SPECTRAL DECOMPOSITION WITH MONOTONIC SPECTRAL RESOLVENTS

BY

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Abstract. The spectral decomposition problem of a Banach space over the complex field entails two kinds of constructive elements: (1) the open sets of the field and (2) the invariant subspaces (under a given linear operator) of the Banach space. The correlation between these two structures, in the framework of a spectral decomposition, is the spectral resolvent concept. Special properties of the spectral resolvent determine special types of spectral decompositions. In this paper, we obtain conditions for a spectral resolvent to have various monotonic properties.

1. Introduction. A spectral decomposition of a Banach space $X$, by a bounded linear operator $T: X \to X$,

(a) expresses $X$ as a finite linear sum of $T$-invariant subspaces $X_i$;

(b) represents $T$ as the sum of its restrictions $T_i = T|X_i$;

(c) localizes the spectrum $\sigma(T_i)$ of each $T_i$ in the closure of a given open set $G_i$, which intersects the spectrum $\sigma(T)$ of $T$.

The relationship between the invariant subspaces $X_i$ and the open sets $G_i$, formalized under the name of spectral resolvent, has been the study of some recent works [1, 2, 8]. In this paper, we investigate conditions under which the spectral resolvent possesses certain specific monotonic properties. Such conditions and subsequence properties infer the corresponding spectral decompositions.

For a bounded linear operator $T$, which maps an abstract Banach space $X$ over the complex field $\mathbb{C}$ into itself, we use the following notation: spectrum $\sigma(T)$, point spectrum $\sigma_p(T)$, resolvent set $\rho(T)$, the unbounded component of the resolvent set $\rho_\infty(T)$, and the resolvent operator $R(\cdot; T)$. If $T$ has the single valued extension property then, for $x \in X$, $\sigma_T(x)$ denotes the local spectrum, $\rho_T(x)$ the local resolvent set and $x(\cdot)$ the local resolvent function.

For a subspace (closed linear manifold) $Y$ of $X$, $T|Y$ is the restriction of $T$ to $Y$ and $T/Y$ is the coinduced operator on the quotient space $X/Y$. $\text{Inv} \ T$ denotes the lattice of the invariant subspaces of $X$ under $T$. $T^*$ is the conjugate of $T$. If $A$ is a subset of $X$ then $A^\perp$ denotes the annihilator of $A$ in the dual space $X^*$. Given a set $S$, we write $\bar{S}$ for the closure, $S^c$ for the complement, $d(\lambda, S)$ for the distance from a point $\lambda$ to $S$, and express by $\text{cov} S$, the collection of all finite open covers of $S$. $\emptyset$ stands for the family of all open subsets of $\mathbb{C}$. An open set $\Delta$ is called a Cauchy
domain if it has a finite number of components and the boundary \( \Gamma = \partial \Delta \) is a
positively oriented finite system of closed, nonintersecting, rectifiable Jordan curves.

Throughout this paper \( T \) is a bounded linear operator mapping the underlying
Banach space \( X \) into itself.

1.1. Definition. A spectral decomposition of \( X \) by \( T \) is a finite system \( \{(G_i, X_i)\} \)
\( \subset \emptyset \times \text{Inv} \, T \), satisfying the following conditions:
(i) \( \{G_i\} \subseteq \text{cov}(T) \);
(ii) \( X = \sum_i X_i \);
(iii) \( \sigma(T \mid X_i) \subseteq \overline{G_i} \), for all \( i \).

1.2. Definition [1]. A map \( E : \emptyset \to \text{Inv} \, T \) is called a spectral resolvent of \( T \) if it
satisfies the following conditions:
(I) \( E(\emptyset) = \{0\} \);
(II) for any \( \{G_i\} \subseteq \text{cov}(T) \), \( \{(G_i, E(G_i))\} \) is a spectral decomposition of \( X \) by \( T \).

Although the spectral resolvent fails to be unique, the properties they have in
common characterize specific types of spectral decompositions. In this vein, we
mention that an operator \( T \) having a spectral resolvent possesses the single valued
extension property [1] and, moreover, it is decomposable [8] in the sense of Foiaş [4].

The following types of invariant subspaces will be involved in our study.

