

STRONGLY COHEN-MACAULAY SCHEMES AND RESIDUAL INTERSECTIONS

BY

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ABSTRACT. This paper studies the local properties of closed subschemes Y in Cohen-Macaulay schemes X such that locally the defining ideal of Y in X has the property that its Koszul homology is Cohen-Macaulay. Whenever this occurs Y is said to be strongly Cohen-Macaulay in X . This paper proves several facts about such embeddings, chiefly with reference to the residual intersections of Y in X . The main result states that any residual intersection of Y in X is again Cohen-Macaulay.

Introduction. Our purpose in this paper is to investigate a property of a closed subscheme Y in a Noetherian Cohen-Macaulay scheme X , which we call *strong Cohen-Macaulayness*. This is a local property which we first describe for a local ring. Let $X = \text{Spec}(R)$, where R is a Cohen-Macaulay local ring, and let $Y = \text{Spec}(R/I)$ ($I = I(Y)$). Fix any (not necessarily minimal) generating set f_1, \dots, f_n of I . Let $H_i(f; R)$ denote the i th homology of the Koszul complex associated to f_1, \dots, f_n . We define Y to be *strongly Cohen-Macaulay* in X if $H_i(f; R)$ are either zero, or Cohen-Macaulay modules and if in addition Y is generically a complete intersection. This last condition means that I_P is generated by a regular sequence for every prime ideal P which is minimal over I .

As $H_0(f; R) = R/I$, Y is necessarily Cohen-Macaulay if it is strongly Cohen-Macaulay in X .

If X is any Noetherian Cohen-Macaulay scheme and Y is any closed subscheme, then we say Y is strongly Cohen-Macaulay in X if this statement is true locally for every $y \in Y$.

The statement “ Y is strongly Cohen-Macaulay in X ” has several good geometric properties. First of all, this property does not depend upon the generating set of the defining ideal of $\mathcal{O}_{Y,y}$ in $\mathcal{O}_{X,y}$. If $X = \text{Spec}(k[Z_1, \dots, Z_n])$ and $X' = \text{Spec}(k[T_1, \dots, T_m])$ where k is a field, and if Y is embedded in both X and X' , then Y is strongly Cohen-Macaulay in X if and only if Y is strongly Cohen-Macaulay in X' . If Y is strongly Cohen-Macaulay in $X = \text{Spec}(k[Z_1, \dots, Z_n])$ and Y' is strongly Cohen-Macaulay in $X' = \text{Spec}(k[T_1, \dots, T_m])$, then $Y \times Y'$ is strongly Cohen-Macaulay in $X \times X'$ (Corollary 1.12).

The most obvious example of a strongly Cohen-Macaulay embedding is the case when Y is a local complete intersection in X , i.e. the defining ideal of Y in X is

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locally generated by a regular sequence. In this case, Y is automatically Cohen-Macaulay (as X is assumed to be), while locally, the higher Koszul homology of the defining ideal of Y is zero (if we choose a minimal generating set for the ideal of Y).

We are motivated by three reasons to separate this class of subschemes from the general class of Cohen-Macaulay subschemes. First of all, strongly Cohen-Macaulay schemes have been recently studied by Simis and Vasconcelos [S-V] who are able to prove many properties concerning the blow-ups of these schemes and also are able to prove isomorphisms of the “residual scheme” of Y and the blow-up of Y (under suitable extra hypothesis). Recall the residual scheme of Y is the scheme $\mathbf{P}(I(Y))$. (See Kleiman [K] for a discussion of these ideas.) In fact Simis and Vasconcelos are able to prove isomorphisms of the homogeneous coordinate rings of these two schemes under very general circumstances. We will discuss their work in detail in §§4 and 5. (See also [Ku] for work concerning when the residual scheme and blow-up are isomorphic.)

The second reason for our interest arises from the fact that a general class of such schemes is found “in nature”. To describe how these schemes are found we first recall the definition of liaison (see Peskine and Szpiro [P-S-1]).

DEFINITION 0.1. Let X be a Cohen-Macaulay Noetherian scheme, and let V_1 and V_2 be two closed subschemes of X . We say V_1 and V_2 are directly linked (written $V_1 \sim V_2$) if there is a complete intersection $Z \subseteq V_1 \cap V_2$ such that

- (i) $\underline{I}(V_2)/\underline{I}(Z) = \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_{V_1}, \mathcal{O}_Z)$, and
- (ii) $\underline{I}(V_1)/\underline{I}(Z) = \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_{V_2}, \mathcal{O}_Z)$.

Here $\underline{I}(Y)$ is the ideal sheaf of a closed subscheme Y of X . If there exist V_1, \dots, V_n such that $V \sim V_1 \sim \dots \sim V_n \sim V'$, we say V and V' are linked in X (written $V \sim V'$).

The following theorem is proved in [Hu-1].

THEOREM 0.2. *Let X be a Gorenstein Noetherian scheme and suppose $Y \subseteq X$ is a closed subscheme. If $Y \sim Z$ where Z is a complete intersection in X , then Y is strongly Cohen-Macaulay in X .*

For the benefit of those readers not conversant with liaison we pause to note two examples of such schemes.

PROPOSITION 0.3. *Let $X = \text{Spec}(R)$ where R is a regular local ring. Suppose $Y \subseteq X$ is a Cohen-Macaulay subscheme. If either*

- (i) $\text{codim } Y = 2$, or
- (ii) $\text{codim } Y = 3$ and Y is Gorenstein,

then Y is strongly Cohen-Macaulay in X .

For in either case $Y \sim Z$, a complete intersection in X . In the situation (i), this latter fact was proved by Peskine and Szpiro [P-S-1], while in case (ii) this fact was proved by J. Watanabe [W].

We will later give some more examples of the type given in Theorem 0.2.

The third reason for separating this class of subschemes is the main reason for our interest in them. Namely, such schemes have very strong residual intersection properties. This author was surprised that these schemes could have such strong

properties. The description of the residual intersection properties enjoyed by strongly Cohen-Macaulay schemes is the main result of this paper. We describe the main theorem in detail here.

Let $X = \text{Spec}(R)$, where R is a Cohen-Macaulay local ring and let $Y = V(I) \subset X$ be a closed subscheme of X . Let $C_k(Y) = C_k \subset X$ be the set of points $x \in X$ such that $v(I_x) \geq k$, where $v(-)$ denotes the least number of generators. We let

$$c_1 = c_1(Y) = \text{codim}(Y \cap C_1(Y)), \quad c_i = c_i(V) = \text{codim}(C_i(Y)) \quad \text{if } i > 1.$$

The codimension of \emptyset is by definition infinity.

DEFINITION 0.4 [A-N, p. 312]. We say Y satisfies condition G_s if $c_i \geq i$ for $i = 1, \dots, s$.

REMARK 0.5. The scheme $Y \subset X$ satisfies G_s if and only if $v(I(Y)_y) \leq \text{codim} \overline{\{y\}}$ for all $y \in Y$ such that $\text{codim} \overline{\{y\}} \leq s - 1$.

We prove the above statement as follows. Assume Y satisfies G_s and let $y \in Y$, $k = \text{codim} \overline{\{y\}} \leq s - 1$. If $v(I_y) > k$, then $y \in C_{k+1}(Y)$. Since $\overline{\{y\}} \subseteq C_{k+1}(Y)$ (this is a closed set), $\text{codim} \overline{\{y\}} \geq k + 1$, contradicting our assumption.

Conversely, suppose $v(I_y) \leq \text{codim} \overline{\{y\}}$ for all $y \in Y$ such that $\text{codim} \overline{\{y\}} \leq s - 1$, and yet $c_i < i$ for some $i \leq s$. This inequality implies there is a $y \in Y$ such that $v(I_y) \geq i$ but $\text{codim} \overline{\{y\}} < i$, which clearly contradicts our assumption. This finishes the proof of Remark 0.5.

In particular we observe that the following three conditions are equivalent (set $n = v(I)$):

- (1) Y satisfies G_∞ .
- (2) $v(I_y) \leq \text{codim} \overline{\{y\}}$ for all $y \in Y$.
- (3) Y satisfies G_n .

Our main result can now be stated.

THEOREM 3.1. *Let Y be a strongly Cohen-Macaulay closed subscheme of a Cohen-Macaulay local scheme $X = \text{Spec}(R)$. Put $k = \text{codim}_X Y$. Let $I =$ the ideal defining Y . Suppose $s > k$ is an integer and let $A = (a_1, \dots, a_s)$ be an ideal such that $A_y = I_y$ if $y \in Y$ and $\text{codim} \overline{\{y\}} \leq s$. Set $J = (A : I)$, $Z = V(J)$. If Y satisfies G_s , then:*

- (i) Z is Cohen-Macaulay, $\text{codim } Z = s$.
- (ii) $Z \cup Y = V(A)$ (scheme-theoretically).
- (iii) $Z \cap Y$ is Cohen-Macaulay of codimension $s + 1$.

The subscheme Z is called a residual intersection of Y in X .

The interested reader should compare this result with Theorem 2.1, p. 312 of [A-N]. In their paper, Artin and Nagata are interested in proving this theorem in the case that Y is a complete intersection in X . However, their theorem is stated too generally. In this general form, the result is incorrect. We give a counterexample (Example 3.3) to the statement of Theorem 2.1 of [A-N]. In fact, a subscheme Y of X which satisfies the conclusion of Theorem 3.1 is very close to being strongly Cohen-Macaulay. In the last section we are able to prove a partial converse.

THEOREM 5.1. *Suppose X is a Gorenstein affine scheme and $Y \subseteq X$ is a closed subscheme which satisfies G_∞ . In addition assume $c_i(Y) \geq i + 2$ if $i \geq \text{codim } Y + 3$.*

Assume that the conclusion of Theorem 3.1 holds for any residual intersection Z of Y . Then Y is strongly Cohen-Macaulay in X .

The proof is obtained by comparing such schemes to those whose defining ideals are generated by d -sequences (see the last section for details). The proof shows that there are many large classes of strongly Cohen-Macaulay subschemes.

Theorem 3.1 has several interesting applications which we discuss in §4. Let V be an r -dimensional vector space over a field k and let W be an s -dimensional vector space over k . Let $D_{r,s}$ be the variety of maps in $\text{Hom}(V, W)$ of rank at most $r - 1$. It is well known that

$$D_{r,s} = \text{Spec}\left(k[X_{ij}]/(\sqrt{I_r}(X))\right)$$

where $X = (x_{ij})$ is an $r \times s$ generic matrix. It is known that $D_{r,s}$ is Cohen-Macaulay and further that $\sqrt{I_r}(X) = I_r(X)$. We show how the Cohen-Macaulayness of $D_{r,s}$ follows from the strong Cohen-Macaulayness of $D_{r,r+1}$. In fact, $D_{r,r+1}$ is a strong Cohen-Macaulay subscheme of $\mathbf{A}_k^{r(r+1)}$, and $D_{r,r+1}$ satisfies G_∞ . Further, the scheme $D_{s-r+1,r}$ is a residual intersection of $D_{r,r+1}$. By Theorem 3.1, $D_{s-r+1,r}$ is Cohen-Macaulay! This example illustrates the theme of this paper; a strongly Cohen-Macaulay scheme Y in X inherently contains a great deal of information about schemes of lower dimension in X , namely those which are residual intersections of Y .

