SUBCONTINUA WITH DEGENERATE TRANCHES
IN HEREDITARILY DECOMPOSABLE CONTINUA1

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ABSTRACT. A hereditarily decomposable, irreducible, metric continuum \( M \) admits a mapping \( f \) onto \([0,1]\) such that each \( f^{-1}(t) \) is a nowhere dense subcontinuum. The sets \( f^{-1}(t) \) are the tranches of \( M \) and \( f^{-1}(t) \) is a tranche of cohesion if \( t \in \{0,1\} \) or \( f^{-1}(t) = \text{Cl}(f^{-1}([0,t])) \cap \text{Cl}(f^{-1}((t,1])) \). The following answer a question of Mahavier and of E. S. Thomas, Jr.

\textbf{Theorem.} Every hereditarily decomposable continuum contains a subcontinuum with a degenerate tranche.

\textbf{Corollary.} If in an irreducible hereditarily decomposable continuum each tranche is nondegenerate then some tranche is not a tranche of cohesion.

The theorem answers a question of Nadler concerning arcwise accessibility in hyperspaces.

1. Introduction. A continuum is a compact connected metric space. A continuum \( M \) is said to be \textit{irreducible} between two points \( p \) and \( q \) if no proper subcontinuum of \( M \) contains both \( p \) and \( q \).

A continuum \( M \) is said to be of \textit{type} \( \lambda \) (see [8, p. 200]) if there exists a map \( \phi \) of \( M \) onto \([0,1]\) such that each point inverse under \( \phi \) is a nowhere dense subcontinuum of \( M \). The sets \( \phi^{-1}(t) \) are called the \textit{tranches} of \( M \). The sets \( \phi^{-1}(0) \) and \( \phi^{-1}(1) \) are called \textit{end-tranches} of \( M \). The tranche \( \phi^{-1}(t) \) is said to be a \textit{tranche of cohesion} if \( t \in \{0,1\} \) or if

\[ \phi^{-1}(t) = \text{Cl}(\phi^{-1}([0,t])) \cap \text{Cl}(\phi^{-1}((t,1])) \].

We denote the closure of a set \( A \) by \( \text{Cl}(A) \) and the boundary of \( A \) by \( \text{Bd}(A) \).

Irreducible continua have been extensively studied, in particular, under the topic of continuous collections. For example, an irreducible continuum which admits a monotone open mapping onto \([0,1]\) is a continuum of type \( \lambda \) and has the additional property that each tranche is a tranche of cohesion. Also, irreducible, hereditarily decomposable continua are of type \( \lambda \).

Thomas in [14] and Mahavier in [9] proved that each hereditarily decomposable arc-like continuum contains a subcontinuum with a degenerate tranche. In the main result of this paper we extend the Thomas and Mahavier result to arbitrary hereditarily decomposable continua. This answers, in the affirmative, Problem 121 in...
To prove the existence of an indecomposable continuum, one constructs a sequence $O_t$ of open covers such that $O_{t,1}$ "folds" in $O_t$. The notion of folding in chain covers is intuitively clear. A large part of this paper is devoted to a definition of folding in covers whose nerves are arbitrary polyhedra.

In 1935, Knaster [6] constructed a monotone, open mapping of a certain irreducible continuum onto $[0,1]$ such that each point inverse is nondegenerate. Dyer proved in [2] (see [7] for a simple proof) that each such mapping has a dense $G_δ$ of indecomposable point inverses. As a corollary to our main theorem, we complement Dyer’s theorem by proving that if $M$ is a continuum of type $λ$, such that each tranche is nondegenerate and is a tranche of cohesion, then $M$ contains indecomposable subcontinua of arbitrarily small diameters. Also, as a corollary to our main result, we obtain an affirmative solution to a question of Nadler [12] concerning arcwise accessibility in hyperspaces.

2. Definitions and preliminaries. We let $M$ be a continuum with a fixed but arbitrary metric $d$. If $𝒰$ is a collection of subsets of $M$ and $A ⊆ M$ we set

$$S^1(A, 𝒰) = S(A, 𝒰) = \bigcup \{ U ∈ 𝒰 | U \cap A \neq \emptyset \}$$

and, inductively,

$$S^n(A, 𝒰) = S(S^{n-1}(A, 𝒰), 𝒰).$$

We let

$$𝒰^* = \{ S(U, 𝒰) | U ∈ 𝒰 \} \quad \text{and} \quad 𝒰^{**} = \{ S^2(U, 𝒰) | U ∈ 𝒰 \}.$$  

If $𝒰$ and $𝒱$ are two collections of subsets of $X$ we say $𝒰$ refines $𝒱$ if for each $U ∈ 𝒰$ there exists $V ∈ 𝒱$ with $U ⊆ V$. If $𝒰 = \{ U_γ | γ ∈ Γ \}$ and $𝒱 = \{ V_γ | γ ∈ Γ \}$ and $U_γ ⊆ V_γ$ for each $γ ∈ Γ$, then $𝒰$ is said to be a precise refinement of $𝒱$.

