COCYCLES AND LOCAL PRODUCT DECOMPOSITION

BY
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Dedicated to Professor Haruo Sunouchi on his 60th birthday

Abstract As an application of cocycles, we establish a relation between the classical Hardy spaces on the real line \( \mathbb{R} \) and simply invariant subspaces on a quotient of the Bohr group. When this result is specialized suitably, it yields the well-known results concerning the elements of invariant subspaces. We also study, by using Gamelin's representation theorem, unitary functions which are the values of cocycles.

1. Preliminaries. Let \( K \) be a compact abelian group, not a circle, dual to a subgroup \( \Gamma \) of the discrete real line \( \mathbb{R} \). For each \( t \) in \( \mathbb{R} \), \( e_t \) is the element of \( K \) defined by \( e_t(x) = e^{2\pi i x t} \) for all \( x \) in \( \Gamma \). Choose and fix a positive \( \gamma \) in \( \Gamma \), and let \( K_{\gamma} \) be the compact subgroup consisting of all \( x \) in \( K \) such that \( x(\gamma) = 1 \). Then \( K \) may be identified measure-theoretically, and almost topologically, with \( K_{\gamma} \times [0, 2\pi/\gamma) \) via the mapping \( y + e_s \) to \( (y, s) \). We suppose for simplicity that \( 2\pi \) lies in \( \Gamma \) in this section, and §§2 and 4. Thus \( K \) may be regarded as \( K_{2\pi} \times [0, 1) \). Let \( \sigma \) and \( \sigma_1 \) be the normalized Haar measures on \( K \) and \( K_{2\pi} \), respectively. Then we may consider \( d\sigma = d\sigma_1 \times dt \) on \( K_{2\pi} \times [0, 1) \).

Our objective in this note, by using this local product decomposition, is to show the fact that a certain class of analytic functions on \( K_{2\pi} \times \mathbb{R} \) has a close connection to simply invariant subspaces on \( K \). In the next section, our characterization of simply invariant subspaces, Theorem 2.1, is obtained. In §3 we investigate the values of cocycles and answer a question of Helson. We close with some remarks in §4.

For any simply invariant subspace \( \mathcal{M} \) of \( L^2(\sigma) \), we define
\[(\mathcal{M})_+ = \bigcap_{\lambda < 0} \chi_\lambda \mathcal{M} \quad \text{and} \quad (\mathcal{M})_- = \text{the closure} \bigcup_{\lambda > 0} \chi_\lambda \mathcal{M},\]
where \( \chi_\lambda \) denotes the character on \( K \) determined by \( \lambda \) in \( \Gamma \). Then \( \mathcal{M} \) is called to be normalized if \( \mathcal{M} = (\mathcal{M})_+ \). Complex-valued functions of modulus one are said to be unitary functions. A cocycle is a unitary Borel function \( A(x, t) \) on \( K \times \mathbb{R} \) which satisfies the cocycle identity
\[A(x, t + u) = A(x, t)A(x + e_t, u)\]
for all \( x \) in \( K \) and \( s, t, u \) in \( R \). A cocycle is trivial (resp. a coboundary) if it has the form \( e^{it'}p(x)p(x + e_t) \) (resp. \( p(x)p(x + e_t) \)) for some \( r \) in \( R \) and some unitary function \( p \) on \( K \). There exists a one-to-one correspondence between normalized simply invariant subspaces and cocycles [4, Chapter 2].

We denote by \( H^p(\sigma) \) and \( H^p(dt) \), \( 0 < p \leq \infty \), the usual Hardy spaces on \( K \) and \( R \), respectively. It is known that \( H^1(dt) \) is the space of all functions in \( L^1(dt) \) whose Fourier transforms vanish on the negative real line. We let \( H^\infty(dt/(1 + t^2)) = H^\infty(dt) \), that is, the space of all the boundary functions of bounded analytic functions in the upper half-plane. The closure of \( H^\infty(dt/(1 + t^2)) \) in \( L^p(dt/(1 + t^2)) \) is denoted by \( H^p(dt/(1 + t^2)) \), \( 0 < p < \infty \). Recall that the class of continuous function \( \phi \) in \( H^p(dt) \) with \( |\phi(t)| = O(t^{-2}) \) (as \( |t| \to \infty \)) is dense in \( H^p(dt) \), \( 0 < p < \infty \) (cf. [3, Chapter II, §3]). Also recall that \( \phi \) lies in \( H^p(dt/(1 + t^2)) \) if and only if \( \phi(t)(t + i)^{-2/p} \) lies in \( H^p(dt) \).

