AMBIENTLY UNIVERSAL SETS IN $E^n$

BY

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ABSTRACT. For each closed set $X$ in $E^n$ of dimension at most $n - 3$, we show that $X$ fails to be ambiently universal with respect to Cantor sets in $E^n$; i.e., we find a Cantor set $Y$ in $E^n$ so that for any self-homeomorphism $h$ of $E^n$, $h(Y)$ is not contained in $X$. This result answers a question posed by H. G. Bothe and completes the understanding of ambiently universal sets in $E^n$.

1. Introduction. Let $M$ and $N$ be subsets of $E^n$. We say that $M$ is ambienply embeddable in $N$ if there is a homeomorphism $h$ of $E^n$ onto itself so that $h(M)$ is a subset of $N$. Let $F$ be a family of sets in some fixed $E^n$ and $U$ some fixed subset of $E^n$. We call $U$ an ambiently universal set for the family $F$ if each set in $F$ is ambiently embeddable in $U$. For $0 \leq m \leq n$, a compact $m$-dimensional subset $X$ in $E^n$ is called a compact ambiently universal $m$-dimensional set if every compactum of dimension $\leq m$ in $E^n$ is ambiently embeddable in $X$.

H. G. Bothe has made an extensive study of the existence of compact ambiently universal sets in $E^n$ [Bo1, Bo2, Bo3]. It seems to be well known that Bothe constructed a one-dimensional continuum in $E^3$ [Bo4] similar to the McMillan-Row continuum [M-R]. However, it does not seem to be well known that his purpose was to exhibit a one-dimensional compactum in $E^3$ that is not ambiently embeddable in the Menger universal curve. We give a summary of Bothe's results.

In each $E^n$, Bothe constructed compact $m$-dimensional sets $M_m^n$, $0 \leq m \leq n$. For $n = 2m + 1$, $M_m^n$ is the Menger universal set [H-W, p. 64]. He then showed that $M_m^n$ is a compact ambiently universal $m$-dimensional set in $E^n$ ($n \neq 3$) if and only if $m > n - 3$. For $E^3$, $M_3^3$ is a compact ambiently universal $m$-dimensional set if and only if $m = 2$ or 3. Bothe defined a condition on a set $X$ in $E^n$ which is now known as the dimension of embedding of $X$ and sometimes written $\text{dem } X$ [Ed]. He then showed that $M_m^n$ is an ambiently universal set for all compact subsets $X$ of $E^n$ for which $\text{dim } X = \text{dem } X$ and $\text{dim } X \leq m$. Bothe showed that there does not exist a compact ambiently universal 0-dimensional set nor a compact ambiently universal 1-dimensional set in $E^3$. His proof that there is no compact ambiently universal 0-dimensional set in $E^3$ answered a question posed by R.H. Bing [Bi]. Interestingly, this problem has received recent attention and a new proof by M. Starbird and his students [S].

Received by the editors May 17, 1982. Presented at the 89th Annual Meeting of the A.M.S. on January 6, 1983 in Denver, Colorado.

1980 Mathematics Subject Classification. Primary 54C25, 57N35; Secondary 57N12, 57N15.

Key words and phrases. Ambiently universal set, ambient embedding, Cantor set, $n$-dimensional Euclidean space.
The question of the existence of compact ambiently universal \( m \)-dimensional sets in \( E^n \) for \( n > 3 \) and \( m \leq n - 3 \) has remained open [Bo, p. 204] until now. Our main theorem states that for each closed set \( X \) in \( E^n \), \( \dim X < n - 3 \), there is a Cantor set \( Y \) in \( E^n \) so that \( Y \) is not ambiently embeddable in \( X \). Hence, it is an easy corollary that there do not exist compact ambiently universal \( m \)-dimensional sets in \( E^n \), \( m \leq n - 3 \).

I thank John Walsh for patiently listening and for suggestions that are reflected in §7.

2. Definitions and notation. We use \( S^n \), \( B^n \), and \( E^n \) to denote the \( n \)-sphere, the \( n \)-ball, and Euclidean \( n \)-space, respectively. We let \( \dim X \) and \( \diam X \) denote the dimension of \( X \) and the diameter of \( X \), respectively. We will assume all manifolds are PL, piecewise-linear, subsets of \( E^n \) whenever possible. We will also assume that PL subsets of \( E^n \) are in general position whenever possible. All of the homology will be done with integer coefficients. If \( M \) is a manifold we let \( \Bd M \) and \( \Int M \) denote the boundary and interior of \( M \), respectively. If \( M \) is a compact orientable 2-manifold we also let \( \Bd M \) denote the boundary of \( M \) oriented in a manner consistent with an orientation on \( M \). If \( J \) is the union of a finite collection of oriented simple closed curves and \( f: J \to E^n - A \) is a map, we say \( f \) links \( A \) in case \( f \), thought of as a 1-cycle, is not null homologous in \( E^n - A \). If \( J \subset E^n - A \) and the inclusion map from \( J \) into \( E^n - A \) links \( A \), we simply say \( J \) links \( A \).