A collection $𝒰$ of sets is said to be taut if $U, V ∈ 𝒰$ with $C(C(U)) \cap C(V) = \emptyset$ implies $U \cap V \neq \emptyset$. The collection $𝒰$ is said to be coherent if $U, V ∈ 𝒰$ implies there exists $U_1 = U, U_2, \ldots , U_n = V$ in $𝒰$ with $U_i \cap U_{i+1} \neq \emptyset$ for each $i = 1, \ldots , n - 1$. If $𝒰$ is a collection of open sets in a set $M$ and $U ∈ 𝒰$ let

$$i(U, 𝒰) = U \setminus C(C(U)) \cup \{ V | U ∪ V ∈ 𝒰 \}.$$  

If $K ⊆ M$ is a continuum we say a collection $𝒰$ of subsets of $M$ strongly irreducibly covers $K$ if $C(C(U)) | U ∈ 𝒰 \}$ is an irreducible cover of $K$. Notice that if $𝒰$ is an irreducible open cover of $K$ then there exists an open, taut, precise refinement $𝒱$ of $𝒰$ such that $𝒱$ covers $K$ and $𝒱$ is a strongly irreducible cover of $K$.

We shall need the following well-known result (see [8, p. 172]):

**Boundary Bumping Theorem.** If $K$ is a component of a proper open subset $U$ of a continuum $X$ then $Bd(U) \cap C(C(K)) \neq \emptyset$.

**Remark.** If $M$ is a continuum which contains no indecomposable continuum of diameter less than $ε$ for some $ε > 0$ then $M$ is one dimensional. To see this, let
$p \in M$ and let $U$ be a closed neighbourhood of $p$ of diameter less than $\epsilon$. Then every component of $U$ has dimension $\leq 1$ by the theorem of Mazurkiewicz [10] that every continuum of dimension $\geq 2$ contains an indecomposable continuum. Let $f: U \to Y$ be the map that identifies components of $U$ to points. Then $\dim Y = 0$ and $f$ is a closed map. Then $\dim U = 1$ by the Hurewicz Theorem [5, VI, 7]. For the sake of geometric intuition the reader may suppose, therefore, that every open cover $\mathcal{U}$ that will be needed for the proofs of the main results of this paper has nerve $\mathfrak{N}(\mathcal{U})$ which is a finite graph.

If $K \subset M$ are continua then by a $M$-$K$-cover $A$ we mean a taut collection of open sets in $K$ which covers and strongly irreducibly covers $K$. If $N \subset M$ are continua and $A$ is a $N$-$K$-cover then $A$ is a $M$-$K$-cover.

Let $A$ be an irreducible open cover of a continuum $M$ and let $U, V \in A$ such that $U \not\subset S^6(V, A)$. Let $\langle U, V \rangle$ be a subcollection of $A$ such that

1. $U, V \subset \langle U, V \rangle$,
2. $\langle U, V \rangle$ is a cover of some subcontinuum $K$ of $M$ such that $U \cap K \neq \emptyset \neq V \cap K$, and
3. if $W \subset \langle U, V \rangle$ is a cover of a subcontinuum $L$ of $M$ such that $L \cap U \neq \emptyset \neq L \cap V$ and $U, V \in W$ then $W = \langle U, V \rangle$.

If $B = \langle U, V \rangle$ we call $U$ the first link of $B$ and write $U = FB$. Similarly, we call $V$ the last link of $B$ and write $V = LB$. We call a $M$-$K$-cover $A$ a $M$-$K$-cover from $FA$ to $LA$ if $FA, LA \in A$ and $A = \langle FA, LA \rangle$.

