ON THE GENERATORS OF THE FIRST HOMOLOGY WITH COMPACT SUPPORTS OF THE WEIERSTRASS FAMILY IN CHARACTERISTIC ZERO

BY

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Abstract. Let \( W_Q = \operatorname{Proj}(Q[g_2, g_3, X, Y, Z]/(\text{homogeneous ideal generated by } -Y^2Z + 4X^3 - g_2XZ^2 - g_3Z^3)) \). This is said to be the Weierstrass Family over the field \( Q \). Then the first homology with compact supports of the Weierstrass Family is computed explicitly, i.e., it is generated by \( \{ C^{-k}dX \otimes dY \}_{k>1} \) and \( \{ X^{-k}dX \otimes dY \}_{k>1} \) over the ring \( Q[g_2, g_3] \), where \( C \) is a polynomial \( Y^2 - 4X^3 + g_2X + g_3 \). When one tensors the homology of the Weierstrass Family with \( \Delta^{-1}Q[g_2, g_3] \), being localized at the discriminant \( \Delta = g_2^3 - 27g_3^2 \), over \( Q[g_2, g_3] \), the first homology is generated by \( C^{-k}dX \otimes dY \) and \( X^{-k}dX \otimes dY \). One also obtains the first homologies with compact supports of singular fibres over \( \varphi = (g_2 = g_3 = 0) \) and \( \varphi = (g_2 = 3, g_3 = 1) \) as corollaries.

Introduction. We wish to compute the \( Q[g_2, g_3] \)-adic homology with compact supports of the Weierstrass Family \( W_Q \), where

\[
W_Q = \operatorname{Proj} \left( \frac{Q[g_2, g_3, X, Y, Z]}{\text{homogeneous ideal generated by } -Y^2Z + 4X^3 - g_2XZ^2 - g_3Z^3} \right).
\]

We regard the graded ring \( Q[g_2, g_3, X, Y, Z] \) as the graded \( Q[g_2, g_3] \)-algebra such that \( X, Y \) and \( Z \) each has degree +1 and all the elements of \( Q[g_2, g_3] \) have degree zero. Let \( U \) be the open subset of \( W_Q \), “the finite points”: \( U = W_Q \cap A^2(\text{Spec}(Q[g_2, g_3])) \). This is the closed subscheme of \( A^2(\text{Spec}(Q[g_2, g_3])) \) given by \( Y^2 = 4X^3 - g_2X - g_3 \). Then we have the long exact sequence of the homology with compact supports, \( \ldots \rightarrow H^c_{h-2}(\{ \text{points at } \infty \}, Q[g_2, g_3]) \rightarrow H^c_{h}(W_Q, Q[g_2, g_3]) \rightarrow H^c_{h}(U, Q[g_2, g_3]) \rightarrow \ldots \). Since \( H^c_{h}(\{ \text{points at } \infty \}, Q[g_2, g_3]) \) vanishes except at \( h = 0 \), we have

\[
H^c_{h}(U, Q[g_2, g_3]) = \begin{cases} H^c_{h}(W_Q, Q[g_2, g_3]), & h \neq 2, \\ Q[g_2, g_3], & h = 2. \end{cases}
\]

Therefore the knowledge of \( H^c_{h}(U, Q[g_2, g_3]), h \geq 0 \), determines the homology groups of all the fibres in the family over the various points \( \varphi \in \text{Spec}(Q[g_2, g_3]) \), i.e.,

\[
E^2_{p, q} = \text{Tor}^Q_{p, q}(H^c_{q}(U, Q[g_2, g_3], K(\varphi)))
\]

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with the abutment $H^0_n(U_\wp, K(\wp))$, where $K(\wp)$ is the characteristic zero residue field at $\wp \in \text{Spec}(\mathbb{Q}[g_2, g_3])$.