3. Antoine’s necklace. We briefly review a specific construction of Antoine’s necklace [A]. A solid torus is a topological space homeomorphic to \( B^2 \times S^1 \). Consider the embedding of four solid tori \( T_1, T_2, T_3, T_4 \) in a solid torus \( T \) as shown in Figure 1. We call this embedding an Antoine embedding.

![Figure 1](image)

We construct Antoine’s necklace, a Cantor set in \( E^3 \), \( A = \bigcap M_i \), where for each nonnegative integer \( i \), \( M_i \) is a collection of \( 4^i \) disjoint solid tori. We let \( M_0 \) be an unknotted solid torus in \( E^3 \). The collection \( M_{i+1} \) is obtained by taking an Antoine embedding of solid tori in each component of \( M_i \). By exercising care so that the diameters of the components of \( M_i \) approach zero as \( i \) gets large, the set \( A = \bigcap M_i \) will be a Cantor set.
4. I-inessential disks with holes and ramification techniques. Let \( H \) be a disk with holes and \( f: H \to M \) a map into a manifold \( M \) so that \( f(\text{Bd} \ H) \subseteq \text{Bd} \ M \). Following Daverman [D2] we call the map \( f \) \( \text{i-inessential} \) (interior inessential) if there is a map \( \tilde{f} \) from \( H \) into \( \text{Bd} \ M \) with \( f|\text{Bd} \ H = \tilde{f}|\text{Bd} \ H \). We now state without proof a relationship between I-inessential maps and Antoine Cantor sets (see [D2] for a more detailed discussion).

**Lemma 4.1.** Let \( H \) be a disk with holes and \( f: H \to M, \ f(\text{Bd} \ H) \subseteq \text{Bd} \ M \), be a map where \( M \) is a component in some stage of the construction of Antoine’s necklace in \( E^3 \). If \( f(H) \) misses the Cantor set, then the map \( f \) is I-inessential.

Let \( M \) be a closed manifold. Consider the manifold \( B^2 \times M \). For a positive integer \( m \), choose \( m \) pairwise disjoint subdisks \( D_1, \ldots, D_m \) of \( \text{Int} \ B^2 \) and form \( m \) “parallel” copies of \( B^2 \times M \) by taking \( D_1 \times M, \ldots, D_m \times M \). We call the set \( \bigcup \bigcup D_i \times M \) an \( m \)-fold ramification of \( B^2 \times M \) [D, Ea].

Let \( \alpha = a_0, a_1, a_2, \ldots \) be a sequence of positive integers. We construct a ramified Antoine’s necklace with respect to \( \alpha \) as the intersection of nested manifolds \( M_0, M_1, M_2, \ldots \). The set \( M_0 \) is a single unknotted solid torus in \( E^3 \). Let \( i \) be a nonnegative integer. The manifold \( M_{2i+1} \) is obtained by taking an \( a_i \)-fold ramification of each component of \( M_{2i} \). The manifold \( M_{2i+2} \) is obtained by taking an Antoine embedding of solid tori in each component of \( M_{2i+1} \). Exercising due care to insure that the diameters of the components get small as \( i \) gets large yields the desired ramified Antoine’s necklace as the intersection of the \( M_i \).

We call the \( M_i \) a **special defining sequence** for the ramified Antoine’s necklace. Notice that in \( M_{2i} \) we can find \( 4^i \) components which we designate by \( M_{2i} \), so that \( M_{2i} \) is embedded in \( M_0 \) in the same manner as the \( i \)th stage of the Antoine necklace construction is embedded in the first stage.

We now give the obvious generalization of Lemma 4.1 to a ramified Antoine’s necklace.

**Lemma 4.2.** Let \( H \) be a disk with holes and \( f: H \to M, \ f(\text{Bd} \ H) \subseteq \text{Bd} \ M \), be a map where \( M \) is a component in some stage of the construction of a ramified Antoine’s necklace in \( E^3 \). If \( f(H) \) misses the Cantor set, then the map \( f \) is I-inessential.

