AMBIENTLY UNIVERSAL SETS IN $E^n$

BY

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ABSTRACT. For each closed set $X$ in $E^n$ of dimension at most $n - 3$, we show that $X$ fails to be ambiently universal with respect to Cantor sets in $E^n$; i.e., we find a Cantor set $Y$ in $E^n$ so that for any self-homeomorphism $h$ of $E^n$, $h(Y)$ is not contained in $X$. This result answers a question posed by H. G. Bothe and completes the understanding of ambiently universal sets in $E^n$.

1. Introduction. Let $M$ and $N$ be subsets of $E^n$. We say that $M$ is ambienly embeddable in $N$ if there is a homeomorphism $h$ of $E^n$ onto itself so that $h(M)$ is a subset of $N$. Let $F$ be a family of sets in some fixed $E^n$ and $U$ some fixed subset of $E^n$. We call $U$ an ambiently universal set for the family $F$ if each set in $F$ is ambienly embeddable in $U$. For $0 \leq m \leq n$, a compact $m$-dimensional subset $X$ in $E^n$ is called a compact ambienly universal $m$-dimensional set if every compactum of dimension $\leq m$ in $E^n$ is ambienly embeddable in $X$.

H. G. Bothe has made an extensive study of the existence of compact ambienly universal sets in $E^n$ [Bo1, Bo2, Bo3]. It seems to be well known that Bothe constructed a one-dimensional continuum in $E^3$ [Bo4] similar to the McMillan-Row continuum [M-R]. However, it does not seem to be well known that his purpose was to exhibit a one-dimensional compactum in $E^3$ that is not ambienly embeddable in the Menger universal curve. We give a summary of Bothe's results.

In each $E^n$, Bothe constructed compact $m$-dimensional sets $M^n_m$, $0 \leq m \leq n$. For $n = 2m + 1$, $M^n_m$ is the Menger universal set [H-W, p. 64]. He then showed that $M^n_m$ is a compact ambienly universal $m$-dimensional set in $E^n$ ($n \neq 3$) if and only if $m > n - 3$. For $E^3$, $M^3_3$ is a compact ambienly universal $m$-dimensional set if and only if $m = 2$ or $3$. Bothe defined a condition on a set $X$ in $E^n$ which is now known as the dimension of embedding of $X$ and sometimes written dem $X$ [Ed]. He then showed that $M^n_m$ is an ambiently universal set for all compact subsets $X$ of $E^n$ for which dem $X = m$ and dim $X \leq m$. Bothe showed that there does not exist a compact ambienly universal 0-dimensional set nor a compact ambienly universal 1-dimensional set in $E^3$. His proof that there is no compact ambienly universal 0-dimensional set in $E^3$ answered a question posed by R.H. Bing [Bi1]. Interestingly, this problem has received recent attention and a new proof by M. Starbird and his students [S].
The question of the existence of compact ambiently universal $m$-dimensional sets in $E^n$ for $n > 3$ and $m \leq n - 3$ has remained open [Bo, p. 204] until now. Our main theorem states that for each closed set $X$ in $E^n$, $\dim X < n - 3$, there is a Cantor set $Y$ in $E^n$ so that $Y$ is not ambiently embeddable in $X$. Hence, it is an easy corollary that there do not exist compact ambiently universal $m$-dimensional sets in $E^n$, $m \leq n - 3$.

I thank John Walsh for patiently listening and for suggestions that are reflected in §7.

2. Definitions and notation. We use $S^n$, $B^n$, and $E^n$ to denote the $n$-sphere, the $n$-ball, and Euclidean $n$-space, respectively. We let $\dim X$ and $\text{diam } X$ denote the dimension of $X$ and the diameter of $X$, respectively. We will assume all manifolds are PL, piecewise-linear, subsets of $E^n$ whenever possible. We will also assume that PL subsets of $E^n$ are in general position whenever possible. All of the homology will be done with integer coefficients. If $M$ is a manifold we let $\text{Bd } M$ and $\text{Int } M$ denote the boundary and interior of $M$, respectively. If $M$ is a compact orientable 2-manifold we also let $\text{Bd } M$ denote the boundary of $M$ oriented in a manner consistent with an orientation on $M$. If $J$ is the union of a finite collection of oriented simple closed curves and $f: J \to E^n - A$ is a map, we say $f$ links $A$ in case $f$, thought of as a 1-cycle, is not null homologous in $E^n - A$. If $J \subseteq E^n - A$ and the inclusion map from $J$ into $E^n - A$ links $A$, we simply say $J$ links $A$.

