ON THE GENERATORS OF THE FIRST HOMOLOGY WITH COMPACT SUPPORTS OF THE WEIERSTRASS FAMILY IN CHARACTERISTIC ZERO

BY

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Abstract. Let \( W_\mathbb{Q} = \text{Proj}(\mathbb{Q}[g_2, g_3, X, Y, Z]/(\text{homogeneous ideal generated by } -Y^2Z + 4X^3 - g_2XZ^2 - g_3Z^3)) \). This is said to be the Weierstrass Family over the field \( \mathbb{Q} \). Then the first homology with compact supports of the Weierstrass Family is computed explicitly, i.e., it is generated by \( \{ C^{-k}dX \wedge dY \}_{k \geq 1} \) and \( \{ X C^{-k}dX \wedge dY \}_{k > 1} \) over the ring \( \mathbb{Q}[g_2, g_3] \), where \( C \) is a polynomial \( Y^2 - 4X^3 + g_2X + g_3 \). When one tensors the homology of the Weierstrass Family with \( \mathbb{Q}[g_2, g_3] \), being localized at the discriminant \( \Delta = g_2^3 - 27g_3^2 \), over \( \mathbb{Q}[g_2, g_3] \), the first homology is generated by \( C^{-1}dX \wedge dY \) and \( XC^{-1}dX \wedge dY \). One also obtains the first homologies with compact supports of singular fibres over \( \varphi = (g_2 = g_3 = 0) \) and \( \varphi = (g_2 = 3, g_3 = 1) \) as corollaries.

Introduction. We wish to compute the \( \mathbb{Q}[g_2, g_3] \)-adic homology with compact supports of the Weierstrass Family \( W_\mathbb{Q} \), where

\[
W_\mathbb{Q} = \text{Proj}
\left( \frac{\mathbb{Q}[g_2, g_3, X, Y, Z]}{\text{homogeneous ideal generated by } -Y^2Z + 4X^3 - g_2XZ^2 - g_3Z^3} \right).
\]

We regard the graded ring \( \mathbb{Q}[g_2, g_3, X, Y, Z] \) as the graded \( \mathbb{Q}[g_2, g_3] \)-algebra such that \( X, Y \) and \( Z \) each has degree +1 and all the elements of \( \mathbb{Q}[g_2, g_3] \) have degree zero. Let \( U \) be the open subset of \( W_\mathbb{Q} \), “the finite points”: \( U = W_\mathbb{Q} \cap \mathbb{A}^2(\text{Spec}(\mathbb{Q}[g_2, g_3])) \). This is the closed subscheme of \( \mathbb{A}^2(\text{Spec}(\mathbb{Q}[g_2, g_3])) \) given by \( Y^2 = 4X^3 - g_2X - g_3 \). Then we have the long exact sequence of the homology with compact supports,

\[
\cdots \to H_{h-2}^c((\text{points at } \infty), \mathbb{Q}[g_2, g_3]) \to H_h^c(W_\mathbb{Q}, \mathbb{Q}[g_2, g_3]) \to H_h^c(U, \mathbb{Q}[g_2, g_3]) \to \cdots.
\]

Since \( H_h^c((\text{points at } \infty), \mathbb{Q}[g_2, g_3]) \) vanishes except at \( h = 0 \), we have

\[
H_h^c(U, \mathbb{Q}[g_2, g_3]) = \begin{cases} H_h^c(W_\mathbb{Q}, \mathbb{Q}[g_2, g_3]), & h \neq 2, \\ \mathbb{Q}[g_2, g_3], & h = 2. \end{cases}
\]

Therefore the knowledge of \( H_h^c(U, \mathbb{Q}[g_2, g_3]), h \geq 0, \) determines the homology groups of all the fibres in the family over the various points \( \varphi \in \text{Spec}(\mathbb{Q}[g_2, g_3]) \), i.e.,

\[
E_2^{p,q} = \text{Tor}^{\mathbb{Q}[g_2, g_3]}_p(H_q^c(U, \mathbb{Q}[g_2, g_3], \mathbb{K}(\varphi)))
\]

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with the abutment $H^n_{\mathfrak{p}}(U_{\mathfrak{p}}, K(\mathfrak{p}))$, where $K(\mathfrak{p})$ is the characteristic zero residue field at $\mathfrak{p} \in \text{Spec}(\mathbb{Q}[g_2, g_3])$.