Let us consider the unequal characteristic case. Suppose that $\wp$ is a complete discrete valuation ring with the quotient field $K$ and residue class field $k$ and suppose that $A$ is an $\wp$-algebra. Let $X$ be a scheme over $A = \left( A \otimes_{k} k \right)_{\text{red}}$. Suppose that $K(\wp)$ is a finite field at $\wp \in \text{Spec}(A)$ and let $W(K(\wp))$ be the complete discrete valuation ring and denote the quotient field of $W(K(\wp))$ by $K_\wp = W(K(\wp)) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then the zeta function of the fibre $X_\wp$ at $\wp$ is given by

$$Z_{X_\wp}(T) = \prod_{p+q = \text{odd}} P_{p,q}(T) \prod_{p+q = \text{even}} P_{p,q}(T)$$

where $P_{p,q}(T)$ is the reverse characteristic polynomial of the endomorphism of

$$E^2_{p,q} = \text{Tor}^{\mathbb{Z}}_{p+q}(\mathbb{Z} \otimes \mathbb{Q}, K_\wp)$$

induced by the $p^r$-th power map, $p^r = \text{card}(K(\wp))$ (see pp. 448–450, [6]). This homological spectral sequence abuts upon $H^0_n(X_\wp, K_\wp)$. Therefore if one knows the lifted $p$-adic homology with compact supports of $X$ over $A$, $H^0_n(X, A \otimes_{\mathbb{Z}} \mathbb{Q})$, $h \geq 0$, and the zeta endomorphisms of these groups, (1) determines the zeta function of every fibre over a finite field in the algebraic family $X$ over the ring $A$. These are the subjects in the forthcoming paper [2].

The main result of the paper is the explicitness of the generation of the first homology with compact supports of the entire Weierstrass Family $W_0$ in the characteristic zero (Theorem 1) and its consequences.

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1. In this section (notations being the same as in the Introduction) we describe explicitly the basis elements over the ring $\mathbb{Q}[g_2, g_3]$ which generate the first homology with compact supports of the Weierstrass Family over the field of rational numbers $\mathbb{Q}$, $H^1_0(U, \mathbb{Q}[g_2, g_3])$. By the definition of the lifted $p$-adic homology with compact supports [6, p. 415], applied to the characteristic zero case, we have

$$H^1_0(U, \mathbb{Q}[g_2, g_3])$$

$$= H^1_0(\text{Spec}(\mathbb{Q}[g_2, g_3]), A^2(\text{Spec}(\mathbb{Q}[g_2, g_3])) - U, \Gamma_{\mathbb{Q}[g_2, g_3]}(\text{Spec}(\mathbb{Q}[g_2, g_3])))$$

If one tensors $H^1_0(U, \mathbb{Q}[g_2, g_3])$ with $\Delta^{-1}\mathbb{Q}[g_2, g_3]$ over $\mathbb{Q}[g_2, g_3]$, one has the free $\Delta^{-1}\mathbb{Q}[g_2, g_3]$-module of rank two, where $\Delta = g_2^3 - 27g_3^2$. This is so because we have the universal coefficients spectral sequence

$$E^2_{0,1} = H^1_0(U, \mathbb{Q}[g_2, g_3]) \otimes_{\mathbb{Q}[g_2, g_3]} \Delta^{-1}\mathbb{Q}[g_2, g_3] \overline{\otimes} H^1_0(U, \Delta^{-1}\mathbb{Q}[g_2, g_3]),$$

and $\Delta^{-1}\mathbb{Q}[g_2, g_3]$ means that the ring $\mathbb{Q}[g_2, g_3]$ is localized at the discriminant $\Delta$. The computation has been made even in the $p$-adic case in [1] for this open subfamily of the Weierstrass Family.
Theorem 1. Consider $U = W_0 \cap \mathbb{A}^2(\text{Spec}(\mathbb{Q}[g_2, g_3]))$, which is the closed affine subscheme of $\mathbb{A}^2(\text{Spec}(\mathbb{Q}[g_2, g_3]))$. Then the first homology with compact supports $H^1_\text{c}(U, \mathbb{Q}[g_2, g_3])$ is generated by $\{C^{-1}dX \wedge dy\}_{i=1}$ and $\{XC^{-1}dX \wedge dY\}_{i=1}$ as a $\mathbb{Q}[g_2, g_3]$-module.