We refer the reader to [4 and 2, Chapter VII] for further details of analyticity on compact abelian groups and to [1 and 3] for results about classical Hardy spaces.

The following lemma is a minor variation of [4, Theorem 17] so the proof will be omitted.

**Lemma 1.1.** Let \( \mathcal{M}_A \) be the normalized simply invariant subspace of \( L^2(\sigma) \) associated with a cocycle \( A \). Then for any \( f \) in \( L^\infty(\sigma) \), the following are equivalent:

(i) \( f \) lies in \( \mathcal{M}_A \);

(ii) the function of \( t, A(y, t)f(y + e_t) \), lies in \( H^\infty(dt/(1 + t^2)) \) for \( \sigma_1 \)-a.a. \( y \) in \( K_{2\pi} \); and

(iii) the function of \( t, A(y, t)f(y + e_t) \), is orthogonal to \( H^1(dt) \) for \( \sigma_1 \)-a.a. \( y \) in \( K_{2\pi} \).

We next consider certain spaces of analytic functions on \( K_{2\pi} \times R \). Let \( \mathcal{H} \) be the space of all bounded Borel functions \( f(y, t) \) on \( K_{2\pi} \times R \) which satisfy

\[(1.2) \text{the function of } f(y, t), \text{ belongs to } H^1(dt) \text{ for } \sigma_1 \text{-a.a. } y \text{ in } K_{2\pi}, \text{ and} \]

\[(1.3) \text{ess.sup} \{|f(y, t)| ; (y, t) \text{ in } K_{2\pi} \times [n, n + 1]| = O(n^{-2}). \]

We denote by \( \mathcal{H}^p, 0 < p < \infty \), the closure of \( \mathcal{H} \cap L^p(d\sigma_1 \times dt) \) in \( L^p(d\sigma_1 \times dt) \), where we use the ordinary metric on \( L^p(d\sigma_1 \times dt) \) when \( 0 < p < 1 \). Let \( B(y, t) \) be a unitary function on \( K_{2\pi} \times R \). Then for each \( f \) in \( \mathcal{H} \), we define a bounded Borel function \( \Phi_B(f) \) on \( K \) by

\[(1.4) \Phi_B(f)(y, s) = \sum_{n=-\infty}^{\infty} B(y - e_n, s + n) f(y - e_n, s + n) \]

for each \((y, s)\) in \( K_{2\pi} \times [0, 1) \). Then \( \Phi_B \) is a linear mapping of \( \mathcal{H} \) into \( L^\infty(\sigma) \). Moreover, for any \( p, 0 < p \leq 1 \), it can be easily seen that the restriction of \( \Phi_B \) to \( \mathcal{H} \cap L^p(d\sigma_1 \times dt) \) may be extended to a bounded linear mapping of \( \mathcal{H}^p \) into \( L^p(\sigma) \) (cf. [7, Lemma 1]).

**2. Cocycles and the space \( \mathcal{H} \).** We may now state our main result.

**Theorem 2.1.** Let \( \mathcal{M}_A \) be the simply invariant subspace of \( L^2(\sigma) \) associated with a cocycle \( A \). Then \( \Phi_A(\mathcal{H}) \) is dense in \( (\mathcal{M}_A)_c \).
PROOF. We first note that, for each $(y, 0) \in K_{2\pi} \times [0, 1), y + e = (y, 0) + e(= (y + e_{[t]}, t - [t]),$ where $[t]$ is the largest integer not exceeding $t$. It follows from the cocycle identity (1.1) that

$$A(y, t) A\left(y - e_{n-[t]}, n + t - [t]\right) = A(y, t) A\left(y - e_{n-[t]}, n - [t]\right) A(y, t)$$

for each $(y, t)$ in $K_{2\pi} \times R$. Hence if $\Phi(y, t)$ lies in $\mathcal{H}$, then we obtain

$$A(y, t) \Phi(y + e) = A(y, t) \Phi(y + e_{[t]}, t - [t])$$

by (1.4). Let $\Phi$ be any function in $H^1(dt)$. Then the property (1.3) assures that