1.3. Definition [5]. A subspace \( Y \) of \( X \) is said to be analytically invariant under \( T \)
if, for every function \( f : D \to X \) analytic on some open \( D \subset C \), the condition
\[
(\lambda - T)f(\lambda) \in Y \quad \text{on} \quad D
\]
implies that \( f(\lambda) \in Y \) on \( D \).

An analytically invariant subspace is also invariant under \( T [6] \).

1.4. Definition [4]. \( Y \in \text{Inv} \, T \) is said to be a spectral maximal space of \( T \) if, for
any \( Z \in \text{Inv} \, T \), the inclusion \( \sigma(T \mid Z) \subseteq \sigma(T \mid Y) \) implies that \( Z \subseteq Y \).

If \( T \) has the single valued extension property then, for any set \( S \subset C \),
\[
X_T(S) = \{ x \in X : \sigma_T(x) \subset S \}
\]
is a linear manifold in \( X \). If \( T \) is a decomposable operator then, for any \( G \in \emptyset \),
\( X_T(G) \) is an analytically invariant subspace under \( T [5] \) and, for any closed \( F \subset C \),
\( X_T(F) \), in particular \( X_T(G) \), is a spectral maximal space of \( T [4] \). Moreover, for a
decomposable \( T \), we have
\[
(1.1) \quad \overline{G \cap \sigma(T)} \subseteq \sigma(T \mid X_T(G)) \subseteq \overline{G \cap \sigma(T)}.
\]

1.5. Definition [9]. \( Y \in \text{Inv} \, T \) is said to be a \( T \)-absorbent space if, for every
\( y \in Y \) and all \( \lambda \in \sigma(T \mid Y) \), the equation \( (\lambda - T)x = y \) has all solutions \( x \), if any,
contained in \( Y \).

If \( T \) has the single valued extension property, then every \( T \)-absorbent space is
analytically invariant under \( T \).

1.6. Proposition [2]. Let \( \{(G_i, X_i)\}_{i=1,2} \) be a spectral decomposition of \( X \) by \( T \) in
terms of \( T \)-absorbent spaces \( X_1 \) and \( X_2 \). Then
\[
\sigma(T \mid X_1 \cap X_2) \subseteq \sigma(T \mid X_1) \cap \sigma(T \mid X_2).
\]
1.7. Proposition. If, for \( X_1, X_2 \in \text{Inv } T, X = X_1 + X_2 \) then
\[
\sigma(T) \subset \sigma(T \mid X_1) \cup \sigma(T \mid X_2) \cup \sigma_p(T).
\]
In particular, if \( T \) has the single valued extension property, then
\[
\sigma(T) \subset \sigma(T \mid X_1) \cup \sigma(T \mid X_2).
\]

Proof. Let \( \lambda \in \rho(T \mid X_1) \cap \rho(T \mid X_2) - \sigma_p(T) \) and \( x \in X \). There is a representation for \( x, x = x_1 + x_2 \) with \( x_i \in X_i, i = 1, 2 \). For \( y_i = R(\lambda; T \mid X_i)x_i, i = 1, 2 \), and \( y = y_1 + y_2 \) we have
\[
(\lambda - T)y = (\lambda - T)y_1 + (\lambda - T)y_2 = x_1 + x_2 = x
\]
and hence \( \lambda - T \) is surjective. Furthermore, since \( \lambda \notin \sigma_p(T) \), we have \( \lambda \in \rho(T) \). The last statement of the proposition follows from [3, Theorem 2]. \( \square \)

Property (1.1) of \( X_T(\cdot) \) has an interesting variant in terms of a spectral resolvent \( T \), expressed by [8, Proposition 16]. For completeness, we recall that property and provide it with a shorter proof.