3. Antoine's necklace. We briefly review a specific construction of Antoine's necklace [A]. A solid torus is a topological space homeomorphic to $B^2 \times S^1$. Consider the embedding of four solid tori $T_1$, $T_2$, $T_3$, $T_4$ in a solid torus $T$ as shown in Figure 1. We call this embedding an Antoine embedding.

![Figure 1](https://example.com/figure1.png)

We construct Antoine's necklace, a Cantor set in $E^3$, $A = \bigcap M_i$, where for each nonnegative integer $i$, $M_i$ is a collection of $4^i$ disjoint solid tori. We let $M_0$ be an unknotted solid torus in $E^3$. The collection $M_{i+1}$ is obtained by taking an Antoine embedding of solid tori in each component of $M_i$. By exercising care so that the diameters of the components of $M_i$ approach zero as $i$ gets large, the set $A = \bigcap M_i$ will be a Cantor set.
4. I-inessential disks with holes and ramification techniques. Let \( H \) be a disk with holes and \( f: H \to M \) a map into a manifold \( M \) so that \( f(\partial H) \subseteq \partial M \). Following Daverman [D2] we call the map \( f \) \textit{i-inessential} (interior inessential) if there is a map \( \tilde{f} \) from \( H \) into \( \partial M \) with \( \tilde{f}|\partial H = f|\partial H \). We now state without proof a relationship between I-inessential maps and Antoine Cantor sets (see [D2] for a more detailed discussion).

**Lemma 4.1.** Let \( H \) be a disk with holes and \( f: H \to M, f(\partial H) \subseteq \partial M, \) be a map where \( M \) is a component in some stage of the construction of Antoine's necklace in \( \mathbb{E}^3 \). If \( f(H) \) misses the Cantor set, then the map \( f \) is I-inessential.

Let \( M \) be a closed manifold. Consider the manifold \( B^2 \times M \). For a positive integer \( m \), choose \( m \) pairwise disjoint subdisks \( D_1, \ldots, D_m \) of \( \text{Int} B^2 \) and form \( m \) "parallel" copies of \( B^2 \times M \) by taking \( D_1 \times M, \ldots, D_m \times M \). We call the set \( \bigcup D_i \times M \) an \( m \)-fold ramification of \( B^2 \times M \) [D, Ea].

Let \( \alpha = a_0, a_1, a_2, \ldots \) be a sequence of positive integers. We construct a ramified Antoine's necklace with respect to \( \alpha \) as the intersection of nested manifolds \( M_0, M_1, M_2, \ldots \). The set \( M_0 \) is a single unknotted solid torus in \( \mathbb{E}^3 \). Let \( i \) be a nonnegative integer. The manifold \( M_{2i+1} \) is obtained by taking an \( a_i \)-fold ramification of each component of \( M_{2i} \). The manifold \( M_{2i+2} \) is obtained by taking an Antoine embedding of solid tori in each component of \( M_{2i+1} \). Exercising due care to insure that the diameters of the components get small as \( i \) gets large yields the desired ramified Antoine's necklace as the intersection of the \( M_i \).

We call the \( M_i \) a \textit{special defining sequence} for the ramified Antoine's necklace. Notice that in \( M_{2i} \) we can find \( 4^i \) components which we designate by \( M_{2i} \), so that \( M_{2i} \) is embedded in \( M_0 \) in the same manner as the \( i \)th stage of the Antoine necklace construction is embedded in the first stage.

We now give the obvious generalization of Lemma 4.1 to a ramified Antoine's necklace.

**Lemma 4.2.** Let \( H \) be a disk with holes and \( f: H \to M, f(\partial H) \subseteq \partial M, \) be a map where \( M \) is a component in some stage of the construction of a ramified Antoine's necklace in \( \mathbb{E}^3 \). If \( f(H) \) misses the Cantor set, then the map \( f \) is I-inessential.

5. There is no ambiently universal Cantor set in \( \mathbb{E}^3 \). In this section we prove our main result in \( \mathbb{E}^3 \). Both [Bo2] and more recently Starbird and his students [S] have proved this theorem. Our proof uses techniques found in both of the previous proofs. This section will serve as a warm-up for the proof in higher dimensions since the strategy is similar. However, some of the techniques will be discarded when we approach the proof in higher dimensions.

**Theorem 5.1.** For each closed 0-dimensional set \( X \) in \( \mathbb{E}^3 \) there is a Cantor set \( Y \) so that \( Y \) is not ambiently embeddable in \( X \).

