EMBEDDING $L^1$ IN $L^1/H^1$

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ABSTRACT. It is proved that $L^1$ is isomorphic to a subspace of $L^1/H^1$. More precisely, there exists a diffuse $\sigma$-algebra $\mathcal{G}$ on the circle such that the corresponding expectation $E: H^\infty \to L^\infty(\mathbb{C})$ is onto. The method consists in studying certain martingales on the product $\mathbb{N}^\mathbb{N}$.

1. Introduction. Let us start by fixing some terminology. As usual, $\mathbb{I}$ will denote the circle equipped with its Haar measure $m$, $H^1_0$ is the subspace of those $f \in L^1(\mathbb{I})$ for which $\hat{f}(n) = 0$ for $n \leq 0$ and $q: L^1 \to L^1/H^1_0$ is the quotient map.

We are interested in the question whether or not there exists a linear embedding of the Banach space $L^1$ in the space $L^1/H^1_0$. We briefly indicate some motivation for this problem. First, it was (and still remains) an open question if the three-space-property holds for $L^1$-embedding, i.e. suppose $X$ a Banach space, $Y$ a subspace of $X$. Is it true that whenever $L^1$ embeds in $X$, it also has to embed in either $Y$ or $X/Y$?

The problem is also unsolved in the particular case $X = L^1$ and $Y$ isomorphic to a dual space. It is not hard to show that an embedding of $L^1$ in $X/Y$ is then equivalent to the existence of a subspace $S$ of $X$, $S$ isomorphic to $L^1$ so that the quotient map $X \to X/Y$ is an isomorphism when restricted to $S$.

In the special situation $X = L^1(\mathbb{I})$ and $Y = H^1_0$, the answer was unknown for some time. There was hope that this may provide a counterexample in view of the following result, due to W. B. Johnson (see [9]).

PROPOSITION 1. No complemented subspace of $L^1/H^1_0$ is isomorphic to $L^1$.

This is a consequence of the fact that any operator $T: L^1/H^1 \to L^1$ maps weakly compact sets onto norm compact sets. Let us sketch the argument.

Consider the identity map $I: L^\infty/H^\infty \to L^1/H^1$. Then $(TI)^*: L^\infty \to H^\infty \to H^1$ is integral and therefore nuclear (since $H^1$ satisfies the Radon-Nikodym property). Consequently, also $TI$ is nuclear. Given now a weakly null sequence $(x_n)_{n=1,2,...}$ in $L^1/H^1$, it follows from the lifting property (see [9] for instance) that $x_n = q(f_n)$ where $(f_n; n = 1,2,...)$ is a relatively weakly compact set in $L^1(\mathbb{I})$. Therefore, for each $\epsilon > 0$, a truncation argument provides a bounded sequence $(g_n)$ in $L^\infty$ such that $\|f_n - g_n\|_1 < \epsilon$ for each $n$. Thus

$$\|Tx_n - TI\tilde{g}_n\| < \|T\| \|x_n - I\tilde{g}_n\| < \epsilon \|T\|.$$
Because $TI$ is nuclear, the set $\{TI(\tilde{g}_n); n = 1, 2, \ldots\}$ is compact for each $\varepsilon > 0$. So we conclude that $\{T_x_n\}$ is compact, as announced.

Using Proposition 1, the following is proved in [2].

**Proposition 2.** There is no almost isometric embedding of the complex $L^1$ space in $L^1/H^1$.

Thus $d(S, L^1) > \gamma > 1$ for each subspace $S$ of $L^1/H_0^1$, where $d$ is the Banach-Mazur distance (see [8, 9] for definitions). This observation allows us to define a natural distortion of $L^1$, by taking

$$||f|| = ||f||_1 + ||q(f)||_1, \quad f \in L^1(\Pi).$$

Say that an operator $T: X \to Y$ is a semiembedding provided $T$ is one-one and maps the closed unit ball of $X$ on a norm-closed subset of $Y$. It can be shown that a semiembedding $T: L^1 \to L^1$ has to fix an $L^1$-copy (i.e. is an isomorphism when restricted to a subspace $S$ of $L^1$, $S$ isomorphic to $L^1$). On the other hand, (see [3]):

**Proposition 3.** The restriction of the quotient map $q: L^1 \to L^1/H_0^1$ to the subspace $L^1_R$ of real functions in $L^1(\Pi)$ is a semiembedding.

No example is known of a semiembedding of $L^1$ in a Banach space $X$ not containing $L^1$.

Our purpose is to prove the existence of a natural embedding of $L^1$ in $L^1/H_0^1$. There exists a diffuse $\sigma$-algebra $\mathcal{E}$ on $\Pi$ so that the restriction of $q$ to the complex $L^1(\mathcal{E})$-space is an isomorphism. More precisely:

**Theorem.** There exists an increasing sequence $(n_k)$ of positive integers, such that if $\mathcal{E}$ is the $\sigma$-algebra on $\Pi$ generated by the functions $\sigma_k(\theta) = \text{sign} \cos n_k \theta$, then the restriction of $q$ to $L^1(\mathcal{E})$ is an isomorphism. Consequently, for this $\sigma$-algebra $\mathcal{E}$, the expectation operator $E: H^\infty \to L^\infty(\mathcal{E})$ is onto.