5. There is no ambiently universal Cantor set in \( E^3 \). In this section we prove our main result in \( E^3 \). Bothe [Bo2] and more recently Starbird and his students [S] have proved this theorem. Our proof uses techniques found in both of the previous proofs. This section will serve as a warm-up for the proof in higher dimensions since the strategy is similar. However, some of the techniques will be discarded when we approach the proof in higher dimensions.

**Theorem 5.1.** For each closed 0-dimensional set \( X \) in \( E^3 \) there is a Cantor set \( Y \) so that \( Y \) is not ambiently embeddable in \( X \).

**Proof.** Let \( X \) be a fixed 0-dimensional set. Let \( T_i, i = 0, 1, 2, \ldots \), be a sequence of all unknotted PL solid tori in \( E^3 \) all of whose vertices have rational coordinates. In \( T_i \) choose \( 4^i + 1 \) disjoint meridional disks. For each disk it is possible to find a
compact 3-manifold that contains the disk in its interior, misses all other disks, and
the boundary of the 3-manifold misses \( X \). Let \( a_i \) be a positive integer so that for any
disk \( D \) of the \( 4^i + 1 \) meridional disks, it is possible to find a compact 3-manifolds \( N \)
so that \( D \subset \text{Int} \ N \), \( N \) misses the other \( 4^i \) meridional disks, \( \text{Bd} \ N \cap X = \emptyset \), and the
number of handles in the 2-manifold \( \text{Bd} \ N \) is less than \( a_i \).

Let \( Y \) be a ramified Antoine's necklace in \( E^3 \) with respect to the sequence \( a_i \), and
let \( M_i \) be a special defining sequence for \( Y \). We now show \( h( Y ) \not\subset X \) for any
homeomorphism \( h \).

Suppose \( h( Y ) \subset X \) for some self-homeomorphism \( h \) of \( E^3 \). Without loss of
generality \( h( M_0 ) \) is a PL solid torus, and we suppose \( h( M_0 ) = T_1 \). To further
simplify the proof we also suppose that \( h \) is the identity homeomorphism.

We choose the union of \( 4^i \) components of \( M_{2i} \), denoted \( \tilde{M}_{2i} \), so that \( \tilde{M}_{2i} \) is
embedded in \( M_0 \) in the same manner as the \( i \)th stage of the Antoine necklace
construction is embedded in the 0th stage. Now each meridional disk in \( M_0 = T_i \)
must contain a meridional simple closed curve of some component of \( \tilde{M}_{2i} \). Hence of
the prechosen \( 4^i + 1 \) meridional disks of \( T_i \), we can find two disks \( D_1 \) and \( D_2 \), and a
component \( W \) of \( \tilde{M}_{2i} \) so that \( W \cap D_1 \) and \( W \cap D_2 \) each contain a meridional simple
closed curve of \( \text{Bd} \ W \).

Let \( N \) be a compact PL 3-manifold so that \( D_1 \subset \text{Int} \ N \), \( N \cap D_2 = \emptyset \), \( \text{Bd} \ N \cap X \)
= \( \emptyset \), and the number of handles of \( \text{Bd} \ N \) is less than \( a_i \). In \( \text{Bd} \ N \) we find a
2-manifold \( M \) such that \( \text{Bd} \ M \) links \( W \). The proof of this fact is somewhat technical
and we defer the proof to the end of this section.

Let \( W(1), W(2), \ldots, W(a_i) \) be the components of \( M_{2i+1} \) that lie inside \( W \). We
assume that \( W(j) \) is in general position with respect to the 2-manifold \( M \). Since the
number of handles of \( M \) is less than the number of handles of \( \text{Bd} \ N \), some one of the
\( W(j) \), which we designate \( W' \), must have the property that \( W' \cap M \) is a 2-manifold
with no handles; i.e., each component of \( W' \cap M \) is a disk with holes. Furthermore,
since \( Y \cap M = \emptyset \), the inclusion of each component of \( W' \cap M \) into \( W' \) is 1-iness-
tessential. This implies that \( \text{Bd} \ M \) does not link \( W' \). Since \( W' \) is “parallel” to \( W \), \( \text{Bd} \ M \)
must link both \( W \) and \( W' \) or neither \( W \) nor \( W' \). We are led to a contradiction from
our supposition that \( h( Y ) \subset X \). We are therefore forced to conclude that \( h( Y ) \not\subset X \).
Our proof is now complete with the exception of the following lemma.