Let us consider the unequal characteristic case. Suppose that $\mathfrak{O}$ is a complete discrete valuation ring with the quotient field $K$ and residue class field $k$ and suppose that $A$ is an $\mathfrak{O}$-algebra. Let $X$ be a scheme over $A = (A \otimes_{\mathfrak{O}} k)_{\text{red}}$. Suppose that $K(\mathfrak{p})$ is a finite field at $\mathfrak{p} \in \text{Spec}(A)$ and let $W(K(\mathfrak{p}))$ be the complete discrete valuation ring and denote the quotient field of $W(K(\mathfrak{p}))$ by $K_p = W(K(\mathfrak{p})) \otimes_{\mathbb{Z}} Q$. Then the zeta function of the fibre $X_p$ at $\mathfrak{p}$ is given by

$$Z_{X_p}(T) = \frac{\prod_{p + q = \text{odd}} P_{p, q}(T)}{\prod_{p + q = \text{even}} P_{p, q}(T)}$$

where $P_{p, q}(T)$ is the reverse characteristic polynomial of the endomorphism of

$$E_{p, q}^2 = \text{Tor}_{p}^{\mathfrak{O} \otimes_{\mathbb{Z}} Q}(H^i_{\mathfrak{p}}(X, A^\dagger \otimes_{\mathbb{Z}} Q), K_p)$$

induced by the $p^r$th power map, $p^r = \text{card}(K(\mathfrak{p}))$ (see pp. 448–450, [6]). This homological spectral sequence abuts upon $H^n_{\mathfrak{p}}(X_p, K_p)$. Therefore if one knows the lifted $p$-adic homology with compact supports of $X$ over $A$, $H^i_{\mathfrak{p}}(X, A^\dagger \otimes_{\mathbb{Z}} Q)$, $i \geq 0$, and the zeta endomorphisms of these groups, (1) determines the zeta function of every fibre over a finite field in the algebraic family $X$ over the ring $A$. These are the subjects in the forthcoming paper [2].

The main result of the paper is the explicitness of the generation of the first homology with compact supports of the entire Weierstrass Family $W_0$ in the characteristic zero (Theorem 1) and its consequences.

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1. In this section (notations being the same as in the Introduction) we describe explicitly the basis elements over the ring $\mathbb{Q}[g_2, g_3]$ which generate the first homology with compact supports of the Weierstrass Family over the field of rational numbers $\mathbb{Q}$, $H^1_{\mathfrak{p}}(U, \mathbb{Q}[g_2, g_3])$. By the definition of the lifted $p$-adic homology with compact supports [6, p. 415], applied to the characteristic zero case, we have

$$H^1_{\mathfrak{p}}(U, \mathbb{Q}[g_2, g_3]) = H^1(\mathbb{A}^2(\text{Spec}(\mathbb{Q}[g_2, g_3])), \mathbb{A}^2(\text{Spec}(\mathbb{Q}[g_2, g_3]))) - U, \Gamma_{\mathfrak{p}[g_2, g_3]}^{\mathfrak{p}}(\text{Spec}(\mathbb{Q}[g_2, g_3])))$$

If one tensors $H^1_{\mathfrak{p}}(U, \mathbb{Q}[g_2, g_3])$ with $\Delta^{-1}\mathbb{Q}[g_2, g_3]$ over $\mathbb{Q}[g_2, g_3]$, one has the free $\Delta^{-1}\mathbb{Q}[g_2, g_3]$-module of rank two, where $\Delta = g_2^3 - 27g_3^2$. This is so because we have the universal coefficients spectral sequence

$$E_{0,1}^2 = H^1_{\mathfrak{p}}(U, \mathbb{Q}[g_2, g_3]) \otimes_{\mathfrak{p}[g_2, g_3]} \Delta^{-1}\mathbb{Q}[g_2, g_3] \Rightarrow H^1_{\mathfrak{p}}(U, \Delta^{-1}\mathbb{Q}[g_2, g_3])$$