The argument presented here is rather delicate. In order to give the reader an idea how it is organised, we briefly outline the proof. We have to introduce the $\sigma$-algebra $\mathcal{E}$ such that the inequality

\[ (*) \quad ||h - E_{\mathcal{E}}[h]||_1 \geq \delta ||h||_1 \]

holds for each $h \in H_0^1$. But choosing the sequence $(n_k)$ sufficiently lacunary, it is enough to verify $(*)$ for functions $h$ with spectrum contained in a set of the form

$$E = \{\sum v_k n_k; |v_k| \leq a_k \text{ for each } k\}$$

where $(a_k)$ is a sequence of positive integers and $(n_k), (a_k)$ satisfy the transference property. Thus the $n_k$-frequencies can be replaced by independent variables. The space $H_0^1 \cap L^1_E$ identifies with a subspace of the space $\mathcal{H} \subset L^1(\Pi^N)$ of those functions $h = \Sigma h_k$ on $\Pi^N$ such that each increment $h_k = h_k(x_1, \ldots, x_k)$ is an $H_0^1$-function in $x_k$. The required inequality now becomes

\[ (**) \quad ||h - E_{\mathcal{E}}[h]||_1 \geq ||h||_1 \]

for $h \in \mathcal{H}$, where $\mathcal{E}$ is a natural diadic product $\sigma$-algebra on $\Pi^N$ (generated by the functions $\sigma_k(x) = \text{sign} \cos x_k$).
This reduction of the problem is worked out in §4. Its purpose is to approach the problem with martingale techniques. The martingale prerequisites are given in §2. To obtain (**) we first prove \( L^1 \)-estimations for certain square functions related to \( h \) (see Lemma 4). These are derived using a “step-by-step” method (explained at the beginning of §5) and an examination of what happens at each increment. More precisely, we have to consider at this point functions of the form \( a + h - b \sigma \), where \( a, b \) are scalars, \( h \in H^1_0 \) and \( \sigma = \text{sign cos} \).

Minorations of the \( L^1 \)-norm of such expressions are given in Propositions 8 and 9 below. It is only at this place that some complex function theory will be involved.

2. Martingale preliminaries. Let \((\mathcal{F}_k)_{k=0,1,2,...}\) be an increasing sequence of \( \sigma \)-algebras on a probability space \((\Omega, \mathcal{F}, P)\) assuming \( \mathcal{F} = \bigvee_{k=1}^{\infty} \mathcal{F}_k \). Denote by \( E_k \) the expectation with respect to \( \mathcal{F}_k \). For \( f \in L^1(\mathcal{F}) \) let

\[
\|E[f]\| = \sup_k |E_k[f]| \quad \text{and} \quad S(f) = \left[ |E_0[f]|^2 + \sum_{k=1}^{\infty} |E_k[f] - E_{k-1}[f]|^2 \right]^{1/2}.
\]

We will use the notation \( C \) to indicate a numerical constant. Let us recall the following result, due to D. Davis (see [7]).

**Proposition 4.** \( C^{-1} \|S(f)\|_1 \leq \|f^*\|_1 \leq C \|S(f)\|_1 \).

The next inequality is probably known, but we include its proof here for the sake of completeness.

**Proposition 5.** Let \((v_k)\) be an adapted sequence of functions; thus \( v_k \) is \( \mathcal{F}_k \)-measurable for each \( k \). Then

\[
\left\| \sum |E_{k-1}[v_k]|^2 \right\|^{1/2} \leq C \left\| \sum |v_k|^2 \right\|^{1/2}.
\]

**Proof.** It is no restriction to assume the \( \mathcal{F}_k \) finite algebras. Moreover, since one may always tensor the \( v_k \) against a Rademacher sequence, we can assume \( E_{k-1}[v_k] = 0 \) and thus \((v_k)\) is an adapted martingale difference sequence. Since, then

\[
\left\| \sum |v_k|^2 \right\|^{1/2} = \|v_k\|_{H^1(\mathcal{F}_k)},
\]

it follows from the atomic decomposition property for \( H^1 \)-functions (see for instance [7, Chapter I]) and convexity, that we may take for \( \sum v_k \) a function of the form (for some positive integer \( j \))

\[
a = \frac{1}{|A|} (\varphi - E_{j-1}[\varphi])
\]

where \( A \) is an \( \mathcal{F}_j \)-atom, supp \( \varphi \subset A \) and \( \|\varphi\|_\infty \leq 1 \). In this case

\[
v_k = E_k[a] - E_{k-1}[a] = 0 \quad \text{for} \ k < j,
\]

\[
= \frac{1}{|A|} \left( E_k[\varphi] - E_{k-1}[\varphi] \right) \quad \text{for} \ k \geq j.
\]
Also, $E_k[\varphi]$ is supported by $A$ for $k \geq j$ and hence $v_k$ for $k > j$. Thus the left side in Proposition 5 is dominated by
\[
\|v_j\|_1 + \left\| \left( \sum_{k > j} E_{k-1}[|v_k|^2] \right)^{1/2} \right\|_1
\]
\[
\leq 2 + \int_A \left( \sum_{k > j} E_{k-1}[|v_k|^2] \right)^{1/2} \text{ (by Cauchy-Schwarz)}
\]
\[
\leq 2 + |A|^{1/2} \left( \int \sum_{k > j} |v_k|^2 \right)^{1/2}
\]
\[
\leq 2 + |A|^{1/2} \|a\|_2 \leq 3,
\]
proving the result.

**Proposition 6.** For $f \in H^1(\mathbb{G}_k)$, one has an inequality
\[
\left( \sum \| (E_k - E_{k-1}) [f] \|_2^2 \right)^{1/2} \leq C \|f\|_1^{1/2} \|f\|_{H^1}^{1/2}.
\]

To prove this, we will first deal with the special case of the Rademacher projection on the Cantor group (in fact, only this will be used later on).

**Proposition 7.** If $D = \{1, -1\}^N$ is the Cantor group and $f \in H^1(D)$, then
\[
(2\|f(k)\|_2)^{1/2} \leq \|f\|_{L^1} \|
\]
where $\hat{f}(k) = \int f(e) e_k$.

