

## TORAL ACTIONS ON 5-MANIFOLDS

BY  
 HAE SOO OH

**ABSTRACT.** We are mainly concerned with closed orientable manifolds of dimension 5 supporting effective three-dimensional torus actions. We obtain a complete classification of simply-connected manifolds of this type and a partial classification for the nonsimply-connected case.

**0. Introduction.** Suppose  $M$  is a closed orientable smooth manifold of dimension  $(n + 2)$  with a smooth and effective  $T^n$ -action. Much work on this type of manifold has been done by Orlik and Raymond [17] and Raymond [22] in the case  $n = 1$ , and by Melvin [10], Orlik and Raymond [18, 19], and Pao [21] in the case  $n = 2$ . In a series of papers, we investigate this type of manifold in the case  $n = 3$ . This paper is the first part of the investigation.

Suppose  $T^3$  acts on a 5-manifold  $M$  so that the set  $Q$  of singular orbits is not void. Then by the slice theorem, the orbit space  $M^*$  is a 2-manifold with  $\partial M^* = Q/T^3$ , as shown in Figure 1 below.

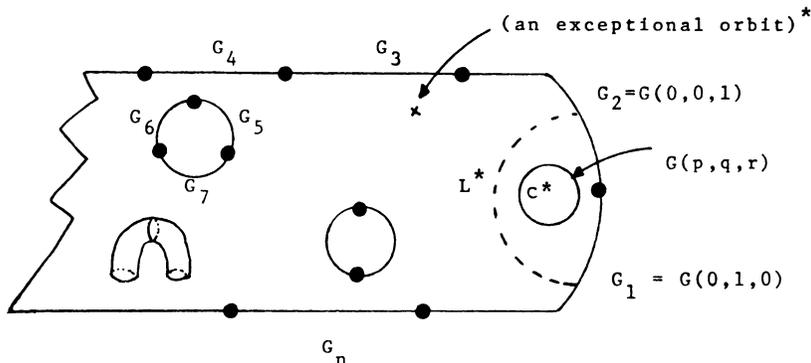


FIGURE 1

Here  $G(p, q, r)$  and  $G_j$  are circle isotropy groups (defined in §1). By applying techniques similar to Orlik and Raymond [19], the 5-manifold  $M$  can be broken down into elementary building blocks  $A(p, q, r)$ ,  $B(a, b, c; p, q, r)$ ,  $C$  and  $D$  (for example, cutting  $M$  along  $L$ , which is  $S^1 \times S^3$ , and adding two copies of  $S^1 \times D^4$

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equivariantly along  $S^1 \times S^3$  results in two new manifolds  $N$  and  $A(p, q, r)$ , where  $N$  has the same orbit space as  $M$  except the boundary component  $c^*$  is deleted; thus  $M$  can be expressed as  $N \#_S A(p, q, r)$ , where the operation “ $\#_S$ ”, which is called an adjacent connected sum in [9], is elaborated in §5). These building blocks are 5-dimensional  $T^3$ -manifolds with orbit spaces as shown in Figure 2.

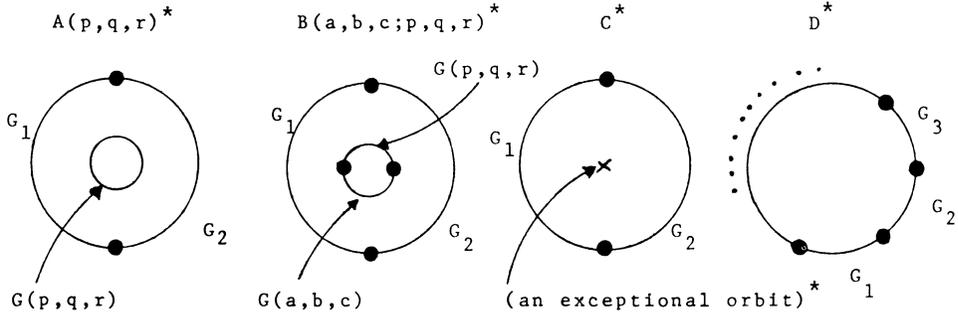


FIGURE 2

It was shown in [15] that  $A(p, q, r)$  [ $B(a, b, c; p, q, r)$ ] can be obtained from  $\#3(S^2 \times S^3) \#(S^1 \times S^4)$  [ $\#5(S^2 \times S^3) \#(S^1 \times S^4)$ ] or

$$(S^2 \tilde{\times} S^3) \# 2(S^2 \times S^3) \#(S^1 \times S^4) \quad [(S^2 \tilde{\times} S^3) \# 4(S^2 \times S^3) \#(S^1 \times S^4)]$$

by a single equivariant surgery of type (4, 2) according as  $w_2 = 0$  or  $w_2 \neq 0$ . Here  $w_2$  is the second Stiefel-Whitney class and  $S^2 \tilde{\times} S^3$  is the nontrivial  $S^3$ -bundle over  $S^2$ . It was also discussed there when  $M$  can be expressed as a connected sum of these building blocks and some well-known manifolds. The 5-manifold  $D$  is studied in this paper.

This paper is organized as follows. In §1, we describe the orbit structure of  $T^n$ -actions on  $(n + 2)$ -manifolds, and also prove some lemmas which are essential tools for our study. We show in §2 that  $\pi_1(D)$  is a finite cycle group. We generalize this to higher dimensions. That is, if  $M$  is an  $(n + 2)$ -dimensional  $T^n$ -manifold with orbit space a 2-disk and no exceptional orbits then  $\pi_1(M)$  is a finite abelian group with at most  $(n - 2)$  generators.

Suppose  $k$  is the number of orbits of type  $T^1$ . Then it is shown in §§3 and 4 that  $D$  is a 5-dimensional lens space if  $k = 3$  and  $D$  is a double mapping cylinder of two circle bundles over lens spaces if  $k = 4$ . Furthermore, if  $k = 4$  and there are two equal circle isotropy groups, then  $D$  is either  $S^2 \times L(p, q)$  or  $S^2 \tilde{\times} L(p, q)$ , the nontrivial  $S^2$ -bundle over a lens space  $L(p, q)$ .

Suppose that an  $(n + 2)$ -manifold  $M$  with an effective  $T^n$ -action is simply connected and that a singular orbit exists. Then there exist no exceptional orbits [2, 18] and the orbit space  $M^*$  is a 2-disk with boundary. In §5, we give examples to show that the classification theorems for simply-connected  $(n + 2)$ -dimensional  $T^n$ -manifolds in [8 and 9] are not valid, and then we obtain a complete classification of simply-connected 5-manifolds with  $T^3$ -actions.

The case of  $n = 3$  has some similarity to the work of Orlik and Raymond. Hence one might expect that 5-manifolds with 3, 4 and 5 orbits of type  $T^1$ , respectively, would be building blocks for the manifold  $D$ . However, we provide examples to show that this is actually not the case.

Throughout this paper, with the exception of §1, manifolds shall always mean closed connected oriented 5-manifolds with orientation-preserving  $T^3$ -actions such that orbit spaces are 2-disks with boundaries, there exist no exceptional orbits, and the isotropy groups span  $T^3$ . All actions are assumed to be smooth and effective. Unless otherwise indicated, the coefficients in all (co)homology will be the integers  $\mathbf{Z}$ .

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I am also grateful to the referee for his various suggestions to reorganize this paper and for suggesting short proofs of Lemmas (3.1) and (5.4).

**1. Definitions and preliminary results.** We shall be concerned with closed orientable smooth manifolds  $M$  of dimension  $(n + 2)$  on which  $T^n$  acts smoothly and effectively,  $n \geq 3$ .

Let  $p: \mathbf{R}^n \rightarrow T^n$  be the universal covering projection defined by  $p(x_1, x_2, \dots, x_n) = (e^{2\pi x_1}, \dots, e^{2\pi x_n})$ . Suppose  $G$  is a circle subgroup of  $T^n$ . Then each component of  $p^{-1}(G)$  is a line containing at least two lattice points. For example, the projection of a line of irrational slope in  $\mathbf{R}^2$  cannot be a circle subgroup of  $T^2$ . Hence it is natural to parameterize a circle subgroup of  $T^n$  by  $G(a_1, a_2, \dots, a_n) = \{(a_1 t, a_2 t, \dots, a_n t), \text{ mod } \mathbf{Z}^n \mid 0 \leq t < 1\}$ . Here  $a_1, a_2, \dots$ , and  $a_n$  are relatively prime integers. For example, we do not allow  $G(0, 2, 0)$ , etc. since  $\text{gcd}(0, 2, 0) = 2 > 1$ .