$$\int_{-\infty}^{\infty} A(y, t) \Phi(y + e) \phi(t) dt = 0$$

for $\sigma$-a.a. $y$ in $K_{2\pi}$. Therefore, by Lemma 1.1, $\Phi(y, t)$ belongs to $\mathfrak{M}_A$. Let $p$ be a function in $\mathfrak{M}_A$ which is orthogonal to $\Phi(y, t)$. We set $g(y, t) = p(y + e)$. Then it can be seen that

$$\int_{-\infty}^{\infty} g(y, t) A(y, t) f(y, t) d\sigma(y) dt = 0$$

for each $f$ in $\mathcal{H}$. This implies that the function of $t, A(y, t) p(y + e)$, lies in $H^2(dt/(1 + t^2))$ for $\sigma$-a.a. $y$ in $K_{2\pi}$. On the other hand, since the function of $t, A(y, t) p(y + e)$, lies in $H^2(dt/(1 + t^2))$, it must be constant. Hence $|p|$ is constant on $K$. From this, we may assume that $p$ is a unitary function on $K$. Thus we have $A(x, t) = p(x) p(x + e)$ and $\mathfrak{M}_A = pH^2(\sigma)$. This completes the proof.

We collect some corollaries following from Theorem 2.1. Recall that a cocycle $A(x, t)$ is continuous if $A(y, t)$ is continuous on $K_{2\pi} \times R$ as a function of $(y, t)$ [4, Chapter 5]. Let $C_0(K_{2\pi} \times R)$ denote the space of all continuous functions on $K_{2\pi} \times R$ which vanish at infinity. We notice that $\mathcal{H} \cap C_0(K_{2\pi} \times R)$ is dense in $\mathcal{H}$ as
a subspace of $L^1(d\sigma_1 \times dt)$, and that $\Phi_A(f)$ lies in $C(K)$ for any $f$ in $\mathcal{H} \cap C_0(K_{2\pi} \times R)$.

These facts easily imply the following.

**Corollary 2.2.** Let $A$ and $\mathcal{M}_A$ be as in Theorem 2.1. If $A$ is continuous, then $(\mathcal{M}_A)_- \cap C(K)$ is dense in $(\mathcal{M}_A)_-$.

We next give another proof of Helson’s existence theorem [4, Theorem 16; 8, 10].

**Corollary 2.3.** Let $A$ and $\mathcal{M}_A$ be as in Theorem 2.1. Then $\mathcal{M}_A$ contains a unitary function.

**Proof.** Define a function $w(t)$ in $L^1(dt)$ by

$$w(t) = \begin{cases} n^{-2} & \text{on } [n, n + 1), |n| \geq 1, \\ 3 \sum_{j=1}^{\infty} j^{-2} & \text{on } [0, 1). \end{cases}$$

It is easy to see that $\log w(t)$ belongs to $L^1(dt/(1 + t^2))$. Hence there is a function $\phi$ in $H^1(dt)$ such that $|\phi(t)| = w(t)$ (cf. [3, Chapter II, Theorem 4.4]). If we set $f(y, t) = \phi(t)$, then $\Phi_A(f)$ lies in $\mathcal{M}_A$ and $|\Phi_A(f)| \geq \sum_{j=1}^{\infty} j^{-2}$ on $K$. Thus it follows from Szegö’s theorem that $\mathcal{M}_A$ contains a unitary function.

**Corollary 2.4.** Let $A$ and $\mathcal{M}_A$ be as in Theorem 2.1. Then there exists a function $f$ in $\mathcal{M}_A$ satisfying:

(i) the function of $t, f(y + e_i)$, can be extended analytically to $\{z; \text{Im } z > -\sqrt{3}/2\}$ for each $\sigma_1$-a.a. $y$ in $K_{2\pi}$;

(ii) $\log |f(x)|$ belongs to $L^1(\sigma)$; and

(iii) for any positive $\lambda$ in $\Gamma$, $f$ does not lie in $\chi_\lambda \mathcal{M}_A$.