Remark 1. For the pair of affine schemes

\[ \mathbb{A}^2(\text{Spec}(\mathbb{Q}[g_2, g_3])) \quad \text{and} \quad \mathbb{A}^2(\text{Spec}(\mathbb{Q}[g_2, g_3])) - U, \]

where $U$ is the closed subscheme corresponding to the polynomial $C = Y^2 - 4X^3 + g_2X + g_3$ in $\mathbb{Q}[g_2, g_3, X, Y, Z]$, there is induced a long exact sequence of hypercohomology groups,

\[ \cdots \rightarrow H^q(\mathbb{A}^2(A), \mathbb{A}^2(A) - U, \Gamma_*^q(\mathbb{A}^2(A))) \rightarrow H^q(\mathbb{A}^2(A), \Gamma_*^q(\mathbb{A}^2(A))) \rightarrow \cdots \]

where $A = \text{Spec}(\mathbb{Q}[g_2, g_3])$.

There are three first-quadrant spectral sequences induced which have the above three hypercohomology groups as their abutments:

\[ 'E_\infty^{p,q} = H^q(\mathbb{A}^2(A) - U, \Gamma_*^q(\mathbb{A}^2(A))), \]

\[ E_2^{p,q} = H^q(\mathbb{A}^2(A), \Gamma_*^q(\mathbb{A}^2(A))), \]

\[ 'E_3^{p,q} = H^q(\mathbb{A}^2(A), \mathbb{A}^2(A) - U, \Gamma_*^q(\mathbb{A}^2(A))). \]

Lemma 1. We have the following isomorphisms: the abutment

\[ 'E_3 = H^3(\mathbb{A}^2(A), \mathbb{A}^2(A) - U, \Gamma_*^q(\mathbb{A}^2(A))) \cong 'E_2^{2,1}, \]

and

\[ 'E_3 \cong 'E_2 = H^2(\mathbb{A}^2(A) - U, \Gamma_*^q(\mathbb{A}^2(A))) \subseteq \text{coker}( 'E_1^{2,0} \leftarrow 'E_1^{1,0}). \]

Proof of Lemma 1. Consider the following diagram (Diagram A) with exact rows.

\[ 0 \rightarrow \ker d_1^{2,0} \rightarrow \ker 'd_1^{2,0} \rightarrow \ker 'd_1^{1,0} \rightarrow \text{coker} 'd_1^{1,0} \rightarrow \text{coker} 'd_1^{2,0} \rightarrow \text{coker} 'd_1^{3,0} \rightarrow \text{coker} 'd_1^{4,0} \rightarrow 0 \]

\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]

\[ \mathcal{E}^2 \quad ' \mathcal{E}^2 \quad ' \mathcal{E}^3 \quad 0 \]

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and since the $Q[g_2, g_3]$-homomorphism

$$d_1^{1,0}: E_1^{1,0} = \Gamma_{Q[g_2, g_3]}(Q[g_2, g_3, X, Y]) \to E_1^{2,0} = \Gamma_{Q[g_2, g_3]}(Q[g_2, g_3, X, Y])$$

is an epimorphism, we have $E^2 \approx E_2^{2,0} \approx 0$. Therefore

$$'E^2 \rightsquigarrow \text{coker} 'd_1^{1,0} \rightsquigarrow \text{coker} ''d_1^{1,1} \approx ''E^3$$

as stated in Lemma 1. Q.E.D.
Hence our computation of the abutment \( E^3 = H^3(A^2(A), A^2(A) - U, \Gamma^*_{\text{c}}(A^2(A))) \) is reduced to compute
\[
\text{coker} \left( \Gamma_{Q[\mathbb{g}_2, \mathbb{g}_3]}(Q[\mathbb{g}_2, \mathbb{g}_3, X, Y, C^{-1}]) \to \Gamma_{Q[\mathbb{g}_2, \mathbb{g}_3]}(Q[\mathbb{g}_2, \mathbb{g}_3, X, Y, C^{-1}]) \right).
\]