1.8. Proposition. If \( T \) has a spectral resolvent \( E \) then, for any \( G \in \mathcal{G} \),
\[
G \cap \sigma(T) \subset \sigma[T \mid E(G)].
\]

Proof. Let \( \lambda \in G \cap \sigma(T) \) be given and let \( H \in \mathcal{G} \) be such that \( (G, H) \in \text{cov } \sigma(T) \) with \( \lambda \notin H \). Then \( X = E(G) + E(H) \) and Proposition 1.7 implies
\[
\sigma(T) \subset \sigma[T \mid E(G)] \cup \sigma[T \mid E(H)].
\]
Since \( \lambda \in [G \cap \sigma(T)] - H \), it follows from (1.4) that \( \lambda \in \sigma[T \mid E(G)] \) and hence inclusion (1.3) holds. \( \square \)

If \( T \) has a spectral resolvent \( E \), then \( T \) has a maximal spectral resolvent \( E_m \) in the sense that, for every \( G \in \mathcal{G} \) and all spectral resolvents \( E \) of \( T \),
\[
E(G) \subset E_m(G) = X_T(G).
\]
Since, clearly \( X_T(G) \subset X_T(G) \), where the inclusion may be proper, some spectral resolvents \( E \) may be such that
\[
X_T(G) \subset E(G) \subset X_T(G) \quad \text{for all } G \in \mathcal{G}.
\]

Condition (1.5) endows \( E \) with some remarkable properties, which will be the topic of the following sections.


2.1. Definition. A spectral resolvent \( E \) is said to be monotonic if \( G_1, G_2 \in \mathcal{G} \) and \( G_1 \subset G_2 \) imply that \( E(G_1) \subset E(G_2) \).

Note that (1.5) is a sufficient condition for a spectral resolvent \( E \) of \( T \) to be monotonic. In fact, if the open sets \( G_1, G_2 \) are such that \( G_1 \subset G_2 \), then (1.5) implies the inclusions
\[
E(G_1) \subset X_T(G_1) \subset X_T(G_2) \subset E(G_2).
\]
2.2. **Theorem.** Let $T$ have a spectral resolvent $E$. If for any pair $G_1, G_2 \in \emptyset$, $E$ satisfies condition

\[(2.1) \quad \sigma[T \mid E(G_1) \cap E(G_2)] \subset \overline{G_1} \cap \overline{G_2} \]

then property (1.5) holds and $E$ is monotonic.

**Proof.** Given $G_1 \in \emptyset$, let $x \in X_T(G_1)$. Choose $G_2 \in \emptyset$ such that $\{G_1, G_2\} \in \text{cov} \sigma(T)$ and $\sigma_T(x) \cap G_2 = \emptyset$ (this is possible because $\sigma_T(x)$ is closed and is contained in $G_1$). To avoid repetitions, we divide the remainder of the proof in two parts.

**Part A.** There is a representation of $x$,

\[x = x_1 + x_2 \quad \text{with } x_i \in E(G_i), i = 1, 2.\]

In view of some elementary properties, the local spectra of $x_1$ and $x_2$ are contained in some pertinent sets

\[(2.2) \quad \sigma_T(x_1) \subset \sigma_T(x) \cup (\overline{G_1} \cap \overline{G_2}), \quad \sigma_T(x_2) \subset \overline{G_1} \cap \overline{G_2}.\]

For $\lambda \in \rho_T(x) \cap (\overline{G_1} \cap \overline{G_2})^c = H$, we have $x(\lambda) = x_1(\lambda) + x_2(\lambda)$. Let $\Delta$ be a Cauchy domain with boundary $\Gamma$ such that $\sigma_T(x) \subset \Delta$ and $\overline{\Delta} \subset (\overline{G_1} \cap \overline{G_2})^c$. The functional calculus gives

\[(2.3) \quad x = \frac{1}{2\pi i} \int_{\Gamma} x(\lambda) \, d\lambda = \frac{1}{2\pi i} \int_{\Gamma} x_i(\lambda) \, d\lambda.\]

For every $\lambda_0 \in \Gamma$, there is a neighborhood $V \subset H$ of $\lambda_0$ and there are functions $f_i: V \to E(G_i) \ (i = 1, 2)$ analytic on $V$ such that

\[(2.4) \quad x_i(\lambda) = f_1(\lambda) + f_2(\lambda) \quad \text{on } V.\]

It follows from

\[(\lambda - T) x_1(\lambda) = x_1 \quad \text{on } \rho_T(x_1),\]

that the function $g: V \to E(G_1) \cap E(G_2)$ defined by

\[g(\lambda) = x_1 - (\lambda - T)f_1(\lambda) = (\lambda - T)f_2(\lambda)\]

is analytic on $V$.