Throughout this paper, we assume that for any subset  $X$  of  $M$ ,  $X^*$  denotes its image in the orbit space  $M^*$  under the orbit map  $q: M \rightarrow M^*$ . Furthermore, if we are given a set  $X^*$  in  $M^*$ , we let  $X = q^{-1}(X^*)$ .

Let us recall that if a compact Lie group  $K$  acts smoothly on a manifold  $M$ , there exists a linear tube about each orbit, and the slice representations are faithful if the action is effective and  $K$  is abelian. We let  $F = F(K, M)$  be the fixed point set of  $M$ ,  $E$  the union of exceptional orbits,  $P$  the union of the principal orbits, and  $Q$  the union of the singular orbits.  $K(x)$  is the orbit containing  $x$ , and  $K_x$  is the isotropy group at  $x$ .

(1.1) LEMMA. *Suppose  $T^n$  acts on an  $(n + 2)$ -manifold  $M$ ,  $n \geq 3$ . Then we have*

- (i)  $F = \emptyset$ .
- (ii) *The number of orbits of type  $T^{n-2}$  is finite.*
- (iii)  $E^*$  *is a finite set.*
- (iv) *For any nontrivial finite subgroup  $H$ , neither  $T^1 \times H$  nor  $T^2 \times H$  can be an isotropy group.*
- (v)  $T^k$ ,  $k \geq 3$ , *cannot be an isotropy group.*

PROOF. (i) and (v) are immediate from Pak [20] and the slice theorem.

The linear slice of an orbit of type  $T^{n-2}$  is a 4-disk  $D^4$ , on which  $T^2$  acts orthogonally. Hence an invariant neighborhood of an orbit of type  $T^{n-2}$  does not contain any other orbit of the same type and hence the compactness of  $M$  yields (ii).

Suppose  $T^n(x)$  is an exceptional orbit. Then  $(T^n)_x$  is a subgroup of  $SO(2) \subset T^n$ . Hence  $(T^n)_x$  is a finite cyclic group  $\mathbf{Z}/q$ . In other words,  $\mathbf{Z}/p \times \mathbf{Z}/q$  cannot be an isotropy group unless  $p$  and  $q$  are relatively prime. Again, the compactness of  $M$  and the slice theorem imply (iii).

An orbit of type  $T^1 \times H$  has a slice  $D^3$  on which  $T^1 \times H$  acts orthogonally. Since the slice representation in this context is faithful,  $T^1 \times H \subset SO(3)$ . But  $T^1 \times H$  cannot be a subgroup of  $SO(3)$  unless  $H$  is a trivial group. Similarly, an isotropy group  $T^2 \times H$  can be regarded as a subgroup of  $SO(4)$ . But we can easily see that  $SO(4)$  does not contain  $(SO(2) \times SO(2)) \times$  (a nontrivial group) as a subgroup. ■

By using the slice theorem, we can easily prove that if  $Q \neq \phi$  then  $Q^*$  is the boundary of a 2-manifold  $M^*$ . Moreover, if  $M$  has an orbit of type  $T^{n-2}$ , then it has at least two orbits of type  $T^{n-2}$ . It is obvious that any two adjacent circle isotropy groups on the boundary of  $M^*$  have trivial intersection.

By the *determinant of  $n$  circle subgroups*  $G(a_{11}, a_{12}, \dots, a_{1n}), \dots$ , and  $G(a_{n1}, a_{n2}, \dots, a_{nn})$  of  $T^n$ , we mean the determinant of the  $n \times n$  matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

(1.2) LEMMA. Two circle subgroups  $G(a_1, a_2, \dots, a_n)$  and  $G(b_1, b_2, \dots, b_n)$  of  $T^n$  have trivial intersection if and only if there exist  $G_i \in \mathbf{Z}^n$ ,  $i = 3, \dots, n$ , such that the determinant of  $G(a_1, a_2, \dots, a_n), G(b_1, b_2, \dots, b_n), G_3, \dots, G_n$  is  $\pm 1$ .

(1.3) COROLLARY.  $G(p, q, r) \cap G(0, 1, 0) = 1$  if and only if  $\gcd(p, r) = 1$ .

PROOF. If  $\gcd(p, r) = 1$ , then there exist integers  $x$  and  $y$  such that  $xr - yp = 1$ . Hence two circle groups have trivial intersection, since we have

$$\det \begin{pmatrix} p & 0 & x \\ q & 1 & 0 \\ r & 0 & y \end{pmatrix} = -1.$$

Conversely, if  $\gcd(p, r) = d > 1$ , then  $p = p'd$  and  $r = r'd$  for some integers  $p'$  and  $r'$ . Hence for any  $(x, y, z) \in \mathbf{Z}^3$ ,

$$\det \begin{pmatrix} p & 0 & x \\ q & 1 & y \\ r & 0 & z \end{pmatrix} = d(p'z - xr') \neq \pm 1. \quad \blacksquare$$

Suppose the determinant of  $n$  circle subgroups of  $T^n$ ,  $G(a_{11}, a_{12}, \dots, a_{1n}), \dots, G(a_{n1}, \dots, a_{nn})$ , is not zero. Then the  $n$  vectors  $(a_{11}, a_{12}, \dots, a_{1n}), \dots, (a_{n1}, \dots, a_{nn})$  span  $\mathbf{R}^n$ . Hence the  $n$  circle subgroups span  $T^n$ . The condition for  $n$  circle subgroups to be the generators of  $T^n$  is the following.

(1.4) LEMMA. *The  $n$  circle subgroups generate  $T^n$ , that is,  $T^n = G(a_{11}, a_{12}, \dots, a_{1n}) \times \dots \times G(a_{n1}, a_{n2}, \dots, a_{nn})$ , if and only if the determinant of  $n$  circle subgroups is  $\pm 1$ .*

One of the most important tools for investigating codimension two toral actions is the cross-sectioning theorem. Orlik and Raymond [18] showed the existence of a cross section for an orbit map in the case of  $n = 2$  under the conditions  $E = \emptyset$  and  $Q \neq \emptyset$ . But one can generalize this to the case of  $n \geq 3$  by applying a similar technique.

(1.5) THEOREM (ORLIK AND RAYMOND). *An orbit map  $M^{n+2} \rightarrow M^{n+2}/T^n$  has a cross section, provided that  $Q \neq \emptyset$  and  $E = \emptyset$ .*

Suppose  $M$  and  $M'$  are  $(n + 2)$ -manifolds with  $T^n$ -actions and  $s$  and  $s'$  are cross sections to each orbit map. If  $f^*$  is a weight-preserving diffeomorphism from  $M^*$  onto  $(M')^*$ , then there is an equivariant diffeomorphism  $f$  which covers  $f^*$ . Furthermore, by applying a similar technique to Orlik and Raymond [19] for exceptional orbits, we can prove an equivariant classification theorem.

(1.6) THEOREM. *Two  $T^n$ -manifolds  $M$  and  $M'$  of dimension  $(n + 2)$  are equivariantly diffeomorphic if and only if there is a weight preserving diffeomorphism from  $M^*$  onto  $(M')^*$ .*

(1.7) REMARK. Suppose  $T^3$  acts on a 5-manifold  $M$  so that the isotropy groups span  $T^k$ ,  $k < 3$ . Then, by (1.6),  $M$  is  $N \times T^1$ , where  $N$  is a 4-manifold with a  $T^2$ -action. Furthermore, if we assume that  $M^*$  is a closed 2-disk and  $E = \emptyset$ , then  $N$  is  $S^3 \times S^1$ , or a connected sum of copies of  $S^4$ ,  $\pm CP^2$ , and  $S^2 \times S^2$  (see Orlik and Raymond [18]).

**2. Fundamental groups.** Recall the orbit space of a manifold under consideration was assumed to be a 2-manifold  $D^*$  described in §0. In this section, we show that the fundamental group of a 5-manifold  $M (= D)$  with a  $T^3$ -action is generated by any orbit of type  $T^1$ .

By the slice theorem, an invariant tubular neighborhood of an orbit of type  $T^1$  is a  $D^4$ -bundle over  $T^1$  with structure group  $T^2$ . It follows from (1.6) that this bundle is trivial (i.e.  $D^4 \times T^1$ ).