**Proof of Theorem 1.** From now on we denote, "\( d \)", instead of the exterior differential, "\( d^1 \)" in the spectral sequence. We have that
\[
(1) \quad d(C^{k-1}X'Y^i dX) = (-2kC^{-k-1}X'Y^{i+1} + jC^{-k}X'Y^{i-1}) dY \wedge dX,
\]
\[
(2) \quad d(C^{-k}X'Y^i dY)
\]
\[
= \left(12kC^{-k-1}X'Y^i - g_2kC^{-k-1}X'Y^i + iC^{-k}X'^{-1}Y^i\right) dX \wedge dY,
\]
in the \( Q[\mathbb{g}_2, \mathbb{g}_3] \)-module \( \Gamma_{Q[\mathbb{g}_2, \mathbb{g}_3]}(Q[\mathbb{g}_2, \mathbb{g}_3, X, Y, C^{-1}]) \), where \( C = Y^2 - 4X^3 + g_2X + g_3 \), \( i, j \) and \( k \) are nonnegative integers. The equations (1) and (2) give the cohomologous relations, which are denoted by "\( \sim \)", as
\[
(3) \quad 2kC^{-k-1}X'Y^i dX \wedge dY \sim jC^{-k}X'Y^i dX \wedge dY
\]
and
\[
(4) \quad (12kC^{-k-1}X^{i+2}Y^j - g_2kC^{-k-1}X'^{-1}Y^i + iC^{-k}X'^{-1}Y^i) dX \wedge dY \sim 0.
\]

Notice that, by Lemma 1:
\[
\varphi^{E^2,1} \sim \varphi^{E^1,1}/\text{Im}(\varphi^{E^1,0} \sim \varphi^{E^1,1})
\]
and
\[
\varphi^{E^1,1} \sim \varphi^{E^1,0}/\text{Im}(\varphi^{E^1,0} \sim \varphi^{E^1,1}),
\]
where \( E^1_{1/0} \sim \Gamma_{Q[\mathbb{g}_2, \mathbb{g}_3]}(Q[\mathbb{g}_2, \mathbb{g}_3, X, Y]) \). Therefore it suffices to consider the integer \( k \geq 1 \) in the equations (1), (2), (3) and (4) above.

If \( j = 0 \) in (3), then \( C^{-k-1}X'Y^i dX \wedge dY \sim 0 \) for all \( i \geq 0 \) and \( k \geq 1 \). But (4) implies that \( C^{-1}X'Y^i dX \wedge dY \sim 0 \) for \( i \geq 0 \) since \( iC^{-1}X'^{-1}Y^i dX \wedge dY \sim g_2C^{-2}X'Y^i dX \wedge dY - 12C^{-2}X'^{-2}Y^i dX \wedge dY \). Therefore,
\[
(5) \quad C^{-k}X'Y^i dX \wedge dY \sim 0 \quad \text{for all integers } i, k \geq 0.
\]

For any odd integer \( j > 1 \) we have \( C^{-k}X'Y^i dX \wedge dY \sim 0 \) by combining (3) and (5) and the repeated use of (4). For example, for \( j = 3 \), we have \( 12kC^{-k-1}X'^{-3}Y^i dX \wedge dY \sim 2C^{-k}X'Y^i dX \wedge dY \), which is cohomologous to zero by (5). Then apply (4) for \( j = 3 \) to get
\[
iC^{-k-1}X'^{-1}Y^3 dX \wedge dY \sim g_2kC^{-k-1}X'^{-3}Y^3 dX \wedge dY - 12kC^{-k-1}X'^{-1}Y^3 dX \wedge dY.
\]