Another application is a new proof of the theorem of Simis and Vasconcelos [S-V] which we have already referred to in this introduction. We review this result. Let $X = \text{Spec}(R)$ where R is a Cohen-Macaulay local ring and let $Y \subseteq X$ be a closed subscheme, $Y = V(I)$. By $G(Y)$ denote the scheme $\text{Spec}(\bigoplus_{n \geq 0} I^n/I^{n+1})$. Set $n = v(I)$, $I = (a_1, \dots, a_n)$, and let $\tilde{Y} = \text{Spec}(R[T_1, \dots, T_n, S]/J)$ where J is the ideal generated by $ST_i - a_i$, and all the polynomials $\sum_{i=1}^n b_i T_i$ such that $\sum b_i a_i = 0$. Of course, \tilde{Y} is closely related to the scheme $\text{Spec}(\text{Sym } I)$. Let $R(Y)$ be $\text{Spec}(\bigoplus_{n \in \mathbf{Z}} I^n)$ where $I^n = R$ if $n \leq 0$. It is clear that there is a closed immersion of $R(Y)$ into \tilde{Y} . An important question is when $\tilde{Y} = R(Y)$. Whenever this occurs, $\text{Sym}(I) = \bigoplus_{n \geq 0} I^n$. If $R(Y) = \tilde{Y}$, then it is well known that Y must satisfy G_∞ . If Y is strongly Cohen-Macaulay in X , Simis and Vasconcelos show the opposite also holds. (In fact they are also able to show the symmetric algebra of I and the Rees algebra of I are isomorphic and Cohen-Macaulay.)

THEOREM 0.6 [S-V]. *Let X be as above, and $Y \subseteq X$ a strongly Cohen-Macaulay closed subscheme of X . Suppose Y satisfies G_∞ . Then*

- (i) $\tilde{Y} = R(Y)$, and
- (ii) \tilde{Y} is Cohen-Macaulay.

Set $n = v(I(Y))$ and let $T = \text{Spec}(R[T_1, \dots, T_n, S])$. Then $R(Y) \subset \tilde{Y} \subset T$, where these are all closed immersions. Let $Y' = V((I, S))$ in T . Then Y' is strongly Cohen-Macaulay in T if Y is so in X , and Y' also will satisfy G_∞ if Y does. We will show that \tilde{Y} is always a residual intersection of Y' . Under these conditions, Theorem 3.1 shows \tilde{Y} is Cohen-Macaulay. Further the condition (ii) of Theorem 3.1 in this case is equivalent to the assertion that $\tilde{Y} = R(Y)$.

One of the key ingredients of Theorem 3.1 is Proposition 1.6, which shows that strongly Cohen-Macaulay subschemes of Cohen-Macaulay schemes have a linkage property which in general is not held by Cohen-Macaulay ideals. Namely,

PROPOSITION 1.6. *Let R be a Cohen-Macaulay local ring and I a strongly Cohen-Macaulay ideal of R . Suppose J is linked to I . Then R/J is Cohen-Macaulay.*

This proposition is *false* if we only assume R/I to be Cohen-Macaulay [**P-S-1**].

As we mentioned above, the last section of the paper is devoted to relating strong Cohen-Macaulayness to d -sequences. We refer the reader to this section for the relevant definitions. Of particular interest is Theorem 5.9, which gives a usable criterion to determine if a scheme Y is strongly Cohen-Macaulay.

1. Preliminaries. In this section we present the basic facts concerning Koszul homology, and prove several basic propositions we will need in the later sections. Our work is entirely local in nature in this section. Throughout this section we fix a Cohen-Macaulay local ring R . If $I = (f_1, \dots, f_n)$ is an ideal of R , the Koszul complex of \underline{f} , denoted $K.(\underline{f}; R)$ is the tensor product of the complexes, $0 \rightarrow R \xrightarrow{f_i} R \rightarrow 0$. (This complex is in degrees 0 and 1.)

We denote the i th homology of this complex by $H_i(\underline{f}; R)$. Thus $H_0(\underline{f}; R) = R/I$. If M is any R -module, by $H_i(\underline{f}; M)$ we denote the i th homology of the complex $K.(\underline{f}; R) \otimes M$.

The homology $H_i(\underline{f}; M)$ depends upon the generating set f_1, \dots, f_n of I . In general, we have the formula

$$(1.1) \quad H_i(\underline{f}, 0; M) = H_i(\underline{f}; M) \oplus H_{i-1}(\underline{f}; M).$$

In addition, the Koszul homology of M with respect to any two *minimal* generating sets of I is isomorphic. The following remark is immediate from (1.1).

REMARK 1.2. The property “ $H_i(\underline{f}; M)$ is Cohen-Macaulay for all $0 \leq i \leq m$ ” is independent of the generating set f_1, \dots, f_n of I . In particular the property “ $H_i(\underline{f}; M)$ is Cohen-Macaulay for all i ” is independent of the generating set of I .

Recall that if $\text{depth}_R M = k$, and $(f_1, \dots, f_n) = I$, then $H_j(\underline{f}; M) = 0$ for $j > n - k$. In addition, $H_{n-k}(\underline{f}; M) \simeq \text{Hom}(M/IM, R/(f_1, \dots, f_k))$ if f_1, \dots, f_k is a regular M -sequence. In particular if $M = R$ and R is Gorenstein, $H_{n-k}(\underline{f}; R) \simeq K_{R/I}$, the canonical module of R/I .

REMARK 1.3 (SEE [Hu-2]). If R is Cohen-Macaulay local and I is an ideal of R , $I = (f_1, \dots, f_n)$, then $\dim H_j(\underline{f}; R) = \dim R/I$ for all j such that $H_j(\underline{f}; R) \neq 0$.

There are two fundamental exact sequences which we will use throughout this paper.

LEMMA 1.4. *Let I be an ideal of a ring R , $I = (f_1, \dots, f_n)$.*

(1) *If f_1 is a regular element of R , then there is an exact sequence,*

$$0 \rightarrow H_i(\underline{f}; R) \rightarrow H_i(\underline{f}; R/f_1R) \rightarrow H_{i-1}(\underline{f}; R) \rightarrow 0.$$

(2) Suppose $(0 : I) \cap I = (0)$. Then there is an exact sequence

$$0 \rightarrow K \rightarrow H_i(\underline{f}; R) \rightarrow H_i(\underline{f}; R/(0 : I)) \rightarrow 0$$

for $i \geq 0$, where K is isomorphic to a direct sum of copies of $(0 : I)$.

PROOF. For (1) we use the exact sequence

$$0 \rightarrow R \xrightarrow{f_1} R \rightarrow R/f_1R \rightarrow 0.$$

This gives a long exact sequence for the Koszul homology, and since $f_1 H_i(\underline{f}; M) = 0$ for any R -module M , this long exact sequence breaks up into the short exact sequences of (1).

The exact sequences of (2) follow similarly from the exact sequence

$$0 \rightarrow (0 : I) \rightarrow R \rightarrow R/(0 : I) \rightarrow 0.$$

We get a long exact sequence on Koszul homology,

$$\begin{aligned} \cdots \rightarrow H_{i+1}(\underline{f}; R) \rightarrow H_{i+1}(\underline{f}; R/(0 : I)) \rightarrow H_i(\underline{f}; (0 : I)) \\ \rightarrow H_i(\underline{f}; R) \rightarrow H_i(\underline{f}; R/(0 : I)) \rightarrow \cdots \end{aligned}$$

Assertion (2) follows from two claims. First, since I annihilates $(0 : I)$, $H_i(\underline{f}; (0 : I)) = (0 : I)^{(i)}$. Secondly, we claim all the maps $H_k(\underline{f}; R) \xrightarrow{i} H_k(\underline{f}; R/(0 : I))$ are surjective.

Suppose $\bar{\alpha}$ is a cycle which represents a class in $H_k(\underline{f}; R/(0 : I))$. The map i is just the restriction of a cycle in $H_k(\underline{f}; R)$ to $H_k(\underline{f}; R/(0 : I))$. Let α be any lifting of $\bar{\alpha}$ to $\Lambda^k F$, where F is a free module on n -generators, and we have identified the Koszul complex with ΛF . We denote the differential by d . Then the fact that $\bar{\alpha}$ is a cycle implies $d\alpha \in (0 : I)(\Lambda^{k-1} F)$. In addition, by definition of the differential d , $d\alpha \in I(\Lambda^{k-1} F)$. As $\Lambda^{k-1} F$ is a free R -module and $(0 : I) \cap I = 0$, $d\alpha = 0$ and so α represents a class in $H_k(\underline{f}; R)$. It follows i is onto.

COROLLARY 1.5. *Let I be an ideal of the Cohen-Macaulay local ring R . Suppose $I = (x_1, \dots, x_n)$ with x_1 a regular element of R . Let “ $'$ ” denote the homomorphism from R to R/Rx_1 . Then I is strongly Cohen-Macaulay in R if and only if I' is strongly Cohen-Macaulay in R' .*

PROOF. This corollary follows at once from the exact sequence of Lemma 1.4(1) and from Remark 1.3.

We now prove the important proposition (see [HU-2]).

PROPOSITION 1.6. *Let R be a Cohen-Macaulay local ring and let I be an ideal of R . Suppose I is strongly Cohen-Macaulay. Let x_1, \dots, x_k be a maximal R -sequence inside I , and place $J = (\underline{x} : I)$. Then R/J is Cohen-Macaulay.*

PROOF. Extend x_1, \dots, x_k to a generating set x_1, \dots, x_n of I . By Corollary 1.5, we may reduce modulo x_1, \dots, x_k and assume that $J = (0 : I)$.

We induct on $\dim R = \dim R/I$ to prove the proposition. If $\dim R = 0$, there is nothing to prove. Otherwise, we may choose a z in R such that z is not a zero-divisor

on R , or on $H_j(\underline{x}; R)$ for all j such that $j \leq n$. (Here we relabel a generating set of $I = (x_1, \dots, x_n)$.)

Using the long exact sequence of Koszul homology determined by the short exact sequence $0 \rightarrow R \xrightarrow{z} R \rightarrow R/zR \rightarrow 0$, we obtain exact sequences,

$$(1) \quad 0 \rightarrow H_i(\underline{x}; R) \xrightarrow{z} H_i(\underline{x}; R) \rightarrow H_i(\underline{x}; R/zR) \rightarrow 0.$$

Further if we denote the homomorphism from R to R/zR by “ $'$ ”, we see from (1) that because I is strongly Cohen-Macaulay in R , I' must be strongly Cohen-Macaulay in R' . Since $\dim R' = \dim R - 1 < \dim R$, by induction $R'/(0 : I')$ is Cohen-Macaulay. Lifting back to R , we see that $R'/(0 : I') = R/(z : (I, z))$ is Cohen-Macaulay. Since z is not a zero-divisor in r , z is not a zero-divisor on $R/(0 : I)$. Consequently, $R/((0 : I), z)$ is Cohen-Macaulay if and only if $R/(0 : I)$ is so. We will finish this proof by establishing the claim

$$(2) \quad ((0 : I), z) = (z : (I, z)).$$

The remarks at the beginning to this section show that $H_n(\underline{x}; R) = (0 : I)$, while $H_n(\underline{x}; R') = H_n(\underline{x}'; R') = (0 : I')$. The exact sequences (1) show that $(0 : I)/z(0 : I) \rightarrow (0 : I')$ is an isomorphism. The map is the obvious one, and takes an element s of $(0 : I)$ to $s' \in (0 : I')$.