(2.1) THEOREM. *If  $M$  is a 5-manifold supporting a  $T^3$ -action, then  $\pi_1(M)$  is a finite cyclic group generated by any orbit of type  $T^1$ .*

PROOF. If  $\alpha$  is an element of  $\pi_1(M)$ , then by the Whitney embedding theorem, there is an embedding  $f: S^1 \rightarrow M$  which represents  $\alpha$ . By the general position theorem,  $f$  is homotopic to an embedding  $g: S^1 \rightarrow P$ , the union of principal orbits. Hence  $j_\#: \pi_1(P) = \pi_1(D^2 \times T^3) \rightarrow \pi_1(M)$  is surjective, where  $j_\#$  is the homomorphism induced by inclusion.

Since it was assumed that the isotropy groups span  $T^3$ , there exist three distinct circle isotropy groups  $G_1 = G(1, 0, 0)$ ,  $G_2 = G(0, 1, 0)$  and  $G_3 = G(a, b, c)$ , whose determinant is not zero. Let  $[G_i x]$  be the homotopy class represented by the circle  $G_i x$ ,  $x \in P$ . Then  $G_i x$  bounds a disk in  $M$  and hence  $f_\#([G_i x]) = 1$  for  $i = 1, 2, 3$ .

We thus have

$$\pi_1(M) = \mathbf{Z}^3 / (\text{kernel } j_{\#}) \subset \mathbf{Z}^3 / \langle [G_1x], [G_2x], [G_3x] \rangle. \quad \blacksquare$$

(2.2) COROLLARY. *If there exist three distinct circle isotropy groups whose determinant is  $\pm 1$ , then  $M$  is simply connected.*

PROOF. Suppose  $G(a_1, b_1, c_1)$ ,  $G(a_2, b_2, c_2)$  and  $G(a_3, b_3, c_3)$  are three circle isotropy groups whose determinant is  $\pm 1$ . Then it follows from (1.4) that  $G(a_1, b_1, c_1) \times G(a_2, b_2, c_2) \times G(a_3, b_3, c_3)$  is isomorphic to  $T^3$ , and hence  $\langle [G(a_1, b_1, c_1)x], [G(a_2, b_2, c_2)x], [G(a_3, b_3, c_3)x] \rangle$  is isomorphic to  $\mathbf{Z}^3$ , where  $x \in P$ . Therefore  $M$  is simply connected.  $\blacksquare$

(2.3) COROLLARY. *If  $M'$  is the manifold obtained by an equivariant surgery along an orbit of type  $T^1$ , then  $M'$  is simply connected.*

PROOF. Suppose the orbit space  $M^*$  is as shown in Figure 3.

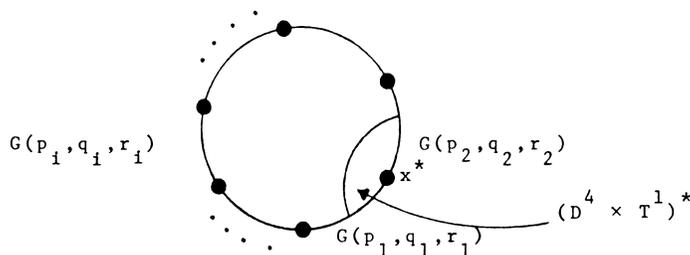


FIGURE 3

Since  $G(p_1, q_1, r_1) \cap G(p_2, q_2, r_2) = 1$ , we can choose relatively prime integers  $a, b$  and  $c$  so that

$$\det A = \det \begin{pmatrix} p_1 & p_2 & a \\ q_1 & q_2 & b \\ r_1 & r_2 & c \end{pmatrix} = 1.$$

Parameterize  $S^5$  by  $\{(z_1, z_2, z_3) \in \mathbf{C}^3 \mid z_1\bar{z}_1 + z_2\bar{z}_2 + z_3\bar{z}_3 = 1\}$  and define a  $T^3$ -action  $\theta$  on  $S^5$  by

$$\theta((p, q, r), (z_1, z_2, z_3)) = (z_1e^{2\pi\bar{p}i}, z_2e^{2\pi\bar{q}i}, z_3e^{2\pi\bar{r}i})$$

where

$$\begin{pmatrix} \bar{p} \\ \bar{q} \\ \bar{r} \end{pmatrix} = A^{-1} \begin{pmatrix} p \\ q \\ r \end{pmatrix} \quad \text{and} \quad (p, q, r) \in T^3.$$

Then the orbit space  $S^5/T^3$  is as shown in Figure 4.

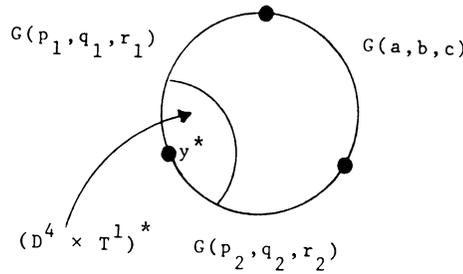


FIGURE 4

Doing an equivariant surgery along an orbit  $T^3(x)$  of type  $T^1$  is just cutting out the interiors of invariant tubular neighborhoods of  $T^3(x)$  and  $T^3(y)$  and pasting along their boundaries. Hence  $M'/T^3$  is as shown in Figure 5. Since the determinant of  $G(p_1, q_1, r_1)$ ,  $G(p_2, q_2, r_2)$  and  $G(a, b, c)$  is 1, by (2.2),  $M$  is simply connected. ■

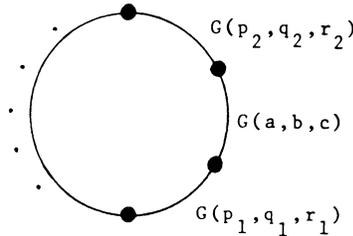


FIGURE 5

(2.4) EXAMPLE. Suppose  $T^3$  acts on a manifold  $M$  of dimension 5 smoothly and effectively so that  $M^*$  is as shown in Figure 6. The orbit  $T^3(x)$  of type  $T^1$  is homeomorphic to  $T^3/(G_1 \times G_2) \approx G(0, 0, 1)$ . By (2.1),  $\pi_1(M)$  is generated by  $[T^3(x)]$  and so is finite cyclic.

$$G_1 = G(1, 0, 0); \quad G_2 = G(0, 1, 0); \quad G_3 = G(2, 0, 3);$$

$$G_4 = G(7, 11, 77); \quad G_5 = G(4, 5, 6).$$

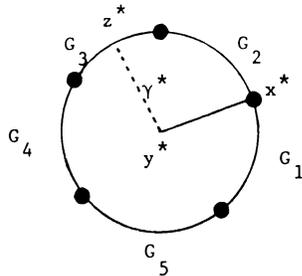


FIGURE 6

Since  $(2, 0, 3) = 2(1, 0, 0) + 0(0, 1, 0) + 3(0, 0, 1)$  as vectors in  $\mathbb{R}^3$ , we have

$$f^\gamma([G(2, 0, 3)]) = [G(2, 0, 3)y] = 2[G(1, 0, 0)y] + 3[G(0, 0, 1)y]$$

where  $f^\gamma: (T^3, 1) \rightarrow (M, y)$  is the evaluation map defined by  $f^\gamma(t) = ty$ .

Choose a path  $\gamma: [0, 1] \rightarrow M$  connecting  $y$  to  $z$ .  $G(2, 0, 3)y$  is homotopic to  $G(2, 0, 3)z = \{z\}$  by the homotopy  $H: G(2, 0, 3) \times I \rightarrow M$  defined by

$$H(g, t) = f^{\gamma(t)}(g) = g\gamma(t).$$

Hence  $[G(2, 0, 3)y] = 0$ . By the same reason,  $[G(1, 0, 0)y] = 0$  and hence  $3[G(0, 0, 1)y] = 0$ . This implies that the generator  $[T^3(x)] = [G(0, 0, 1)y]$  has order dividing 3.

Similarly, the relation  $(7, 11, 77) = 7(1, 0, 0) + 11(0, 1, 0) + 77(0, 0, 1)$  implies that the generator  $[G(0, 0, 1)y]$  has order dividing 77.

Since  $\text{gcd}(3, 77) = 1$ ,  $M$  is simply connected. But the determinant of any three isotropy groups of  $G_1, G_2, G_3, G_4$  and  $G_5$  is different from  $\pm 1$ .

This example shows that the converse of (2.2) is not true in general. The existence of a 5-manifold  $M$  with the given orbit data is a consequence of the cross-sectioning theorem. We shall discuss it in (4.7).

(2.5) REMARK. By the techniques similar to those in this section, (2.1) and (2.2) can be extended to higher-dimensional manifolds.