**Proof.** Let \( X \) be a fixed 0-dimensional set. Let \( T_i, i = 0, 1, 2, \ldots, \) be a sequence of all unknotted PL solid tori in \( \mathbb{E}^3 \) all of whose vertices have rational coordinates. In \( T_i \) choose \( 4^i + 1 \) disjoint meridional disks. For each disk it is possible to find a
compact 3-manifold that contains the disk in its interior, misses all other disks, and
the boundary of the 3-manifold misses \( X \). Let \( a \) be a positive integer so that for any
disk \( D \) of the \( 4^i + 1 \) meridional disks, it is possible to find a compact 3-manifolds \( N \)
so that \( D \subset \text{Int} \ N, N \) misses the other \( 4^i \) meridional disks, \( \text{Bd} \ N \cap X = \emptyset \), and the
number of handles in the 2-manifold \( \text{Bd} \ N \) is less than \( a \).

Let \( Y \) be a ramified Antoine's necklace in \( E^3 \) with respect to the sequence \( a \), and
let \( M \) be a special defining sequence for \( Y \). We now show \( h(Y) \not\subset X \) for any
homeomorphism \( h \).

Suppose \( h(Y) \subset X \) for some self-homeomorphism \( h \) of \( E^3 \). Without loss of
generality \( h(M_0) \) is a PL solid torus, and we suppose \( h(M_0) = T_i \). To further
simplify the proof we also suppose that \( h \) is the identity homeomorphism.

We choose the union of \( 4^i \) components of \( M_{2i} \), denoted \( \hat{M}_{2i} \), so that \( \hat{M}_{2i} \)
embedded in \( M_0 \) in the same manner as the \( i \)th stage of the Antoine necklace
construction is embedded in the 0th stage. Now each meridional disk in \( M_0 = T_i \)
must contain a meridional simple closed curve of some component of \( \hat{M}_{2i} \). Hence of
the prechosen \( 4^i + 1 \) meridional disks of \( T_i \), we can find two disks \( D_1 \) and \( D_2 \), and a
component \( W \) of \( \hat{M}_{2i} \) so that \( W \cap D_1 \) and \( W \cap D_2 \) each contain a meridional simple
closed curve of \( \text{Bd} \ W \).

Let \( N \) be a compact PL 3-manifold so that \( D_1 \subset \text{Int} \ N, N \cap D_2 = \emptyset, \text{Bd} \ N \cap X = \emptyset \), and the number of handles of \( \text{Bd} \ N \) is less than \( a \). In \( \text{Bd} \ N \) we find a
2-manifold \( M \) such that \( \text{Bd} \ M \) links \( W \). The proof of this fact is somewhat technical
and we defer the proof to the end of this section.

Let \( W(1), W(2), \ldots, W(a) \) be the components of \( M_{2i+1} \) that lie inside \( W \). We
assume that \( W(j) \) is in general position with respect to the 2-manifold \( M \). Since the
number of handles of \( M \) is less than the number of handles of \( \text{Bd} \ N \), some one of the
\( W(j) \), which we designate \( W' \), must have the property that \( W' \cap M \) is a 2-manifold
with no handles; i.e., each component of \( W' \cap M \) is a disk with holes. Furthermore,
since \( Y \cap M = \emptyset \), the inclusion of each component of \( W' \cap M \) into \( W' \) is 1-inessential.
This implies that \( \text{Bd} \ M \) does not link \( W' \). Since \( W' \) is "parallel" to \( W \), \( \text{Bd} \ M \)
must link both \( W \) and \( W' \) or neither \( W \) nor \( W' \). We are led to a contradiction from
our supposition that \( h(Y) \subset X \). We are therefore forced to conclude that \( h(Y) \not\subset X \).
Our proof is now complete with the exception of the following lemma.

**Lemma 5.2.** Let \( D_1 \) and \( D_2 \) be disks in \( E^3 \), \( N \) a compact 3-manifold \( D_1 \subset \text{Int} \ N, N \cap D_2 = \emptyset \), and \( W \) a solid torus so that \( W \cap D_1 \) and \( W \cap D_2 \) each contain a
meridional simple closed curve of \( \text{Bd} \ W \). Then there is a 2-manifold \( M \) contained in
\( \text{Bd} \ N \) so that \( \text{Bd} \ M \) links \( W \).