**Proof.** Let $h(y, t) = 4/(2t - 1 + \sqrt{3}i)^2$. Then $|h(y, t)| \geq 1$ on $K_{2\pi} \times [0, 1]$, and $|h(y, t)| < 1$ otherwise. So by Theorem 2.1 we may choose an integer $m$ such that the function $g_1 = \Phi_A(h^m)$ in $\mathcal{M}_A$ satisfies (i) and $|g_1(y, s)| > 1$ on $K_{2\pi} \times [\frac{1}{6}, \frac{5}{6})$. It may be assumed that $g_1$ has property (iii). Similarly, we can construct a function $g_2$ in $\mathcal{M}_A$ which satisfies (i) and $|g_2(y, s)| > 1$ on $K_{2\pi} \times \{(0, \frac{1}{6}) \cup (\frac{5}{6}, 1)\}$. It follows from Jensen’s inequality and Fubini’s theorem that

$$\frac{1}{2\pi} \int_{K_{2\pi}} \int_{0}^{2\pi} \log |g_1(x)| + e^{i\theta}g_2(x) \, d\sigma(x) \, d\theta \geq \int_{K} \max(\log |g_1(x)|, \log |g_2(x)|) \, d\sigma(x) \geq 0.$$

Thus there exists a $\theta$ in $[0, 2\pi)$ for which $f = g_1 + e^{i\theta}g_2$ has the desired properties.

The above corollary is the main step in the proof of [4, Theorem 26].

We may easily choose a unitary function $B$ on $K_{2\pi} \times R$ such that the closure $[\Phi_B(\mathcal{H})_2]_2$ of $\Phi_B(\mathcal{H})$ in $L^2(\sigma)$ is a doubly invariant subspace, so it is worthwhile to note a condition under which $[\Phi_B(\mathcal{H})_2]_2$ is simply invariant.
THEOREM 2.5. Let $B$ be a unitary function on $K_{2\pi} \times \mathbb{R}$. Then $[\Phi_B(\mathcal{H})]_2$ is simply invariant if and only if there exists a cocycle $A(x, t)$ for which

(2.1) the function of $t$, $A(y, t)B(y, t)$, belongs to $H^\omega(dt/(1 + t^2))$ for $\sigma_1$-a.a. $y$ in $K_{2\pi}$.

PROOF. Suppose that there exists a cocycle $A$ with property (2.1). Then $A\tilde{B}f$ lies in $\mathcal{H}$ for each $f$ in $\mathcal{H}$. Since $\Phi_B(f) = \Phi_A(A\tilde{B}f)$, it follows from Theorem 2.1 that $[\Phi_B(\mathcal{H})]_2$ is a simply invariant subspace. Conversely, suppose that $[\Phi_B(\mathcal{H})]_2$ is simply invariant, and let $A$ be the cocycle of $([\Phi_B(\mathcal{H})]_2)_+$. Let $\phi h$ be a function in $\mathcal{H}$ which is the product of a function $\phi$ in $H^1(dt)$ times a function $h$ in $C(K_{2\pi})$. We notice that

$$A(y, t) = A(y, [t] - n)A(y + e_{[t] - n}, t - [t] + n)$$

and

$$\Phi_B(\phi h)(y + e_{[t]}) = \Phi_B(\phi h)(y + e_{[t]}, t - [t])$$

$$= \sum_{n=-\infty}^{\infty} B(y + e_{[t] - n}, t - [t] + n)\phi(t - [t] + n)h(y + e_{[t] - n}).$$

It follows from Lemma 1.1 and an argument similar to the proof of Theorem 2.1 that

$$\int_{-\infty}^{\infty} A(y, t)\Phi_B(\phi h)(y + e_{[t]})\psi(t) dt$$

$$= \sum_{m=-\infty}^{\infty} A(y, -m)h(y - e_{-m})$$

$$\times \int_{-\infty}^{\infty} A(y - e_{m}, t + m)B(y - e_{m}, t + m)\phi(t + m)\psi(t) dt$$

$$= 0$$

for each $\psi$ in $H^1(dt)$. Since $h$ is arbitrary in $C(K_{2\pi})$, we have

$$\int_{-\infty}^{\infty} A(y - e_{m}, t + m)B(y - e_{m}, t + m)\phi(t + m)\psi(t) dt = 0$$

for each integer $m$. This implies that $A$ satisfies (2.1).