(2.1)' If  $M$  is an  $(n + 2)$ -manifold with a  $T^n$ -action, then  $\pi_1(M)$  is a finite abelian group with at most  $(n - 2)$  generators.

(2.2)' If there exist  $n$ -distinct circle isotropy groups whose determinant is  $\pm 1$ , then  $M$  is simply connected.

**3. 5-manifolds with three orbits of type  $T^1$ .** It was assumed that the isotropy groups span  $T^3$ . Thus the number of orbits of type  $T^1$  is at least three. In this section, we show that a 5-manifold with three orbits of type  $T^1$  is a 5-dimensional lens space  $L(p, q, r)$ .

(3.1) LEMMA. *A 5-dimensional lens space  $L(p, q, r)$  admits a  $T^3$ -action with orbit space*

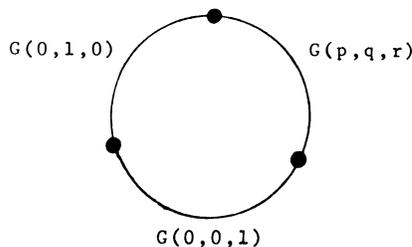


FIGURE 7

PROOF. Define a  $T^3$ -action on  $S^5$  by the matrix,

$$\begin{pmatrix} 1 & 0 & 0 \\ q & 1 & 0 \\ r & 0 & 1 \end{pmatrix}^{-1}$$

(see 2.3), and set  $K = \{(j/p, 0, 0) \mid 0 \leq j < p\} \subset T^3$ . Factoring  $S^5$  by the  $K$ -action inside the  $T^3$ -action, we obtain a 5-dimensional lens space  $L(p, -q, -r)$

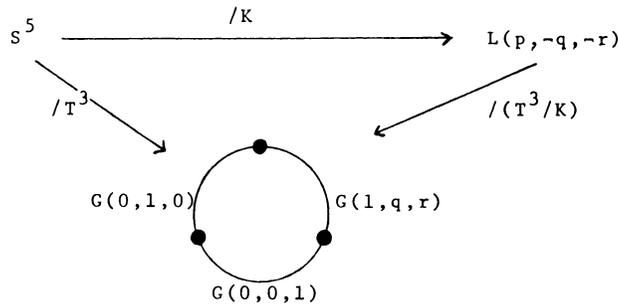


FIGURE 8

Now the isomorphism  $T^3/K \rightarrow T^3$  defined by  $(x, y, z) + K \rightarrow (px, y, z)$  gives rise to an action of  $T^3$  on  $L(p, -q, -r)$  with orbit space as shown in Figure 7. Since  $L(p, -q, -r)$  is homeomorphic to  $L(p, q, r)$  we have the conclusion. ■

Suppose a 5-manifold  $M$  has a  $T^3$ -action with orbit space

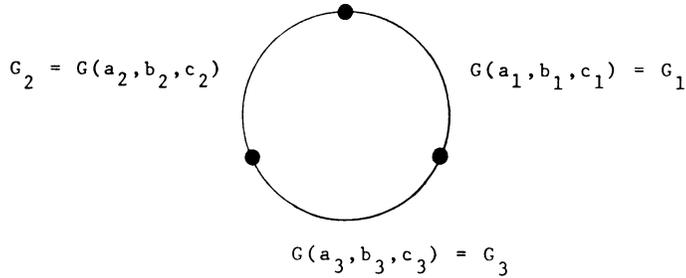


FIGURE 9

Then there exists an automorphism  $f$  of  $T^3$  which maps  $G_1, G_2$  and  $G_3$  to  $G(p, q, r), G(0, 1, 0)$  and  $G(0, 0, 1)$ , respectively. Hence (3.1) and (1.6) yield the following.

(3.2) THEOREM. A 5-manifold  $M$  with three orbits of type  $T^1$  is a lens space  $L(p, q, r)$ , where  $p, q, r$  are the integers described above.

4. 5-manifolds with four orbits of type  $T^1$ . In this section, we determine 5-manifolds with four orbits of type  $T^1$ . Reparameterizing (if necessary), we may assume the orbit space of this manifold is as shown in Figure 10.

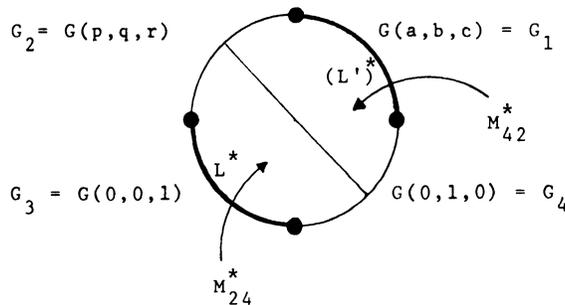


FIGURE 10

(4.1) LEMMA. *The space  $L$  sitting over  $L^*$  is the lens space  $L(p, q)$  (notice:  $p = -\det(G_2, G_3, G_4)$ ) and the space  $M_{24}$  sitting over  $M_{24}^*$  is a  $D^2$ -bundle over  $L$  with structure group  $SO(2)$ . Hence  $M$  is a double mapping cylinder over two circle bundles over lens spaces  $L$  and  $L'$ .*

PROOF. Let  $L = q^{-1}(L^*)$ , where  $q$  is the orbit map. Then  $L$  is a 3-manifold with a noneffective  $T^3$ -action and  $L/T^3$  is

$$G_2 \times G_3 \bullet \xrightarrow{G_3} \bullet G_3 \times G_4.$$

Factoring  $T^3$  by  $G_3$ ,  $L$  is also a  $T^2$ -manifold with orbit space

$$G(p, q) \bullet \xrightarrow{\quad} \bullet G(0, 1).$$

Hence it follows from [18] that  $L$  is a lens space  $L(p, q)$ .

Since the action was assumed to be smooth, it is possible to choose a Riemannian metric so that  $T^3$  acts as a group of isometries. By Bredon [2, VI, 2.2],  $M_{24}$  is an invariant tubular neighborhood of  $L$ . Now  $G(0, 0, 1)$  acts on the normal bundle freely away from  $L$ . Thus  $M_{24}$  is a 2-disk bundle over  $L$  with structure group  $G_3$ . ■

By using the techniques similar to those in (2.4) we have the following.

(4.2) COROLLARY. *Suppose  $d_i$  is the determinant of the adjacent triple  $G_i, G_{i+1}, G_{i+2}$ , and  $n$  is the number of orbits of type  $T^1$ . If  $\pi_1(M) = \mathbf{Z}/m$ , then  $\gcd(d_1, \dots, d_n) \geq m$  and if  $\gcd(d_1, \dots, d_n) = 1$ , then  $M$  is simply connected.*

Suppose the number of orbits of type  $T^1$  is four. Then  $\partial M^*$  is divided into four arcs by the orbits of type  $T^1$ . We may assume that the orbit data are as in Figure 11.

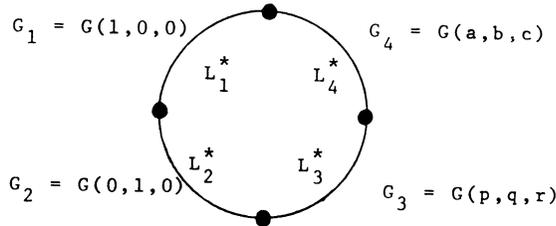


FIGURE 11

If  $L_2$  is  $S^1 \times S^2$ , then by (4.1),  $\det(G_1, G_2, G_3) = r = 0$  and hence by (1.3),  $p$  should be  $\pm 1$ . Since it was assumed that the isotropy groups span  $T^3$ ,  $c$  must not be zero. If  $L_1$  is also  $S^1 \times S^2$  then by (4.1),  $\det(G_1, G_2, G_4) = c$  is zero which is a contradiction. By a similar reason,  $L_3$  cannot be  $S^1 \times S^3$ . Thus we have

(4.3) LEMMA. *If one of four arcs is  $(S^1 \times S^2)^*$ , then neither of the two arcs next to this is  $(S^1 \times S^2)^*$ .*

(4.4) LEMMA. (i) *There exist two equal circle isotropy groups if and only if two arcs out of four correspond to  $S^1 \times S^2$ .*

(ii) *If there are two equal circle isotropy groups, then in fact  $M$  is an  $S^2$ -bundle over a lens space ( $\neq S^1 \times S^2$ ) with structure group  $SO(2)$ .*

PROOF. (i) Suppose two arcs correspond to  $S^1 \times S^2$ . Then it follows from (4.3) that either  $L_1$  and  $L_3$  or  $L_2$  and  $L_4$  are  $S^1 \times S^2$ . Say  $L_2$  and  $L_4$  are  $S^1 \times S^2$ . Then the orbit space  $M^*$  is as shown in Figure 12.