We obtain

$$\frac{((0 : I), z)}{(z)} \xrightarrow{\sim} \frac{(z : (I, z))}{(z)}.$$

This isomorphism proves the claim, and hence proves the proposition.

We remark that the proposition is *false* if one only assumes R/I to be Cohen-Macaulay. See Example 1.8 in [P-S-1]. This proposition and the next give us the main tools needed in the induction to prove Theorem 3.1.

PROPOSITION 1.7 (SEE [Hu-2]). *Let R be a Cohen-Macaulay local ring and let I be a strongly Cohen-Macaulay ideal of R . Suppose x_1, \dots, x_k are a maximal regular sequence contained in I , and let $J = (x_1, \dots, x_k; I)$. If*

- (1) $J \cap I = (x_1, \dots, x_k)$, and
- (2) $\text{ht}(J + I) \geq k + 1$,

then I' is strongly Cohen-Macaulay in R' , where “ $'$ ” denotes the homomorphism from R to R/J .

PROOF. By Corollary 1.5, we may assume $k = 0$, i.e. $J = (0 : I)$. Let f_1, \dots, f_n generate I . By Proposition 1.6, R/J is Cohen-Macaulay; in particular by Remark 1.3, $\dim H_j(f'; R) = \dim R'/I' = \dim R/I + J < \dim R/J$ by assumption (2) of the proposition.

Set $d = \dim R/I = \dim R/J = \dim R$. It is enough to show $\text{depth } H_j(f'; R') \geq d - 1$. The exact sequence $0 \rightarrow J \rightarrow R \rightarrow R' \rightarrow 0$, together with Proposition 1.6, show that $\text{depth } J = \text{depth } R = d$. From Lemma 1.4(2), there are exact sequences

$$0 \rightarrow K \rightarrow H_i(\underline{f}; R) \rightarrow H_i(\underline{f}'; R') \rightarrow 0$$

for every $i \geq 0$. Since K is isomorphic to a direct sum of copies of J , $\text{depth } K = d$. By assumption $\text{depth } H_i(\underline{f}; R) = d$. It easily follows that $\text{depth } H_i(\underline{f}'; R') \geq d - 1$.

We remark that assumption (2) of Proposition 1.7 follows from assumption (1) provided R_P is Gorenstein for all prime ideals P such that P is minimal over I .

REMARK 1.8. We have seen that if z is a nonzero-divisor on R/I and R ($I = (f_1, \dots, f_n)$), then

$$H_i(f, z; R) \simeq H_i(f; R) \otimes_R R/Rz.$$

It easily follows that each of the following statements implies the other two. Let z_1, \dots, z_k be a regular sequence on R and on R/I .

- (i) I is strongly Cohen-Macaulay in R .
- (ii) (I, z_1, \dots, z_k) is strongly Cohen-Macaulay in R .
- (iii) I' is strongly Cohen-Macaulay in R' , where $R' = R/(z_1, \dots, z_k)$.

Suppose $\text{Spec}(T)$ is embedded in the spectrum of two different polynomial rings, $R_1 = k[T_1, \dots, T_n]$ and $R_2 = k[Y_1, \dots, Y_m]$, so that there is an isomorphism, $R_1/I_1 \simeq T \simeq R_2/I_2$.

We claim I_1 is strongly Cohen-Macaulay in R_1 if and only if I_2 is strongly Cohen-Macaulay in R_2 . Denote the tensor product $R_1 \otimes_k R_2$ by S , and let π be the isomorphism $R_1/I_1 \rightarrow R_2/I_2$. It easily follows that in S

$$(I_1 S, T_i - \pi(T_i)) = (I_2 S, Y_j - \pi^{-1}(Y_j))$$

and that $\{T_i - \pi(T_i)\}$ form a regular sequence on $S/I_1 S$ and S and that $Y_j - \pi^{-1}(Y_j)$ form a regular sequence on $S/I_2 S$ and S . The above equivalences now imply our claim.

Next we wish to discuss how strong Cohen-Macaulayness behaves under products. Our goal is to prove

THEOREM 1.9. *Let R be a regular local ring, and I and J two ideals of R such that $Y = \text{Spec}(R/I)$ and $Z = \text{Spec}(R/J)$ are strongly Cohen-Macaulay in $X = \text{Spec}(R)$. Assume $I \cap J = IJ$. Then $Y \cap Z$ is strongly Cohen-Macaulay in X .*

Although this theorem is essentially the Künneth formula, there are several details which must be checked. We separate some of these details in two lemmas.

LEMMA 1.10. *Suppose R is a regular local ring, and I and J are two ideals of R such that $R/I, R/J$ are Cohen-Macaulay and $I \cap J = IJ$. If M is a finite maximal Cohen-Macaulay R/I -module and N is a finite maximal Cohen-Macaulay R/J -module, then $\text{Tor}_i^R(M, N) = 0$ for all $i \geq 1$. Further $M \otimes_R N$ is a maximal Cohen-Macaulay module of $R/(I + J)$.*

PROOF. Our assumption reads, $\text{Tor}_1^R(R/I, R/J) = 0$. By rigidity [L],

$$\text{Tor}_i^R(R/I, R/J) = 0 \quad \text{for all } i \geq 1.$$

Let

$$F: 0 \rightarrow F_k \xrightarrow{f_k} \dots \xrightarrow{f_1} F_0 \rightarrow M \rightarrow 0$$

be a free resolution for M over R . Here $k = \text{ht } I$. To show $F \otimes N$ is exact, it is enough by the Buchsbaum-Eisenbud criterion [B-E] to prove $\text{grade}_{I(f_j)} N \geq j$, where $I(f_j)$ is the ideal generated by the maximal nonvanishing minors of the map f_j .

However, as M is perfect, $\sqrt{I} = \sqrt{I}(f_j)$ for all j , $1 \leq j \leq k$. Hence we must show $\text{grade}_J N \geq k$. As N is a maximal Cohen-Macaulay module for R/J , it is enough to prove $\text{grade}_J R/J \geq k$. However, if we take a free resolution for R/I and tensor it with R/J , the condition $\text{Tor}_i(R/I, R/J) = 0, i \geq 1$, forces $\text{grade}_J R/J \geq k$.

Next we prove the last statement of Lemma 1.10. Let F be a free resolution of M over R and let G be a free resolution of N over R . The tensor product $F \otimes_R G$ furnishes a free resolution of $M \otimes_R N$ since $\text{Tor}_i^R(M, N) = 0$ by the above argument. The length of this resolution is $\text{ht } I + \text{ht } J = \text{ht}(I + J)$ (by the assumption $\text{Tor}_i^R(R/I, R/J) = 0$). This proves $\text{depth } M \otimes_R N = \dim R - \text{ht}(I + J) = \dim R/I + J$.

LEMMA 1.11. *Let F be a finite free complex over a Cohen-Macaulay local ring R and suppose M is an R -module with $\text{Tor}_i(M, H_j(F)) = 0$ for all $i \geq 1$ and all $j \geq 0$. Let $Z_n = \text{nth cycles of } F$ and $B_n = \text{nth boundaries}$. Then $\text{Tor}_i(M, Z_n) = 0 = \text{Tor}_i(M, B_n)$ for all $i \geq 1$ and all n .*

PROOF. $Z_n = \ker(F_n \rightarrow F_{n-1})$, and $B_n = \text{im}(F_{n+1} \rightarrow F_n)$. There are exact sequences

$$(1) \quad 0 \rightarrow B_n \rightarrow Z_n \rightarrow H_n \rightarrow 0, \quad n \geq 0,$$

$$(2) \quad 0 \rightarrow Z_n \rightarrow F_n \rightarrow B_{n-1} \rightarrow 0, \quad n \geq 1,$$

where $H_n = H_n(F)$.

We induct on n to show $\text{Tor}_i(M, Z_n) = \text{Tor}_i(M, B_n) = 0$ for all i . If $n = 0$, there is an exact sequence as in (1), $0 \rightarrow B_0 \rightarrow Z_0 \rightarrow H_0 \rightarrow 0$. Since $Z_0 = F_0$ and $\text{Tor}_i(M, F_0) = \text{Tor}_i(M, H_0) = 0$ for all $i \geq 1$, it easily follows $\text{Tor}_i(M, B_0) = 0$ for all $i \geq 1$.

Assume $n > 0$ and we have shown our claim for $n - 1$. Then the vanishing of Tor for B_{n-1} and F_n implies $\text{Tor}_i(M, Z_n) = 0$ for all $i \geq 1$ from the exact sequence (2); now the vanishing for H_n and Z_n give $\text{Tor}_i(M, B_n) = 0$ for all $i \geq 1$ by the exact sequence (1).

We are now able to prove Theorem 1.9. We just copy the proof of the Künneth formula found in [M, pp. 166–168]; however the assumptions are too strong in the above theorem for our case so we must make some modifications. Rather than repeat the proof we simply note the salient points. First of all, our assumption that $\text{Tor}_1^R(R/I, R/J) = 0$ forces $\text{Tor}_i^R(R/I, R/J) = 0$ for all $i \geq 1$ by rigidity. By hypothesis, R/I and R/J are strongly Cohen-Macaulay in R ; hence $H_i(I; R)$ and $H_i(J; R)$ are maximal Cohen-Macaulay modules (or zero) for R/I and R/J respectively. (Here by $H_i(I; R)$ we mean fix some (any) generating set for I and take Koszul homology.) Lemma 1.10 now implies $\text{Tor}_i^R(H_j(I; R), H_k(J; R)) = 0$ for all $i \geq 1, j \geq 0, k \geq 0$.

Let K be the Koszul complex for I (with respect to a fixed generating set) and let L be the Koszul complex for J . Let $C_n = C_n(K)$ be the cycles of K , and $D_n = B_{n-1}(K)$ be the boundaries of K . By Lemma 1.11, $\text{Tor}_i(C_n, H_k(J; R)) = \text{Tor}_i(D_n, H_k(J; R))$ for all $i \geq 1, k \geq 0$, and $n \geq 0$.

There is a short exact sequence of complexes, $0 \rightarrow C. \rightarrow K. \rightarrow D. \rightarrow 0$, which remain exact after tensoring with $L.$. To continue the proof as in [M], we need to show $\text{Tor}_1(C_m, H_q(L.)) = 0$, which we have shown, and we need to observe that

$$H_n(L.) \otimes D_m = H_n(L. \otimes D_m).$$

The n th homology of $L.$ is described by the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \downarrow & & \\
 0 & \rightarrow & B_n & \rightarrow & C_n & \rightarrow & H_n \rightarrow 0 \\
 & & \uparrow & & \downarrow & & \\
 & & L_{n+1} & \rightarrow & L_n & & \\
 & & & & \downarrow & & \\
 & & & & L_{n-1} & &
 \end{array}$$

(see [M, Lemma 10.3, p. 167]).