**Lemma 5.2.** Let \( D_1 \) and \( D_2 \) be disks in \( E^3 \), \( N \) a compact 3-manifold \( D_1 \subset \text{Int} \ N \),
\( N \cap D_2 = \emptyset \), and \( W \) a solid torus so that \( W \cap D_1 \) and \( W \cap D_2 \) each contain a
meridional simple closed curve of \( \text{Bd} \ W \). Then there is a 2-manifold \( M \) contained in
\( \text{Bd} \ N \) so that \( \text{Bd} \ M \) links \( W \).

**Proof.** Let \( \tilde{W} \) be a small regular neighborhood of \( W \) so that \( \text{Bd} \ \tilde{W} \) is in general
position with \( \text{Bd} \ N \). Let \( J_1 \) and \( J_2 \) be meridional simple closed curves of \( \tilde{W} \) so that
\( J_1 \subset \text{Int} \ N \), \( J_2 \cap N = \emptyset \), and \( J_1, J_2 \) each bound homologically in the complement of
\( \text{Bd} \ N \). Choose an annulus \( A \) in \( \text{Bd} \ \tilde{W} \) with boundary components \( J_1 \) and \( J_2 \). Let \( J \) be
the collection of all simple closed curves, of \( A \cap \text{Bd} \ N \). Orient the simple closed
curves of \( J \) consistent with some orientation on \( A \cap N \). Hence, as a 1-cycle, \( J \) is
homologous to \( J_1 \) in \( \text{Bd} \ \tilde{W} \), and \( J \) links \( W \) in \( E^3 \). Using \( A \cap N \) it is easy to see that \( J \)
bounds a 2-cycle in $N$. Similarly, $J$ bounds a 2-cycle in closure $(E^3 - N)$. Therefore, a Mayer-Vietoris argument shows that $J$ also bounds homologically in $\text{Bd} \ N$.

We now use geometric interpretations of homology theory to show the existence of the desired 2-manifold $M$ with $\text{Bd} \ M \subset J$. We assume the $\text{Bd} \ N$ has an oriented triangulation so that $J$ is contained in the 1-skeleton. By simplicial homology theory $J = \partial \Sigma \alpha_i \sigma_i$ where $\alpha_i$ are integers and the $\sigma_i$ are all of the finitely many oriented 2-simplexes of the triangulation of $\text{Bd} \ N$. Let $\alpha = \max \{\alpha_i\}$ and set $M' = \bigcup \{\sigma_i | \alpha_i = \alpha\}$. One readily checks that $M'$ is a 2-manifold, $\text{Bd} \ M' \subset J$, and $\text{Bd} \ M'$ is oriented in the same manner as $J$. If $\text{Bd} \ M'$ links $W$, let $M = M'$; otherwise, consider $J' = J - \text{Bd} \ M'$. Then $J'$ must link $W$ and be null homologous in $\text{Bd} \ N$. Since $J'$ has fewer components than $J$, an inductive argument on the number of components of $J$ yields the desired manifold $M$.

6. Bing Cantor sets. R. H. Bing’s proof that the sewing of two Alexander horned spheres yields $S^3 \setminus B^2$ consisted in showing that a Cantor set could be described in $E^3$ as the intersection of manifolds $M_i, i = 0, 1, 2, \ldots$. The manifold $M_0$ is an unknotted solid torus. Each component of $M_i$ is a solid torus and contains two components of $M_{i+1}$ which are embedded as shown in Figure 2. Bing’s clever proof showed that the $M_i$ could be chosen in such a manner that the diameters of the components of $M_i$ tend to zero as $i$ gets large. The intersection of the $M_i$ yields a Cantor set in $E^3$ which we call a Bing Cantor set.