But the right-hand side is cohomologous to zero from the above result. If \( i = 0 \) in (4), we then have
\[
(6) \quad 12kC^{-k-1}X'^{-3}Y^i dX \wedge dY \sim g_2kC^{-k-1}X'^{-1}Y^i dX \wedge dY
\]
for all integers \( k \geq 1 \) and \( j \geq 0 \). Especially we have, for \( j = 0 \), \( 12kC^{-k-1}X'^{-2}Y^i dX \wedge dY \sim g_2kC^{-k-1}X'^{-1}Y^i dX \wedge dY \). Then it can be plainly seen that
\[
(C^{k} dX \wedge dY)_{k \geq 1}, \quad (X C^{k-1} dX \wedge dY)_{k \geq 1} \quad \text{and} \quad (X' C^{k-1} dX \wedge dY)_{i \geq 2}
\]
generate all the elements of the type $X'C^{-k}dX \wedge dY$ for integers $i \geq 0$ and $k \geq 0$ over the ring $\mathbb{Q}[g_2, g_3]$ from equations (3) and (4). In particular, $X^2C^{-1}dX \wedge dY \sim X^2Y^{-2}C^{-2}dX \wedge dY$ by (3) for letting $i = 2$, $j = 1$ and $k = 1$, but $X^2Y^{-2}C^{-2}dX \wedge dY \sim Y^{-2}C^{-2}dX \wedge dY$ by (4) for $i = 0$, $j = 2$ and $k = 1$; furthermore, $Y^{-2}C^{-2}dX \wedge dY$ is cohomologous to $C^{-1}dX \wedge dY$ from (3) for $i = 0$, $j = 1$ and $k = 1$. Hence we have established that $X^2C^{-1}dX \wedge dY \sim C^{-1}dX \wedge dY$. Next we claim that all the elements of the type $(X'C^{-1}dX \wedge dY)_{i \geq 3}$ are generated by the two elements $C^{-1}dX \wedge dY$ and $XC^{-1}dX \wedge dY$ over the ring $\mathbb{Q}[g_2, g_3]$. We have the following recursive formula for integers $i \geq 3$ from (3) and (4):

$$
4X'C^{-1}dX \wedge dY \sim g_2 \left( \frac{1}{12(i-2)} + 1 \right) X^{i-2}C^{-1}dX \wedge dY
$$

$$
+ \left( g_3 - \frac{1}{i-2} \right) X^{i-3}C^{-1}dX \wedge dY.
$$

Therefore it follows from this recursive formula that $(X'C^{-1}dX \wedge dY)_{i \geq 3}$ are generated by $C^{-1}dX \wedge dY$ and $XC^{-1}dX \wedge dY$ over $\mathbb{Q}[g_2, g_3]$. We have established the statement of Theorem 1 for the elements $X'C^{-1}dX \wedge dY$ with $i \geq 1$, $j = 0$ and $k \geq 1$. Now we need consider the elements $X^iY'C^{-k}dX \wedge dY$ for $j = 1, 2, 3, \ldots$. As noted before, we know that if $j$ is an odd integer, $X^iY'C^{-k}dX \wedge dY \sim 0$. If $j$ is an even integer, the repeated use of (3) and (4) for the elements $X^iY'C^{-k}dX \wedge dY$, $i \geq 1$ and $j \geq 1$, provides the generation of the first homology with compact supports $H_1(U, \mathbb{Q}[g_2, g_3])$ of the Weierstrass Family by the elements $(C^{-k}dX \wedge dY)_{k \geq 1}$ and $(XC^{-k}dX \wedge dY)_{k \geq 1}$. Q.E.D.

**Proposition 1.** Assumptions and notations being the same as in Theorem 1, $H_1(U, \mathbb{Q}[g_2, g_3]) \otimes_{\mathbb{Q}[g_2, g_3]} (\Delta^{-1} \mathbb{Q}[g_2, g_3])$ is a free $(\Delta^{-1} \mathbb{Q}[g_2, g_3])$-module of rank two, i.e., it is generated by $XC^{-1}dX \wedge dY$ and $C^{-1}dX \wedge dY$, where $\Delta$ is the discriminant, $\Delta = g_2^2 - 27g_3^2$, and $\Delta^{-1} \mathbb{Q}[g_2, g_3]$ is localized at the discriminant $\Delta$.

