g'}. \tag{116}$$

From (83) and (2) we obtain

$$m(r, \alpha) = S(r, f). \tag{117}$$

From (115) with $C = \frac{1}{2}$ we have the following two conditions.

(118)

(A) If z_0 is a simple a-point (b-point) of f then z_0 is a double a-point (b-point) of g. In view of (96) we will find from inspection of (116) that α is analytic at z_0.

(B) If z_0 is a multiple a-point (b-point) of f, then z_0 is a multiple a-point (b-point) of g.

Also, we will find from inspection of (116) that α is analytic at simple poles of f and g that are not zeros of $e^w - 1$.

Therefore we can deduce from (116) that if z_0 is a pole of α then z_0 will satisfy at least one of the following conditions:

(i) z_0 is either an a-point, a b-point, a pole, or a zero of f and g and is multiple for both f and g;

(ii) $e^{w(z_0)} = 1$ and $f(z_0) \neq a, b$;

(iii) $f'(z_0) = 0$ and $f(z_0) \neq 0, a, b$;

(iv) $g'(z_0) = 0$ and $g(z_0) \neq 0, a, b$.

By combining (i), (ii), (iii), (iv), and (116) with (17), (87), and (22) we see that

$$N(r, \alpha) = S(r, f). \tag{119}$$

From (119) and (117) we have

$$T(r, \alpha) = S(r, f). \tag{120}$$

Now suppose that z_0 is a simple a-point of f and a double a-point of g. Then α is analytic at z_0 from (118), and $f'(z_0) = aw'(z_0)$ from (82). Thus if we apply (24) to (116) we can deduce that

$$\alpha(z_0) = \frac{2aw'(z_0)}{a-b} + \frac{w''(z_0) + (w'(z_0))^2}{w'(z_0)}. \tag{121}$$

If $\alpha \not\equiv 2aw'/(a-b) + w''/w' + w'$ (note that $w' \not\equiv 0$ from (82)), then it follows from (121), (118), (120), (84), and (17) that

$$\begin{aligned}\overline{N}(r, a) &\leqslant \overline{N}\left(r, \alpha - \frac{2aw'}{a-b} - \frac{w''}{w'} - w', 0\right) + \overline{N}_4(r, a)\\ &\leqslant T\left(r, \alpha - \frac{2aw'}{a-b} - \frac{w''}{w'} - w'\right) + S(r, f) = S(r, f),\end{aligned}$$

which contradicts (60). Therefore

$$\alpha \equiv \frac{2aw'}{a-b} + \frac{w''}{w'} + w'.$$

Since α is symmetric in a and b in (116), this implies that we also have

$$\alpha \equiv \frac{2bw'}{b-a} + \frac{w''}{w'} + w'.$$

This implies that $a + b = 0$ which contradicts our hypothesis. We have therefore shown that (113) holds.

We can now complete the proof of Theorem 2. Suppose that $\alpha_1' \equiv \beta_1$ where β_1 is given in (98). We note that α_1 and α_2 are symmetric in a and b in (88) and (89). Then α_1' is symmetric in a and b. Since $\alpha_1' \equiv \beta_1$ it follows that β_1 is symmetric in a and b; that is $\beta_1 \equiv \beta_2$ where β_2 is β_1 in (98) with "a" and "b" interchanged ("a" and "b" are interchanged in h_1 also). If we appropriately use (99) in the identity $\beta_1 \equiv \beta_2$ and

cancel identical terms in β_1 and β_2 we will then obtain

$$\begin{aligned}\frac{4b(w')^2}{a-b} &+ \frac{8bw'}{a-b}\left(\alpha_2 - \frac{1}{2}\frac{w''}{w'} - \frac{1}{2}w'\right) + \frac{a}{a-b}w'' \\ &- \frac{3}{4}\left(4\alpha_1\frac{a+b}{a-b}w' - 2\frac{a+b}{a-b}w''\right) - \frac{2aw'\alpha_1}{a-b} \\ \equiv \frac{4a(w')^2}{b-a} &+ \frac{8aw'}{b-a}\left(\alpha_2 - \frac{1}{2}\frac{w''}{w'} - \frac{1}{2}w'\right) + \frac{b}{b-a}w'' \\ &- \frac{3}{4}\left(4\alpha_1\frac{b+a}{b-a}w' - 2\frac{b+a}{b-a}w''\right) - \frac{2bw'\alpha_1}{b-a}.\end{aligned}$$

This identity reduces to $(a+b)w'(\alpha_1 - \alpha_2) \equiv 0$. Since $a + b \neq 0$ from our hypothesis, $w' \not\equiv 0$ from (82), and $\alpha_1 \not\equiv \alpha_2$ from (113), we have a contradiction. Therefore

$$\alpha_1' \not\equiv \beta_1. \tag{122}$$

Since α_1 is symmetric in a and b we can also deduce that

$$\alpha_1 \not\equiv \frac{w''}{w'} + \frac{bw'}{b-a}. \tag{123}$$

If $\overline{N}_{12}(r, a)$ "counts" only those a-points that are simple for f and multiple for g, then from (97), (100), (122), and (123) we obtain

$$\overline{N}_{12}(r, a) \leq \overline{N}(r, \alpha_1' - \beta_1, 0) + \overline{N}\left(r, \alpha_1 - \frac{w''}{w'} - \frac{bw'}{b-a}, 0\right). \tag{124}$$

From (124), (84), (94), (95), (98), and (99) it follows that

$$\overline{N}_{12}(r, a) = S(r, f). \tag{125}$$

If $\overline{N}_{21}(r, a)$ "counts" only those a-points that are multiple for f and simple for g, then we can deduce that

$$\overline{N}_{21}(r, a) = S(r, f). \tag{126}$$

But then from (126), (125), (62), and (17) we obtain that $\overline{N}(r, a) = S(r, f)$, which contradicts (60). This contradiction shows that our original assumption that either a or b is shared by **DM** cannot be true. The proof of Theorem 2 is now complete.