**Proof.** Let \( \tilde{W} \) be a small regular neighborhood of \( W \) so that \( \text{Bd} \tilde{W} \) is in general
position with \( \text{Bd} \ N \). Let \( J_1 \) and \( J_2 \) be meridional simple closed curves of \( \tilde{W} \) so that
\( J_1 \subset \text{Int} \ N, J_2 \cap N = \emptyset \), and \( J_1, J_2 \) each bound homologically in the complement of
\( \text{Bd} \ N \). Choose an annulus \( A \) in \( \text{Bd} \tilde{W} \) with boundary components \( J_1 \) and \( J_2 \). Let \( J \) be
the collection of all simple closed curves, of \( A \cap \text{Bd} \ N \). Orient the simple closed
curves of \( J \) consistent with some orientation on \( A \cap N \). Hence, as a 1-cycle, \( J \) is
homologous to \( J_1 \) in \( \text{Bd} \tilde{W} \), and \( J \) links \( W \) in \( E^3 \). Using \( A \cap N \) it is easy to see that \( J \)
bounds a 2-cycle in $N$. Similarly, $J$ bounds a 2-cycle in closure $(E^3 - N)$. Therefore, a Mayer-Vietoris argument shows that $J$ also bounds homologically in $\text{Bd} N$.

We now use geometric interpretations of homology theory to show the existence of the desired 2-manifold $M$ with $\text{Bd} M \subset J$. We assume the $\text{Bd} N$ has an oriented triangulation so that $J$ is contained in the 1-skeleton. By simplicial homology theory $J = \partial \Sigma \alpha_i \sigma_i$ where $\alpha_i$ are integers and the $\sigma_i$ are all of the finitely many oriented 2-simplexes of the triangulation of $\text{Bd} N$. Let $\alpha = \max\{\alpha_i\}$ and set $M' = \bigcup \{\sigma_i | \alpha_i = \alpha\}$. One readily checks that $M'$ is a 2-manifold, $\text{Bd} M' \subset J$, and $\text{Bd} M'$ is oriented in the same manner as $J$. If $\text{Bd} M'$ links $W$, let $M = M'$; otherwise, consider $J' = J - \text{Bd} M'$. Then $J'$ must link $W$ and be null homologous in $\text{Bd} N$. Since $J'$ has fewer components than $J$, an inductive argument on the number of components of $J$ yields the desired manifold $M$.

6. Bing Cantor sets. R. H. Bing's proof that the sewing of two Alexander horned spheres yields $S^3 [Bi_2]$ consisted in showing that a Cantor set could be described in $E^3$ as the intersection of manifolds $M_i$, $i = 0, 1, 2, \ldots$. The manifold $M_0$ is an unknotted solid torus. Each component of $M_i$ is a solid torus and contains two components of $M_{i+1}$ which are embedded as shown in Figure 2. Bing's clever proof showed that the $M_i$ could be chosen in such a manner that the diameters of the components of $M_i$ tend to zero as $i$ gets large. The intersection of the $M_i$ yields a Cantor set in $E^3$ which we call a Bing Cantor set.

Consider the 3-cell $A$ in $E^3_+ = \{(x, y, z) \in E^3 | z \geq 0\}$ that contains two properly embedded arcs $A_1$ and $A_2$ as shown in Figure 3. Notice that Figure 2 can be obtained from Figure 3 by reflecting through the $x$-$y$ plane and thickening the resulting simple closed curves. By spinning $E^3_+$ into $E^n$ ($n \geq 3$) [C-D] we then obtain as the spin of $A$ a manifold $T$ homeomorphic to $B^2 \times S^{n-2}$ that contains the spin of $A_1$ and the spin of $A_2$, two geometrically linked $(n-2)$-spheres, denoted by $S_1$ and $S_2$, in the interior of $T$. Notice that there are obvious $(n-1)$-cells $D_1, D_2$ in $\text{Int} T$ so that $S_i = \text{Bd} D_i$ ($i = 1, 2$) and $D_1 \cap S_j$ $[(i, j) = (1, 2), (2, 1)]$ is homeomorphic to $S^{n-3}$. Observe that $D_1 \cup S_2$ contains a core of $T$ so that any map $f: X \to E^n - (D_1 \cup S_2)$ is homotopic to a map to $E^n - \text{Int} T$, the homotopy fixing points in $f^{-1}(E^n - \text{Int} T)$. For $n > 3$ it is also true that a map $g: S^1 \to D_i - S_j$ $[(i, j) = (1, 2), (2, 1)]$ is null homotopic in $D_i - S_j$ if and only if $g$ does not link $S_j$ in $E^n$. This last fact is not true for $n = 3$. 
Figure 3