and $\Delta^{-1}\mathbb{Q}[g_2, g_3]$ means that the ring $\mathbb{Q}[g_2, g_3]$ is localized at the discriminant $\Delta$. The computation has been made even in the $p$-adic case in [1] for this open subfamily of the Weierstrass Family.
**Theorem 1.** Consider \( U = W_0 \cap A^2(\text{Spec}(\mathbb{Q}[g_2, g_3])) \), which is the closed affine subscheme of \( A^2(\text{Spec}(\mathbb{Q}[g_2, g_3])) \). Then the first homology with compact supports \( H^i(U, \mathbb{Q}[g_2, g_3]) \) is generated by \( \{C^{-i}dx \wedge dy\} \) and \( \{X^{-j}dx \wedge dY\} \) as a \( \mathbb{Q}[g_2, g_3] \)-module.

**Remark 1.** For the pair of affine schemes

\[
A^2(\text{Spec}(\mathbb{Q}[g_2, g_3])) \quad \text{and} \quad A^2(\text{Spec}(\mathbb{Q}[g_2, g_3])) - U,
\]

where \( U \) is the closed subscheme corresponding to the polynomial \( C = Y^2 - 4X^3 + g_2X + g_3 \) in \( \mathbb{Q}[g_2, g_3, X, Y, Z] \), there is induced a long exact sequence of hyper-cohomology groups,

\[
\cdots \rightarrow H^n(A^2(A), A^2(A) - U, \Gamma^*(A^2(A))) \rightarrow H^n(A^2(A), \Gamma^*(A^2(A))) \rightarrow H^n(A^2(A) - U, \Gamma^*(A^2(A))) \rightarrow \cdots
\]

where \( A = \text{Spec}(\mathbb{Q}[g_2, g_3]) \).

There are three first-quadrant spectral sequences induced which have the above three hypercohomology groups as their abutments:

\[
\begin{align*}
\text{"E}_2^{p,q} &= H^q(A^2(A) - U, \Gamma^*_A(A^2(A))), \\
\text{E}_2^{p,q} &= H^q(A^2(A), \Gamma^*_A(A^2(A))), \\
\text{"E}_2^{p,q} &= H^q(A^2(A), A^2(A) - U, \Gamma^*_A(A^2(A))).
\end{align*}
\]

**Lemma 1.** We have the following isomorphisms: the abutment

\[
\text{"E}_3 = H^3(A^2(A), A^2(A) - U, \Gamma^*_A(A^2(A))) \cong \text{"E}_2^{2,1},
\]

and

\[
\text{"E}_3 \rightarrow \text{"E}_2 = H^2(A^2(A) - U, \Gamma^*_A(A^2(A))) \rightarrow \text{coker}(\text{"E}_2^{1,0} \rightarrow \text{"E}_2^{1,0}).
\]

**Proof of Lemma 1.** Consider the following diagram (Diagram A) with exact rows.

We denote the structure sheaf of the affine scheme \( A^2(A) = A^2(\text{Spec}(\mathbb{Q}[g_2, g_3])) \) by \( \mathfrak{O}_{A^2(A)} \). Therefore, we have \( \text{"E}_2^{p,q} = 0 \) unless \( q = 1 \), which is abutting \( \text{"E}_3 = H^3(A^2(A), \mathfrak{O}_{A^2(A)} - U, \Gamma^*_A(A^2(A))) \). Then the isomorphism \( \text{"E}_2^{2,1} \rightarrow \text{"E}_3 \) in Lemma 1 follows. Furthermore, this diagram can be rewritten as Diagram B. The remaining two isomorphisms in Lemma 1 are obtained from the well-known lemma in homological algebra, i.e., from Diagram B with the exact rows we have the induced exact sequence

\[
0 \rightarrow \ker d_2^1 \rightarrow \ker d_1^0 \rightarrow \ker d_2^0 \rightarrow \text{coker } d_1^0 \rightarrow \text{coker } d_2^0 \rightarrow \text{coker } d_1^1 \rightarrow \text{coker } d_2^1 \rightarrow 0
\]

\[
\epsilon^2 \rightarrow \epsilon^2 \rightarrow \cdots
\]
and since the $\mathbb{Q}[g_2, g_3]$-homomorphism