**Figure 2**

Consider the 3-cell $A$ in $E^3_+ = \{(x, y, z) \in E^3 | z \geq 0\}$ that contains two properly embedded arcs $A_1$ and $A_2$ as shown in Figure 3. Notice that Figure 2 can be obtained from Figure 3 by reflecting through the $x$-$y$ plane and thickening the resulting simple closed curves. By spinning $E^3_+$ into $E^n (n \geq 3)$ [C-D] we then obtain as the spin of $A$ a manifold $T$ homeomorphic to $B^2 \times S^{n-2}$ that contains the spin of $A_1$ and the spin of $A_2$, two geometrically linked $(n-2)$-spheres, denoted by $S_1$ and $S_2$, in the interior of $T$. Notice that there are obvious $(n-1)$-cells $D_1, D_2$ in $\text{Int} T$ so that $S_i = \text{Bd} \ D_i (i = 1, 2)$ and $D_i \cap S_j [(i, j) = (1, 2), (2, 1)]$ is homeomorphic to $S^{n-3}$. Observe that $D_1 \cup S_2$ contains a core of $T$ so that any map $f: X \to E^n - (D_1 \cup S_2)$ is homotopic to a map to $E^n - \text{Int} T$, the homotopy fixing points in $f^{-1}(E^n - \text{Int} T)$. For $n > 3$ it is also true that a map $g: S^1 \to D_i - S_j [(i, j) = (1, 2), (2, 1)]$ is null homotopic in $D_i - S_j$ if and only if $g$ does not link $S_j$ in $E^n$. This last fact is not true for $n = 3$. 
Let $T_1$ and $T_2$ be disjoint regular neighborhoods of $S_1$ and $S_2$ in $\text{Int } T$. We construct generalizations of the Bing Cantor set in $E^n$ as the intersection of nested manifolds $W_i, i = 0, 1, 2, \ldots$. The manifold $W_0$ is an unknotted $B^2 \times S^{n-2}$ in $E^n$, and each component of $W_i$ contains two components of $W_{i+1}$ which are embedded in $W_i$ in the same manner as $T_1 \cup T_2$ is embedded in $T$. It is a consequence of [C-D, §8] that the $W_i$ may be chosen so that the diameters of the components tend to zero as $i$ gets large. We also call the intersection of the $W_i$ a Bing Cantor set.

Of course the ramification process of §4 can be applied to Bing Cantor sets to obtain ramified Bing Cantor sets.

7. The 1-inessential property revisited. Let $H$ be a disk with holes and $f: H \to M$, $f(\text{Bd } H) \subset \text{Bd } M$, a map where $M$ is a component in some stage of the construction of a Bing Cantor set in $E^n$ ($n > 3$). The arguments of §4 can be generalized to show that if $f(H)$ misses the Bing Cantor set, then the map $f$ is 1-inessential. However, we will have need of the above fact when $H$ is a compact 2-dimensional polyhedron that behaves like a disk with holes. By a 2-dimensional polyhedron we will always mean a polyhedron that is strictly 2-dimensional, i.e., each open subset is 2-dimensional.

Definition 7.1. Let $P$ be a compact 2-dimensional polyhedron. For some fixed triangulation $T$ of $P$, let boundary of $P$ be the union of all 1-simplexes of $T$ that are the face of exactly one 2-simplex of $T$. Clearly this is independent of the choice of the triangulation since we could also define the boundary to be the closure of the set \{\(x \in P \mid H_2(P, P - x) = 0\)\} as is done in defining the boundary of a homology manifold [Sp, p. 277]. We let $\text{Bd } P$ denote the boundary of $P$, and we define the interior of $P$, denoted $\text{Int } P$, to be $P - \text{Bd } P$.

Definition 7.2. Let $Q$ be a compact 2-dimensional polyhedron. We call $Q$ a pseudo disk with holes if for each $\alpha \in H_1(Q)$ there is a nonzero integer $m$ so that $m\alpha$ is in the image of $H_1(\text{Bd } Q)$ under the inclusion induced homomorphism.

Definition 7.3. Let $f$ be a map of a pseudo disk with holes $Q$ into a manifold $M$ so that $f(\text{Bd } Q) \subset \text{Bd } M$. We call the map $f$ 1-inessential (interior inessential) if there is a map $\tilde{f}$ from $Q$ into $\text{Bd } M$ with $\tilde{f}|\text{Bd } Q = f|\text{Bd } Q$. Otherwise the map $f$ is I-essential.

Theorem 7.4. Let $Q$ be a pseudo disk with holes. If $P$ is a compact 2-dimensional subpolyhedron of $Q$ so that $\text{Int } P$ is an open subset of $Q$, then $P$ is a pseudo disk with holes.

Proof. Let $\gamma$ be a 1-cycle in $P$. By hypothesis there is a nonzero integer $m$ so that $m\gamma$ is homologous in $Q$ to a 1-cycle $\beta$ of $\text{Bd } Q$. Notice that $\text{Bd } Q \subset Q - \text{Int } P$. 

Consider the following Mayer-Vietoris sequence:

\[ \rightarrow H_1(\text{Bd } P) \rightarrow H_1(\text{P}) \oplus H_1(\text{Q - Int } P) \rightarrow H_1(\text{Q}) \rightarrow \cdot \]

Since the element represented by \( m\gamma \oplus (-\beta) \) is sent to zero, we can find a 1-cycle \( \delta \) in \( \text{Bd } P \) that is homologous to \( m\gamma \) in \( P \).