**Proof.** We will use the theorem of [6] on the BMO-distance of a BMO-function to $L^\infty$ (in the diadic setting). The result asserts, in particular, that for $\varphi \in \text{BMO}(D)$, $\text{dist}_{\text{BMO}}(\varphi, L^\infty) = 0$ and $\varepsilon > 0$, there exists a decomposition $\varphi = \alpha + \beta$ such that
\[
\|\alpha\|_{\text{BMO}} \leq C_1 \varepsilon \quad \text{and} \quad \|\beta\|_{\infty} \leq C_2 \max(\varepsilon, \lambda_0(\varepsilon))
\]
where $\lambda_0 = \lambda_0(\varepsilon)$ has to satisfy
\[
\sup_I \frac{1}{|I|} \left| \{ x \in I, |\varphi(x) - \varphi_I| > \lambda \} \right| \leq e^{-\lambda/\varepsilon}
\]
whenever $\lambda > \lambda_0$ ($\varphi_I = |I|^{-1} \int_I \varphi$).

Now take $\varphi = \sum a_k e_k$ with $\sum |a_k|^2 = 1$. It follows from the distribution property of Rademacher that for each diadic interval $I$,
\[
\left| \{ \alpha \in I; |\varphi(x) - \varphi_I| > \lambda \} \right| \leq C e^{-c\lambda^2}|I|,
\]
for numerical constants $c > 0$, $C < \infty$. Hence $\text{dist}_{\text{BMO}}(\varphi, L^\infty) = 0$ and $\lambda_0(\varepsilon) \sim 1/\varepsilon$.

Decomposing $\varphi = \alpha + \beta$ as above, we get
\[
|\langle f, \varphi \rangle| \leq |\langle f, \alpha \rangle| + |\langle f, \beta \rangle| \leq C_1 \|f\|_{H^1} + C_2 \|f\|_1.
\]
Taking supremum over $\varphi$ and choosing $\varepsilon = \|f\|_1^{1/2} \|f\|_{H^1}^{1/2}$, the inequality follows.
PROOF OF PROPOSITION 6. Assume \( f \) real and estimate
\[
\left( \sum_{k=1}^{K} \| (E_k - E_{k-1})[f] \|_1^2 \right)^{1/2}.
\]

Define for each \( k \),
\[
\sigma_k = \text{sign} \Delta f_k \quad \text{and} \quad b_k = \frac{1}{2} (\sigma_k - E_{k-1}[\sigma_k]).
\]

Then
\[
\|f\|_1 \geq \int \int |f| \prod_{k=1}^{K} (1 + \epsilon_k b_k) \, d\varepsilon (d\omega) \geq \frac{1}{2} \int \sum_{k=1}^{K} \epsilon_k \Phi_k(\varepsilon) \, d\varepsilon
\]
where
\[
\Phi_k(\varepsilon) = \int \prod_{j=1}^{k-1} (1 + \epsilon_j b_j) |\Delta f_k| \, d\omega.
\]

Application of Proposition 7 to the function \( \sum \epsilon_k \Phi_k(\varepsilon) \) then gives
\[
\left( \sum_{k=1}^{K} \| \Delta f_k \|_1^2 \right)^{1/2} \leq C \|f\|_1^{-1/2} \left[ \int \left( \sum |\phi_k(\varepsilon)|^2 \right)^{1/2} \, d\varepsilon \right]^{1/2}
\]
\[
\leq C \|f\|_1^{1/2} \left[ \int \int S(f) \prod (1 + \epsilon_j b_j) \, d\omega \, d\varepsilon \right]^{1/2}
\]
\[
= C \|f\|_1^{1/2} \|f\|_{H^1}^{1/2}
\]
as announced.

REMARK. The author is grateful to P. W. Jones for outlining a more explicit procedure to obtain the decomposition used in the proof of Proposition 7.

3. Some inequalities involving \( H^1_0 \)-functions. The purpose of this section is to prove the following results.

PROPOSITION 8. For \( a \in C \) and \( h \in H^1_0 \), one has
\[
\|a + h\|_1 \geq \left( \|a\|^2 + \delta^2 \|h\|^2 \right)^{1/2}
\]
where \( \delta > 0 \) is a fixed constant.

PROPOSITION 9. There exists \( \delta > 0 \) such that for \( a \in C, b \in C \) and \( h \in H^1_0 \),
\[
(i) \quad \|a + h - b\sigma\|_1 \geq \left\{ \|a\|^2 + \delta^2 \left[ \frac{\text{Re}(\langle h, \sigma \rangle (\langle h, \sigma \rangle - b))}{|\langle h, \sigma \rangle| + |b|} \right] \right\}^{1/2},
\]
\[
(ii) \quad \|a + h - \langle h, \sigma \rangle \sigma\|_1 \geq \left\{ \|a\|^2 + \delta^2 \|h\|_1 - \langle h, \sigma \rangle \sigma^2 \right\}^{1/2}
\]
where \( \sigma = \text{sign} \cos \theta \) and \( h_\varepsilon(\theta) = \sum_{n=1}^{\infty} \varepsilon(n) \cos n\theta \).
It is clear that it suffices to prove Propositions 8 and 9, with $a = 1$.

**Proof of Proposition 8.** Factoring $1 + h$ gives $1 + h = (1 + g_1)(1 + g_2)$ where $g_1, g_2 \in H_0^2$ and

$$
\| 1 + h \|_1 = \left( 1 + \| g_1 \|_2^2 \right)^{1/2} \left( 1 + \| g_2 \|_2^2 \right)^{1/2}.
$$

Since $| h | \leq | g_1 | + | g_2 | + | g_1 || g_2 |$ the result follows from the majorations

$$
\left\| \left( 1 + | g_i |^2 \right)^{1/2} \right\|_1 \leq \left\| \left( 1 + | g_i |^2 \right)^{1/2} \right\|_2 = \left( 1 + \| g_i \|_2^2 \right)^{1/2} \leq \| 1 + h \|_1 \quad (i = 1, 2)
$$

and

$$
\left\| \left( 1 + | g_1 |^2 | g_2 |^2 \right)^{1/2} \right\|_1 \leq 1 + \| g_1 g_2 \|_1 \leq 1 + \| g_1 \|_2 \| g_2 \|_2 \leq \| 1 + h \|_1.
$$

Also to obtain Proposition 9, we will use the $L^2$-theory. Our argument here is, however, more complicated. This is the only point where explicit constructions of $H^\infty$-functions appear.