Tensor this diagram with D_m . Since $\text{Tor}_1(D_m, H_n(L.)) = 0$ by above, the middle row stays exact. $L_{n+1} \otimes D_m \rightarrow B_n \otimes D_m$ remains onto since tensor is right exact. Finally the middle column exact sequence gives a short exact sequence $0 \rightarrow C_n \rightarrow L_n \rightarrow B_{n-1} \rightarrow 0$. If $\text{Tor}_1(D_m, B_{n-1}) = 0$ then the sequence

$$0 \rightarrow C_n \otimes D_m \rightarrow L_n \otimes D_m \rightarrow L_{n-1} \otimes D_m$$

will remain exact. Since we have shown $\text{Tor}_i(D_m, H_q(L.)) = 0$, we may apply Lemma 1.11 to conclude $\text{Tor}_1(D_m, B_{n-1}) = 0$. This finishes the outline of the proof of the Künneth formula. We conclude,

$$(3) \quad \sum_{m+q=n} H_m(I; R) \otimes H_q(J; R) = H_n(I + J; R).$$

Here note the homology of $K. \otimes L.$ is precisely $H_n(I + J; R)$ while we have shown $\text{Tor}_1^R(H_m(K.), H_q(L.)) = 0$.

However by Lemma 1.10, $H_m(I; R) \otimes H_q(J; R)$ are maximal Cohen-Macaulay modules for $R/I + J$. It now follows that $\text{Spec}(R/I + J)$ is strongly Cohen-Macaulay in $\text{Spec}(R)$.

COROLLARY 1.12. *Let $X = \text{Spec}(k[X_1, \dots, X_n])$ and $X' = \text{Spec}(k[T_1, \dots, T_m])$. Suppose $Y \subset X$ and $Y' \subset X'$ are strongly Cohen-Macaulay closed subschemes of X and X' respectively. Then $Y \times Y'$ is strongly Cohen-Macaulay in $X \times X'$.*

PROOF. Let $I = I(Y)$, $J = I(Y')$. Then $I(Y \times Y')$ is given by the ideal generated by $I + J$ in $k[X_1, \dots, X_n, T_1, \dots, T_m] = S$. If we show $IS \cap JS = IJS$, then we may apply Theorem 1.9. (As by definition we need only check strong Cohen-Macaulayness locally.) However, $IS \cap JS = I \cdot JS$ is clear.

2. Examples and remarks. In this section we present some known examples of strongly Cohen-Macaulay schemes and further give a simple criterion to decide if a scheme is *not* strongly Cohen-Macaulay. We also give some remarks concerning the duality of Koszul homology which we will later need. Finally we show that a conjecture of Vasconcelos is true for strongly Cohen-Macaulay schemes. Much of

the first part of this section appears in [Hu-1]; therefore we do not give proofs for these results.

The largest known class of strongly Cohen-Macaulay subschemes comes from liaison as we noted in the introduction. As we stated there, this class includes perfect subschemes of codimension two in Cohen-Macaulay schemes and perfect Gorenstein schemes of codimension three in Gorenstein schemes. We list another such example here.

EXAMPLE 2.1. Let X be a generic n by $n + 1$ matrix and Y an $n + 1$ by 1 matrix. J. Herzog [He-2] gave a resolution for the ideal $I_1(XY) + I_n(X) = J$ in the polynomial ring $k[X, Y] = R$. In fact R/J is Gorenstein of height $n + 1$. It was shown in [Hu-1] that J is in the linkage class of a complete intersection. Consequently R/J is strongly Cohen-Macaulay in R .

Two other classes of examples are also known to be strongly Cohen-Macaulay.

PROPOSITION 2.2. *Let R be a Cohen-Macaulay local ring, and I an ideal of R such that:*

- (i) R/I is Cohen-Macaulay.
- (ii) I is generically a complete intersection.
- (iii) $v(I) = \text{ht } I + 1$, where in general $v(M)$ is the least number of generators of the R -module M .

Then R/I is strongly Cohen-Macaulay in R .

PROOF. Let $I = (a_1, \dots, a_k, a_{k+1})$ where a_1, \dots, a_k is a regular sequence which generically generates I . From §1, we see that $H_i(\underline{a}; R) = 0$ for $i > 1$, while $H_0(\underline{a}; R) = R/I$ is Cohen-Macaulay. Hence it remains to show $H_1(\underline{a}; R)$ is Cohen-Macaulay. This module is isomorphic to $(a_1, \dots, a_k : a_{k+1}) / (a_1, \dots, a_k)$. Consider the exact sequence,

$$(1) \quad 0 \rightarrow ((a_1, \dots, a_k) : a_{k+1}) / (a_1, \dots, a_k) \rightarrow R / (a_1, \dots, a_k) \rightarrow R / (a_1, \dots, a_k : a_{k+1}) \rightarrow 0.$$

To show the left-hand module of (1) is Cohen-Macaulay, it suffices to show the right-hand module is Cohen-Macaulay. (All these modules have the same dimension.) This fact, however, follows from the exact sequence

$$0 \rightarrow R / ((a_1, \dots, a_k) : a_{k+1}) \rightarrow R / (a_1, \dots, a_k) \rightarrow R / (a_1, \dots, a_{k+1}) \rightarrow 0.$$

EXAMPLE 2.3. If R is Gorenstein local and if I is an ideal satisfying

- (1) $v(I) = \text{ht } I + 2$,
- (2) R/I is Cohen-Macaulay,

then Avramov and Herzog [A-H] showed R/I is strongly Cohen-Macaulay in R . In particular, if $\dim R/I = 2$ and R/I is normal, then I is strongly Cohen-Macaulay if $v(I) \leq \dim R$. For example, the defining ideal I of the surface $k[s^8, s^5t^3, s^4t^4, s^3t^5, t^8]$ is generated by $Z^2 - XV, YU - XV, Y^3 - ZUX, XU^2 - ZY^2$, and $U^3 - ZYV$. If $R = k[X, Y, Z, U, V]$, then R/I is two dimensional normal and $v(I) = 5 \leq \dim R$. Hence R/I is strongly Cohen-Macaulay in R .

In [Hu-1] a criterion involving the module T_2 was given which gives a strong necessary condition for a scheme to be strongly Cohen-Macaulay. Let R be as usual,

and set $I = (f_1, \dots, f_n)$. There is an exact sequence, $H_1(\underline{f}; R) \xrightarrow{\beta} (R/I)^n \xrightarrow{\alpha} I/I^2 \rightarrow 0$. The maps α and β are defined as follows: If $(\bar{r}_1, \dots, \bar{r}_n) \in (R/I)^n$, $\alpha((\bar{r}_1, \dots, \bar{r}_n)) = \sum r_i f_i \pmod{I^2}$. If (s_1, \dots, s_n) represents the class of an element in $H_1(\underline{f}; R)$ (that is, $\sum_{i=1}^n s_i f_i = 0$), then $\beta((s_1, \dots, s_n)) = (\bar{s}_1, \dots, \bar{s}_n)$.

The kernel of β is by definition $T_2^{X/Y} = T_2(X = \text{Spec}(R), Y = \text{Spec}(R/I))$.

The following proposition can be found in [Hu-2].

PROPOSITION 2.4. *Suppose Y is strongly Cohen-Macaulay in X . Then $T_2^{X/Y} = 0$.*

Simis and Vasconcelos showed in [S-V] that T_2 also is equal to $\ker(\text{Sym}_2 I \rightarrow I^2)$. From this remark and from Proposition 2.4, the following example was given in [Hu-1].

EXAMPLE 2.5. Let X be a generic $r \times s$ matrix ($r \leq s$). Then $I_t(X)$ is not strongly Cohen-Macaulay if

- (i) $1 < t < r$, or
- (ii) $t = r < s - 1$.

In other cases, $I_t(X)$ is strongly Cohen-Macaulay.

Finally we wish to show how a conjecture of Vasconcelos is verified for strongly Cohen-Macaulay varieties (see [V]).

PROPOSITION 2.6. *Suppose R is a local ring and I an ideal such that $H_i(\underline{f}; R)$ are Cohen-Macaulay for all i (for some $(\underline{f}) = I$). If $\text{pd}_R I < \infty$ and $\text{pd}_{R/I} I/I^2 < \infty$, then I is generated by a regular sequence.*

PROOF. Notice that we do not need to assume I is generically a complete intersection. In fact, if $P \supseteq I$ is a minimal prime of I , then all our assumptions still hold for I_P . In this case, since $\dim(R/I)_P = 0$, and $\text{pd } I_P/I_P^2 < \infty$, I_P/I_P^2 is free. Now Vasconcelos has proved that I_P is generated by a regular sequence. By Proposition 2.4, $T_2 = 0$. Hence we have an exact sequence,

$$0 \rightarrow H_1(\underline{f}; R) \rightarrow (R/I)^n \rightarrow I/I^2 \rightarrow 0.$$

We conclude that $\text{pd}_{R/I} H_1(\underline{f}; R) < \infty$. As $H_1(\underline{f}; R)$ is by assumption Cohen-Macaulay, $H_1(\underline{f}; R)$ is free. Now a result of Gulliksen [G-L] shows I is generated by a regular sequence.

Applying this proposition to the examples of this section recovers many of the results of [V].

Next we wish to investigate some duality which holds on the Koszul complex. In [He-1] the following remark was essentially proved: Suppose R is a Gorenstein local ring and I an ideal of R , $I = (a_1, \dots, a_n)$. Set

$$d(I) = d = \text{deviation of } I = n - \text{ht } I.$$

If $H_0(\underline{a}; R), \dots, H_j(\underline{a}; R)$ are Cohen-Macaulay modules, then so are $H_d(\underline{a}; R), \dots, H_{d-j}(\underline{a}; R)$ and

$$H_{d-j}(\underline{a}; R) \simeq \text{Hom}(H_j(\underline{a}; R), H_d(\underline{a}; R)).$$

There is always a pairing $H_i(\underline{a}; R) \times H_{d-i}(\underline{a}; R) \rightarrow H_d(\underline{a}; R)$ given by the exterior product on the Koszul homology. If R is Gorenstein, $H_d(\underline{a}; R) \simeq K_{R/I}$, the canonical module of R/I .

PROPOSITION 2.7. *Suppose $H_i(\underline{a}; R)$ are reflexive modules for every i , $0 \leq i \leq d$. Then the pairing above is perfect, that is,*

$$H_i(\underline{a}; R) = \text{Hom}(H_{d-i}(\underline{a}; R), H_d(\underline{a}; R))$$

and

$$H_{d-i}(\underline{a}; R) = \text{Hom}(H_i(\underline{a}; R), H_d(\underline{a}; R)).$$

PROOF. We denote $H_j(\underline{a}; R)$ by H_j for simplicity, and we only show the first equality. From the pairing, there is a map, $H_i \xrightarrow{f} \text{Hom}(H_{d-i}, H_d)$. By assumption, both of these modules are reflexive and so it is enough to show f is an isomorphism in codimension at most one. In this case the modules $(H_i)_p$ are Cohen-Macaulay modules and we may apply the result of [He-1] to conclude f is an isomorphism.