**Part B.** Since $V \subset (\overline{G_1} \cap \overline{G_2})^c \subset \rho[T \mid E(G_1) \cap E(G_2)]$, the function $h: V \to E(G_1) \cap E(G_2)$ defined by

\[h(\lambda) = R[\lambda; T \mid E(G_1) \cap E(G_2)] g(\lambda)\]

is analytic on $V$. We have

\[(\lambda - T)h(\lambda) = g(\lambda) = (\lambda - T)f_2(\lambda) \quad \text{on } V\]

and hence the single valued extension property of $T$ implies that

\[f_2(\lambda) = h(\lambda) \in E(G_1) \cap E(G_2) \subset E(G_1) \quad \text{on } V.\]

Thus, by (2.4) $x_i(\lambda) \in E(G_i)$ on $V$ and, in particular, $x_1(\lambda_0) \in E(G_1)$. Since $\lambda_0$ is arbitrary on $\Gamma$, it follows from (2.3) that $x \in E(G_1)$. Thus, $X_T(G_1) \subset E(G_1)$ and this establishes (1.5). Consequently, $E$ is a monotonic spectral resolvent. \qed
2.3. Corollary. Let $E$ be a spectral resolvent of $T$. If for each $G \subseteq \emptyset$, any one of the following conditions holds, then $E$ is monotone.

1. $\sigma[T^* | E(G)^{-1}] \subseteq G^c$;
2. $\sigma[T/E(G)] \subseteq G^c$;
3. $E(G)$ is analytically invariant;
4. $E(G)$ is $T$-absorbent.

Proof. Conditions (1)–(3) are equivalent [1]. Moreover, since $T$ has the single valued extension property, every $T$-absorbent space is analytically invariant under $T$. Thus, it suffices to prove the statement of the corollary under hypothesis (4). Given $G_1, G_2 \subseteq \emptyset$, Proposition 1.6 implies

$$\sigma[T \cap E(G_1) \cap E(G_2)] \subseteq \sigma[T \cap E(G_1)] \cap \sigma[T \cap E(G_2)] \subseteq \overline{G_1} \cap \overline{G_2}.$$ 

Now, Theorem 2.2 concludes the proof. 

2.4. Corollary. Let $T$ have a spectral resolvent $E$. If $\sigma(T)$ has empty interior and $\rho_\infty(T) = \rho(T)$ (in particular, if $\sigma(T)$ is contained on an open Jordan curve), then $E$ is monotone.

Proof. It suffices to show that for every $G \subseteq \emptyset$, $E(G)$ is analytically invariant under $T$. Let $f: D \to \mathbb{C}$ be analytic on an open $D \subseteq \mathbb{C}$ such that for every $G \subseteq \emptyset$, 

$$\frac{(\lambda - T)f(\lambda)}{\lambda - T} \in E(G) \text{ on } D.$$ 

Since $\sigma(T)$ has empty interior, $D - \sigma(T)$ is a nonempty open set. Then, since $\rho_\infty(T) = \rho(T)$, we have 

$$f(\lambda) = R(\lambda; T)(\lambda - T)f(\lambda) \in E(G) \text{ for all } \lambda \in D - \sigma(T)$$ 

and $f(\lambda) \in E(G)$ on $D$, by analytic continuation. 

As a summary of this section, the “spectral inclusion property” (1.5) and the “spectral invariance property” (2.1) proved to be sufficient conditions for a spectral resolvent $E$ to be monotone. By strengthening the monotone spectral resolvent concept, (1.5) is heightened to a necessary and sufficient condition for the validity of the new monotone attribute of a spectral resolvent.

3. Strongly monotone spectral resolvents.

3.1. Definition. A spectral resolvent $E$ is said to be strongly monotone if $G, G_1, G_2 \subseteq \emptyset$ and $\overline{G_1} \cap \overline{G_2} \subseteq G$ imply $E(G_1) \cap E(G_2) \subseteq E(G)$.