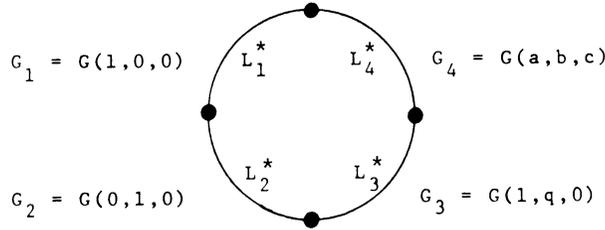


FIGURE 12

Since  $L_4$  is  $S^1 \times S^2$ , by (4.1),  $\det(G_3, G_4, G_1) = qc = 0$ . Since the isotropy groups span  $T^3$ ,  $c$  is not zero and hence  $q = 0$ . So  $G_1 = G(1, 0, 0) = G_3$ . Conversely, if  $G_1 = G_3$ , then  $\det(G_1, G_2, G_3) = 0$  and  $\det(G_3, G_4, G_1) = 0$ . Hence  $L_2$  and  $L_4$  are  $S^1 \times S^2$ .

(ii) If  $G_1$  is equal to  $G_3$  then by (i),  $L_2$  and  $L_4$  are  $S^1 \times S^2$  and hence by (4.3), neither  $L_1$  nor  $L_3$  is  $S^1 \times S^2$ . By (4.1),  $L_1$  is a lens space  $L(m, n)$  and  $L_2$  is  $-L(m, n)$ . Hence  $M$  is an  $S^2$ -bundle over  $L(m, n)$ ,  $m \neq 0$ . ■

Let  $\xi$  be an  $S^2$ -bundle over a lens space  $X$  with structure group  $SO(2)$ . Then we may consider  $\xi$  as an  $SO(3)$ -bundle. Since the Euler class of  $\xi$  is zero and  $w_1(\xi) \in H^1(X; \mathbf{Z}/2)$  is 0,  $w_2(\xi)$ , the second Stiefel-Whitney class, is the only obstruction class. Since  $\pi_i(B\text{spin}(3)) = \pi_{i-1}(\text{spin}(3))$  and  $\pi_{i-1}(\text{spin}(3)) = \pi_{i-1}(S^3) = 0$  for  $i \leq 3$ , we have  $[X, B\text{spin}(3)] = 0$ , since  $\dim X \leq 3$ . It is known that  $\xi$  admits a spin structure if and only if  $w_2(\xi) = 0$ .

Since  $\dim X \leq 3$ , by attaching cells to  $BSO(3)$ , we can construct a  $K(\mathbf{Z}/2, 2)$ -space such that  $[X, BSO(3)] \rightarrow [X, K(\mathbf{Z}/2, 2)]$  is a bijection. That is, we have  $[X, BSO(3)] \approx [X, K(\mathbf{Z}/2, 2)] = H^2(X; \mathbf{Z}/2) = 0$  or  $\mathbf{Z}/2$ . Hence there is at most one nontrivial  $S^2$ -bundle over  $X$ . We claim that  $\xi$  is not always trivial. In the Bockstein sequence associated with

$$0 \rightarrow \mathbf{Z} \xrightarrow{\times 2} \mathbf{Z} \rightarrow \mathbf{Z}/2 \rightarrow 0,$$

$$H^2(X; \mathbf{Z}) \rightarrow H^2(X; \mathbf{Z}) \rightarrow H^2(X; \mathbf{Z}/2) \rightarrow H^3(X; \mathbf{Z}) \xrightarrow{\alpha} H^3(X; \mathbf{Z}),$$

the homomorphism  $\alpha$  is injective. So if  $w_2(\xi) \neq 0$ , then there is an  $f \in H^2(X; \mathbf{Z})$  such that  $f/(\mathbf{Z}/2) = w_2(\xi) \neq 0$ . In other words,  $\xi$  has a nonzero chern class and hence  $\xi$  is nontrivial. Thus we have

(4.5) THEOREM. *If a 5-manifold  $M$  has a  $T^3$ -action such that the number of orbits of type  $T^1$  is four and there are two equal circle isotropy groups, then  $M$  is either  $S^2 \times L(m, n)$  or  $S^2 \tilde{\times} L(m, n)$ . Here  $S^2 \tilde{\times} L(m, n)$  is the nontrivial  $S^2$ -bundle over the lens space  $L(m, n)$  and  $m, n$  are integers described in (4.1) and (4.4).*

5-manifolds with  $T^3$ -actions have some similarities to those studied by Orlik and Raymond [18, 19] in dimension 4. Hence, one might expect that the 5-manifolds

with 3, 4 and 5 orbits of type  $T^1$ , respectively, could be elementary building blocks for a 5-manifold with orbit space  $D^*$  from §0. But the following example shows that this is not the case.

(4.6) EXAMPLE. Suppose  $M$  is a  $T^3$ -manifold of dimension 5 with orbit space as shown in Figure 14. Then  $M$  is simply connected and any two nonadjacent isotropy groups have nontrivial intersection. Hence, even if  $M$  is simply connected,  $M$  cannot be broken down into simpler pieces.

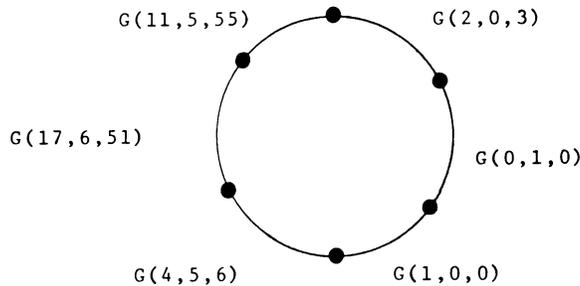


FIGURE 13

PROOF. Since  $\gcd(55, 3) = 1$ , by an argument similar to (2.4),  $M$  is simply connected. By (1.2), any two circle isotropy groups  $G(p, q, r)$  and  $G(p', q', r')$  have trivial intersection if and only if there are relatively prime integers  $x, y, z$  such that

$$\det \begin{pmatrix} p & p' & x \\ q & q' & y \\ r & r' & z \end{pmatrix} = 1.$$

This condition is equivalent to  $\gcd(qr' - rq', pr' - rp', pq' - qp') = 1$ . However we can see that  $\gcd(qr' - rq', pr' - rp', pq' - qp') > 1$  unless  $G(p, q, r)$  and  $G(p', q', r')$  are adjacent to each other. Hence any arc connecting two points which are not in adjacent arcs in  $M^*$  does not correspond to  $S^1 \times S^3$  in  $M$ . ■

The orbit data of (4.6) are legally weighted (that is, any two adjacent circle isotropy groups have trivial intersection). Hence, the following remark implies the existence of a  $T^3$ -manifold  $M$  with the given orbit data.

(4.7) REMARK. In [7], a simply-connected manifold of dimension  $(n + 2)$ , supporting an effective  $T^n$ -action, was constructed. Their construction is beautiful but complicated. We now give a simpler construction of this type of manifold. Let  $T^n = G(1, 1, \dots, 0) \times G(0, 1, 0, \dots, 0) \times \dots \times G(0, 0, \dots, 0, 1)$  and  $D^2$  be a closed 2-disk. Divide  $\partial D^2$  into  $n$  arcs  $L_1^*, L_2^*, \dots, L_n^*$ . Let  $M$  be the quotient space  $(T^n \times D^2)/\sim$  by the relation “ $\sim$ ” on  $T^n \times D^2$  defined by

$$\begin{cases} (g, x) \sim (0, x) & \text{if } (g, x) \in G_i \times L_i^*, \\ (g, x) \sim (0, x) & \text{for } g \in G_i \times G_{i+1} \text{ if } \{x\} = L_i^* \cap L_{i+1}^*, \\ (g, x) \sim (g, x) & \text{otherwise, where } G_i = G(0, \dots, 0, 1^i, 0, \dots, 0). \end{cases}$$

Then  $M$  admits an effective  $T^n$ -action in a natural way and the orbit space  $M^*$  with respect to the action is shown in Figure 14. (1.6) can be applied to show that  $M$  is actually an  $(n + 2)$ -manifold. Moreover, (2.5) yields the simple connectivity of  $M$ .