**Lemma 1.** Given a measurable subset $A$ of $\Pi$, there exist $H^\infty$-functions $\varphi$ and $\psi$ satisfying the following conditions:

(i) $| \varphi | + | \psi | \leq 1$,
(ii) $\Re \psi$ is an even function on $\Pi$,
(iii) $| \varphi - 1/8 | < 1/100$ on the set $A$,
(iv) $\| \varphi \|_1 \leq C | A |$,
(v) $\| \Re \psi - 1 \|_1 \leq C | A |$.

**Proof.** Fix some (large) $M > 0$ and define the following $H^\infty$-functions:

$$
\tau(z) = -M \int_A \frac{e^{i\theta} + z}{e^{i\theta} - z} m(d\theta), \quad \varphi = \frac{1}{8} \left( 1 - e^\tau \right)^2,
$$

$$
\psi(z) = \exp \left\{ \int \log(1 - \alpha(\theta)) \frac{e^{i\theta} + z}{e^{i\theta} - z} m(d\theta) \right\},
$$

where $\alpha(\theta) = | \varphi(e^{i\theta}) | \vee | \varphi(e^{-i\theta}) |$.

Notice that this makes sense, because $e^\tau$ has boundary value $e^{-M(\chi_A + i\mathcal{H}(\chi_A))}$ ($\mathcal{H} = \text{Hilbert-transform}$) and therefore $\| \alpha \|_\infty \leq \frac{1}{2}$.

(i) is obvious. On $\Pi$, we have $\Re \psi = (1 - \alpha) \cos \mathcal{H}(\log(1 - \alpha))$ and thus an even function. Since $| \varphi - \frac{1}{8} \alpha | \leq \frac{3}{8} | e^\tau |$ and thus $| \varphi - \frac{1}{8} | < e^{-M}$ on $A$ (iii) holds for $M$ large enough. Because on $\Pi$

$$
8 | \varphi | \leq \chi_A + M^2 | \mathcal{H}(\chi_A) |^2,
$$

(iv) follows. Finally,

$$
| 1 - \Re \psi | \leq | \alpha | + \frac{1}{2} | \mathcal{H}(\log(1 - \alpha)) |^2, \quad \| 1 - \Re \psi \|_1 \leq 4 \| \varphi \|_1
$$

and hence (v).

We refer the reader to [4, Proposition 1.6] for the following Marcinkiewicz type decomposition.
Lemma 2. There is a constant \( C < \infty \) such that for given \( h \in H^1_0 \) and \( \lambda > 0 \), there exists \( h_\lambda \in H_0^{2\lambda} \) satisfying:

(i) \( |h_\lambda| \leq C |h| \),

(ii) \( \|h_\lambda\|_\infty \leq C \lambda \),

(iii) \( \| h - h_\lambda \|_1 \leq C \| |h| > \lambda \| |h| \).

Let \( h \) be as in Proposition 9. For \( \lambda > 0 \), define \( A_\lambda = [|h| > \lambda] \). Application of Lemma 1 to the set \( A_\lambda \) provides \( H^\infty \)-functions \( \varphi_\lambda, \psi_\lambda \). We are now ready to prove Lemma 3.

Proof. First, since \( 1 - b \sigma \) is even and \( \text{Im} \psi_\lambda \) odd, we find

\[
\| 1 + h - b \sigma \|_1 \geq \| (1 + h - b \sigma) \varphi_\lambda \|_1 + \left| \int (1 - b \sigma) \psi_\lambda \right|
\]

\[
\geq \frac{1}{9} \int_{A_\lambda} |h| - (1 + |b|) + \left| \int (1 - b \sigma) \text{Re} \psi_\lambda \right|
\]

\[
\geq \frac{1}{9} \int_{A_\lambda} |h| - \frac{1}{9} (1 + |b|)|A_\lambda| + 1 - (1 + |b|) \| 1 - \text{Re} \psi_\lambda \|_1
\]

\[
\geq \frac{1}{9} \int_{A_\lambda} |h| - C(1 + |b|)|A_\lambda| + 1
\]

for some constant \( C \). Thus, choosing \( K \) large enough, we get

\[
(\ast) \quad \| 1 + h - b \sigma \|_1 \geq 1 + \frac{1}{10} \int_{A_\lambda} |h|.
\]

Fix some small constant \( \delta > 0 \). Since we always have

\[
\| 1 + af \|_1 \leq \| 1 + f \|_1 \quad \text{for} \ 0 \leq a \leq 1 \quad \text{and} \ f \ \text{of mean} \ 0,
\]

it follows that

\[
\| 1 + h - b \sigma \|_1 \geq \| 1 + \delta \lambda^{-1}(h - b \sigma) \|_1 \geq \| 1 + \delta \lambda^{-1}(h_\lambda - b \sigma) \|_1 - \delta \lambda^{-1}\| h - h_\lambda \|_1.
\]

Because \( \delta \lambda^{-1}|h_\lambda - b \sigma| \ll 1 \) the inequality

\[
(1 + t)^{1/2} \geq 1 + t/3 \quad \text{for} \ 0 \leq t \leq 1
\]

yields

\[
|1 + \delta \lambda^{-1}(h_\lambda - b \sigma)| \geq [1 + \delta \lambda^{-1} \text{Re}(h_\lambda - b \sigma)] [1 + \frac{1}{12} \delta^2 \lambda^{-2} (\text{Im}(h_\lambda - b \sigma))^2].
\]

Therefore, also

\[
(\ast\ast) \quad \| 1 + h - b \sigma \|_1 \geq 1 + \frac{1}{20} \delta^2 \lambda^{-2} \int_{A_\lambda} \text{Im}^2(h_\lambda - b \sigma) - c \delta \lambda^{-1} \int_{A_\lambda} |h|.
\]

The required minoration clearly follows combining \( (\ast) \) and \( (\ast\ast) \).