Let $T_1$ and $T_2$ be disjoint regular neighborhoods of $S_1$ and $S_2$ in $\text{Int} \, T$. We construct generalizations of the Bing Cantor set in $E^n$ as the intersection of nested manifolds $W_i$, $i = 0, 1, 2, \ldots$. The manifold $W_0$ is an unknotted $B^2 \times S^{n-2}$ in $E^n$, and each component of $W_i$ contains two components of $W_{i+1}$ which are embedded in $W_i$ in the same manner as $T_1 \cup T_2$ is embedded in $T$. It is a consequence of [C-D, §8] that the $W_i$ may be chosen so that the diameters of the components tend to zero as $i$ gets large. We also call the intersection of the $W_i$ a Bing Cantor set.

Of course the ramification process of §4 can be applied to Bing Cantor sets to obtain ramified Bing Cantor sets.

7. The $I$-inessential property revisited. Let $H$ be a disk with holes and $f: H \to M$, $f(\text{Bd} \, H) \subset \text{Bd} \, M$, a map where $M$ is a component in some stage of the construction of a Bing Cantor set in $E^n$ ($n > 3$). The arguments of §4 can be generalized to show that if $f(H)$ misses the Bing Cantor set, then the map $f$ is $I$-inessential. However, we will have need of the above fact when $H$ is a compact 2-dimensional polyhedron that behaves like a disk with holes. By a 2-dimensional polyhedron we will always mean a polyhedron that is strictly 2-dimensional, i.e., each open subset is 2-dimensional.

**Definition 7.1.** Let $P$ be a compact 2-dimensional polyhedron. For some fixed triangulation $T$ of $P$, let boundary of $P$ be the union of all 1-simplexes of $T$ that are the face of exactly one 2-simplex of $T$. Clearly this is independent of the choice of the triangulation since we could also define the boundary to be the closure of the set \{ $x \in P \mid H_2(P, P - x) = 0$ \} as is done in defining the boundary of a homology manifold [Sp, p. 277]. We let $\text{Bd} \, P$ denote the boundary of $P$, and we define the interior of $P$, denoted $\text{Int} \, P$, to be $P - \text{Bd} \, P$.

**Definition 7.2.** Let $Q$ be a compact 2-dimensional polyhedron. We call $Q$ a *pseudo disk with holes* if for each $a \in H_1(Q)$ there is a nonzero integer $m$ so that $ma$ is in the image of $H_1(\text{Bd} \, Q)$ under the inclusion induced homomorphism.

**Definition 7.3.** Let $f$ be a map of a pseudo disk with holes $Q$ into a manifold $M$ so that $f(\text{Bd} \, Q) \subset \text{Bd} \, M$. We call the map $f$ *$I$-inessential* (interior inessential) if there is a map $\tilde{f}$ from $Q$ into $\text{Bd} \, M$ with $\tilde{f}| \text{Bd} \, Q = f| \text{Bd} \, Q$. Otherwise the map $f$ is $I$-essential.

**Theorem 7.4.** Let $Q$ be a pseudo disk with holes. If $P$ is a compact 2-dimensional subpolyhedron of $Q$ so that $\text{Int} \, P$ is an open subset of $Q$, then $P$ is a pseudo disk with holes.

**Proof.** Let $\gamma$ be a 1-cycle in $P$. By hypothesis there is a nonzero integer $m$ so that $m \gamma$ is homologous in $Q$ to a 1-cycle $\beta$ of $\text{Bd} \, Q$. Notice that $\text{Bd} \, Q \subset Q - \text{Int} \, P$. 
Consider the following Mayer-Vietoris sequence:
\[ \rightarrow H_1(\text{Bd } P) \rightarrow H_1(\text{P}) \oplus H_1(\text{Q} - \text{Int } P) \rightarrow H_1(\text{Q}) \rightarrow . \]

Since the element represented by \( m\gamma \oplus (-\beta) \) is sent to zero, we can find a 1-cycle \( \delta \) in \( \text{Bd } P \) that is homologous to \( m\gamma \) in \( P \).

**THEOREM 7.5.** Let \( P \) be a compact 2-dimensional polyhedron. Suppose \( Q_1, Q_2, \ldots, Q_n \) are disjoint 2-dimensional subpolyhedra with \( \text{Int } Q_i \) an open subset of \( P \) for each \( i \). If \( n > \text{rank } H_1(P) \), then some \( Q_i \) is a pseudo disk with holes.