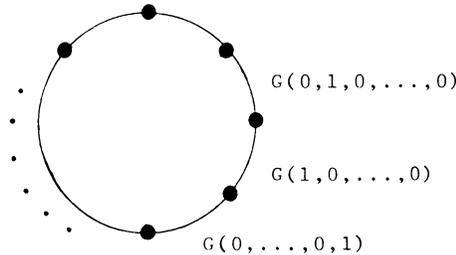


FIGURE 14

By using techniques similar to this, one can show that for any legally weighted orientable 2-manifold  $M^*$  there is an  $(n + 2)$ -dimensional  $T^n$ -manifold  $M$  with orbit space  $M^*$ .

**5. Simply-connected 5-manifolds with  $T^3$ -actions.** McGavran claimed to have classified simply-connected  $(n + 2)$ -manifolds supporting effective  $T^n$ -action, for  $n = 3$  and 4 [8] and for  $n \geq 5$  in [9]. In this section, we shall give examples to show that there are some gaps in his claims and then we give a complete classification theorem for simply-connected 5-manifolds with  $T^3$ -actions. A classification theorem for simply-connected 6-manifolds with  $T^4$ -actions was proved in [16].

The manifolds in question do not, in general, satisfy his lemma ((2.1) of [8]). Even if one restricts to manifolds nice enough to satisfy his lemma, it will, in general, be impossible to obtain his conclusions for several reasons. First, the simple-connectivity is not inherited in the inductive step. Secondly, even if it is inherited, an equivariant replacement may create, contrary to his claims, a manifold with nonvanishing second Stiefel-Whitney class.

For the sake of completeness we prove the following.

(5.1) LEMMA. *Suppose the second Stiefel-Whitney class of a 5-manifold  $M$  is not zero and let  $M'$  be the manifold resulting from a surgery of type (2, 4) in the sense of Milnor [12 or 13]. Then  $w_2(M')$  is also not zero.*

PROOF. Let  $W$  be the trace of the surgery. Then it follows the duality that

$$H^2(W, M') = H_4(W, M) = H_4(M \cup 2\text{-cell}, M) = 0.$$

Hence the cohomology exact sequence for  $(W, M')$  gives rise to the injectivity of  $j^*$ :  $H^2(W) \rightarrow H^2(M')$ . Since  $w_2(W) \neq 0$ , by the naturality of Stiefel-Whitney class,  $w_2(M') = j^*(w_2(W)) \neq 0$ . ■

The classification theorem for simply-connected 5-dimensional  $T^3$ -manifolds in [8] does not include any manifolds with nonvanishing second Stiefel-Whitney class. Contrary to this we have the following.

(5.2) THEOREM. *There exist simply-connected 5-dimensional  $T^3$ -manifolds with non-vanishing second Stiefel-Whitney classes.*

PROOF. From the definition of the 5-dimensional lens space  $L(2k, 1, 1)$ , we have a commutative diagram

$$\begin{array}{ccc}
 S^5 & \xrightarrow{/(\mathbf{Z}/2)} & \mathbf{R}P^5 = L(2, 1, 1) \\
 \searrow /(\mathbf{Z}/2k) & & \downarrow /(\mathbf{Z}/k) \\
 & & L(2k, 1, 1)
 \end{array}$$

Since  $w_2(\mathbf{R}P^5) \neq 0$ , we have  $w_2(L(2k, 1, 1)) \neq 0$ . It follows from (3.1) that  $L(2k, 1, 1)$  has a  $T^3$ -action with orbit space shown in Figure 15. Suppose  $M'$  is the manifold resulting from equivariant surgery along an orbit  $T^3(x)$  of type  $T^1$ . Then  $M'$  has three circle isotropy groups whose determinant is  $\pm 1$  (see the proof of (2.3)). Hence by (2.3),  $M'$  is simply connected and by (5.1),  $M'$  has nonvanishing second Stiefel-Whitney class. ■

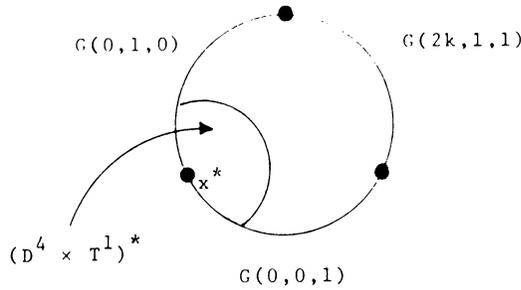


FIGURE 15

Suppose  $M_1$  and  $M_2$  are 5-manifolds. Then the connected sum  $M_1 \# M_2$  is obtained by removing the interior of a 5-ball from each and attaching the resulting manifolds together along their boundaries. By modifying Goldstein-Lininger's construction on manifolds [6], we now introduce another construction as follows. Suppose  $M_1$  and  $M_2$  are  $T^3$ -manifolds. The interior of an invariant tubular neighborhood  $S^1 \times D^4$  of an orbit of type  $T^1$  can be removed from  $M_1$  and  $M_2$ . The resulting manifolds can be attached equivariantly along  $S^1 \times S^3$  to obtain a new manifold  $M_1 \#_S M_2$  with a  $T^3$ -action.

Now we give examples which show that the crucial lemma [8, Lemma 2.1] is not true.

(5.3) COUNTEREXAMPLES. (i) By (3.1), lens spaces  $L(4, 1, 1)$  and  $L(7, 1, 1)$  have  $T^3$ -actions with respective orbit spaces shown in Figure 16. Let  $M = L(4, 1, 1) \#_S L(7, 1, 1)$ . Then the orbit space  $M^*$  is as shown in Figure 17. Let  $N_1 = L(4, 1, 1) - \text{int}(D^4 \times T^1)$  and  $N_2 = L(7, 1, 1) - \text{int}(D^4 \times T^1)$ . Then  $\pi_1(N_1) = \{a \mid a^4 = 1\}$  and  $\pi_1(N_2) = \{b \mid b^7 = 1\}$ . Hence by the Van Kampen Theorem,  $M$  is simply connected. Since the determinants of adjacent triples are 3,  $-4$ , 3 and 7, it follows from (1.4) that no adjacent triples generate  $T^3$ .

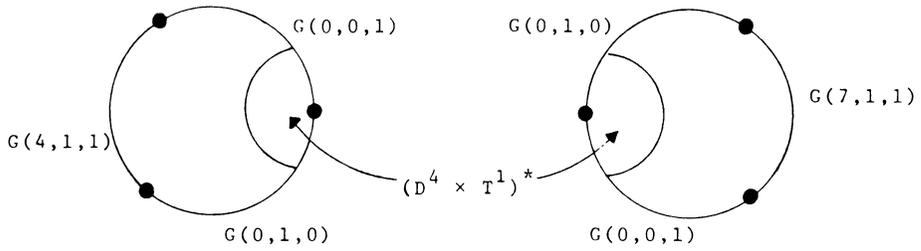


FIGURE 16

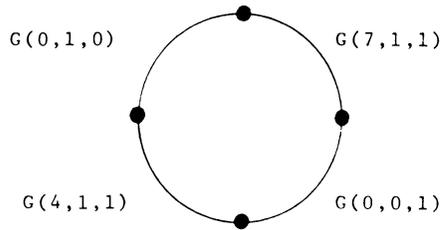


FIGURE 17

(ii) Suppose  $M$  is a 6-dimensional  $T^4$ -manifold with orbit space, shown in Figure 18, where  $G_1 = G(1, 0, 0, 0)$ ,  $G_2 = G(0, 1, 0, 0)$ ,  $G_3 = G(2, 1, 6, 9)$ ,  $G_4 = G(0, 1, 0, 0)$ ,  $G_5 = G(0, 0, 0, 1)$ ,  $G_6 = G(6, 3, 2, 4)$ ,  $G_7 = G(0, 0, 1, 0)$ ,  $G_8 = G(0, 1, 0, 0)$  and  $G_9 = G(6, 3, 11, 44)$ . Then  $M$  is simply connected, since  $\det(G_1, G_2, G_5, G_7) = 1$ . By (1.4),  $G_i, G_{i+1}, G_{i+2}$  generate  $T^3$  if and only if there exists an element  $X$  of  $\mathbf{Z}^4$  such that  $\det(G_i, G_{i+1}, G_{i+2}, X) = \pm 1$ . But  $\det(G_i, G_{i+1}, G_{i+2}, X) \neq \pm 1$  for any  $X \in \mathbf{Z}^4$  and hence there is no adjacent triple which generates  $T^3$ . ■

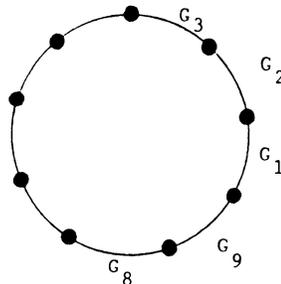


FIGURE 18

In his classification of simply-connected 5-manifolds [1], Barden showed that  $H_2(M)$  and  $i(M)$  determined by  $w_2(M)$  form a complete set of invariants for the diffeomorphism classification. Now we are going to identify simply-connected 5-manifolds with  $T^3$ -actions by using Barden's classification theorem. Hence we need to compute  $H_2(M)$ .