Proof of Proposition 9. First

\[
\| 1 + h - b \sigma \|_1 \geq d(b \sigma, H^1) \geq \frac{|b|}{2\pi} \left| \int_{-\pi}^{\pi} \sigma(\theta) e^{i\theta} d\theta \right| = \frac{2}{\pi} |b|
\]
and hence, also,
\[ \|1 + h - b\alpha\|_1 \geq \frac{1}{2}\|1 + h\|_1 \geq \frac{1}{2}\|h\|_1. \]

Notice that the right member of (i), (ii) is bounded by \(1 + 2\delta\|h\|_1\). Since now \(\|1 + h - b\alpha\|_1 \geq \frac{1}{2}\|h\|_1 + \frac{1}{2}\|b\|\), it follows that (i) (resp. (ii)) are satisfied for \(|b| \geq 6\) (resp. \(|\langle h, \sigma \rangle| \geq 6\)). Hence, we may assume \(|b| \leq M\) in (i), \(|\langle h, \sigma \rangle| \leq M\) in (ii) where \(M\) is some numerical constant.

Fix a constant \(\lambda > KM\) and put \(k = h_\lambda\) for simplicity. Using again Lemma 2(iii), the right member of (i) can be majorized by
\[
\left[1 + 2\delta^2\left(|\text{Re}\langle h, \sigma \rangle|^2 + |\text{Im}(\langle h, \sigma \rangle - b)|^2\right)\right]^{1/2}
\leq \left[1 + 2\delta^2\left(|\text{Re}\langle k, \sigma \rangle|^2 + |\text{Im}(\langle k, \sigma \rangle - b)|^2\right)\right]^{1/2} + 2\delta C \int_{A_\lambda} |h|.
\]

Taking Lemma 3 into account, we see that it suffices to check the inequality
\[
|\text{Re}\langle k, \sigma \rangle|^2 + |\text{Im}(\langle k, \sigma \rangle - b)|^2 \leq \|\text{Im}(k - b\sigma)\|_2^2
\]
which is straightforward:
\[
\|\text{Im}(k - b\sigma)\|_2^2 = \frac{1}{2} \sum_{n>0} |\text{Im}\hat{k}(n) - 2\text{Im} b\hat{\sigma}(n)|^2 + \frac{1}{2} \sum_{n>0} |\text{Re}\hat{k}(n)|^2
\]
while
\[
|\text{Re}\langle k, \sigma \rangle| \leq \sum_{n>0} |\text{Re}\hat{k}(n)||\hat{\sigma}(n)| \leq \frac{1}{\sqrt{2}} \left(\sum |\text{Re}\hat{k}(n)|^2\right)^{1/2},
\]
\[
|\text{Im}(\langle k, \sigma \rangle - b)| \leq \sum_{n>0} |\text{Im}\hat{k}(n) - 2\text{Im} b\hat{\sigma}(n)||\hat{\sigma}(n)|
\leq \frac{1}{\sqrt{2}} \left(\sum |\text{Im}\hat{k}(n) - 2\text{Im} b\hat{\sigma}(n)|^2\right)^{1/2}.
\]

For the right member of (ii), a similar reasoning reduces the question to the verification of
\[
\int |k_x - \langle k, \sigma \rangle\sigma|^2 \leq \|\text{Im}(k - b\sigma)\|_2^2,
\]
which the reader will easily do.

4. Reduction of the problem. In this section, we will reduce the problem of proving that certain elements of \(L^1(\Pi)\) normed by the quotient norm \(L^1/H^1\) to the verification of an inequality for certain functions in \(L^1(\Pi^N)\), where \(\Pi^N = \Pi \times \Pi \times \cdots\) is the product group. Denote by \(E_k (k = 1, 2, \ldots)\) the expectation with respect to the \(k\) first variables \((x_1, x_2, \ldots, x_k)\), where \(x = (x_1, x_2, \ldots)\) is the product variable.

We consider the subspace \(\mathcal{H}\) of \(L^1(\Pi^N)\) of those functions \(h\) such that for each \(k\) the difference \(E_k[h] - E_{k-1}[h]\) is an \(H^1_0\)-function with respect to \(x_k\). Thus \(h\) is of the form
\[
h = \sum h_k\quad \text{where} \quad h_k = \sum_{n>0} \hat{h}_k(n)e^{in x_k}
\]
and the \(\hat{h}_k(n)\) are functions of \(x_1, \ldots, x_{k-1}\).
Again let \( \sigma = \text{sign} \cos \) and \( \sigma_k(x) = \sigma(x_k) \) for each \( k \). Let \( \mathcal{F} \) be the \( \sigma \)-algebra on \( \Pi^N \) generated by the \( \sigma_k \). In the next section, we show the following

**Proposition 10.** There is a constant \( c > 0 \) s.t. \( \| h - E_{\sigma}[h] \|_1 \geq c \| h \|_1 \) for all \( h \in \mathcal{F} \).

This fact obviously implies

**Corollary 11.** \( \inf_{h \in \mathcal{F}} \| f - h \|_1 \geq c \| f \|_1 \) for all \( f \in L^1(\mathcal{F}) \).

For \( a, n \) positive integers, \( \mathcal{F}_a \) will be the Fejér kernel

\[
F_a(\theta) = \sum_{|j| \leq a} \frac{a + 1 - |j|}{a + 1} e^{ij\theta}
\]

and \( F_{a,n}(\theta) = F_a(n\theta) \).