Using the duality and a multiplicities argument as in [Hu-5] one may show

PROPOSITION 2.8. *Let R be a local Gorenstein ring, $I = (a_1, \dots, a_n)$ an ideal with $d(I) = d \leq 4$. If $H_1(\underline{a}; R)$ and $H_0(\underline{a}; R)$ are Cohen-Macaulay, then $H_j(\underline{a}; R)$ are Cohen-Macaulay for all $j \geq 0$.*

COROLLARY 2.9. *Suppose R is local Gorenstein and I is an ideal generically a complete intersection with R/I Cohen-Macaulay. Set $X = \text{Spec}(R)$, $Y = V(I)$. If $T_2^{X/Y} = 0$, and $d(I) \leq 4$, then Y is strongly Cohen-Macaulay in X if and only if the conormal module, I/I^2 , satisfies Serre's condition S_j where $j = \dim Y - 1$.*

PROOF. By the proposition above it is enough to prove H_1 is Cohen-Macaulay. Since $T_2^{X/Y} = 0$, there is an exact sequence,

$$0 \rightarrow H_1 \rightarrow (R/I)^n \rightarrow I/I^2 \rightarrow 0.$$

The result easily follows from standard depth considerations.

3. Main result. We refer the reader to the introduction for the definition of condition G_s . The purpose of this section is to prove

THEOREM 3.1. *Let $X = \text{Spec}(R)$, where R is a Cohen-Macaulay local ring, and let $Y = V(I)$ be a strongly Cohen-Macaulay closed subscheme of X . Assume Y satisfies G_s . Let a_1, \dots, a_s be in I such that $(a_1, \dots, a_s)_P = I_P$ if $\text{ht } P \leq s$ and $P \in Y$. Set $Z = V(J)$ where $J = (a_1, \dots, a_s; I)$, and we assume $\text{codim } Z \geq s$. Let $W = V(a_1, \dots, a_s)$. Then:*

- (i) Z is Cohen-Macaulay and $\text{codim } Z = s$.
- (ii) $W = Z \cup Y$.
- (iii) $Z \cap Y$ is strongly Cohen-Macaulay in Z ; in particular, $Z \cap Y$ is Cohen-Macaulay.
- (iv) $\text{depth } R/(a_1, \dots, a_s) \geq \dim R - s$.

We make a few remarks before beginning the proof of Theorem 3.1. The ideals J arising in this fashion are called *residual intersections* by Artin and Nagata in their paper [A-N]. In that paper, Theorem 3.1 is stated [A-N, Theorem 2.1, p. 312] more generally; however this more general statement is incorrect as we shall show by a counterexample (Example 3.3). The statement (iii) of the theorem is necessary for the

induction. Statements (i) and (iii) may be thought of as generalizations of Propositions 1.6 and 1.7 respectively. These propositions only dealt with the case $s = \text{ht } I$, in which case J is linked to I .

We will prove the theorem by induction on s , beginning with the case $s = k = \text{ht } I$. In this case, (i) follows immediately from Proposition 1.6, while (iv) is also immediate as in this case a_1, \dots, a_k is a regular sequence. (Notice $\text{ht}(a_1, \dots, a_k) = \text{ht } I$ as $(a_1, \dots, a_k)_P = I_P$ for all primes P minimal over I .) We show (ii). Let $(a_1, \dots, a_k) = q_1 \cap \dots \cap q_m$ be a primary decomposition of (a_1, \dots, a_k) , and put $Q_k = \sqrt{q_k}$. To show $J \cap I = (a_1, \dots, a_k)$ it suffices to prove $J \cap I \subseteq q_i$ for each $1 \leq i \leq m$. As $J \cap I \subseteq (a_1, \dots, a_k)$, it is enough to prove $J + I \not\subseteq Q_i$. However, if $I \subseteq Q_i$, then by assumption $I_{Q_i} = (a_1, \dots, a_k)_{Q_i}$; so that $J_{Q_i} = ((a_1, \dots, a_k)_{Q_i} : I_{Q_i}) \not\subseteq Q_i$. This shows both (ii) and the fact that $\text{ht}(I + J) \geq k + 1$. The assertion (iii) now follows from Proposition 1.7.

Henceforth we assume $s > k$ and assume the theorem is true for all values strictly less than s . We use a lemma which is a general position argument and is found in [A-N, Lemma 2.3, p. 312].

LEMMA 3.2. *Suppose $s > k$. The generators a_1, \dots, a_s may be chosen in such a way that $(a_1, \dots, a_{s-1})_P = I_P$ for all prime ideals $P \supseteq I$ such that $\text{ht } P \leq s - 1$. Further, $\text{height}(a_1, \dots, a_{s-1}; I)$ is at least $s - 1$.*

We now suppose we have chosen a_1, \dots, a_s as in Lemma 3.2. Set

$$J_1 = (a_1, \dots, a_{s-1}; I).$$

By induction we may assume:

- (1) R/J_1 is Cohen-Macaulay, $\text{ht } J_1 = s - 1$,
- (2) $J_1 \cap I = (a_1, \dots, a_{s-1})$,
- (3) $I + J_1/J_1$ is strongly Cohen-Macaulay, and
- (4) $\text{depth } R/(a_1, \dots, a_{s-1}) \geq n - s + 1$.

Let “ $'$ ” denote the map from R to R/J_1 , and set $J^* = ((J_1, a_s) : I)$. We first claim that $J^* = J$. It is clear that $J \subseteq J^*$ since $J \cap I \subseteq (a_1, \dots, a_s) \subseteq (J_1, a_s)$. To show the converse we need to prove $J^*I \subseteq (a_1, \dots, a_s)$. By definition, $J^*I \subseteq (J_1, a_s)$ and so $J^*I \subseteq (J_1, a_s) \cap I$. Let $w \in J_1$ be such that $w + ra_s \in I$. Then $w \in I \cap J_1 = (a_1, \dots, a_{s-1})$. It follows that $(J_1, a_s) \cap I = (a_1, \dots, a_s)$. Therefore, $J^*I \subseteq (a_1, \dots, a_s)$ and $J = J^*$.

We next prove that $\text{ht}(J_1, a_s) = s$. If not, there is a prime $Q \supseteq (J_1, a_s)$, $\text{ht } Q = s - 1$. Then, as $IJ \subseteq (a_1, \dots, a_s) \subseteq (J_1, a_s) \subseteq Q$, either $I \subseteq Q$ or $J \subseteq Q$. However, as $(a_1, \dots, a_{s-1})_P = I_P$ at every prime of height $\leq s - 1$, $I \not\subseteq Q$. Hence $J \subseteq Q$. However $\text{ht } J \geq s$. Thus $\text{ht}(J_1, a_s) = s$. Since $\text{ht } J_1 = s - 1$ and R/J_1 is Cohen-Macaulay, a'_s is not a zero-divisor in R/J_1 .

By the induction hypothesis, I' is strongly Cohen-Macaulay in R' . By Proposition 1.6 it follows that $(J^*)' = (a'_s : I')$ satisfies the condition $R'/(J^*)'$ is Cohen-Macaulay. As $R'/(J^*)' = R/J^* = R/J$, we have shown (i).

We next show (ii). First, we will show $\text{ht}(I + J) \geq s + 1$. This is clear from our assumption that $(a_1, \dots, a_s)_P = I_P$ for every prime ideal $P \supseteq I$ such that $\text{ht } P \leq s$.

Since $(J_1, a_s) \cap I = (a_1, \dots, a_s)$, to show $J \cap I = (a_1, \dots, a_s)$, it suffices to prove $J \cap I \subseteq (J_1, a_s)$. As $\text{ht}(J + I) \geq s + 1$ and (J_1, a_s) is a height s ideal such that $R/(J_1, a_s)$ is Cohen-Macaulay, it is enough to prove $J \cdot I \subseteq (J_1, a_s)$. However, $J = J^* = ((J_1, a_s): I)$.

To prove (iii) we use Proposition 1.7. We apply this proposition to the ideals I' and $(J^*)' = J' = (a'_s: I')$ in R' . We need to verify

- (1) $J' \cap I' = (a'_s)$, and
- (2) $\text{ht}(J' + I') \geq 2$.

However, we have shown both of these conditions in the preceding argument. We conclude $I' + J'/J'$ is strongly Cohen-Macaulay in R/J' . Since $J_1 \subset J$, this says $I + J/J$ is strongly Cohen-Macaulay in R/J , which proves (iii).

To prove (iv) we emulate [A-N]. Consider the exact sequences,

$$(3) \quad 0 \rightarrow (a_1, \dots, a_{s-1}) \rightarrow (a_1, \dots, a_s) \rightarrow (a_s)/(a_s) \cap (a_1, \dots, a_{s-1}) \rightarrow 0,$$

$$(4) \quad 0 \rightarrow (a_1, \dots, a_{s-1}) \rightarrow R \rightarrow R/(a_1, \dots, a_{s-1}) \rightarrow 0.$$

As $\text{depth } R/(a_1, \dots, a_{s-1}) \geq n - s + 1$, where $n = \dim R$, from (4) we obtain

$$\text{depth}(a_1, \dots, a_{s-1}) \geq n - s + 2.$$

We next evaluate $(a_s)/(a_s) \cap (a_1, \dots, a_{s-1})$. Mapping R onto this module we obtain

$$R/(a_1, \dots, a_{s-1}: a_s) \simeq (a_s)/(a_s) \cap (a_1, \dots, a_{s-1}).$$

Notice $(a_1, \dots, a_{s-1}: I) = J_1 \subseteq (a_1, \dots, a_{s-1}: a_s)$. In fact we claim these two ideals are equal. As $a_s(a_1, \dots, a_{s-1}: a_s) \subseteq J_1$ and a_s is not a zero-divisor on R/J_1 , $(a_1, \dots, a_{s-1}: a_s) \subseteq J_1$. We know $\text{depth } R/J_1 = n - s + 1$. It now follows from (3) that $\text{depth}(a_1, \dots, a_s) \geq n - s + 1$. Consequently, $\text{depth } R/(a_1, \dots, a_s) \geq n - s$. This completes the proof of Theorem 3.1.

Finally, in this section we wish to give an example to show that the hypothesis of "strongly Cohen-Macaulay" is needed. This example provides a counterexample to Theorem 2.1 of [A-N]; however we note that the Corollary 2.2 in [A-N] is correct since in this case the ideal in question is a complete intersection.

EXAMPLE 3.3. Let X and Y be generic two by two matrices over a field k ; set $R = k[X, Y]$. (We adjoin the entries of X and Y to k .) Let $I = I_1(XY) + I_2(X) + I_2(Y)$. It is known that R/I is Cohen-Macaulay of codimension 3. However, it is also known,

$$I \cap I_1(X) \cap I_1(Y) = I_1(XY).$$

The ideal $I_1(XY)$ is generated by four equations, and of course, $\text{ht}(I_1(X) \cap I_1(Y)) = 4$. In addition I satisfies G_∞ . If we could apply Theorem 3.1 to I , we could conclude $R/(I_1(X) \cap I_1(Y))$ is Cohen-Macaulay. As this conclusion is obviously false, we have obtained the desired counterexample.

Observe that $v(I) = 6 = \text{ht } I + 3$. Hence Proposition 2.2 and Example 2.3 cannot be extended any further.

4. Applications. We apply Theorem 3.1 to two different situations. The results of this section were already known. However the use of Theorem 3.1 gives very simple proofs, and what this author felt were very appealing proofs. These applications demonstrate some of the remarkable information which can be found in a strongly Cohen-Macaulay scheme. Furthermore Theorem 3.1 and the results here can be seen as a generalization of liaison.