**PROOF.** Suppose each \( Q_i \) is not a pseudo disk with holes. Choose \( \gamma_i \) a 1-cycle in \( Q_i \) so that no nonzero multiple of \( \gamma_i \) represents an element of \( H_1(Q_i) \) that is the image of \( H_1(\text{Bd } Q_i) \) under the inclusion induced homomorphism. Since \( n > \text{rank } H_1(P) \), there exist integers \( m_1, m_2, \ldots, m_n \), not all zero, so that \( m_1\gamma_1 + m_2\gamma_2 + \cdots + m_n\gamma_n \) is null homologous in \( P \). Without loss of generality, assume \( m_1 \neq 0 \). A Mayer-Vietoris argument applied to \( Q \) and \( P - \text{Int } Q_1 \) shows there is a 1-cycle \( \beta \) in \( Q_1 \cap (P - \text{Int } Q_1) = \text{Bd } Q_1 \) that is homologous to \( m_1\gamma_1 \) in \( Q_1 \). This contradicts the choice of \( \gamma_1 \) and the theorem is proved.

Let \( T, T_1, T_2, S_1, S_2, D_1, D_2 \) be defined as in §6.

**THEOREM 7.6.** Let \( f: Q \rightarrow T \subset E^n \) \( (n > 3) \) be an I-essential map from a pseudo disk with holes so that \( f \) is in general position with respect to \( T_1 \cup T_2 \). Let \( Q_i = f^{-1}(T_i), i = 1, 2 \). Then \( f \mid Q_i: Q_i \rightarrow T_i \) is an I-essential map from a pseudo disk with holes for \( i = 1 \) or \( i = 2 \).

**PROOF.** By Theorem 7.4 one easily checks that \( Q_i \) is either empty or a pseudo disk with holes for each \( i \). If \( f \mid Q_i \) fails to be I-essential for each \( i \), then we may find a new map \( f_1: Q \rightarrow T - (S_1 \cup S_2) \) that agrees with \( f \) on \( \text{Bd } Q \) and is in general position with respect to the \((n - 1)\)-cell \( D_1 \).

Let \( \Gamma = f_1^{-1}(D_1) \). We now show \( f_1 \mid \Gamma: \Gamma \rightarrow \text{Int } D_1 - S_2 \) induces the trivial homomorphism between the first homology groups. Suppose not. Then there is a 1-cycle \( \gamma \) in \( \Gamma \) so that \( f_1(\gamma) \) is a 1-cycle in \( \text{Int } D_1 \) that links \( D_1 \cap S_2 \). Hence \( f_1(\gamma) \) links \( S_2 \) in \( E^n \). Since \( Q \) is a pseudo disk with holes, there is a 1-cycle \( \beta \) in \( \text{Bd } Q \) so that \( \beta \) is homologous to \( m\gamma \) in \( Q \) for some nonzero integer \( m \). Since \( m \neq 0 \), \( mf_1(\gamma) \) also links \( S_2 \). Because the support of \( f_1(\beta) \) is in \( \text{Bd } M \), \( f_1(\beta) \) does not link \( S_2 \). But this is impossible since \( f_1(\beta) \) is homologous to \( mf(\gamma) \) in \( E^n - S_2 \). Hence, \( f_1 \) induces the trivial homomorphism on first homology.

Since the fundamental group of \( \text{Int } D_1 - S_2 \) is the same as the first homology group, we find that \( f_1 \mid \Gamma: \Gamma \rightarrow \text{Int } D_1 - S_2 \) is homotopic to a constant map. Using this fact and the collar structure on \( D_1 \), we can find a new map \( f_2 \) of \( Q \) into \( T \) that agrees with \( f_1 \) on \( \text{Bd } Q \) and misses \( D_1 \cup S_2 \). Recall that \( D_1 \cup S_2 \) contains a core of \( T \), and we can modify \( f_2 \) to get a map \( f_3: Q \rightarrow \text{Bd } T \) agreeing with \( f_2 \) on \( \text{Bd } Q \). This contradicts the fact that \( f \) is I-essential, and our theorem is proved.

Theorem 7.6 and standard techniques now give our required generalization of the first paragraph of this section which we now state.
Theorem 7.7. Let $Q$ be a pseudo disk with holes and $f: Q \to M$, $f(\text{Bd } Q) \subset \text{Bd } M$, a map where $M$ is a component in some stage of the construction of a (ramified) Bing Cantor set in $E^n$ ($n > 3$). If $f(Q)$ misses the Cantor set, then the map $f$ is $I$-inessential.