3. Cocycles and unitary functions. Let $A(x, t)$ be a cocycle on $K$. In [5, §4], Helson has shown that if the function of $x$, $A_u(x) = A(x, u)$, lies in $H^2(\sigma)$, then it must be constant. This odd result grew out of a basic problem concerning spectrum of cocycles. We provide, by using Gamelin's representation theorem, some remarks on this theorem.

In this section we do not assume $2\pi$ belongs to $\Gamma$. Let $K_{\gamma}$ be as in §1 for a positive $\gamma$ in $\Gamma$, and put $u = 2\pi/\gamma$. We denote by $\mathcal{U}(K_{\gamma})$ and $\mathcal{U}(K)$ the classes of all unitary functions on $K_{\gamma}$ and $K$, respectively. We first recall the definition of cocycles introduced by Gamelin (see [2, Chapter VII, §11]). For any $\beta$ in $\mathcal{U}(K_{\gamma})$ the cocycle
$B_p(x, t)$ is given explicitly for positive $t$ by

$$B_p(y, t) = \begin{cases} 1 & \text{on } K_\gamma \times [0, u), \\ \prod_{j=0}^{m-1} \beta(y + e_ju) & \text{on } K_\gamma \times [mu, (m+1)u) \end{cases}$$

for each positive integer $m$, and $B_p(y + e_s, t) = B_p(y, s + t)$ for $s$ in $[0, u)$. Then $B_p$ is trivial if and only if there is an $f$ in $\mathcal{U}(K_\gamma)$ and some $r$ in $R$ for which $\beta$ can be expressed in the form $\beta(y) = e^{rf(y)}f(y + e_u)$ for a.a. $y$ in $K_\gamma$ [4, Chapter 4, §9].

Gamelin’s representation theorem [2, Chapter VII, Theorem 11.1] asserts that every cocycle $A$ on $K$ has the factorization $A = B_\beta C$, where $\beta$ is a function in $\mathcal{U}(K_\gamma)$, and $C$ is a coboundary.

The following theorem shows vaguely which unitary functions on $K$ are the values of cocycles and settles a question posed by Helson [5, §1]: Is the class of all $A_1$ on $K$ different from the class of all $A_2$?

**Theorem 3.1.** For any positive $u$ in $R$, let $\gamma = 2\pi/u$, and let $\{A_u\}$ denote the class of all the values $A_u(x) = A(x, u)$ of cocycles $A$. Then we obtain the following properties:

(i) if $\gamma$ belongs to $\Gamma$, then every $A_u$ in $\{A_u\}$ has the form

$$A_u(x) = \beta(y)q(x)q(x + e_u)$$

for $\sigma$-a.a. $x = (y, s)$ in $K_\gamma \times [0, u)$, where $\beta$ is a function in $\mathcal{U}(K_\gamma)$ and $q$ is a function in $\mathcal{U}(K)$;

(ii) for each positive integer $m$, $\{A_u\}$ contains $\{A_{mu}\}$; and

(iii) if $\gamma$ belongs to $\Gamma$, then for any $v$ in $(0, u)$, there exists an $A_v$ in $\{A_v\}$ which does not lie in $\{A_u\}$.

**Proof.** (i) is a direct consequence of Gamelin’s representation theorem, so it is enough to show (ii) and (iii). We notice that if $A(x, t)$ is a cocycle, then so is $A(x + x_0, t)$ for any fixed $x_0$ in $K$, and the product of two cocycles is also a cocycle. For any positive integer $m$, we set

$$B(x, t) = A(x, t)A(x + e_u, t) \cdots A(x + e_{(m-1)u}, t).$$

Then it follows from the cocycle identity (1.1) that $B(x, t)$ is a cocycle which satisfies $B(x, u) = A(x, mu)$. Thus we have (ii). On the other hand, by Gamelin’s representation theorem, we may choose a function $\beta$ in $\mathcal{U}(K_\gamma)$ for which $B_\beta$ is a nontrivial cocycle. We now show that for each $v$ in $(0, u)$, $B_\beta(x, v)$ cannot belong to $\{A_u\}$. By Definition (3.1), it can be seen that