$$d_{1,0}^1: E_1^{1,0} = \Gamma_{\mathbb{Q}[g_2, g_3]}(\mathbb{Q}[g_2, g_3, X, Y]) \rightarrow E_2^{2,0} = \Gamma_{\mathbb{Q}[g_2, g_3]}(\mathbb{Q}[g_2, g_3, X, Y])$$

is an epimorphism, we have $E_2^2 \approx E_2^{2,0} \approx 0$. Therefore

$$\text{coker } d_{1,0}^1 \approx \text{coker } d_{1,1}^1 \approx E_3^3$$

as stated in Lemma 1. Q.E.D.
Hence our computation of the abutment \( E^3 = H^3(\mathbb{A}^2(A), \mathbb{A}^2(A) - U, \Gamma^*_\alpha(\mathbb{A}^2(A))) \) is reduced to compute
\[
\text{coker} \left( \Gamma^1_{Q[g_2, g_3]}(Q[g_2, g_3, X, Y, C^{-1}]) \right) \rightarrow \Gamma^1_{Q[g_2, g_3]}(Q[g_2, g_3, X, Y, C^{-1}]).
\]

**Proof of Theorem 1.** From now on we denote, \( \text{“d”} \), instead of the exterior differential, \( \text{“d,0”} \) in the spectral sequence. We have that
\[
(1) \quad d(C^{-k}X^iY^j dX) = (-2kC^{-k-1}X^iY^{j+1} + jC^{-k}X^iY^{j-1}) dY \wedge dX,
\]
\[
(2) \quad d(C^{-k}X^iY^j dY) = (32kC^{-k-1}X^iY^j + g_2C^{-k-1}X^iY^j + iC^{-k}X^{j-1}Y^j) dX \wedge dY,
\]
in the \( Q[g_2, g_3] \)-module \( \Gamma^2_{Q[g_2, g_3]}(Q[g_2, g_3, X, Y, C^{-1}]) \), where \( C = Y^2 - 4X^3 + g_2X + g_3 \), \( i, j \) and \( k \) are nonnegative integers. The equations (1) and (2) give the cohomologous relations, which are denoted by \( \text{“~”} \), as
\[
2kC^{-k-1}X^iY^{j+1} dX \wedge dY \sim jC^{-k}X^iY^{j-1} dX \wedge dY
\]
and
\[
(12kC^{-k-1}X^iY^{j+1} - g_2C^{-k-1}X^iY^j + iC^{-k}X^{j-1}Y^j) dX \wedge dY \sim 0.
\]
Notice that, by Lemma 1:
\[
\text{“} E^{2,1} \Rightarrow \text{“} E^{2,1} \rangle \text{Im}(\text{“} E^{1,1} \Rightarrow \text{“} E^{2,1})
\]
and
\[
\text{“} E^{1,1} \Rightarrow \text{“} E^{1,0} \rangle \text{Im}(\text{“} E^{1,0} \Rightarrow \text{“} E^{1,0}),
\]
where \( E^{2,1}_i \approx \Gamma^1_{Q[g_2, g_3]}(Q[g_2, g_3, X, Y]) \). Therefore it suffices to consider the integer \( k > 1 \) in the equations (1), (2), (3), and (4) above.