**Theorem 7.5.** Let \( P \) be a compact 2-dimensional polyhedron. Suppose \( Q_1, Q_2, \ldots, Q_n \) are disjoint 2-dimensional subpolyhedra with \( \text{Int } Q_i \) an open subset of \( P \) for each \( i \). If \( n > \text{rank } H_1(P) \), then some \( Q_i \) is a pseudo disk with holes.

**Proof.** Suppose each \( Q_i \) is not a pseudo disk with holes. Choose \( \gamma_i \) a 1-cycle in \( Q_i \) so that no nonzero multiple of \( \gamma_i \) represents an element of \( H_1(Q_i) \) that is the image of \( H_1(\text{Bd } Q_i) \) under the inclusion induced homomorphism. Since \( n > \text{rank } H_1(P) \), there exist integers \( m_1, m_2, \ldots, m_n \), not all zero, so that \( m_1\gamma_1 + m_2\gamma_2 + \cdots + m_n\gamma_n \) is null homologous in \( P \). Without loss of generality, assume \( m_1 \neq 0 \). A Mayer-Vietoris argument applied to \( Q \) and \( P - \text{Int } Q_1 \) shows there is a 1-cycle \( \beta \) in \( Q_1 \cap (P - \text{Int } Q_1) = \text{Bd } Q_1 \) that is homologous to \( m_1\gamma_1 \) in \( Q_1 \). This contradicts the choice of \( \gamma_1 \) and the theorem is proved.

Let \( T, T_1, T_2, S_1, S_2, D_1, D_2 \) be defined as in §6.

**Theorem 7.6.** Let \( f: Q \to T \subset E^n \) be an \( I \)-essential map from a pseudo disk with holes so that \( f \) is in general position with respect to \( T_1 \cup T_2 \). Let \( Q_i = f^{-1}(T_i) \), \( i = 1, 2 \). Then \( f|Q_i: Q_i \to T_i \) is an \( I \)-essential map from a pseudo disk with holes for \( i = 1 \) or \( i = 2 \).

**Proof.** By Theorem 7.4 one easily checks that \( Q_i \) is either empty or a pseudo disk with holes for each \( i \). If \( f|Q_i \) fails to be \( I \)-essential for each \( i \), then we may find a new map \( f_i: Q \to T - (S_1 \cup S_2) \) that agrees with \( f \) on \( \text{Bd } Q \) and is in general position with respect to the \((n - 1)\)-cell \( D_i \).

Let \( \Gamma = f_i^{-1}(D_i) \). We now show \( f_i|\Gamma: \Gamma \to \text{Int } D_1 - S_2 \) induces the trivial homomorphism between the first homology groups. Suppose not. Then there is a 1-cycle \( \gamma \) in \( \Gamma \) so that \( f_i(\gamma) \) is a 1-cycle in \( \text{Int } D_1 \) that links \( D_1 \cap S_2 \). Hence \( f_i(\gamma) \) links \( S_2 \) in \( E^n \). Since \( Q \) is a pseudo disk with holes, there is a 1-cycle \( \beta \) in \( \text{Bd } Q \) so that \( \beta \) is homologous to \( m\gamma \) in \( Q \) for some nonzero integer \( m \). Since \( m \neq 0 \), \( mf_i(\gamma) \) also links \( S_2 \). Because the support of \( f_i(\beta) \) is in \( \text{Bd } M, f_i(\beta) \) does not link \( S_2 \). But this is impossible since \( f_i(\beta) \) is homologous to \( mf_i(\gamma) \) in \( E^n - S_2 \). Hence, \( f_i \) induces the trivial homomorphism on first homology.

Since the fundamental group of \( \text{Int } D_1 - S_2 \) is the same as the first homology group, we find that \( f_i|\Gamma: \Gamma \to \text{Int } D_1 - S_2 \) is homotopic to a constant map. Using this fact and the collar structure on \( D_1 \), we can find a new map \( f_2 \) of \( Q \) into \( T \) that agrees with \( f_1 \) on \( \text{Bd } Q \) and misses \( D_1 \cup S_2 \). Recall that \( D_1 \cup S_2 \) contains a core of \( T \), and we can modify \( f_2 \) to get a map \( f_3: Q \to \text{Bd } T \) agreeing with \( f_2 \) on \( \text{Bd } Q \). This contradicts the fact that \( f \) is \( I \)-essential, and our theorem is proved.