$$B_\beta(x, v) = \begin{cases} 1 & \text{for } x = (y, s) \text{ in } K_\gamma \times [0, u - v), \\ \beta(y) & \text{for } x = (y, s) \text{ in } K_\gamma \times [u - v, u). \end{cases}$$
Suppose to the contrary that $B_{\beta}(x, v)$ belongs to $\{A_u\}$. Then by (i) there are functions $\alpha$ in $\mathfrak{U}(K_{\gamma})$ and $p$ in $\mathfrak{U}(K)$ such that

$$B_{\beta}(x, v) = \alpha(y)p(x)p(x + e_u)$$

for $\sigma$-a.a. $x = (y, s)$ in $K_{\gamma} \times [0, u)$. Therefore it follows from (3.2) and Fubini’s theorem that there is an $s$ in $[0, u - v)$ such that $\alpha(y) = p((y, s))p((y + e_u, s))$ for a.a. $y$ in $K_{\gamma}$. From this fact, we can easily see that $\beta(y) = \delta(y)\delta(y + e_u)$ for some $\delta$ in $\mathfrak{U}(K_{\gamma})$, so $B_{\beta}$ must be trivial. Thus we have a contradiction, and this completes the proof.

From (ii) and (iii) of Theorem 3.1 we have

**COROLLARY 3.2.** If $\pi$ belongs to $\Gamma$, then $\{A_1\}$ contains strictly $\{A_2\}$.

**4. Remarks.** We recall that a Borel function $f$ on $K_{2\pi} \times R$ is **automorphic** if $f(y, s + 1) = f(y + e_t, s)$ a.a. on $K_{2\pi} \times R$, and any Borel function on $K = K_{2\pi} \times [0, 1)$ can be extended uniquely to be automorphic on $K_{2\pi} \times R$ [2, Chapter VII, §6].

(a) Let $\mathcal{H}^\infty(\sigma)$ be the space of all automorphic extensions of functions in $H^\infty(\sigma)$. The following question is interesting and probably difficult:

*For any cocycle $A$, does there exist an $f$ in $\mathfrak{U}^1$ for which $(f + \Phi^{-1}_A(0)).\mathcal{H}^\infty(\sigma)$ is dense in $\mathfrak{H}^1$?*

This is related to the old problem of whether every simply invariant subspace is generated by one of its elements. Indeed, if we could choose such a function $f$, then $\Phi_A(f)$ would be a single generator of $(\mathfrak{M}_A)\cup$.

(b) We know that the dual space of $H^p(\sigma), 0 < p < 1$, has dimension one [7, 9]. By the argument of [7], Theorem 2.1 provides an extension of this result:

*Let $\mathfrak{M}$ be a simply invariant subspace of $L^p(\sigma), 0 < p < 1$. Then the dual space of $\mathfrak{M}$ has at most dimension one.*

(c) Let $\mathfrak{M}$ be a simply invariant subspace of $L^2(\sigma)$. For any $f$ in $\mathfrak{M}$, let $\tilde{f}$ denote the automorphic extension of $f$ to $K_{2\pi} \times R$. Then we may easily verify that there exists a unitary function $q$ on $K_{2\pi} \times R$ such that the closure $[\tilde{f}\mathfrak{H}]_1$ of $\tilde{f}\mathfrak{H}$ in $L^1(d\sigma_t \times dt)$ coincides with $q\mathfrak{H}^1$. Let $p(x) = p(y, s)$ be the restriction of $q$ to $K_{2\pi} \times [0, 1)$, and set $\beta(y, s) = q(y, s + 1)q(y + e_1, s)$. Since $\tilde{f}$ is automorphic, $\beta$ defines a unitary function on $K_{2\pi}$. We denote by $C_{\beta}$ the cocycle defined by (3.1). Then it can be seen that the cocycle $C_{\beta}(x, t)p(x)p(x + e_t)$ corresponds to the simply invariant subspace generated by $f$ (cf. [5, §3]). Similarly, let $\mathfrak{M}$ denote the space of all automorphic extensions of functions in $\mathfrak{M}$. Then it is not hard to see that $[\mathfrak{M} \cdot \mathfrak{K}]_1 = q\mathfrak{H}^1$ for some unitary function $q$ on $K_{2\pi} \times R$. Thus in the same manner, we may find the cocycle associated with $\mathfrak{M}$. This provides another naive definition of cocycles (cf. [4, Chapter 2]).

**REFERENCES**


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