Theorem 7.8. Let $M$ be a PL $n$-manifold in $E^n$ and $f: P \to E^n$ a map from a polyhedron. Suppose $Q = f^{-1}(M)$ is a subpolyhedron of $P$ that is a pseudo disk with holes whose interior is open in $P$ and $f|Q: Q \to M$ is $I$-inessential. If $\gamma$ is a $1$-cycle in $P - Q$ that bounds homologically in $P$, then $f(\gamma)$ bounds homologically in $E^n - M$.

Proof. Since $f|Q$ is $I$-inessential and $\text{Int } Q$ is open in $P$, we can find a new map $f': P \to E^n - \text{Int } T$, agreeing with $f$ on $P - \text{Int } Q$. Since $\gamma$ bounds in $P$, $f'(\gamma)$ bounds in $f'(P)$. Hence $f'(\gamma)$ bounds in $E^n - \text{Int } T$ and, therefore, in $E^n - T$. But $f(\gamma) = f'(\gamma)$ and our proof is complete.

8. Linking Bing Cantor sets. Let $T, T'_1, T'_2, S_1, S_2, D_1, D_2$ be as defined in §6.

Theorem 8.1. Let $f: M \to E^n$ ($n > 3$) be a PL map in general position from a compact orientable $2$-manifold $M$ so that $f|\text{Bd } M$ links $\text{Int } T$. Then there is a simple closed curve $J$ in $M$ so that $f|J$ links either $T_1$ or $T_2$.

Proof. We first consider the case where $f(M)$ misses $T_1 \cup T_2$. We assume that $f$ is in general position with respect to $D_1$ so that $f^{-1}(D_1)$ is a finite collection of disjoint simple closed curves in $M$. If $f$ restricted to any of the simple closed curves links $T_2$ we are done. Otherwise, $f$ restricted to each simple closed curve is null homotopic in $D_1 - S_2$, and the manifold $M$ can be surgered to obtain a new manifold $M'$ and a map $f': M' \to E^n$ so that $\text{Bd } M = \text{Bd } M'$, $f|\text{Bd } M = f'|\text{Bd } M'$, and $f'(M')$ misses $D_1 \cup S_2$. But this implies that $f|\text{Bd } M = f|\text{Bd } M'$ does not link $\text{Int } T$ which is a contradiction.

If $f(M)$ meets $T_1 \cup T_2$, we take small regular neighborhoods $T'_1$ and $T'_2$ of $T_1$ and $T_2$, respectively, so that $f$ is in general position with respect to $T'_1$ and $T'_2$. Hence, $f^{-1}(\text{Bd } T'_1 \cup \text{Bd } T'_2)$ is a finite collection of disjoint simple closed curves. If $f$ restricted to any of these simple closed curves is not null homotopic in $\text{Bd } T'_1 \cup \text{Bd } T'_2$, we are done (requires $n > 3$). If $f$ restricted to each simple closed curve is null homotopic in $\text{Bd } T'_1 \cup \text{Bd } T'_2$, then $M$ can be surgered on its interior to obtain a $2$-manifold $M'$ and a map $f': M' \to E^n$ so that $\text{Bd } M = \text{Bd } M'$, $f|M \cap M' = f'|M \cap M'$, and $f'(M')$ misses $\text{Bd } T'_1 \cup \text{Bd } T'_2$. By ignoring any components of $M'$ that are sent by $f$ to $\text{Int}(T'_1 \cup T'_2)$, we may assume that $f'(M')$ misses $T_1 \cup T_2$. By the previous case there is a simple closed curve $J$ in $M'$ so that $f'|J$ links $T_1$ or $T_2$. The simple closed curve can be pushed off the disks of $M' - M$ so that $f'|J = f|J$, and the proof is complete.

Let $W_0, W_1, \ldots, W_m$ be nested manifolds in $E^n$ as given in the construction of the Bing Cantor set in §6.