Theorem 7.6 and standard techniques now give our required generalization of the first paragraph of this section which we now state.
Theorem 7.7. Let $Q$ be a pseudo disk with holes and $f: Q \to M, f(\text{Bd } Q) \subset \text{Bd } M$, a map where $M$ is a component in some stage of the construction of a (ramified) Bing Cantor set in $E^n$ ($n > 3$). If $f(Q)$ misses the Cantor set, then the map $f$ is I-inessential.

Theorem 7.8. Let $M$ be a PL $n$-manifold in $E^n$ and $f: P \to E^n$ a map from a polyhedron. Suppose $Q = f^{-1}(M)$ is a subpolyhedron of $P$ that is a pseudo disk with holes whose interior is open in $P$ and $f|_Q: Q \to M$ is I-inessential. If $\gamma$ is a 1-cycle in $P - Q$ that bounds homologically in $P$, then $f(\gamma)$ bounds homologically in $E^n - M$.

Proof. Since $f|_Q$ is I-inessential and Int $Q$ is open in $P$, we can find a new map $f': P \to E^n - \text{Int } T$, agreeing with $f$ on $P - \text{Int } Q$. Since $\gamma$ bounds in $P$, $f'(\gamma)$ bounds in $f'(P)$. Hence $f'(\gamma)$ bounds in $E^n - \text{Int } T$ and, therefore, in $E^n - T$. But $f(\gamma) = f'(\gamma)$ and our proof is complete.

8. Linking Bing Cantor sets. Let $T, T_1, T_2, S_1, S_2, D_1, D_2$ be as defined in §6.

Theorem 8.1. Let $f: M \to E^n$ ($n > 3$) be a PL map in general position from a compact orientable 2-manifold $M$ so that $f|\text{Bd } M$ links $\text{Int } T$. Then there is a simple closed curve $J$ in $M$ so that $f|J$ links either $T_1$ or $T_2$.

Proof. We first consider the case where $f(M)$ misses $T_1 \cup T_2$. We assume that $f$ is in general position with respect to $D_1$ so that $f^{-1}(D_1)$ is a finite collection of disjoint simple closed curves in $M$. If $f$ restricted to any of the simple closed curves links $T_2$ we are done. Otherwise, $f$ restricted to each simple closed curve is null homotopic in $D_1 - S_2$, and the manifold $M$ can be surgered to obtain a new manifold $M'$ and a map $f': M' \to E^n$ so that $\text{Bd } M = \text{Bd } M'$, $f|\text{Bd } M = f'|\text{Bd } M'$, and $f'(M')$ misses $D_1 \cup S_2$. But this implies that $f|\text{Bd } M = f|\text{Bd } M'$ does not link $\text{Int } T$ which is a contradiction.

If $f(M)$ meets $T_1 \cup T_2$, we take small regular neighborhoods $T_1'$ and $T_2'$ of $T_1$ and $T_2$, respectively, so that $f$ is in general position with respect to $T_1'$ and $T_2'$. Hence, $f^{-1}(\text{Bd } T_1' \cup \text{Bd } T_2')$ is a finite collection of disjoint simple closed curves. If $f$ restricted to any of these simple closed curves is not null homotopic in $\text{Bd } T_1' \cup \text{Bd } T_2'$, we are done (requires $n > 3$). If $f$ restricted to each simple closed curve is null homotopic in $\text{Bd } T_1' \cup \text{Bd } T_2'$, then $M$ can be surgered on its interior to obtain a 2-manifold $M'$ and a map $f': M' \to E^n$ so that $\text{Bd } M = \text{Bd } M'$, $f|M \cap M' = f'|M \cap M'$, and $f'(M')$ misses $\text{Bd } T_1' \cup \text{Bd } T_2'$. By ignoring any components of $M'$ that are sent by $f$ to $\text{Int}(T_1' \cup T_2')$, we may assume that $f'(M')$ misses $T_1 \cup T_2$. By the previous case there is a simple closed curve $J$ in $M'$ so that $f'|J$ links $T_1$ or $T_2$. The simple closed curve can be pushed off the disks of $M' - M$ so that $f'|J = f|J$, and the proof is complete.

Let $W_0, W_1, \ldots, W_m$ be nested manifolds in $E^n$ as given in the construction of the Bing Cantor set in §6.