(5.4) LEMMA. Let  $M$  be a  $T^3$ -manifold of dimension 5 and suppose  $k$  is the number of orbits of type  $T^1$ . Then  $H_2(M) \approx \mathbf{Z}^{k-3}$ .

PROOF. Recall  $P = q^{-1}(\text{int } M^*)$  and  $Q = q^{-1}(\partial M^*)$ , where  $q$  is the orbit map  $M \rightarrow M^*$ . Then  $P \approx D^2 \times T^3$ . Parameterize  $T^3 = T_1^1 \times T_2^1 \times T_3^1$  by  $\{(x, y, z) \mid 0 \leq x, y, z \leq 1\}$ . Suppose that  $j$  denotes the inclusion map from  $T^3$  into  $M$  and that  $j_{12}, j_{13}, j_{23}, f, g$  and  $h$  denote the restrictions of  $j$  to  $T_1^1 \times T_2^1, T_1^1 \times T_3^1, T_2^1 \times T_3^1, T_1^1, T_2^1$  and  $T_3^1$ , respectively.

By the Künneth formula,  $H_2(T^3) = H_1(T_1^1) \otimes H_1(T_2^1) + H_1(T_1^1) \otimes H_1(T_3^1) + H_1(T_2^1) \otimes H_1(T_3^1)$ . Since the isotropy groups span  $T^3$ , we can assume that  $T_1^1 = G(1, 0, 0)$  and  $T_2^1 = G(0, 1, 0)$  are isotropy groups and hence, by an argument similar to that of (2.4),  $f_*: H_1(T_1^1) \rightarrow H_1(M)$  and  $g_*: H_1(T_2^1) \rightarrow H_1(M)$  are zero-maps. Since the Künneth formula is functorial, we have a commutative diagram

$$\begin{array}{ccc} H_1(T_1^1) \otimes H_1(T_3^1) & \begin{array}{c} \times \\ \rightarrow \\ \approx \end{array} & H_2(T_1^1 \times T_3^1) \\ f_* \otimes h_* \downarrow & & \downarrow (f \times h)_* \\ H_1(M) \otimes H_1(M) & \begin{array}{c} \times \\ \rightarrow \end{array} & H_2(M \times M) \end{array}$$

Since  $f_*$  is the zero-map, so is  $(f \times h)_*$ . On the other hand, the left half of the following diagram is homotopy commutative by a homotopy defined by  $H(x, y, t) = j_{13}(x, t + (1 - t)y) \times j_{13}(t + (1 - t)x, y)$ .

$$\begin{array}{ccccc} T_1^1 \times T_3^1 & \xrightarrow{f \times h} & M \times M & \xrightarrow{\text{proj}} & M \\ & \searrow j_{13} & \uparrow \Delta & \nearrow c & \\ & & M & & \end{array}$$

h.c.

Here  $\Delta$  is the diagonal map. Since  $\Delta_*(j_{13})_* = (f \times h)_*$  is the zero-map and  $\Delta_*$  is injective,  $(j_{13})_*$  is also the zero-map. Similarly,  $(j_{12})_*$  and  $(j_{23})_*$  are zero-maps and hence  $j_*: H_2(P) \rightarrow H_2(M)$  is also the zero-map.

By Poincaré duality,  $H_2(M, P) \approx H^3(Q)$ . By an argument similar to that of (4.1),  $Q$  is a chain of several lens spaces. Hence by the Mayer-Vietoris sequence, we have  $H^3(Q) \approx \mathbf{Z}^k$ . Thus the homology sequence of the pair  $(M, P)$  yields

$$\begin{array}{ccccccc} H_2(P) & \xrightarrow{j_*} & H_2(M) & \rightarrow & H_2(M, P) & \rightarrow & H_1(P) \rightarrow H_1(M) \\ & \parallel & & & \text{''} & & \text{''} & \text{''} \\ & \text{0-map} & & & \mathbf{Z}^k & & \mathbf{Z}^3 & \mathbf{Z}/m \end{array}$$

and the result follows. ■

The  $S^3$ -bundles over  $S^2$  with structure group  $\text{SO}(4)$  are classified by  $\pi_1(\text{SO}(4)) = \mathbf{Z}/2$ . Suppose  $S^2 \tilde{\times} S^3$  denotes the nontrivial  $S^3$ -bundle over  $S^2$ . Then  $S^2 \tilde{\times} S^3$  also admits a  $T^3$ -action. In fact, in the proof of (5.2), by performing an equivariant surgery on  $L(2, 1, 1)$ , we constructed a  $T^3$ -manifold  $M'$  with orbit space shown in Figure 19. It follows from (2.2), (5.1) and (5.4) that  $\pi_1(M') = 1$ ,  $w_2(M') \neq 0$  and  $H_2(M') = \mathbf{Z}$ . By [1, (1.1) and (2.3)],  $M'$  must be the nontrivial  $S^3$ -bundle over  $S^2$ . Hence  $S^2 \tilde{\times} S^3$  admits a  $T^3$ -action.

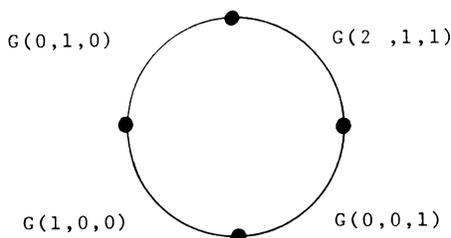


FIGURE 19

(5.5) THEOREM. Suppose  $M$  is a simply-connected 5-manifold with a  $T^3$ -action and the number of orbits of type  $T^1$  is  $k$ . Then we have

$$\begin{aligned} M &\approx S^5 \quad \text{if } k = 3, \\ M &\approx \#(k - 3)(S^2 \times S^3) \quad \text{if } w_2(M) = 0, \\ M &\approx (S^2 \tilde{\times} S^3) \# (k - 4)(S^2 \times S^3) \quad \text{if } w_2(M) \neq 0. \end{aligned}$$

PROOF. (3.2), (5.4) and [1, (1.1) and (2.3)] yield the results. ■

(5.6) REMARK. (1) For any integer  $k \geq 3$ ,  $(S^2 \tilde{\times} S^3) \# (k - 4)(S^2 \times S^3)$  and  $\#(k - 3)(S^2 \times S^3)$  actually admit  $T^3$ -actions (that is, every simply-connected 5-manifold  $M$  with  $H_2(M)$  torsion free admits a  $T^3$ -action).

PROOF. (a) Let  $\tilde{M}_k$  be the manifold obtained by performing successively an equivariant surgery of type (2, 4) on  $L(2, 1, 1)$  ( $k - 3$ ) times, this is,

$$\tilde{M}_k = L(2, 1, 1) \underbrace{\#_S S^5 \#_S S^5 \#_S \cdots \#_S S^5}_{(k-3) \text{ copies}}$$

Then  $\tilde{M}_k$  has  $k$  orbits of type  $T^1$ . Moreover, it follows from (2.3) and (5.1) that  $\tilde{M}_k$  is a simply-connected  $T^3$ -manifold with nonvanishing second Stiefel-Whitney class. Thus we have  $\tilde{M}_k \approx (S^2 \tilde{\times} S^3) \# (k - 4)(S^2 \times S^3)$ .

(b) It follows from [18] that the lens space  $L(2, 1)$  admits a  $T^2$ -action so that its orbit space is

$$G(2, 1) \bullet \text{---} \bullet G(0, 1).$$

Hence  $S^2 \times L(2, 1)$  admits a  $T^3$ -action with orbit space as shown in Figure 20. Similarly,  $S^2 \times S^3 (= M_4)$  also admits a  $T^3$ -action so that  $M_4/T^3$  is as shown in Figure 21.