If \( j = 0 \) in (3), then \( C^{-k-1}X^iYdX \wedge dY \sim 0 \) for all \( i > 0 \) and \( k > 1 \). But (4) implies that \( C^{-i-1}X^iYdX \wedge dY \sim 0 \) for all \( i \) since \( iC^{-i-1}X^iYdX \wedge dY \sim g_2C^{-2}X^iYdX \wedge dY - 12C^{-2}X^{i+1}YdX \wedge dY. \) Therefore,
\[
C^{-k}X^iYdX \wedge dY \sim 0 \quad \text{for all integers } i, k > 0.
\]
For any odd integer \( j > 1 \) we have \( C^{-k}X^iYdX \wedge dY \sim 0 \) by combining (3) and (5) and the repeated use of (4). For example, for \( j = 3 \), we have \( 12kC^{-k-1}X^iY^3dX \wedge dY \sim 2C^{-k}X^iY^3dX \wedge dY \), which is cohomologous to zero by (5). Then apply (4) for \( j = 3 \) to get
\[
iC^{-k-1}X^iY^3dX \wedge dY \sim g_2kC^{-k-1}X^iY^3dX \wedge dY - 12C^{-k-1}X^{i+3}Y^3dX \wedge dY.
\]
generate all the elements of the type $X'C^{k-1}dX \wedge dY$ for integers $i \geq 0$ and $k \geq 0$ over the ring $\mathbb{Q}[g_2, g_3]$ from equations (3) and (4). In particular, $X^2C^{i-1}dX \wedge dY \sim X^2Y^2C^{-2}dX \wedge dY$ by (3) for letting $i = 2$, $j = 1$ and $k = 1$, but $X^2Y^2C^{-2}dX \wedge dY \sim Y^2C^{-2}dX \wedge dY$ by (4) for $i = 0$, $j = 2$ and $k = 1$; furthermore, $Y^2C^{-2}dX \wedge dY$ is cohomologous to $C^{-1}dX \wedge dY$ from (3) for $i = 0$, $j = 1$ and $k = 1$. Hence we have established that $X^2C^{-1}dX \wedge dY \sim C^{-1}dX \wedge dY$. Next we claim that all the elements of the type $(X'C^{i-1}dX \wedge dY)_{i \geq 3}$ are generated by the two elements $C^{i-1}dX \wedge dY$ and $XC^{-i}dX \wedge dY$ over the ring $\mathbb{Q}[g_2, g_3]$. We have the following recursive formula for integers $i \geq 3$ from (3) and (4):

$$4X'C^{i-1}dX \wedge dY \sim g_2 \left( \frac{1}{12(i-2)} + 1 \right) X^{i-2}C^{-1}dX \wedge dY$$

$$+ \left( g_3 - \frac{1}{i-2} \right) X^{i-3}C^{-1}dX \wedge dY.$$

Therefore it follows from this recursive formula that $(X'C^{i-1}dX \wedge dY)_{i \geq 3}$ are generated by $C^{i-1}dX \wedge dY$ and $XC^{-i}dX \wedge dY$ over $\mathbb{Q}[g_2, g_3]$. We have established the statement of Theorem 1 for the elements $X'i'Y'C^{k-1}dX \wedge dY$ with $i \geq 1$, $j = 0$ and $k \geq 1$. Now we need consider the elements $X'i'Y'C^{k-1}dX \wedge dY$ for $j = 1, 2, 3, \ldots$. As noted before, we know that if $j$ is an odd integer, $X'i'Y'C^{k-1}dX \wedge dY \sim 0$. If $j$ is an even integer, the repeated use of (3) and (4) for the elements $X'i'Y'C^{k-1}dX \wedge dY$, $i \geq 1$ and $j \geq 1$, provides the generation of the first homology with compact supports $H_i(U, \mathbb{Q}[g_2, g_3])$ of the Weierstrass Family by the elements $(C^{-k}dX \wedge dY)_{k \geq 1}$ and $(XC^{-k}dX \wedge dY)_{k \geq 1}$. Q.E.D.

**Proposition 1.** Assumptions and notations being the same as in Theorem 1, $H_i(U, \mathbb{Q}[g_2, g_3]) \otimes _{\mathbb{Q}[g_2, g_3]} (\Delta^{-1}\mathbb{Q}[g_2, g_3])$ is a free $(\Delta^{-1}\mathbb{Q}[g_2, g_3])$-module of rank two, i.e., it is generated by $XC^{-k}dX \wedge dY$ and $C^{-1}dX \wedge dY$, where $\Delta$ is the discriminant, $\Delta = g_2^3 - 27g_3^2$, and $\Delta^{-1}\mathbb{Q}[g_2, g_3]$ is localized at the discriminant $\Delta$.