**Proof of Proposition 1.** For any integer $i \geq 2$ we have

$$
C^{-i-1} = C^{-i}(Y^2 - 4X^3 + g_2X + g_3),
$$

where $dX \wedge dY$ is omitted for simplicity, and from equations (3), (4) and (6) we have the following cohomologous relation for $i \geq 2$:

$$
\frac{6i - 11}{6(i-1)} C^{-i-1} \sim \frac{2g_2}{3} XC^{-i} + g_3 C^{-i}.
$$

Similarly, one has the corresponding formula for $XC^{-i-1}$ by the equations (3), (4) and (6):

$$
\frac{6i - 13}{6(i-1)} XC^{-i-1} \sim \frac{g_2^2}{18} C^{-i} + g_3 XC^{-i}.
$$
We finally have for $i \geq 2$,
\[
C^{-i}dX \wedge dY \sim \frac{18}{\Delta} \left\{ \frac{g_2(6i - 13)}{6(i - 1)} X^{(-i - 1)} dX \wedge dY \right\} \left\{ \frac{g_2(6i - 11)}{4(i - 1)} C^{-i}dX \wedge dY \right\}
\]
from equations (1.1) and (1.2).

Equations (1.3) and (1.1) prove that $H^i(U_\varphi, \mathbb{Q}[g_2, g_3]) \otimes \mathbb{Q}[g_2, g_3] (\Delta^{-1} \mathbb{Q}[g_2, g_3])$ is generated by $X^{(-i)} dX \wedge dY$ and $X^{-i - 1} dX \wedge dY$ as a $(\Delta^{-1} \mathbb{Q}[g_2, g_3])$-module. Q.E.D.

**Corollary 1.** Let $V_0^0$ be the closed subfamily defined by “$g_2 = 0$” of the whole Weierstrass family $W_0^0$. Then the first homology with compact supports,

\[
H^i_f(V_0^0 \cap \Lambda^2(\text{Spec} \mathbb{Q}[g_3]), \mathbb{Q}[g_3]),
\]

is generated by \{\$X^{-k} dX \wedge dY\}_{k \geq 1}$ and \{\$X^{-k} dX \wedge dY\}_{k \geq 1}$ as a $\mathbb{Q}[g_3]$-module.

**Proof.** In (1.1) and (1.2) in the proof of Proposition 1, we have the following corresponding equations for the closed subfamily $V_0^0$ defined by “$g_2 = 0$”:

\[
(1.1)^0 \quad \frac{12i - 22}{12(i - 1)} C^{(-i - 1)} \sim g_3 C^{(-i)};
\]

\[
(1.2)^0 \quad \frac{6i - 13}{6(i - 1)} X^{(-i - 1)} \sim g_3 X^{(-i)}.
\]

Then the statement of Corollary 1 follows plainly from (1.1)$^0$ and (1.2)$^0$. Q.E.D.

**Note 1.** The equations (1.1)$^0$ and (1.2)$^0$ also show that Corollaries 2 and 3 are true.

**Corollary 2.** The first homology with compact supports of the singular fibre $U_\varphi$ over a point $\varphi = (g_2 = 0, g_3 = 0) \in \text{Spec} \mathbb{Q}[g_2, g_3])$, a projective line with a cusp (or $\varphi = (g_3 = 0) \in \text{Spec} \mathbb{Q}(g_3))$, $H^i_f(U_\varphi, \mathbb{Q})$, is trivial.

**Corollary 3.** Notations being the same as in Proposition 1,

\[
H^i_f(V_0^0 \cap \Lambda^2(\text{Spec} \mathbb{Q}[g_3]), \mathbb{Q}[g_3]) \otimes \mathbb{Q}(g_3) (\Delta^{-1} \mathbb{Q}[g_3])
\]

is generated by the two elements $C^{-1} dX \wedge dY$ and $X^{-1} dX \wedge dY$, where $g_3^{-1} \mathbb{Q}[g_3]$ means the localization of the ring $\mathbb{Q}[g_3]$ at $g_3$.