REMARK. The proof of Theorem E can be naturally incorporated into the proof of Theorem 2. Indeed, if $a = -b$ then ϕ_1 in (72) is identically γ_1 in (26) and ϕ_2 (see (81)) is identically γ_2 in (39).

6. A proof of Theorem B with these methods. There are many ways to prove Theorem B with the methods in this paper, and we shall now give one of them.

With no loss of generality we can assume that all the shared values are finite. Consider the following four functions:

$$\mu_i = \frac{f''}{f'} - 2\frac{f'}{f-a_i} - \frac{g''}{g'} + 2\frac{g'}{g-a_i}, \qquad i = 1, 2, 3, 4. \tag{127}$$

Then from (2) we obtain

$$m(r, \mu_i) = S(r, f) \quad \text{for } i = 1, 2, 3, 4. \tag{128}$$

Since each a_k is shared **CM** it follows from (127) that each μ_i is analytic at the a_k-points of f and g for $k = 1, 2, 3, 4$. If z_0 is a simple pole of f or a simple pole of g or both, then it follows from (127) that each μ_i is analytic at z_0. Thus from inspection of (127) we can deduce from (4) (with $c = \infty$) and (22) that

$$\begin{aligned} N(r, \mu_i) &\leq N(r, f) - \overline{N}(r, f) + N(r, g) - \overline{N}(r, g) \\ &\quad + N_0(r, f', 0) + N_0(r, g', 0) \\ &= S(r, f) \quad \text{for } i = 1, 2, 3, 4. \end{aligned} \tag{129}$$

From (128) and (129) we have

$$T(r, \mu_i) = S(r, f) \quad \text{for } i = 1, 2, 3, 4. \tag{130}$$

Now let z_0 be a simple a_i-point of f and g. Formula (24), when applied to (127), shows that

$$\mu_i(z_0) = 0. \tag{131}$$

Suppose that $\mu_j \not\equiv 0$ for some particular j. Then from (131), (130), and (17) we obtain

$$\overline{N}(r, a_j) \leq \overline{N}(r, \mu_j, 0) + \overline{N}_4(r, a_j) \leq T(r, \mu_j) + S(r, f) = S(r, f). \tag{132}$$

Hence from (2) (or the second fundamental theorem), it follows from (132) that at least two of the functions μ_1, μ_2, μ_3, μ_4 must be identically zero. With no loss in generality we can assume that $\mu_1 \equiv \mu_2 \equiv 0$.

From integration of $\mu_1 \equiv 0$ and $\mu_2 \equiv 0$ we obtain from (127) that

$$\frac{f'(g - a_i)^2}{g'(f - a_i)^2} \equiv C_i \quad \text{for } i = 1, 2,$$

where C_1 and C_2 are nonzero constants. From these two identities we can obtain that

$$\frac{f - a_1}{g - a_1} \equiv C \frac{f - a_2}{g - a_2} \tag{133}$$

for some nonzero constant C. Thus

$$f \equiv \frac{(a_1 - Ca_2)g + a_1 a_2 (C - 1)}{(1 - C)g + Ca_1 - a_2}. \tag{134}$$

If $C = 1$ then $f \equiv g$. Hence $C \neq 1$. Then a_3 and a_4 must be Picard values of f and g from (133).

We will now finish the proof with Nevanlinna's reasoning [**5**, p. 124]. Set $f = L(g)$ where L is the Möbius transformation given by (134). Since $L(z) \not\equiv z$ (because $f \not\equiv g$), L can have at most two fixed points. Of course neither f nor g could have another Picard value besides a_3 and a_4. Since f and g share a_1, a_2, a_3, a_4, it then follows that

$$L(a_1) = a_1, \quad L(a_2) = a_2, \quad L(a_3) = a_4, \quad \text{and} \quad L(a_4) = a_3.$$

The last two equations are

$$(1 - C)(a_4 a_3 + a_1 a_2) = a_1 a_3 - Ca_2 a_3 + a_2 a_4 - Ca_1 a_4, \tag{135}$$

and

$$(1 - C)(a_4a_3 + a_1a_2) = a_1a_4 - Ca_2a_4 + a_2a_3 - Ca_1a_3, \tag{136}$$

respectively. Combining (135) and (136) we obtain $(1 + C)(a_1 - a_2)(a_3 - a_4) = 0$, which yields $C = -1$. By substituting $C = -1$ into (135) we will get

$$(a_3, a_4, a_1, a_2) = \frac{(a_2 - a_3)(a_1 - a_4)}{(a_2 - a_4)(a_1 - a_3)} = -1.$$

This completes our proof of Theorem B.

Note that from (134) and $C = -1$ we obtain

$$f \equiv \frac{(a_1 + a_2)g - 2a_1a_2}{2g - (a_1 + a_2)}.$$

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