Theorem 8.2. Let $A_0 \subset A_1 \subset \cdots \subset A_m$ be absolute neighborhood retracts in $E^n$ ($n > 3$) so that the inclusion $A_{i-1} \subset A_i$ induces the trivial map on first homology, $1 \leq i \leq m$. If $f_0: S^1 \to A_0$ is a map that links $W_0$ in $E^n$, then there is a map $f_i: S^1 \to A_i$ that links some component of $W_i$. 

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PROOF. The proof is by induction. Suppose $f_i : S^1 \to A_i$ is given so that $f_i$ links some component $T$ of $W_i$. Since the inclusion from $A_i$ into $A_{i+1}$ is trivial on first homology, we find an orientable 2-manifold $M$ and a map $f : M \to A_{i+1}$ so that $f|_{\text{Bd } M}$ links $T$. Let $T_1$, $T_2$ be small regular neighborhoods of the components $T_1$, $T_2$ of $W_{i+1}$ in $T$, respectively. Let $\tilde{f}$ be a close approximation to $f$ that is in general position, the closeness will be stipulated later. By Theorem 8.2 there is a simple closed curve $J$ in $M$ so that $\tilde{f}|_{\text{J}}$ links $T_1$ or $T_2$. Since $A_{i+1}$ is an absolute neighborhood retract, then $\tilde{f}|_{\text{J}}$ is homotopic to a map $g : J \to A_{i+1}$. We now assume $f$ is close enough to $\tilde{f}$ so that the image of the homotopy misses $T_1 \cup T_2$. Let $h$ be a homeomorphism from $S^1$ to $J$. Then $f_{i+1} = g \circ h$ is the desired map.

9. The $n$-dimensional theorem ($n > 3$).

THEOREM 9.1. Each closed set $X$ in $E^n$ ($n > 3$) of dimension at most $(n - 3)$ fails to be ambiently universal with respect to the family of Cantor sets in $E^n$.

PROOF. Let $J_0, J_1, J_2, \ldots$ be the collection of all polygonal simple closed curves in $E^n - X$ all of whose vertices have rational coordinates. For each $i$ set $A'_0 = J_i$ and find compact 2-dimensional polyhedra $A'_1 \subset A'_2 \subset \cdots \subset A'_i \subset P_i$ in $E^n - X$ so that each has boundary $J_i$ and the inclusion from $A'_{i-1}$ into $A'_i$ and from $A'_i$ into $P_i$ is trivial on first homology for all $j, 1 \leq j \leq i$. It may be helpful to think of $P_i$ as a "grope" as described by J. W. Cannon [C]. The construction of $P_i$ depends on the fact that a 1-cycle in $E^n - X$ bounds homologically in $E^n - X$ [H-W].

Let $a_i = \text{rank } H_1(P_i) + 1$. Let $Y$ be a ramified Bing Cantor set with respect to the sequence $a_0, a_1, a_2, \ldots$. Let $M_0, M_1, \ldots$ be a special defining sequence for $Y$. Recall that $M_{2i+1}$ is obtained by taking an $a_i$-fold ramification of each component of $M_{2i}$. The manifold $M_{2i+2}$ has two components in each component of $M_{2i+1}$ that are embedded in the way that $T_1 \cup T_2$ is embedded in $T$ in §6.

We now suppose that $h : E^n \to E^n$ is a homeomorphism so that $h(Y) \subset X$. We assume, for simplicity, in what follows that $h$ is the identity homeomorphism. Some polygonal simple closed curve $J_i$ must link $M_0$. In $M_{2i}$ we find $2^i$ components, designated $M_{2i}$, that lie in $M_0$ in the same manner as $W_i$ is embedded in $W_0$ in the Bing Cantor set construction. By Theorem 8.2 there is a map $f : S^1 \to A'_i$ that links some component $V$ of $M_{2i}$. Let $V(1), V(2), \ldots, V(a_i)$ be the components of $M_{2i+1}$ in $V$. We assume $P_i$ is in general position with each $V(j)$. For each $j, Q_j = V(j) \cap P_i$ is a two complex, $\text{Bd } Q_j \subset \text{Bd } V_j$, and $\text{Int } Q_j$ is open in $P_i$. Since $a_i > \text{rank } H_1(P_i)$, Theorem 7.5 shows that $Q_j$ is a pseudo disk with holes for some fixed $j$. Since $Q_j$ misses $X$ (and therefore $Y$), the inclusion of $Q_j$ into $V(j)$ is I-inessential. Recall that the inclusion of $A'_i$ into $P_i$ is trivial on first homology. Therefore, we can invoke Theorem 7.8 to see that the map $f$ does not link $V(j)$. However, this implies that $f$ does not link $V$, a contradiction.

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