Evidently, every strongly monotone spectral resolvent is monotone. As an example, if $T$ has a spectral resolvent $E$ then its maximal spectral resolvent $E_m$ is strongly monotone. Indeed, $G, G_1, G_2 \subseteq \emptyset$ and $\overline{G_1} \cap \overline{G_2} \subseteq G$ imply 

$$E_m(G_1) \cap E_m(G_2) = X_T(\overline{G_1} \cap \overline{G_2}) = X_T(\overline{G_1} \cap \overline{G_2}) \subseteq X_T(G) = E_m(G).$$ 

3.2. Theorem. Let $E$ be a spectral resolvent of $T$. $E$ is strongly monotone if and only if (1.5) holds for every $G \subseteq \emptyset$.

Proof. We only have to prove the “only if” part. Assume that $E$ is strongly monotone. Given $G \subseteq \emptyset$, let $x \in X_T(G)$. Let $\{G_1, G_2\} \subseteq \text{cov } \sigma(T)$ be such that 

$$\sigma_T(x) \subseteq G_1 \subseteq \overline{G} \subseteq G \text{ and } \sigma_T(x) \cap \overline{G_2} = \emptyset.$$
Follow verbatim Part A of the proof of Theorem 2.2. Let $K \in \emptyset$ be such that
\[ \overline{G}_1 \cap \overline{G}_2 \subset K \subset \overline{K} \subset G, \quad \overline{K} \cap \sigma_T(x) = \emptyset \quad \text{and} \quad V \cap \overline{K} = \emptyset. \]
$E$ being strongly monotone, we have $g(\lambda) \in E(K)$ on $V$. The function $h: V \to E(K)$
defined by $h(\lambda) = R[\lambda; T|E(K)]g(\lambda)$ is analytic on $V$ and
\[ (\lambda - T)h(\lambda) = (\lambda - T)f_2(\lambda) \quad \text{on} \ V. \]
By the single valued extension property of $T$,
\[ f_2(\lambda) = h(\lambda) \in E(K) \quad \text{on} \ V. \]
$E$ being monotone, we have
\[ x_1(\lambda) \in E(G_1) + E(K) \subset E(G) \quad \text{on} \ V \]
and, in particular, $x_i(\lambda_0) \in E(G)$. Since $\lambda_0$ is arbitrary on $\Gamma$, it follows from (2.3)
that $x \in E(G)$. Since $x$ is arbitrary in $X_T(G)$, the proof concludes with $X_T(G) \subset E(G)$. \square

Another characterization of a strongly monotone spectral resolvent involves the
range of the local resolvent function.

3.3. THEOREM. Let $E$ be a spectral resolvent of $T$. The following assertions are
equivalent:

(i) $E$ is strongly monotone;
(ii) $G_1, G_2 \in \emptyset$, $\overline{G}_1 \subset G_2$ and $x \in E(G_1)$ imply \{x(\lambda): \lambda \in \rho_T(x)\} \subset E(G_2).

PROOF. (i) $\Rightarrow$ (ii): Let $G_1, G_2 \in \emptyset$ be such that $\overline{G}_1 \subset G_2$. By Theorem 3.2, we have
\[ E(G_1) \subset X_T(\overline{G}_1) \subset X_T(G_2) \subset E(G_2). \]
Let $x \in E(G_1)$ be given. Then $x \in X_T(\overline{G}_1)$ and since $X_T(\overline{G}_1)$ is a spectral
maximal space of $T$, (3.1) implies
\[ \{x(\lambda): \lambda \in \rho_T(x)\} \subset X_T(\overline{G}_1) \subset E(G_2). \]