**Proof of Proposition 1.** For any integer $i \geq 2$ we have

$$C^{-i-1} = C^{-i}(Y^2 - 4X^3 + g_2X + g_3),$$

where $dX \wedge dY$ is omitted for simplicity, and from equations (3), (4) and (6) we have the following cohomologous relation for $i \geq 2$:

$$\frac{6i - 11}{6(i-1)} C^{-(i-1)} \sim \frac{2g_2}{3} XC^{-i} + g_3C^{-i}.$$

Similarly, one has the corresponding formula for $XC^{-(i-1)}$ by the equations (3), (4) and (6):

$$\frac{6i - 13}{6(i-1)} XC^{-(i-1)} \sim \frac{g_2^2}{18} C^{-i} + g_3 XC^{-i}.$$
We finally have for \( i \geq 2 \),

\[
C^{-i}dX \wedge dY \sim \frac{18}{\Delta} \left\{ \frac{g_2(6i - 13)}{6(i - 1)} \, XC^{-(i-1)}dX \wedge dY - \frac{g_3(6i - 11)}{4(i - 1)} \, C^{-(i-1)}dX \wedge dY \right\}
\]

from equations (1.1) and (1.2).

Equations (1.3) and (1.1) prove that \( H^1_f(U, \mathbb{Q}(g_2, g_3)) \otimes_{\mathbb{Q}(g_2, g_3)} (\Delta^{-1} \mathbb{Q}(g_2, g_3)) \) is generated by \( XC^{-1}dX \wedge dY \) and \( C^{-1}dX \wedge dY \) as a \((\Delta^{-1} \mathbb{Q}(g_2, g_3))-\text{module}\).  
Q.E.D.

**Corollary 1.** Let \( V^0 \) be the closed subfamily defined by \( g_2 = 0 \) of the whole Weierstrass Family \( W^0 \). Then the first homology with compact supports,

\[
H^1_f(V^0 \cap \mathbb{A}^2(\text{Spec} \mathbb{Q}(g_3)), \mathbb{Q}(g_3)),
\]

is generated by \( \{C^{-k}dX \wedge dY\}_{k \geq 1} \) and \( \{XC^{-k}dX \wedge dY\}_{k \geq 1} \) as a \( \mathbb{Q}(g_3) \)-module.

**Proof.** In (1.1) and (1.2) in the proof of Proposition 1, we have the following corresponding equations for the closed subfamily \( V^0 \) defined by \( g_2 = 0 \):

\[
(1.1)^0 \quad \frac{12i - 22}{12(i - 1)} \, C^{-(i-1)} \sim g_3 C^{-i},
\]

\[
(1.2)^0 \quad \frac{6i - 13}{6(i - 1)} \, XC^{-(i-1)} \sim g_3 XC^{-i}.
\]

Then the statement of Corollary 1 follows plainly from (1.1)^0 and (1.2)^0.  
Q.E.D.

**Note 1.** The equations (1.1)^0 and (1.2)^0 also show that Corollaries 2 and 3 are true.

**Corollary 2.** The first homology with compact supports of the singular fibre \( U_\varphi \) over a point \( \varphi = (g_2 = 0, g_3 = 0) \in \text{Spec} \mathbb{Q}(g_2, g_3) \), a projective line with a cusp (or \( \varphi = (g_3 = 0) \in \text{Spec} \mathbb{Q}(g_3) \)), \( H^1_f(U_\varphi, \mathbb{Q}) \), is trivial.

**Corollary 3.** Notations being the same as in Proposition 1,

\[
H^1_f(V^0 \cap \mathbb{A}^2(\text{Spec} \mathbb{Q}(g_3)), \mathbb{Q}(g_3)) \otimes_{\mathbb{Q}(g_3)} (g_3^{-1} \mathbb{Q}(g_3))
\]

is generated by the two elements \( C^{-1}dX \wedge dY \) and \( XC^{-1}dX \wedge dY \), where \( g_3^{-1} \mathbb{Q}(g_3) \) means the localization of the ring \( \mathbb{Q}(g_3) \) at \( g_3 \).