We consider sequences of positive integers \( (n_k) \), \( (a_k) \) satisfying the following conditions: (\( \mathcal{F} \) denotes again the \( \sigma \)-algebra on \( \Pi \) generated by the functions \( \sigma(n_k\theta) \).)

(i) The transference property, i.e. let \( E = \{ \sum v_k n_k; (v_k) \in F \} \) where \( F \) is the subset \( \{(v_k), \| v_k \| \leq a_k \} \) of the dual group of \( \Pi^N \). Then the operator

\[
T: L^1_E(\Pi) \to L^1(\mathcal{F}), \quad T(f)(x) = \sum_{(v_k) \in F} f(\sum v_k n_k) e^{i(\sum v_k x_k)}
\]

satisfies

\[
\frac{1}{2} \| f \|_1 \leq \| T(f) \|_1 \leq 2 \| f \|_1.
\]

Moreover, \( T(f) \in \mathcal{F} \) for \( f \in L^1_E \cap H_0^1 \).

(ii) Defining for each \( k \),

\[
\xi_k = \sigma * F_{a_k}, \quad K = \prod_k F_{a_k,n_k},
\]

\[
R(\theta, \psi) = \prod \left[ 1 + \xi_k(n_k\theta) \sigma(n_k\psi) \right],
\]

one has

(a) \( \| \xi_k - \sigma \|_1 \leq \varepsilon \),

(b) \( \| K \|_1 = 1 \).

For \( f \in L^1(\mathcal{F}) \),

(\( \gamma \)) \( \| f - f * K \|_1 \leq \varepsilon \| f \|_1 \),

(\( \delta \)) \( \| f - R(f) \|_1 \leq \varepsilon \| f \|_1 \) where \( R(f)(\theta) = \int f(\psi) R(\theta, \psi) m(d\psi) \) (where \( \varepsilon > 0 \) is a small constant).

The reader will easily convince himself that the realisation of these conditions is straightforward. Details on the transference property can be found in [1].

Let us now show that the sequence \( (n_k) \) satisfies the Theorem. Thus, fix \( f \in L^1(\mathcal{F}) \) and \( h \in H_0^1 \). We get, by (ii),

\[
\| f - h \|_1 \geq \| f * K - h * K \|_1 \geq \| R(f) - h * K \|_1 - 2\varepsilon \| f \|_1.
\]

Notice that \( R(f) \in L^1_E \). By (i),

\[
\| R(f) - h * K \|_1 \geq \frac{1}{2} \| T(R(f)) - h \|_1,
\]

where \( h_1 = T(h * K) \in \mathcal{F} \).
Now

\[ T(R(f))(x) = \int f(\psi) \prod \left[ 1 + \xi_k(x_k)\sigma(n_k\psi) \right] m(d\psi). \]

By (ii)(a), we see that for any \((\pm 1)-sequence (\tau_k)\)

\[ \left\| \bigotimes (1 + \tau_k \xi_k) - \prod (1 + \tau_k \sigma_k) \right\|_1 < \varepsilon \]

implying that

\[ \left\| T(R(f)) - f_1 \right\| \leq 2\varepsilon\|f\| \quad \text{where } f_1 = E[T(R(f))]. \]

It follows then from Corollary 11 that

\[ \left\| f - h \right\|_1 \geq \frac{1}{2}\|f_1 - h_1\|_1 - 3\varepsilon\|f\|_1 \geq \frac{c}{2}\|f_1\|_1 - 3\varepsilon\|f\|_1 \]

\[ \geq \frac{c}{2}\|T(R(f))\|_1 - 4\varepsilon\|f\|_1 \geq \frac{c}{4}\|f\|_1 - 5\varepsilon\|f\|_1 \geq c\|f\|_1 \]

taking \(\varepsilon > 0\) small enough.

5. Proof of the Theorem. It remains to prove Proposition 10. So fix \(h = \Sigma h_k \in \mathcal{F}\) where

\[ h_k = \sum_{n>0} \hat{h}_k(n)(x_1, \ldots, x_k) e^{inx_k}. \]

We also define

\[ [h_k]_e = \sum \hat{h}_k(n) \cos nx_k, \]
\[ [h_k]_o = \sum \hat{h}_k(n) \sin nx_k, \]
\[ \langle h_k, \sigma_k \rangle = \sum \hat{h}_k(n) \delta(n) \]

(which is thus a function of \(x_1, \ldots, x_k\)). If \(f = E_{\varphi}[h]\), then \(f = \Sigma b_k \cdot \sigma_k\), where \(b_k = b_k(x_1, \ldots, x_k) = E_{\varphi}\langle \hat{h}_k, \sigma_k \rangle\).

Using E. Stein’s theorem on \(H^1\)-multipliers (see [11]), it is easily seen that

\[ \|h\|_1 \sim \|S(h)\|_1 \quad (S = \text{square function with respect to the natural decomposition}). \]

We give a direct proof of this fact, based on Proposition 8.