(ii) $\Rightarrow$ (i): Let $G \subset C$ be an open set and let $x \in X_T(G)$. Choose $G_1 \in \emptyset$ such that
\[ \sigma_T(x) \subset G_1 \subset \overline{G}_1 \subset G. \]
Let $G_2 \in \emptyset$ satisfy conditions
\[ \sigma(x) \cap G_1 = \emptyset, \quad \sigma(x) \cap G_2 = \emptyset. \]
Then $x$ has a representation $x = x_1 + x_2$ with $x_i \in E(G_i)$, $i = 1, 2$. As obtained in
an earlier proof, we have (2.2)
\[ \sigma_T(x_1) \subset \sigma_T(x) \cup (\overline{G}_1 \cap \overline{G}_2), \quad \sigma_T(x_2) \subset \overline{G}_1 \cap \overline{G}_2. \]
Let $\Delta$ be a Cauchy domain with boundary $\Gamma \subset \rho_T(x) \cap (\overline{G}_1 \cap \overline{G}_2)^c$, such that
\[ \sigma_T(x) \subset \Delta \quad \text{and} \quad \overline{\Delta} \cap (\overline{G}_1 \cap \overline{G}_2) = \emptyset. \]
Then
\[ x = \frac{1}{2\pi i} \int_{\Gamma} x(\lambda) \, d\lambda = \frac{1}{2\pi i} \int_{\Gamma} x_1(\lambda) \, d\lambda. \]
Since $x_1 \in E(G_1)$ and $\overline{G}_1 \subset G$, hypothesis (ii) implies
\[ \{x_1(\lambda): \lambda \in \rho_T(x)\} \subset E(G). \]
Then, by (3.2), $x \in E(G)$ and hence $X_T(G) \subset E(G)$. Now, Theorem 3.2 concludes
the proof. \square
A further characterization of a strongly monotonic spectral resolvent can be obtained in terms of a localization property of the spectral resolvent. The following definition generalizes the concept of “almost localized spectrum” [10].

3.4. Definition. A spectral resolvent £ is said to be almost localized if $G, G_1, G_2 \in \emptyset$ and $\overline{G} \subseteq G_1 \cup G_2$ imply $E(G) \subseteq E(G_1) + E(G_2)$.

The following result is due to Radjabalipour [7].

3.5. Proposition. If $T$ is decomposable then, for every closed set $F$ and $(H_1, H_2) \in \text{cov } F$, the following inclusion holds:

$$X_T(F) \subseteq X_T(\overline{H_1}) + X_T(\overline{H_2}).$$

Since, for every open cover $(H_1, H_2)$ of $F$, there is $(G_1, G_2) \in \text{cov } F$ with $\overline{H_1} \subseteq G_1$ and $\overline{H_2} \subseteq G_2$, property (3.3) can be expressed as

$$X_T(F) \subseteq X_T(G_1) + X_T(G_2).$$

3.6. Theorem. Let $T$ have a spectral resolvent $E$. Then $E$ is strongly monotonic if and only if $E$ is almost localized.

Proof. In view of Theorem 3.2, we have to show that the following conditions are equivalent:

(i) $X_T(G) \subseteq E(G)$ for all $G \in \emptyset$;

(ii) $G, G_1, G_2 \in \emptyset$ and $\overline{G} \subseteq G_1 \cup G_2$ imply $E(G) \subseteq E(G_1) + E(G_2)$.

(i) $\Rightarrow$ (ii): Let $G, G_1, G_2 \in \emptyset$ be such that $\overline{G} \subseteq G_1 \cup G_2$. Since $T$ is decomposable, (3.4) implies

$$E(G) \subseteq X_T(\overline{G}) \subseteq X_T(G_1) + X_T(G_2) \subseteq E(G_1) + E(G_2).$$

(ii) $\Rightarrow$ (i): Given $G \in \emptyset$, let $x \in X_T(G)$. Further, let $H_0$ be a relatively compact, open neighborhood of $\sigma(T)$. Then

$$x \in X = E(H_0) \quad \text{and} \quad \sigma_T(x) \subseteq \sigma(T) \subseteq H_0.$$

Let $\varepsilon$ be arbitrary, with $0 < \varepsilon < \sup_{\lambda \in \sigma(T)} d[\lambda, \sigma_T(x)]$. Define the open sets

$$H = \{ \lambda \in \mathbb{C}: d[\lambda, \sigma_T(x)] < \varepsilon \}, \quad H' = \left\{ \lambda \in \mathbb{C}: d(\lambda, H_0) < \frac{\varepsilon}{6} \right\}.$$