We begin by recovering a classical result on determinantal varieties. We adopt the notation of the introduction, so that $D_{r,s}$ is the (affine) scheme whose closed points give maps from an r -dimensional space to an s -dimensional space having rank at most $r - 1$. (Here $r \leq s$.) As noted before $D_{r,s}$ embeds in \mathbf{A}_k^s and it is easily seen that $D_{r,s} = \text{Spec}(k[X_{ij}]/\sqrt{I_r}(X))$. We wish to prove

THEOREM 4.1. *The scheme $D_{r,s}$ is Cohen-Macaulay.*

PROOF. It is well known that it suffices to prove this statement at the origin. Accordingly, let $X = \text{Spec}(k[X_{ij}]_{(X_{ij})})$, and $Y \subseteq X$ be the closed subscheme with $I(Y) = \sqrt{I_r}(X)$. We must prove Y is Cohen-Macaulay.

We first extend the base X . Let X' be $\mathbf{A}_k^{s(s+1)}$ localized at the origin. We think of the coordinates of $\mathbf{A}_k^{s(s+1)}$ as an $s \times (s + 1)$ matrix $A = (z_{ij})$, with the right-most r columns of A giving the map of $X \rightarrow X'$. (We set $B =$ right-most r columns.) In X' we let Z be the subscheme given by the ideal $I(Z) = I_s(A)$. As codimension $Z = 2$, and Z is perfect, by Proposition 0.3, Z is strongly Cohen-Macaulay in X' . Further Z satisfies G_∞ .

Let $Y' = i(Y)$ where $i: X \rightarrow X'$. As i is a faithfully flat morphism, it is enough to prove Y' is Cohen-Macaulay. In fact we show Y' is a residual intersection of Z .

Let a_i be the $s \times s$ matrix of A determined by deleting the i th column, and consider the ideal $(a_1, \dots, a_{s-r+1}: I(Z)) = J$. Let $T = V(J)$. We will show $T = Y'$.

By elementary linear algebra one can observe that $I(Z) \cdot I_r(B) \subseteq (a_1, \dots, a_{s-r+1})$. Hence, $I_r(B) \subseteq J$. On the other hand, since $I(Z)$ contains a nonzero-divisor modulo the ideal $\sqrt{I_r}(B)$ (which is prime and $= I(Y')$), we see that $J \subseteq \sqrt{I_r}(B)$. Therefore, $Y' = T$, as topological spaces.

In particular, $\text{codim } T = \text{codim } Y' = s - r + 1$. It follows that T is a residual intersection of Z and consequently T is Cohen-Macaulay. It remains to show $Y' = T$ as schemes, i.e. that $\sqrt{I_r}(B) = (a_1, \dots, a_{s-r+1}: I_s(A))$. Since T is Cohen-Macaulay, it has no embedded components. We only need to show T is an integral scheme, and it suffices to do this at its unique generic point, $P = \sqrt{I_r}(B)$. However after localizing at P , $(I_r(B))_P = (a_1, \dots, a_{s-r+1})_P = P_P$. Hence, $I_r(B) = (a_1, \dots, a_{s-r+1}: I_s(A))$ and so $Y' = T$ is Cohen-Macaulay.

Next we wish to give an alternate proof of the theorem below of Simis and Vasconcelos by viewing the Rees algebra of an ideal I as a residual intersection of R/I .

THEOREM 4.2 [S-V]. *Let R be a Cohen-Macaulay local ring and let I be an ideal of R such that $\text{Spec}(R/I) = Y$ is strongly Cohen-Macaulay in $X = \text{Spec}(R)$. Suppose Y*

satisfies G_∞ . Then:

- (1) $\text{Sym}(I) \simeq \bigoplus_{n \geq 0} I^n$.
- (2) $\bigoplus_{n \geq 0} I^n$ is Cohen-Macaulay.
- (3) $\text{Sym}(I/I^2) = \bigoplus_{n \geq 0} I^n/I^{n+1}$.
- (4) $\bigoplus_{n \geq 0} I^n/I^{n+1}$ is Cohen-Macaulay.

In general, (1) holds if and only if (3) holds [Va], while (2) implies (4) [Hu-3].

We will actually only recover (1), (3) and (4). We wish to fix some notation. We let $T = \text{Spec}(R[It, t^{-1}])$ where $R[It, t^{-1}]$ is viewed as a subring of the ring $R[t, t^{-1}]$. (The variable t is algebraically independent over R .) This ring is called the Rees algebra of R . If $G = \text{Spec}(\bigoplus_{n \geq 0} I^n/I^{n+1})$, the associated ‘‘canonical’’ scheme of Y , then $G = V(t^{-1}) \subseteq T$.

We also let $S = \text{Spec}(\text{Sym}(I))$, the scheme associated to the symmetric algebra of I , and let $W = \text{Spec}(R[T_1, \dots, T_n, U])$ where $n = v(I)$. S is a closed subscheme of W defined by the vanishing of the equations U , and $\sum_{i=1}^n b_i T_i$ where $\sum_{i=1}^n b_i a_i = 0$ (where $I = (a_1, \dots, a_n)$ is a minimal generating set of I). Set $T' = \text{Spec}(R[T_1, \dots, T_n, U]/J)$ where J is the ideal generated by the equations $\sum_{i=1}^n b_i T_i$ ($\sum_{i=1}^n b_i a_i = 0$), and the equations $UT_i - a_i$. There is an obvious surjection

$$\frac{R[T_1, \dots, T_n, U]}{J} \rightarrow R[It, t^{-1}] \rightarrow 0$$

by sending T_i to $a_i t$ and U to t^{-1} .

This gives a closed immersion of T into T' . Set

$$Y' = \text{Spec}(R[T_1, \dots, T_n, U]/R[T_1, \dots, T_n, U](I, U)).$$

We will show that in general T' is a residual intersection of Y' . Further, under the conditions of Theorem 3.1, $T = T'$ and is Cohen-Macaulay. Finally, we interpret the equality $T = T'$ as equivalent to the equality $\text{Sym}(I) = \bigoplus_{n \geq 0} I^n$.

We proceed to show

PROPOSITION 4.3. *T' is a residual intersection of Y' if $\text{codim } Y \geq 1$, and if Y satisfies G_∞ .*

Set $b_i = UT_i - a_i$. We will prove that $((b_1, \dots, b_n): (I, U)) = J$. First we prove J is contained in the ideal on the left-hand side. This follows from the equations

$$(1) \quad \left(\sum_{i=1}^n c_i T_i \right) U = \sum_{i=1}^n c_i (T_i U - a_i)$$

and

$$(2) \quad \left(\sum_{i=1}^n c_i T_i \right) a_j = \sum_{i=1}^n c_i (T_i a_j - a_i T_j),$$

both of which are valid as $\sum_{i=1}^n c_i a_i = 0$. Furthermore, $T_i a_j - a_i T_j = (UT_i - a_i)T_j - (UT_j - a_j)T_i$. Conversely, suppose $F = F(T_1, \dots, T_n, U) \in (b: (I, U))$. Then there is an equation,

$$(3) \quad FU = \sum_{i=1}^n G_i b_i = \sum_{i=1}^n G_i (UT_i - a_i).$$

Therefore,

$$(4) \quad U\left(F - \sum_{i=1}^n G_i T_i\right) = - \sum_{i=1}^n a_i G_i.$$

Write $G_i = G'_i + UG''_i$ where $G'_i \in R[T_1, \dots, T_n]$. Equation (4) becomes

$$U\left(F - \sum_{i=1}^n G_i T_i + \sum_{i=1}^n a_i G''_i\right) = \sum_{i=1}^n a_i G'_i.$$

Since $\sum_{i=1}^n a_i G'_i \in R[T_1, \dots, T_n]$, we see that

$$(5) \quad \sum_{i=1}^n a_i G'_i = 0,$$

and

$$(6) \quad F = \sum_{i=1}^n G_i T_i - \sum_{i=1}^n a_i G''_i = \sum_{i=1}^n (UT_i - a_i)G''_i + \sum_{i=1}^n G'_i T_i.$$

Therefore modulo J , $F = \sum_{i=1}^n G'_i T_i$. From equation (5), $\sum_{i=1}^n a_i G'_i = 0$. As the map from R to $R[T_1, \dots, T_n]$ is faithfully flat, there exist relations ($1 \leq j \leq N$), $\sum_{i=1}^n a_i c_{ij} = 0$ in R , and polynomials H_j in $R[T_1, \dots, T_n]$ so that $G'_i = \sum_{j=1}^N c_{ij} H_j$. Then

$$F \equiv \sum_{i=1}^n T_i \sum_{j=1}^N c_{ij} H_j \equiv \sum_{j=1}^N H_j \left(\sum_{i=1}^n c_{ij} T_i \right) \equiv 0 \pmod{J},$$

since $\sum c_{ij} a_i = 0$.

It follows that $J = (b_1, \dots, b_n : (I, U))$. Therefore T' is a residual intersection of Y provided $\text{codim } T' = n$. This follows from the lemma below.

LEMMA 4.4. *Suppose I satisfies G_∞ , and $\text{ht } I > 0$. Then*

$$\dim \text{Sym}(I) = \dim \bigoplus_{n \geq 0} I^n = \dim R + 1.$$

PROOF. The latter equality is well known. Let $I = (a_1, \dots, a_n)$. There are surjections:

$$\begin{array}{ccc} R[T_1, \dots, T_n] & \xrightarrow{f} & \text{Sym}(I) \rightarrow 0 \\ & \searrow & \downarrow g \\ & & \bigoplus_{n \geq 0} I^n \\ & & \downarrow \\ & & 0 \end{array}$$

Set $A = \ker f$, $P = \ker(f \circ g)$. P is a prime ideal. We need to show $\text{ht } A = \text{ht } P$. Choose any minimal prime over $(A : P) \cap R$, and localize at it. It is enough to prove $\text{ht } A = \text{ht } P$ in this localized ring. Thus we may assume R is local with maximal ideal m , and $m^N P \subseteq A$, for some N . It follows that $\text{ht } A = \min(\text{ht } P, \text{ht } m)$; however $\text{ht } m = \dim R$, while $\text{ht } P = \dim R + v(I) - (\dim R + 1) = v(I) - 1$. Since $\dim R \geq v(I)$ because I satisfies G_∞ , we obtain $\text{ht } A = \text{ht } P$ as we required.

It now easily follows that $\text{ht } J = n$ and consequently T' is a residual intersection of Y' .

We are now able to give an alternate proof of Theorem 4.2(1), (3) and (4). Suppose our notations and assumptions are as in the statement of this theorem. By Proposition 4.3, T' is a residual intersection of Y' . Since Y' is strongly Cohen-Macaulay and satisfies G_∞ , by Theorem 3.1 we conclude T' is Cohen-Macaulay. Further $(I, U) = I(Y')$ is not contained in the zero-divisors of $I(T') = J$. Therefore, $J: (I, U) = J$.