**Remark 2.** For a point \( \varphi \neq (g_3 = 0) \), \( H^1_f(U_\varphi, \mathbb{K}(\varphi)) \) is generated by \( C^{-1}dX \wedge dY \) and \( XC^{-1}dX \wedge dY \) as a \( \mathbb{K}(\varphi)-\text{vector space} \) and where \( \mathbb{K}(\varphi) \) is the characteristic zero residue field, i.e., \( U_\varphi \) is an elliptic curve. Note that the open subfamily of the Weierstrass Family over \( \mathbb{Z}/P\mathbb{Z} \) defined by \( \Delta = 0 \) has been computed explicitly using the hypercohomology of a flat lifting with coefficients in the \( \hat{\mathcal{Z}} \) of sheaves of differential forms, \( H^1_f(U, (\Delta^{-1} \hat{\mathcal{Z}}_p[g_2, g_3])^\dagger \otimes \mathcal{Z} \mathbb{Q}) \), where \( (\Delta^{-1} \hat{\mathcal{Z}}_p[g_2, g_3])^\dagger \) is the \( \dagger \) of the localization of the ring \( \hat{\mathcal{Z}}_p[g_2, g_3] \) at the discriminant \( \Delta = g_2^3 - 27g_3^2 \), see [1]. The following universal coefficient spectral sequence explains the relationship between Corollary 2 and Theorem 1.

\[
E^2_{p,q} = \text{Tor}^2_{\mathbb{Q}(g_2, g_3)}(H^q_f(U, \mathbb{Q}(g_2, g_3)), \mathbb{K}(\varphi))\text{ with the abutment } H^q_f(U_\varphi, \mathbb{K}(\varphi)), \text{ where } \varphi = (g_2 = g_3 = 0) \in \text{Spec} \mathbb{Q}(g_2, g_3) \text{ and } \mathbb{K} = \mathbb{K}(\varphi).
Corollary 4. Let $V^3_Q$ be the closed subfamily of the Weierstrass Family $W_Q$, defined by “$g_2 = 3$”. Then $H_i(Y^3_Q \cap A^2(\text{Spec} Q[g_3]), Q[g_3])$ is generated by $(C^{-1}dX \wedge dY)_{k \geq 1}$ and $(XC^{-1}dX \wedge dY)_{k \geq 1}$ as a $Q[g_3]$-module. Moreover the first homology with compact supports of the singular fibre over the point $\varphi = (g_3 = 1)$ in the base $\text{Spec}(Q[g_3])$, a projective line with an ordinary double point over $K(\varphi)$, is generated by one element as a $K(\varphi)$-vector space. One can then take either $C^{-1}dX \wedge dY$ or $XC^{-1}dX \wedge dY$ as the base element for the vector space.

Proof. We only need prove the latter statement. From equations (1.1) and (1.2), we have (1.1)$_i^3$ and (1.2)$_i^3$ as follows:

\begin{align*}
(1.1)_i^3 & \quad \frac{6i - 11}{6(i - 1)} C^{-(i-1)} \sim 2 XC^{-i} + C^{-i}, \\
(1.2)_i^3 & \quad \frac{6i - 13}{6(i - 1)} XC^{-(i-1)} \sim \frac{1}{2} C^{-i} + XC^{-i}.
\end{align*}

Then we have $2(6i - 13)XC^{-(i-1)} \sim (6i - 11)C^{-(i-1)}$ for $i \geq 2$. Hence this vector space is one dimensional and the statement of Corollary 4 follows. Q.E.D.

Note 2. For the closed subfamily $V^3_Q$ of the Weierstrass Family we have the following equations (1.1)$_3$, (1.2)$_3$ and (1.3)$_3$:

\begin{align*}
(1.1)_3 & \quad \frac{6i - 11}{6(i - 1)} C^{-(i-1)} \sim 2 XC^{-i} + g_3 C^{-i}, \\
(1.2)_3 & \quad \frac{6(i - 13)}{6(i - 1)} XC^{-(i-1)} \sim \frac{1}{2} C^{-i} + g_3 XC^{-i}, \\
(1.3)_3 & \quad (g_3^2 - 1)C^{-i} \sim \frac{1}{6(i - 1)} \left( g_3(6i - 11)C^{-(i-1)} - 2(6i - 13)XC^{-(i-1)} \right),
\end{align*}

for integers $i \geq 2$.

Note 3. This paper has been entirely in characteristic zero. The case of nonzero characteristic $p \neq 2, 3$ will appear in a forthcoming paper [2], which is a generalization of the paper [1], where an open subfamily “$\Delta \neq 0$” of the Weierstrass Family was studied.

References