Fix \(1 > \varepsilon > 0\) and a positive sequence \((s_k)_{k=1,2,\ldots} \in L^\infty(\mathbb{N})\) satisfying

\[ \|\Sigma s_k^2\|^{1/2} \|_{\infty} \leq \varepsilon. \]

Fixing a positive integer \(K\), we get, using Proposition 8,

\[ \left\| \Sigma_{K}[h] \right\|_1 = \left\| \Sigma_{K-1}[h] \right\|_1 + h_k \]

\[ \geq \left\| \left( \left( \Sigma_{K-1}[h] \right)^2 + \delta^2 |h_k|^2 \right)^{1/2} \right\|_1 \]

\[ \geq \left\| \Sigma_{K-1}[h] \right\|_1 (1 - s_K^2)^{1/2} + \delta \left\| h_k s_K \right\|_1 \]

\[ \geq \left\| \Sigma_{K-1}[h] \right\|_1 + \delta \left\| h_k s_K \right\|_1 - \left\| \Sigma_{K-1}[h] s_K^2 \right\|_1. \]

Iterating,

\[ \|h\|_1 \geq \delta \sum \left\| h_k s_k \right\|_1 - \sum \left\| \Sigma_{K-1}[h] s_k^2 \right\|_1 \]

\[ \geq \delta \sum \left\| h_k s_k \right\|_1 - \varepsilon^2 \max_k \left\| \Sigma_{K}[h] \right\|_1. \]
Taking supremum over the sequences \((s_k)\), it follows that

\[
\|h\|_1 \Rightarrow \delta \|S(h)\|_1 - \varepsilon^2 \max_k \|E_k[h]\|_1
\]

and choosing

\[
e^2 = \frac{\|h\|_1}{\max_k \|E_k[h]\|_1},
\]

we get

\[
\|S(h)\|_1 \leq \delta^{-1} \|h\|_1^{1/2} \max_k \|E_k[h]\|_1^{1/2}.
\]

Hence, by Proposition 4, \(\|S(h)\|_1 \leq \delta^{-2} \|h\|_1\) as required.

Before continuing, notice that since \(\mathcal{F}\)-expectation is a contraction, \(\|S(f)\|_1 \leq \|S(f)\|_1\). Since for each \(k\), \(\cdots \|E_k[h_x]\|\), application of Proposition 5 yields

\[
\left\| \left( \sum \left| \langle h_k, \sigma_k \rangle \right|^2 \right)^{1/2} \right\|_1 \leq C \|h\|_1.
\]

If we now apply the previous procedure using Proposition 9, the following inequalities are derived.

**Lemma 4.**

\[
\begin{align*}
(1) \quad \left\| \left\{ \sum_k \left| \frac{\text{Re} \left( \langle h_k, \sigma_k \rangle (\langle h_k, \sigma_k \rangle - b_k) \right)}{\| \langle h_k, \sigma_k \rangle \| + |b_k|} \right|^2 \right\}^{1/2} \right\|_1 & \leq C \|h - f\|_1^{1/2} \|h\|_1^{1/2}, \\
(2) \quad \left\| \left\{ \sum_k \|h_k\|^2 - \langle h_k, \sigma_k \rangle \sigma_k \right\}^{1/2} \right\|_1 & \leq C \|h - \sum \langle h_k, \sigma_k \rangle \sigma_k \|_1^{1/2} \|h\|_1^{1/2}.
\end{align*}
\]

**Proof.** Let us show how \((1)\) follows from Proposition 9(i). The argument for \((2)\) is analogous. Fix \(0 < \varepsilon < 1\) and a sequence \((s_k)_{k=1,2,\ldots}\) of positive \(L^\infty\)-functions on \(\prod^N\) satisfying \(\|(\sum s_k^2)^{1/2}\|_\infty \leq \varepsilon\). Fix an integer \(k\) and apply Proposition 9(i) in the variable \(x_k\). We get

\[
\|E_k[h - f]\|_1 = \|E_{k-1}[h - f]\|_1 + \|h_k - b_k\|_\sigma_1
\]

\[
\geq \left\| \left\{ \text{Re} \left( \langle h_k, \sigma_k \rangle (\langle h_k, \sigma_k \rangle - b_k) \right) \right\}^{1/2} \right\|_1
\]

\[
\geq \|E_{k-1}[h - f]\|_1 + \delta \left\| \text{Re} \left( \langle h_k, \sigma_k \rangle (\langle h_k, \sigma_k \rangle - b_k) \right) \right\|_1
\]

\[
- \|E_{k-1}[h - f]\| s_k^2_1.
\]

Iterating and using the same considerations as in the beginning of this section it follows that the left member of \((1)\) is dominated by

\[
\delta^{-1} \varepsilon^{-1} \|h - f\|_1 + \text{const.} \varepsilon \|S(h - f)\|_1,
\]

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and hence, choosing $\epsilon$ appropriately, by the right member of (I). We first make use of (I) to show

**Lemma 5.** $\|\sum (\langle h_k, \sigma_k \rangle - b_k^2)^{1/2}\|_1 \leq C\|h - f\|_1^{1/4}\|h\|_1^{3/4}$.

**Proof.** Write

$$2 \left( \frac{\text{Re} \langle h_k, \sigma_k \rangle \langle h_k, \sigma_k \rangle - h_k}{|\langle h_k, \sigma_k \rangle| + |b_k|} \right) = \xi_k - |b_k|$$

where

$$\xi_k = \frac{|\langle h_k, \sigma_k \rangle - b_k|^2}{|\langle h_k, \sigma_k \rangle| + |b_k|} + |\langle h_k, \sigma_k \rangle|.$$ 

By the triangle inequality, the left side of (I) dominates

$$\left\| \left( \sum |\xi_k|^2 \right)^{1/2} \right\|_1 - \left\| \left( \sum |b_k|^2 \right)^{1/2} \right\|_1.$$ 

Also, since $b_k = E\langle h_k, \sigma_k \rangle$,

$$\left\| \left( \sum |b_k|^2 \right)^{1/2} \right\|_1 \leq \left\| \left( \sum |\langle h_k, \sigma_k \rangle|^2 \right)^{1/2} \right\|_1.$$ 