Let $A = I(T)$, $A \subseteq R[T_1, \dots, T_n, U]$. It is easily observed that A is generated by the equations $UT_i - a_i$, plus all homogeneous polynomials $F(T_1, \dots, T_n)$ with coefficients in R such that $F(a_1, \dots, a_n) = 0$. In particular, if $\deg F = d$, then

$$(*) \quad (I, U)^d F(T_1, \dots, T_n) \subseteq (UT_i - a_i).$$

From (*) it follows that there exists an $M > 0$ such that $A = ((UT_i - a_i): (I, U)^M)$. However, in this case $A = (J: (I, U)^{M-1}) = J$. Hence $T = T'$ is Cohen-Macaulay. Since $\text{Spec}(\text{Sym}(I/I^2)) = V(U)$ in T' and $\text{Spec}(\oplus I^n/I^{n+1}) = V(U)$ in T , we obtain statements (3) and (4) of Theorem 4.2. As we observed, Valla [Va] has shown that (3) implies statement (1). In general we do not know if statements (3) and (4) also imply (2); however we do not know of any counterexample either.

Finally, note that the applications in §§3 and 4 of the paper of Artin and Nagata [A-N] now are valid with the hypothesis of strongly Cohen-Macaulay. We do not list these here, but refer the reader to the above paper.

5. A partial converse. Our purpose in this section is twofold. First we relate the residual intersection properties of Theorem 3.1 to the theory of d -sequences and approximation complexes, and secondly we use the latter theories to prove a converse to Theorem 3.1. An important ingredient is the use of the duality of the Koszul homology as we outlined in Proposition 2.7.

Suppose now that $X = \text{Spec}(R)$, where R is a Gorenstein local ring and assume $Y \subseteq X$ is a closed subscheme satisfying G_∞ . We will say Y is *residually Cohen-Macaulay* in X whenever $a_1, \dots, a_n \in I(Y)$ such that $\text{codim}(a_1, \dots, a_n: I(Y)) \geq n$, then $Z = \text{Spec}(R/(a_1, \dots, a_n: I(Y)))$ is Cohen-Macaulay of codimension n , $V(a_1, \dots, a_n) = Z \cup Y$, and $Z \cap Y$ is Cohen-Macaulay of codimension $n + 1$.

The main question of this section is the following: when is a residually Cohen-Macaulay scheme strongly Cohen-Macaulay? We will prove

THEOREM 5.1. *Let $X = \text{Spec}(R)$, where R is a Gorenstein local ring and suppose $Y \subseteq X$ is a closed subscheme satisfying G_∞ which is residually Cohen-Macaulay. Assume $c_i(Y) \geq i + 2$ if $i \geq \text{codim } Y + 3$. Then Y is strongly Cohen-Macaulay in X .*

Our first step in proving Theorem 5.1 is to relate the condition of being residually Cohen-Macaulay to the property of being generated by a d -sequence.

DEFINITION [Hu-4]. Let x_1, \dots, x_n be elements of a ring R . x_1, \dots, x_n are said to be a d -sequence if

- (1) $x_i \notin (x_1, \dots, \hat{x}_i, \dots, x_n)$, and
 - (2) x_i is not a zero-divisor on $R/(x_0, \dots, x_{i-1}: x_k)$ for all $i \geq 0$, and all $k \geq i$.
- (Here we set $x_0 = 0$.)

An ideal J is said to be *related* to the d -sequence x_1, \dots, x_n if J is either of the form $J = (x_0, \dots, x_{i-1}; x_i)$ or $J = (x_0, \dots, x_{i-1}; x_i) + (x_1, \dots, x_n)$, where $1 \leq i \leq n$. The d -sequence is said to be a *Cohen-Macaulay d -sequence* (or a C-M d -sequence for short) if R/J is Cohen-Macaulay for every related ideal J .

D -sequences were studied originally in [Hu-4] to study the properties of powers of an ideal. It is known that if an ideal I is generated by a d -sequence, then $\text{Sym}(I) \simeq \bigoplus_{n \geq 0} I^n$. Further, if I is generated by a Cohen-Macaulay d -sequence then $\text{Sym}(I)$ is Cohen-Macaulay. An equivalent definition of a d -sequence is the condition (1) + the condition (2'),

$$(x_0, \dots, x_{i-1}; x_i) \cap I = (x_0, \dots, x_{i-1})$$

for all $1 \leq i \leq n$ ($I = (x_1, \dots, x_n)$).

The following proposition is now apparent.

PROPOSITION 5.2. *Suppose $X = \text{Spec}(R)$, where R is a Cohen-Macaulay local ring and let $Y \subseteq X$ be a closed subscheme satisfying G_∞ which is residually Cohen-Macaulay. Then $I(Y) = I$ can be generated by a Cohen-Macaulay d -sequence.*

PROOF. Since Y satisfies G_∞ , we may choose minimal generators a_1, \dots, a_n of $I(Y)$ in such a way that $(a_1, \dots, a_s)_P = I_P$ for all prime ideals $P \in Y$ such that $\text{ht } P \leq s$ (see Lemma 3.2). Then the assumption that Y is residually Cohen-Macaulay forces $(a_0, \dots, a_{i-1}; I) \cap I = (a_0, \dots, a_{i-1})$, and $R/(a_0, \dots, a_{i-1}; I)$ to be Cohen-Macaulay. Further, a_i is not a zero-divisor modulo $(a_0, \dots, a_{i-1}; I)$. Since

$$(a_0, \dots, a_{i-1}) \subseteq (a_0, \dots, a_{i-1}; I) \subseteq (a_0, \dots, a_{i-1}; a_i),$$

it follows that $(a_0, \dots, a_{i-1}; I) = (a_0, \dots, a_{i-1}; a_i)$. By definition, a_1, \dots, a_n are a Cohen-Macaulay d -sequence.

We need to make some remarks concerning d -sequences (see [Hu-4] for proofs).

REMARK 5.3. If $0 \leq i \leq n - 1$, and a_1, \dots, a_n form a d -sequence, then the image of a_{i+1}, \dots, a_n in $R/((a_0, \dots, a_i); I)$ form a d -sequence ($I = (a_1, \dots, a_n)$).

REMARK 5.4. Any d -sequence satisfies G_∞ . In fact, any d -sequence is analytically independent. In particular, if a_1, \dots, a_n is a d -sequence in R , $n \leq \dim R$.

REMARK 5.5. If $I = (a_1, \dots, a_n)$ where a_1, \dots, a_n is a d -sequence and $\text{grade } I = k$, then a_1, \dots, a_k form an R -sequence.

D -sequences are intimately connected with the study of the \mathfrak{M} -complex of [S-V], which played an essential role in the proofs in [S-V] of Theorem 4.2. Part of the relation between d -sequences and the \mathfrak{M} -complex was studied in [Hu-2], but in the works [HSV-1 and HSV-2], Herzog, Simis, and Vasconcelos brought to light the full connection.

In fact d -sequences play a role for the \mathfrak{M} -complex similar to the role regular sequences play for the Koszul complex. We recall the definition of the \mathfrak{M} -complex: let $\underline{x} = x_1, \dots, x_n$ be a sequence of elements of R , and $L_{r,s} = K_r \otimes_R S_s$ be the double complex with differentials ∂, ∂' where $K_r = \Lambda^r F = r$ th exterior power of F , a free module with basis e_1, \dots, e_n , and $S_s = \text{Sym}_s(F)$. The differentials ∂ and ∂' are respectively obtained by viewing $L_{r,s}$ as either $K(x; R) \otimes \text{Sym}(F)$, the Koszul maps determining ∂ , or viewing the complex as the Koszul complex associated to the

images of e_1, \dots, e_n in $\text{Sym}_1(F)$. The two differentials commute, and the double complex gives rise to a complex $\mathfrak{N} = \mathfrak{N}(x; R) = \{H_i \otimes_R S, \partial^i\}$, where $H_i =$ homology of the Koszul complex of x_1, \dots, x_n , and where $S = \text{Sym}(F)$.

One of the main results of [HSV-2] is the following theorem.

THEOREM. *Let (R, m) be a local ring with infinite residue class field, and let I be an ideal. Then the following are equivalent:*

- (1) $\mathfrak{N}(I; R)$ is acyclic,
- (2) I is generated by a d -sequence.

We should remark that the homology of \mathfrak{N} is independent of the generating set of I . Moreover, in [HSV-2] the same results are shown to hold if we replace R by an arbitrary finitely generated module M . Herzog, Simis, and Vasconcelos go on to relate the acyclicity of \mathfrak{N} with the property of having a linear resolution. We refer the reader to [HSV-2] for details.

The key result in proving the converse of Theorem 3.1 is the next proposition which was almost proved in [Hu-2, Theorem 2.1].

PROPOSITION 5.6. *Let R be a Cohen-Macaulay local ring and let x_1, \dots, x_n be a Cohen-Macaulay d -sequence. Then,*

$$\text{depth } H_i(x; R) \geq \dim R - n + i$$

for all $i, 1 \leq i \leq n - k, k = \text{ht } I$.

PROOF. We induct on n . Let $I = (x_1, \dots, x_n)$.

Case 1. grade $I > 0$. We show in this case that we may always reduce to a smaller n . Set $k = \text{grade } I = \text{ht } I$. We are assuming $k \geq 1$. By Remark 5.5, x_1, \dots, x_k form an R -sequence. Let “ $-$ ” denote the map from R to R/Rx_1 . By Lemma 1.4(1), there are exact sequences,

$$0 \rightarrow H_i(I; R) \rightarrow H_i(\bar{I}; \bar{R}) \rightarrow H_{i-1}(I; R) \rightarrow 0$$

where the middle homology is taken with respect to $0, \bar{x}_2, \dots, \bar{x}_n$. By (1.1), the middle homology is isomorphic to

$$H_i(\bar{x}_2, \dots, \bar{x}_n; R) \oplus H_{i-1}(\bar{x}_2, \dots, \bar{x}_n; R).$$

By induction on n , we may assume the depth of this module to be at least $\dim R - n + i - 1$. We now induct on j to show $\text{depth } H_{n-j}(I; R) \geq \dim R - n + (n - j)$. The first nonzero homology is when $j = k$, and in this case, $H_{n-k}(I; R) \simeq ((x_1, \dots, x_k): I)/(x_1, \dots, x_k)$. By assumption, $R/((x_1, \dots, x_k): I)$ is Cohen-Macaulay, clearly of depth $R/I = \dim R - k$. It follows from the exact sequence

$$0 \rightarrow ((x_1, \dots, x_k): I)/(x_1, \dots, x_k) \rightarrow R/(x_1, \dots, x_k) \rightarrow R/((x_1, \dots, x_k): I) \rightarrow 0$$

that $\text{depth } H_{n-k}(I; R) \geq \text{depth } R - k$. However, R is Cohen-Macaulay so that $\text{depth } R = \dim R$. Thus our assertion is verified for $j = k$.

Suppose $j > k$. Consider the exact sequence

$$0 \rightarrow H_{n-j+1}(I; R) \rightarrow H_{n-j+1}(\bar{I}; \bar{R}) \rightarrow H_{n-j}(I; R) \rightarrow 0.$$

By our induction on n , $\text{depth } H_{n-j+1}(\bar{I}; \bar{R}) \geq \dim R - j + 1$. By our induction on j , $\text{depth } H_{n-j+1}(I; R) \geq \dim R - j + 1$. It now follows $\text{depth } H_{n-j}(I; R) \geq \dim R - j$.