Theorem 8.2. Let $A_0 \subset A_1 \subset \cdots \subset A_m$ be absolute neighborhood retracts in $E^n$ ($n > 3$) so that the inclusion $A_{i-1} \subset A_i$ induces the trivial map on first homology, $1 \leq i \leq m$. If $f_0: S^1 \to A_0$ is a map that links $W_0$ in $E^n$, then there is a map $f_i: S^1 \to A_i$ that links some component of $W_i$. 
Proof. The proof is by induction. Suppose \( f_i: S^1 \to A_i \) is given so that \( f_i \) links some component \( T \) of \( W_i \). Since the inclusion from \( A_i \) into \( A_{i+1} \) is trivial on first homology, we find an orientable 2-manifold \( M \) and a map \( f: M \to A_{i+1} \) so that \( f|_{\partial M} \) links \( T \). Let \( T_1, T_2 \) be small regular neighborhoods of the components \( T_1, T_2 \) of \( W_{i+1} \) in \( T \), respectively. Let \( \tilde{f} \) be a close approximation to \( f \) that is in general position, the closeness will be stipulated later. By Theorem 8.2 there is a simple closed curve \( J \) in \( M \) so that \( \tilde{f}|J \) links \( T_1 \) or \( T_2 \). Since \( A_{i+1} \) is an absolute neighborhood retract, then \( g|J \) is homotopic to a map \( g: J \to A_{i+1} \). We now assume \( f \) is close enough to \( \tilde{f} \) so that the image of the homotopy misses \( T_1 \cup T_2 \). Let \( h \) be a homeomorphism from \( S^1 \) to \( J \). Then \( f_{i+1} = g \circ h \) is the desired map.

9. The \( n \)-dimensional theorem \((n > 3)\).

Theorem 9.1. Each closed set \( X \) in \( E^n \) \((n > 3)\) of dimension at most \((n - 3)\) fails to be ambiently universal with respect to the family of Cantor sets in \( E^n \).

Proof. Let \( J_0, J_1, J_2, \ldots \) be the collection of all polygonal simple closed curves in \( E^n - X \) all of whose vertices have rational coordinates. For each \( i \) set \( A'_0 = J_i \) and find compact 2-dimensional polyhedra \( A'_1 \subset A'_2 \subset \cdots \subset A'_i \subset P_i \) in \( E^n - X \) so that each has boundary \( J_i \) and the inclusion from \( A'_{i-1} \) into \( A'_i \) and from \( A'_i \) into \( P_i \) is trivial on first homology for all \( j, 1 \leq j \leq i \). It may be helpful to think of \( P_i \) as a "grope" as described by J. W. Cannon [C]. The construction of \( P_i \) depends on the fact that a 1-cycle in \( E^n - X \) bounds homologically in \( E^n - X[H-W] \).

Let \( a_i = \text{rank } H_1(P_i) + 1 \). Let \( Y \) be a ramified Bing Cantor set with respect to the sequence \( a_0, a_1, a_2, \ldots \). Let \( M_0, M_1, \ldots \) be a special defining sequence for \( Y \). Recall that \( M_{2i+1} \) is obtained by taking an \( a_i \)-fold ramification of each component of \( M_{2i} \). The manifold \( M_{2i+2} \) has two components in each component of \( M_{2i+1} \) that are embedded in the way that \( T_1 \cup T_2 \) is embedded in \( T \) in §6.

We now suppose that \( h: E^n \to E^n \) is a homeomorphism so that \( h(Y) \subset X \). We assume, for simplicity, in what follows that \( h \) is the identity homeomorphism. Some polygonal simple closed curve \( J_i \) must link \( M_0 \). In \( M_{2i} \) find \( 2^i \) components, designated \( M_{2i} \), that lie in \( M_0 \) in the same manner as \( W_i \) is embedded in \( W_0 \) in the Bing Cantor set construction. By Theorem 8.2 there is a map \( f: S^1 \to A'_i \) that links some component \( V \) of \( M_{2i} \). Let \( V(1), V(2), \ldots, V(a_i) \) be the components of \( M_{2i+1} \) in \( V \). We assume \( P_i \) is in general position with each \( V(j) \). For each \( j, Q_j = V(j) \cap P_i \) is a two complex, \( \text{Bd } Q_j \subset \text{Bd } V_j \), and \( \text{Int } Q_j \) is open in \( P_i \). Since \( a_i > \text{rank } H_1(P_i) \), Theorem 7.5 shows that \( Q_j \) is a pseudo disk with holes for some fixed \( j \). Since \( Q_j \) misses \( X \) (and therefore \( Y \)), the inclusion of \( Q_j \) into \( V(j) \) is I-inessential. Recall that the inclusion of \( A'_i \) into \( P_i \) is trivial on first homology. Therefore, we can invoke Theorem 7.8 to see that the map \( f \) does not link \( V(j) \). However, this implies that \( f \) does not link \( V \), a contradiction.

References


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