Write

$$\left[ \sum (\xi_k^2 - |\langle h_k, \sigma_k \rangle|^2) \right]^{1/2}$$

$$= \left[ \left( \sum \xi_k^2 \right)^{1/2} + \left( \sum |\langle h_k, \sigma_k \rangle|^2 \right)^{1/2} \right]^{1/2} \left[ \left( \sum \xi_k^2 \right)^{1/2} - \left( \sum |\langle h_k, \sigma_k \rangle|^2 \right)^{1/2} \right]^{1/2}$$

and apply Cauchy-Schwarz. From (I) and previous observations

$$\left\| \left( \sum (\xi_k^2 - |\langle h_k, \sigma_k \rangle|^2) \right)^{1/2} \right\|_1 \leq C\|h\|_1^{1/2}\|h - f\|_1^{1/4}\|h\|_1^{3/4} = C\|h - f\|_1^{1/4}\|h\|_1^{3/4}.$$ 

Since for each $k$,

$$\xi_k^2 - |\langle h_k, \sigma_k \rangle|^2 = (\xi_k + |\langle h_k, \sigma_k \rangle|) \frac{|\langle h_k, \sigma_k \rangle - b_k|^2}{|\langle h_k, \sigma_k \rangle| + |b_k|} \geq C|\langle h_k, \sigma_k \rangle - b_k|^2.$$

Lemma 5 is proved.

The left side of Lemma 5 dominates $\|f - \sum \langle h_k, \sigma_k \rangle \sigma_k\|_1$.

**Lemma 6.** $\|\Sigma[h_k]_0\|_1$ and $\|\Sigma \left[ |h_k| \right] - b_k \sigma_k^2 \right]^{1/2}\|_1 \leq C\|h - f\|_1^{1/8}\|h\|_1^{7/8}$.

**Proof.** Since $\Sigma[h_k]_0 = h - \Sigma[h_k]_c$, the first inequality is a consequence of the second. Write

$$\left\| \left[ \sum \left[ |h_k| \right] - b_k \sigma_k^2 \right]^{1/2} \right\|_1 \leq \left\| \left[ \sum \left[ |h_k| \right] - \langle h_k, \sigma_k \rangle \sigma_k^2 \right]^{1/2} \right\|_1 + \left\| \left[ \sum \langle h_k, \sigma_k \rangle - b_k \right]^{1/2} \right\|_1,$$
which by Lemmas 4(II) and 5 is estimated by

\[ C\|h - \sum \langle h_k, \sigma_k \rangle \sigma_k \|_1^{1/2} \|h\|_1^{1/2} + C\|h - f\|_1^{1/4}\|h\|_1^{3/4} \leq C\|h - f\|_1^{1/8}\|h\|_1^{7/8}. \]

Define for \( u \in L^1(\mathbb{N}) \),

\[ (u)_e(x) = \int_D u(e_1x_1, e_2x_2, \ldots) \, d\varepsilon \]

(= the natural projection on the even part in \( x_1, x_2, \ldots \)).

**Lemma 7.** \( \|\sum_k (\hat{h}_k(n))_e(x) \sin nx_k \|_2 \leq C\|h - f\|_1^{1/16}\|h\|_1^{15/16} \).

**Proof.** At this point, we will make use of Proposition 7. Fix \( x \in \mathbb{N} \) and remark that the sequence of functions in \( e \in D \),

\[ [h_k]_0(e_1x_1, e_2x_2, \ldots), \]

is a martingale difference sequence.

Moreover, the \( k \) th Rademacher coefficient is clearly given by

\[ \sum_{n>0} (\hat{h}_k(n))_e(x) \sin nx_k \]

and Proposition 7 yields

\[ \left[ \sum_k \left( \sum_{n>0} (\hat{h}_k(n))_e(x) \sin nx_k \right)^2 \right]^{1/2} \leq C\left[ \int \left| \sum_k [h_k]_0(e \cdot x) \right|^2 \, d\varepsilon \left[ \int \left[ \sum_k \left| [h_k]_0 e \cdot x \right|^2 \right]^{1/2} \, d\varepsilon \right]^{1/2}. \]

Integration in \( x \), application of Cauchy-Schwarz and Lemma 6, gives

\[ (+) \left\| \sum_k \left( \sum_{n>0} (\hat{h}_k(n))_e \sin nx_k \right)^2 \right\|_2 \leq C\|h - f\|_1^{1/16}\|h\|_1^{15/16} \left[ \sum_k \left| [h_k]_0 \right|^2 \right]^{1/2}. \]

Also

\[ \left\| \sum_k \left( [h_k]_0 \right)^2 \right\|_1 \leq C\|h\|_1. \]

On the other hand, we can multiply the \( k \) th increment in the left member of \((+)\) by \( \sin x_k \) and then take \( \mathbb{E}_{k-1} \)-expectation. Proposition 5 shows that

\[ \left\| \sum_k \left( \hat{h}_k(n) \right)_e \right\|_1^{1/2} \leq C\|h - f\|_1^{1/16}\|h\|_1^{15/16}, \]

proving Lemma 7.

Now, rewriting

\[ \left[ \sum_k \left( [h_k]_e - b_k\sigma_k \right)^2 \right]^{1/2} = \left[ \sum_k \sum_{n>0} \hat{h}_k(n) \cos nx_k - b_k\sigma_k \right]^2 \]
multiplication of the $k$th increment by $\cos x_k$ and taking $E_{k-1}$-expectation yields (by Proposition 5 and Lemma 6)

$$\left\| \sum_k \left\{ \frac{1}{2} \hat{h}_k(1) - \frac{2}{\pi} b_k \right\}^{2^{1/2}} \right\|_1 \leq C \| h - f \|_1^{1/8} \| h \|^{7/8}_1.$$ 

Since $b_k = (b_k)_e$, a convexity argument allows us to replace, in a previous inequality, $\hat{h}_k(1)$ by $(\hat{h}_k(1))_e$. Combining with Lemma 7, we conclude

$$C^{-1} \| f \|_1 \leq \left( \sum |b_k|^2 \right)^{1/2} \leq C \| h - f \|_1^{1/16} \| h \|_1^{15/16}, \quad \| f \|_1 \leq C \| h - f \|_1,$$

and thus Proposition 10.

REFERENCES