Case 2. grade $I = 0$. Let “ $-$ ” denote the homomorphism from R to $R/(0 : I)$. We know $(0 : I) \cap I = (0)$. By Lemma 1.4(2) there are exact sequences

$$0 \rightarrow \bigoplus (0 : I) \rightarrow H_i(I; R) \rightarrow H_i(\bar{I}, \bar{R}) \rightarrow 0.$$

Since we are assuming $R/(0 : I)$ is Cohen-Macaulay (since $(0 : I)$ is a related ideal), it follows that $(0 : I)$ is also Cohen-Macaulay. As $(0 : I) = (0 : x_1) = (0 : x_1^2)$, grade $\bar{I} \geq 1$, and we may apply Case 1 to conclude $\text{depth } H_i(\bar{I}, \bar{R}) \geq \dim \bar{R} - n + i$. Since $\text{depth}(0 : I) = \text{depth } R/I = \text{depth } I$, it easily follows from the exact sequence above that $\text{depth } H_i(I; R) \geq \dim R - n + i$. (Note that the images of x_1, \dots, x_n form a d -sequence in $R/(0 : I)$ by Remark 5.3.)

COROLLARY 5.7. *Let R be a Gorenstein local ring and let x_1, \dots, x_n be a C-M d -sequence. Set $I = (x_1, \dots, x_n)$, set $Y = \text{Spec}(R/I)$. Suppose $c_i(Y) \geq i + 1$ if $i \geq \text{codim } Y + 2$. Then $H_i(\underline{x}; R)$ are reflexive for all $0 \leq i \leq d(I)$.*

PROOF. First observe that the condition, “ $H_i(\underline{x}; R)$ is reflexive for all $i, 0 \leq i \leq d(I)$ ” is independent of the generating set chosen. We need to prove $H_i(\underline{x}; R)_p$ is isomorphic to $H_i(\underline{x}; R)_p^{**}$ if $\text{ht } P/I \leq 1$, and further show $\text{depth}(H_i(\underline{x}; R))_p \geq 2$ if $\text{ht } P/I \geq 2$. Suppose $\text{ht } P/I \leq 1$. In this case since Y satisfies G_∞ , $v(I_p) \leq \text{ht } I + 1$. As we may choose whatever generating set we want to show that $H_i(\underline{x}; R)_p$ are reflexive, we choose a minimal generating set of \underline{x}' of I_p . Then, $H_i(\underline{x}'; R_p) = 0$ for $i \geq 2$ while $H_1(\underline{x}'; R_p) = 0$, or is the canonical module of R/I . It follows that $(H_i(\underline{x}; R))_p$ is reflexive if $\text{ht } P/I \leq 1$.

Assume $\text{ht } P/I \geq 2$, and localize at P . We assume R is local with maximal ideal P . We need to show $\text{depth } H_i(\underline{x}; R) \geq 2$ for all $1 \leq i \leq d(I)$. We use Proposition 5.6. If $i \geq 2$, then since

$$\text{depth } H_i(\underline{x}; R) \geq \dim R - v(I) + i \geq 2$$

there is nothing to show. If $i = 1$, it suffices to show $\dim R > v(I)$. If not, $v(I) = \dim R$. Set $n = v(I)$, and consider $c_n(Y)$. Since $\dim R \geq \dim R/I + 2$, our assumption tells us that $c_n(Y) \geq n + 1$. As $n = \dim R$, this is a contradiction. Therefore $\dim R > v(I)$, and $\text{depth } H_1(\underline{x}; R) \geq 2$.

In particular, under the conditions of Corollary 5.7, there is a pairing,

$$H_i(\underline{x}; R) \times H_{d-i}(\underline{x}; R) \rightarrow H_d(\underline{x}; R) = K_{R/I}.$$

Here $d = d(I)$ (see Proposition 2.7).

We will use this duality of the Koszul homology along with Proposition 5.6 to prove under fairly weak conditions that the Koszul homology must be Cohen-Macaulay.

The key to this attempt is the following lemma. Although it is essentially found in [H-0 and FFGR, Corollary 2.6], it does not seem to be well known in the form below. However it is extremely useful.

LEMMA 5.8. Let R be a local Cohen-Macaulay ring with canonical module K_R . Let “ \checkmark ” denote $\text{Hom}(-, K_R)$. Suppose M_p is Cohen-Macaulay if $\text{ht } p \leq 2$ and assume

$$(*) \quad \text{depth } M_p + \text{depth}(M^\checkmark)_p \geq \dim R_p + 2$$

for all prime ideals p of R of height at least three with $M_p \neq 0$. Then M is a maximal Cohen-Macaulay module.

The proof is a straightforward use of duality, but for completeness we give the proof here.

Since $\text{Hom}_R(M, K_R)_p = \text{Hom}(M_p, (K_R)_p) = \text{Hom}_{R_p}(M_p, K_{R_p})$, the assumptions and statement of the lemma remain valid under localization. We may therefore assume that M_p is Cohen-Macaulay if $p \neq m$, the maximal ideal of R . Further we may assume $\dim R \geq 3$ and $M_p \neq 0$.

Let F be a free resolution of M , and consider the complex F^\checkmark . The cohomology of F^\checkmark is $\text{Ext}_R^i(M, K_R)$, which in turn is dual to $H_m^{d-i}(M)$, if $d = \dim R$.

We wish to prove $H_m^i(M) = 0$ for $i < d$. Set $k = \text{depth } M$. Then $H_m^i(M) = 0$ for $i < k$. We have assumed

$$\text{depth } M^\checkmark \geq d + 2 - k.$$

In addition $\text{Ext}_R^i(M, K_R)$ has finite length if $i \geq 1$, since M_p is Cohen-Macaulay if $p \neq m$ (or $M_p = 0$). Consider the complex

$$(1) \quad 0 \rightarrow M^\checkmark \rightarrow F_0^\checkmark \rightarrow F_1^\checkmark \rightarrow \cdots \rightarrow F_j^\checkmark \rightarrow \cdots$$

We are in a position to apply the lemma of acyclicity of Peskine and Szpiro [P-S-2]. As $\text{depth } M^\checkmark \geq d + 2 - k$, and $\text{depth } F_i^\checkmark \geq d$ for all i , we may conclude the complex (1) is exact up to $j = d - k + 1$, i.e. that $\text{Ext}^i(M, K_R) = 0$ if $1 \leq i \leq d - k$. In this case, $H_m^{d-i}(M) = 0$ for $1 \leq i \leq d - k$, so that $H_m^j(M) = 0$ for $k \leq j < d$. It follows that M is Cohen-Macaulay.

We observe that the well-known criterion of Hartshorne and Ogus follows at once from this lemma.

PROPOSITION [H-O]. Let R be a local ring, quotient of a Gorenstein ring and assume R is factorial. If

$$\text{depth } R_p \geq \frac{1}{2}(\dim R_p) + 1$$

for all $p \in \text{Spec}(R)$, then R is Cohen-Macaulay.

PROOF. We may assume R is a quotient of a Gorenstein local ring S with $\dim S = \dim R$. By Lemma 5.8 if

$$(2) \quad \text{depth } R_p + \text{depth}(K_R)_p \geq \dim R_p + 2$$

for all $p \in \text{Spec}(R)$ of height at least 3, then R is Cohen-Macaulay. Since R is factorial, it is well known that $K_R = R$. Therefore (2) reads

$$(3) \quad \text{depth } R_p \geq \frac{1}{2}(\dim R_p) + 1.$$

THEOREM 5.9. Let R be a Gorenstein local ring, and let x_1, \dots, x_n be a Cohen-Macaulay d -sequence. Set $I = (x_1, \dots, x_n)$, $Y = \text{Spec}(R/I)$. Suppose $c_i(Y) \geq i + 2$ if $i \geq \text{codim } Y + 3$. Then Y is strongly Cohen-Macaulay in $X = \text{Spec}(R)$.

PROOF. We induct on $\dim R$. We may assume that $H_i(\underline{x}; R)_p$ is Cohen-Macaulay if $p \neq m$, the maximal ideal of R . Furthermore we may assume $\dim R/I \geq 3$, since if $\dim R/I \leq 2$, then Example 2.3 and Proposition 2.2 show that Y will be strongly Cohen-Macaulay. By Corollary 5.7, we may assume (with $d = d(I)$)

$$\begin{aligned} H_i(\underline{x}; R) &= \text{Hom}(H_{d-i}(\underline{x}; R), H_d(\underline{x}; R)), \\ H_{d-i}(\underline{x}; R) &= \text{Hom}(H_i(\underline{x}; R), H_d(\underline{x}; R)). \end{aligned}$$

By using Lemma 5.8, it is enough to prove

$$\text{depth } H_i(\underline{x}; R) + \text{depth } H_{d-i}(\underline{x}; R) \geq \dim R/I + 2$$

since $H_d(\underline{x}; R) = K_{R/I}$. However, applying Proposition 5.6 shows that

$$\begin{aligned} \text{depth } H_i(\underline{x}; R) + \text{depth } H_{d-i}(\underline{x}; R) &\geq \dim R - v(I) + i + \dim R - v(I) + d - i \\ &= 2 \dim R - 2v(I) + d. \end{aligned}$$

It remains to show

$$2 \dim R - 2v(I) + (v(I) - \text{ht } I) \geq \dim R - \text{ht } I + 2,$$

i.e. that $\dim R \geq v(I) + 2$. Set $n = v(I)$. As we may assume $d \geq 3$, by assumption $c_n(Y) \geq n + 2$. In particular, $\dim R \geq n + 2$ which finishes the proof of the theorem.

We may now prove (and restate)

THEOREM 5.1 (PARTIAL CONVERSE TO THEOREM 3.1). *Let $X = \text{Spec}(R)$, where R is a Gorenstein local ring and suppose $Y \subseteq X$ is a closed subscheme satisfying G_∞ which is residually Cohen-Macaulay. Assume $c_i(Y) \geq i + 2$ if $i \geq \text{codim } Y + 3$. Then Y is strongly Cohen-Macaulay in X .*

PROOF. By Proposition 5.2, $I = I(Y)$ can be generated by a Cohen-Macaulay d -sequence. We now simply apply Theorem 5.9.

We close with an example to point out the Gorenstein condition on the base ring is necessary.

EXAMPLE 5.10. Let $R = k[X_{ij}]/I_2(X)$, where $X = (X_{ij})$ is a 2×3 matrix. Denote the image of X_{ij} in R by x_{ij} , and let $I = (x_{11}, x_{12}, x_{13})$. It is easily observed that I is generated by a Cohen-Macaulay d -sequence. Further, $C_2(R/I) = \{m\}$, where $m = (x_{ij})$. Thus,

$$c_2(R/I) = c_2(R/I) = 4, \quad c_i(R/I) = \infty \quad \text{for } i \geq 3.$$

In particular, $c_i(Y) \geq i + 2$ if $i \geq \text{codim } Y + 3 = 4$. Also R/I satisfies G_∞ (as it must since I is generated by a d -sequence). However, one can compute that $\text{depth } H_1(x_{11}, x_{12}, x_{13}; R) = 2$.

Therefore $H_1(x_{11}, x_{12}, x_{13}; R)$ is not Cohen-Macaulay. However, R is not Gorenstein. This example also shows that Example 2.3 cannot be improved by removing the assumption that R be Gorenstein. Notice $v(I) = 3 = \text{ht } I + 2$.

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