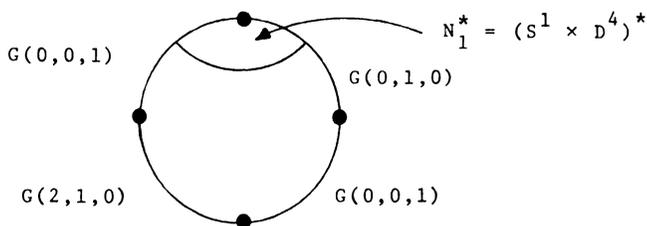


FIGURE 20

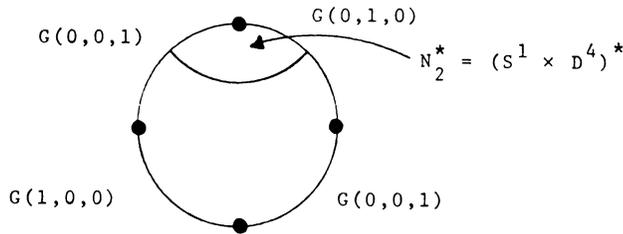


FIGURE 21

Let  $M_6$  be the manifold constructed by gluing  $P = (S^2 \times L(2, 1)) - \text{int } N_1$  and  $Q = (S^2 \times S^3) - \text{int } N_2$  equivariantly along their boundaries (this is,  $M_6 = (S^2 \times L(2, 1)) \#_S (S^2 \times S^3)$ ). Then it follows from (2.2) that  $M_6$  is simply connected. Furthermore,  $w_2(M_6)$  is zero. In fact, by the Mayer-Vietoris sequence of  $P$  and  $Q$  with  $\mathbf{Z}/2$  coefficients, we have an exact sequence

$$0 \rightarrow \mathbf{Z}/2 \rightarrow \mathbf{Z}/2 \rightarrow H^2(M_6) \xrightarrow{k^*} H^2(P) \oplus H^2(Q) \rightarrow$$

Thus  $k^*$  is injective and hence  $k^*(w_2(M_6)) = w_2(P) + w_2(Q) = 0$  implies  $w_2(M_6) = 0$ . Since  $M_6$  is a  $T^3$ -manifold with 6 orbits of type  $T^1$ ,  $M_6$  is  $\#3(S^2 \times S^3)$ .

By applying a similar argument,  $(S^2 \times L(2, 1)) \#_S S^5 (= M_5$ , the manifold resulting from equivariant surgery on  $S^2 \times L(2, 1)$  along an orbit of type  $T^1$ ) is a simply-connected  $T^3$ -manifold with  $w_2(M_5) = 0$  and with 5 orbits of type  $T^1$ . Hence  $M_5$  is  $\#2(S^2 \times S^3)$ .

For  $k \geq 7$ , a simply-connected  $T^3$ -manifold with  $w_2(M_k) = 0$  and  $k$  orbits of type  $T^1$  can be constructed inductively by the algorithm

$$\begin{aligned} M_{2i-1} \#_S (S^2 \times L(2, 1)) &= M_{2i+1}, \\ M_{2i-2} \#_S (S^2 \times L(2, 1)) &= M_{2i}, \quad \text{where } i \geq 3. \end{aligned}$$

(2) It is worthwhile to note that the connected sums in (5.5) are not equivariant. Pak [20] showed that if  $T^n$  acts on an  $(n + 1)$ -manifold  $M$ , then  $M$  is diffeomorphic to either  $T^{n+1}$  or  $L(p, q) \times T^{n-2}$  for  $n \geq 3$ . Hence if  $M$  is an  $(n + 2)$ -dimensional  $T^n$ -manifold,  $n \geq 3$ , then  $M$  cannot have an equivariant connected sum decomposition.

(5.7) COROLLARY. *If  $M$  is a 5-manifold with a  $T^3$ -action, then  $M$  bounds a 6-manifold.*

PROOF. Suppose  $M'$  is the manifold resulting from equivariant surgery on  $M$  along an orbit of type  $T^1$ . Then by (2.3),  $M'$  is a simply-connected 5-manifold with a  $T^3$ -action. Hence it follows from (5.5) that the Stiefel-Whitney numbers are all zero. Since the Stiefel-Whitney numbers are cobordism invariants, the Stiefel-Whitney numbers of  $M$  are all zero and hence  $M$  bounds a 6-manifold. ■

In concluding, we generalize (5.2) to higher-dimensional manifolds. Contrary to the results in [9], we have the following.

(5.8) REMARK. There exist simply-connected  $(n + 2)$ -manifolds with  $T^n$ -actions and nonvanishing second Stiefel-Whitney classes.

PROOF. Suppose the statement is true for  $n = k$  and suppose  $M^{k+2}$  is a manifold of this type.

Let  $M = M^{k+2} \times S^1$  and define a  $T^{k+1}$ -action on  $M$  by product. By the Cartan formula,  $w_2(M) = w_2(M^{k+2}) \times w_0(S^1) \neq 0$ . Let  $N = M - \text{int}(D^4 \times T^{k-1})$  and  $j: N \rightarrow M$  be the inclusion map. By the universal coefficient theorem,  $w_2(M): H_2(M; \mathbf{Z}/2) \rightarrow \mathbf{Z}/2$  and this map is nontrivial.

By the homology sequence of the pair  $(M, N)$ ,

$$\begin{array}{ccccc} \rightarrow & H_2(N; \mathbf{Z}_2) & \xrightarrow{j_*} & H_2(M; \mathbf{Z}_2) & \longrightarrow & H_2(M, N; \mathbf{Z}_2) & \rightarrow \\ & \searrow w_2(N) & & \downarrow w_2(M) & & \cong & \\ & & & \mathbf{Z}_2 & & H_2(D^4 \times T^{k-1}, S^3 \times T^{k-1}; \mathbf{Z}_2) & \\ & & & & & \cong & \\ & & & & & 0 & \end{array}$$

we have  $w_2(N) = w_2(M) \cdot j_*$  and  $j_*$  is surjective. Hence,  $w_2(M) \neq 0$  yields  $w_2(N) \neq 0$ .

We may assume that the orbit space of  $M$  is as shown in Figure 22.

$$\begin{aligned} G_1 &= G(1, 0, 0, \dots, 0), & G_2 &= G(0, 1, 0, \dots, 0), \\ G_3 &= G(a_{31}, a_{32}, \dots, a_{3k}, 0), \dots, & G_m &= G(a_{m1}, a_{m2}, \dots, a_{mk}, 0). \end{aligned}$$

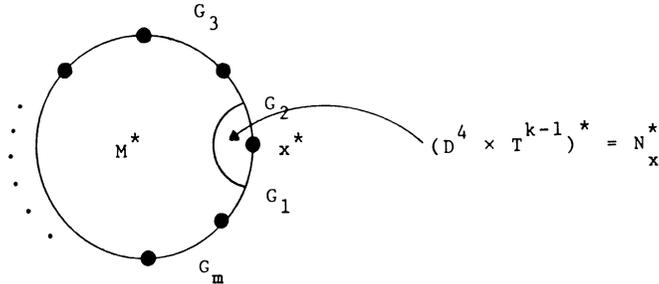


FIGURE 22

Suppose  $M_1$  is a  $(k + 3)$ -dimensional  $T^{k+1}$ -manifold with orbit space, shown in Figure 23. (Existence of such a manifold was shown in (4.7).)

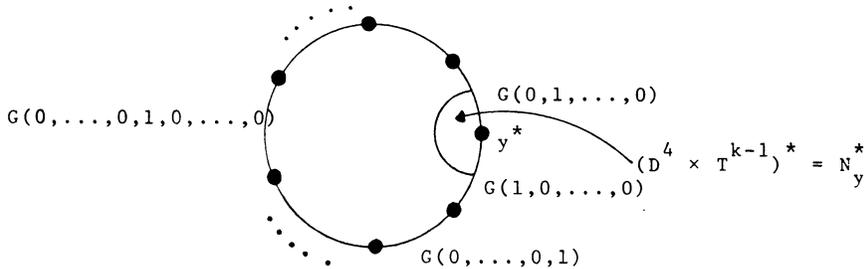


FIGURE 23

Cutting out the interiors of  $N_x$  and  $N_y$  from  $M$  and  $M_1$  respectively and gluing the resulting spaces together along their boundaries, we obtain a  $(k + 3)$ -manifold  $\overline{M}$  with a  $T^{k+1}$ -action. By (2.5),  $\overline{M}$  is simply connected and  $w_2(\overline{M}) \neq 0$ , since  $w_2(N) \neq 0$ .

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