1. Note. Let $A$ be an irreducible open cover of a continuum $M$ and suppose $U, V \in A$ such that $U \not\subset S^6(V, A)$. Since $A$ is finite there exists a $\langle U, V \rangle$ which need not, however, be unique. If $N$ is a continuum in $M$ such that $N \subset \bigcup \langle U, V \rangle$ and $N \cap U \neq \emptyset \neq N \cap V$, let $\langle\langle U, V \rangle\rangle$ be a $M$-$N$-cover which is a precise refinement of $\langle U, V \rangle$ (we can show as before that $\langle\langle U, V \rangle\rangle$ exists). For each $W \in \langle U, V \rangle$ let $W'$ be the element of $\langle\langle U, V \rangle\rangle$ which corresponds to $W$. Then $\langle\langle U, V \rangle\rangle$ is a $M$-$N$-cover from $U'$ to $V'$. It suffices to show that if $K$ is a continuum in $\bigcup \langle\langle U, V \rangle\rangle$ such that $K \cap U' \neq \emptyset \neq K \cap V'$ and $O \subset \langle U, V \rangle \setminus \{U, V\}$, then $K \not\subset \bigcup \langle\langle U, V \rangle\rangle \setminus \{O'\}$. If $K \subset \bigcup \langle\langle U, V \rangle\rangle \setminus \{O\}$, then $K \subset \bigcup \langle\langle U, V \rangle\rangle \setminus \{O\}$ and $K \cap U \neq \emptyset \neq K \cap V$, which contradicts the definition of $\langle U, V \rangle$.

We say that a $N$-$K$-cover $B$ is embedded in a $M$-$N$-cover $A$ if $\{S^3(U, B) \mid U \in B\}$ refines $A$. If $A$ is a $M$-$N$-cover from $FA$ to $LA$, then we say a $N$-$K$-cover $B$ is embedded in $A$ from $FA$ to $LA$ if $B$ is embedded in $A$, $B$ is a $N$-$K$-cover from $FB$ to $LB$, $C_1(FB) \subset i(FA, A)$ and $C_1(U) \subset i(LA, A)$ for some $U \in B$.

REMARK. If $B$ is a $N$-$K$-cover embedded in a $M$-$N$-cover $A$ from $FA$ to $LA$ then for each $U \in A$ there exists $W \in B$ such that $C_1(W) \subset U$.

PROOF. Without loss of generality $FA \neq U \neq LA$. Let $V \in B$ such that $V \subset i(LA, A)$. Since $B$ is an irreducible cover of $K \subset N$, $FB \subset i(FA, A)$, $V \subset i(LA, A)$ and $A$ is a $M$-$N$-cover from $FA$ to $LA$, there exists $x \in K \cap U \setminus \{T \in A \mid T \neq U\}$. Let $W \in B$ such that $x \in W$. Then $C_1(W) \subset U$.

If it follows from the above Remark that if $B$ is a $N$-$K$-cover embedded in the $M$-$N$-cover $A$ from $FA$ to $LA$ and $C$ is a $K$-$L$-cover embedded in $B$ from $FB$ to $LB$, then $C$ is embedded in $A$ from $FA$ to $LA$. 

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2. Note. If $A$ is a $M$-$N$-cover from $FA$ to $LA$, then there exists for each $\varepsilon > 0$, by Note 1, a $N$-$K$-cover $B$ of mesh less than $\varepsilon$ embedded in $A$ from $FA$ to $LA$ and $\text{Cl}(LB) \subseteq i(LA, A)$.

If $A$ is a $M$-$N$-cover from $FA$ to $LA$ then by an endpiece $T$ of $A$ we mean a coherent subcollection of $A$ which contains $\{W \in A \mid W \subseteq S^3(LA, A)\}$ and such that $FA \cap \bigcup T = \emptyset$. By $fT$ we denote

$$\{W \in T \mid W \cap Z \neq \emptyset \text{ for some } Z \in A \setminus T\},$$

and we call $fT$ the first links of $T$.

Let $B$ be a $N$-$K$-cover embedded in a $M$-$N$-cover $A$ from $FA$ to $LA$. Let $S$ be an endpiece of $A$ and let $T$ be an endpiece of $B$. We say that $T$ folds in $S$ if $\text{Cl}(LB) \subseteq \bigcup S$, $\text{Cl}(\bigcup T) \subseteq \bigcup S$, $\text{Cl}(\bigcup T) \subseteq \bigcup S$, and no coherent subcollection of $\{W \in T \mid W \subseteq i(LA, A)\}$ contains both $LB$ and an element of $fT$.