For every $\lambda \in \overline{H'} \cap H^c$, let $D_\lambda = \{ \mu \in \mathbb{C}: |\mu - \lambda| < \varepsilon/3 \}$. Then $\{ D_\lambda : \lambda \in \overline{H'} \cap H^c \}$ is an open cover of $\overline{H'} \cap H^c$. Since $\overline{H'} \cap H^c$ is compact, there is a finite collection $\{ \lambda_1, \lambda_2, \ldots, \lambda_n \} \subseteq \overline{H'} \cap H^c$ such that

$$\overline{H'} \cap H^c \subseteq \bigcup_{i=1}^{n} D_{\lambda_i}, \quad \text{where } D_{\lambda} = D_{\lambda_i} \text{ for } \lambda = \lambda_i.$$

For $1 \leq i \leq n$, define

$$K_i = \left\{ \mu \in \mathbb{C}: |\mu - \lambda_i| < \frac{2}{3} \varepsilon \right\}, \quad \Delta_i = \left\{ \mu \in \mathbb{C}: |\mu - \lambda_i| < \frac{\varepsilon}{2} \right\}.$$

Clearly, $K_i \cap \sigma_T(x) = \emptyset$, $1 \leq i \leq n$. Put

$$H_i = \left\{ \lambda \in \mathbb{C}: d(\lambda, H_0) < \frac{\varepsilon}{9n} \right\} - \Delta_i.$$
It is easy to see that $\overline{H}_1 \cap \overline{D}_1 = \emptyset$. Since

$$\overline{H}_0 \subset H_1 \cup \overline{\Delta}_1 \subset H_1 \cup K_1,$$

we have

$$x \in E(H_0) \subset E(H_1) + E(K_1).$$

For $G_1 = H_1$, $G_2 = K_1$, follow Part A of the proof of Theorem 2.2. Note that the boundary $\Gamma$ of the Cauchy domain $\Delta$ in Part A, verifies inclusions

$$\Gamma \subset \rho_\infty[T | E(K_1)] \subset \rho[T | E(H_1) \cap E(K_1)].$$

The function $h: V \to E(H_1) \cap E(K_1)$, defined by

$$h(\lambda) = R[\lambda; T | E(H_1) \cap E(K_1)] g(\lambda)$$

verifies equality

$$(\lambda - T) h(\lambda) = (\lambda - T) f_2(\lambda) \quad \text{on } V,$$

which implies

$$f_2(\lambda) = h(\lambda) \in E(H_1) \cap E(K_1) \quad \text{on } V.$$

Thus, with reference to Part A, (2.4) implies that $x_1(\lambda) \in E(H_1)$ on $V$, and hence $x_1(\lambda_0) \in E(H_1)$. $\lambda_0 \in \Gamma$ being arbitrary, $x \in E(H_1)$ by (2.3).

Inductively, define

$$H_k = \{ \lambda \in C: d(\lambda, H_{k-1}) < \varepsilon/9n \} - \overline{\Delta}, \quad 1 \leq k \leq n.$$  

Then $\{H_k, K_k\}$ covers $\overline{H}_{k-1}$ and $\overline{H}_k \cap \overline{D_i} = \emptyset$, $1 \leq i \leq k$. In view of hypothesis (ii), $E(H_{k-1}) \subset E(H_k) + E(K_k)$, and the hypothesis $x \in E(H_{k-1})$ of the induction gives $x \in E(H_k) + E(K_k)$. As for $k = 1$, by using Part A of the proof of Theorem 2.2 and a conveniently defined function $h: V \to E(H_k) \cap E(K_k)$, we obtain $x \in E(H_k)$. Thus, by the inductive process, we obtain an open set $H_n$ with the properties

$$x \in E(H_n) \quad \text{and} \quad \overline{H}_n \subset H' - \left( \bigcup_{i=1}^n \overline{D}_i \right) \subset H.$$  

$E$ being monotonic, $E(H_n) \subset E(H)$ and hence $x \in E(H)$. Since $\varepsilon$ is arbitrarily small, we may choose it such that $\overline{H} \subset G$. Then $E(H) \subset E(G)$ and hence $x \in E(G)$. Since $x \in \mathcal{X}_T(G)$ is arbitrary, we obtain $\overline{\mathcal{X}_T(G)} \subset E(G)$. \qed

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