**Remark 2.** For a point $\varphi \neq (g_3 = 0)$, $H^i_f(U_\varphi, \mathbb{K}(\varphi))$ is generated by $C^{-1} dX \wedge dY$ and $X^{-1} dX \wedge dY$ as a $\mathbb{K}(\varphi)$-vector space and where $\mathbb{K}(\varphi)$ is the characteristic zero residue field, i.e., $U_\varphi$ is an elliptic curve. Note that the open subfamily of the Weierstrass family over $\mathbb{Z}/P\mathbb{Z}$ defined by “$\Delta \neq 0$” has been computed explicitly using the hypercohomology of a flat lifting with coefficients in the $\mathbb{Z}$ of sheaves of differential forms, $H^i(U, (\Delta^{-1} \mathbb{Z}_p g_2, g_3)) \otimes \mathbb{Z} \mathbb{Q}$, where $(\Delta^{-1} \mathbb{Z}_p g_2, g_3))$ is the $\mathbb{Z}$ of the localization of the ring $\mathbb{Z}_p g_2, g_3$ at the discriminant $\Delta = g_2^3 - 27g_3^2$, see [1]. The following universal coefficient spectral sequence explains the relationship between Corollary 2 and Theorem 1.

\[
E^2_{p,q} = \text{Tor}^\mathbb{Q}(\bar{\mathbb{Z}}, \mathbb{Q})(H^i_f(U, \mathbb{Q}[g_2, g_3], \mathbb{K}(\varphi)), \mathbb{K}(\varphi)) \text{ with the abutment } H^i_n(U_\varphi, \mathbb{K}(\varphi)),
\]

where $\varphi = (g_2 = g_3 = 0) \in \text{Spec} \mathbb{Q}(g_2, g_3)$ and $\mathbb{K} = \mathbb{K}(\varphi)$. 
Corollary 4. Let $\mathcal{V}_Q^3$ be the closed subfamily of the Weierstrass Family $\mathcal{W}_Q$, defined by $g_2 = 3$. Then $H^i_!(\mathcal{V}_Q^3 \cap A^2(\text{Spec } \mathbb{Q}[g_3]), \mathbb{Q}[g_3])$ is generated by $(C^{-i}dX \wedge dY)_{k>1}$ and $(XC^{-i}dX \wedge dY)_{k>1}$ as a $\mathbb{Q}[g_3]$-module. Moreover the first homology with compact supports of the singular fibre over the point $\varphi = (g_3 = 1)$ in the base $\text{Spec}(\mathbb{Q}[g_3])$, a projective line with an ordinary double point over $K(\varphi)$, is generated by one element as a $K(\varphi)$-vector space. One can then take either $C^{-i}dX \wedge dY$ or $XC^{-i}dX \wedge dY$ as the base element for the vector space.

Proof. We only need prove the latter statement. From equations (1.1) and (1.2), we have (1.1)$^3$ and (1.2)$^3$ as follows:

\begin{align*}
(1.1)^3 & \quad \frac{6i-11}{6(i-1)} C^{-(i-1)} \sim 2XC^{-i} + C^{-i}, \\
(1.2)^3 & \quad \frac{6i-13}{6(i-1)} XC^{-(i-1)} \sim \frac{1}{2} C^{-i} + XC^{-i}.
\end{align*}

Then we have $2(6i-13)XC^{-(i-1)} \sim (6i-11)C^{-(i-1)}$ for $i \geq 2$. Hence this vector space is one dimensional and the statement of Corollary 4 follows. Q.E.D.

Note 2. For the closed subfamily $\mathcal{V}_Q^3$ of the Weierstrass Family we have the following equations (1.1)$^3$, (1.2)$^3$ and (1.3)$^3$:

\begin{align*}
(1.1)^3 & \quad \frac{6i-11}{6(i-1)} C^{-(i-1)} \sim 2XC^{-i} + g_3C^{-i}, \\
(1.2)^3 & \quad \frac{6(i-13)}{6(i-1)} XC^{-(i-1)} \sim \frac{1}{2} C^{-i} + g_3XC^{-i}, \\
(1.3)^3 & \quad \frac{(g_3^2-1)C^{-i}}{6(i-1)} \sim \frac{1}{6(i-1)} \left( g_3(6i-11)C^{-(i-1)} - 2(6i-13)XC^{-(i-1)} \right),
\end{align*}

for integers $i \geq 2$.

Note 3. This paper has been entirely in characteristic zero. The case of nonzero characteristic $p \neq 2, 3$ will appear in a forthcoming paper [2], which is a generalization of the paper [1], where an open subfamily $\Delta \neq 0$ of the Weierstrass Family was studied.

References


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