3. Lemma. Let $B$ be a $N$-$K$-cover embedded in a $M$-$N$-cover $A$ from $FA$ to $LA$, let $S$ be an endpiece of $A$ and suppose $\{W \in B \mid \text{Cl}(W) \subseteq i(LA, A)\}$ contains at least two maximal distinct coherent subcollections $P$ and $Q$ and elements $U \in P$ and $V \in Q$ such that $\text{Cl}(U \cup V) \subseteq \bigcup S$. Then there exists a $K$-$L$-cover $C$ embedded in $A$ from $FA$ to $LA$ and an endpiece $T$ of $C$ such that $T$ folds in $S$.

Proof. Let $\{U \in B \mid \text{Cl}(U) \subseteq i(LA, A)\} = R \cup R'$, where $R$ is the maximal coherent subcollection of $\{U \in B \mid \text{Cl}(U) \subseteq i(LA, A)\}$ which contains $FB$ and $R \cap R' = \emptyset$. Let $R'' = \{U \in R' \mid \text{Cl}(U) \subseteq \bigcup S\}$.

If $R'' = R'$ let $U \in R''$ such that $\text{Cl}(U) \subseteq \bigcup S$. By Note 1 choose $(FB, U) \subset B$, a continuum $L \subset K \cap \bigcup \langle FB, U \rangle$ such that $L \cap FB \neq \emptyset \neq L \cap U$, and a $K$-$L$-cover $C = \langle\langle FB, U\rangle\rangle$ from $FC$ to $LC$ with $FC \subset FB$ and $LC \subset U$. Then $C$ is embedded in $A$ from $FA$ to $LA$. Let $T$ be the maximal coherent subcollection of $C$ which contains $LC$ and such that $\text{Cl}(\bigcup T) \subseteq \bigcup S$. Then $\text{Cl}(\bigcup T) \subseteq \bigcup S \cap \bigcup R$.

To see this, let $W \in fT$. There exists $Z \in C \setminus T$ such that $Z \cap W \neq \emptyset$. Let $V \in A$ such that $S^3(Z, C) \subset V$. Since $\text{Cl}(Z) \subset V$, $V \not\subseteq S$. Let $V_1, \ldots, V_n \in S$ such that $\text{Cl}(W) \subset V_1 \cup \cdots \cup V_n$. Then $V_i \cap V \neq \emptyset$ for each $i$ and, hence, $V_i \in S$. Hence, $T$ folds in $S$.

If $R'' \neq R'$, let $x_0 \in K \cap i(FB, B)$. Then by the Boundary Bumping Theorem there exists a continuum $K' \subset K \cap (\bigcup R \cup LA \cup \bigcup R'')$ such that $x_0 \in K'$ and $\text{Cl}(S(K', B)) \cap \bigcup R' \not\subseteq \bigcup S$.

Let $B' \subset R'' \cup (A \setminus R')$ be an irreducible cover of $K'$. Let $U \in B' \cap R''$ such that $\text{Cl}(S(U, B)) \not\subseteq \bigcup S$. Notice $U \subset V$ for some $V \in A \setminus S$. Thus, $\text{Cl}(U) \subset \bigcup S$. Choose $(FB, U) \subset B'$, a continuum $L \subset K' \cap \bigcup (FB, U) \subset K$ such that $L \cap FB \neq \emptyset \neq L \cap U$, and a $K$-$L$-cover $C = \langle\langle FB, U\rangle\rangle$ from $FC$ to $LC$ such that $FC \subset FB$ and $LC \subset U$. Then $C$ is embedded in $A$ from $FA$ to $LA$. Let $T$ be the maximal coherent collection of $C$ which contains $LC$ and such that $\text{Cl}(\bigcup T) \subset \bigcup S$. As above $T$ folds in $S$. 
The following is a variation on a theorem of Rogers [13] and Bellamy [1]:

4. Lemma. Let $M_0$ be a continuum. Suppose $B_1, B_2, \ldots$ is a sequence such that $B_{i+1}$ is a $M_i$-$M_{i+1}$-cover embedded in the $M_i$-$M_r$-cover $B_i$ from $FB_i$ to $LB_i$, and $T_1, T_2, \ldots$ is a sequence such that $T_i$ is an endpiece of $B_i$ and $T_{i+1}$ folds in $T_i$. Then $\bigcap_{i=1}^{\infty} \left( \bigcup \{U \in T_i\} \right)$ contains an indecomposable continuum.

Proof. By the Boundary Bumping Theorem there exists a continuum $K_i$ in $\bigcup T_i$ such that $K_i$ meets $fT_i$ and $LB_i$. Then $K_i$ also meets $fT_j$ and $LB_j$ for $j < i$. Without loss of generality $\lim K_i = K$. Then $K$ is a continuum,

$$K \subset \bigcap_{i=1}^{\infty} C_1 \left( \bigcup \{U \in T_i\} \right) = \bigcap_{i=1}^{\infty} \left( \bigcup \{U \in T_i\} \right)$$

and $K$ meets $C_1(\bigcup fT_i)$ and $C_1(LB_i)$ for each $i$. So $K \cap \bigcup fT_i \neq \emptyset$ for each $i$. Let $h: [0, 1] \to [0, 1]$ be defined by

$$h(x) = \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2}], \\ 2 - 2x & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Define $f_1: C_1(\bigcup T_i) \to [0, 1]$ to be a continuous function such that $f_1^{-1}(0) = C_1(\bigcup fT_i)$ and $f_1^{-1}(1) = C_1(LB_i)$. Define $f_2: C_1(\bigcup T_2) \to [0, 1]$ so that if $R$ is the union of the maximal coherent subcollections of $\{U \in B_2 \mid U \not\subseteq i(LB_1, B_1)\}$ which meets $\bigcup fT_2$, then

$$f_2(x) = \begin{cases} \frac{1}{2} f_1(x) & \text{if } x \in C_1(\bigcup R) \cap C_1(\bigcup T_2), \\ \frac{1}{2} & \text{if } x \in f_1^{-1}(1) \cap C_1(\bigcup T_2), \\ 1 - \frac{1}{2} f_1(x) & \text{if } x \in C_1(\bigcup T_2) \setminus (C_1(\bigcup R) \cup f_1^{-1}(1)). \end{cases}$$

Notice that $LB_2 \cap C_1(\bigcup R) = \emptyset$ and $K \cap \bigcup fT_2 \neq \emptyset \neq K \cap C_1(LB_2)$, so $f_2(K) = [0, 1]$. Then $f_1(x) = h \circ f_2(x)$ for each $x \in C_1(\bigcup T_2)$. By induction we define continuous functions $f_i: C_1(\bigcup T_i) \to [0, 1]$ such that $f_i(x) = h \circ f_{i+1}(x)$ for each positive integer $i$ and for each $x \in C_1(\bigcup T_{i+1})$ and $f_i(K) = [0, 1]$. Then $f = \lim f_i | K$ is a mapping of $K \subset \bigcap_{i=1}^{\infty} \left( \bigcup \{U \in T_i\} \right)$ onto Knaster’s indecomposable continuum $Y = \lim_{i \to \infty} (I_i, h_i)$, where $I_i = [0, 1]$ and $h_i = h$ for all $i$ and $j$ (as in the proof of Bellamy [1 Theorem, p. 305]). Since $f$ maps $K$ onto $Y$, $K$ contains an indecomposable continuum.

5. Lemma. Let $M$ be a continuum which contains no indecomposable subcontinuum of diameter less than $\varepsilon$ for some $\varepsilon > 0$. Let $A$ be a $N$-$M$-cover of mesh less than $\varepsilon$ from $FA$ to $LA$. Then there exists a $N_n$-$M$-cover $B$ embedded in $A$ from $FA$ to $LA$, and an endpiece $T$ of $B$ such that $C_1(\bigcup T) \subset i(LA, A)$, and such that if $C$ is a $N_r$-$K$-cover embedded in $B$ from $FB$ to $LB$ then no endpiece of $C$ folds in $T$. Moreover, $B$ can be chosen to have arbitrarily small mesh.

Proof. As in Note 2 there is a $N$-$L$-cover $B_i$ of arbitrarily small mesh embedded in $A$ from $FA$ to $LA$ and an endpiece $T_i$ of $B_i$ such that $C_1(\bigcup T_i) \subset LA$. The lemma now follows by contradiction from Lemma 4.
3. The main results. The first three results in this section were proved by Thomas in [14] and Mahavier in [9] for the special case of arc-like continua.

6. Theorem. If $M$ is a continuum which contains no indecomposable subcontinuum of diameter less than $\varepsilon$ for some $\varepsilon > 0$ and $x, y \in M$, then there exists a subcontinuum $K$ of $M$ irreducible from $p$ to $q$ such that $K$ is locally connected at $q$, $d(x, p) < \varepsilon$ and $d(y, q) < \varepsilon$. In particular, if $K$ is irreducible from $p$ to $q'$ then $q' = q$.

Proof. Let $N_0 = M$. Let $\emptyset$ be a $M$-$N_0$-cover of mesh less than $\min(\varepsilon, 1)$ such that $y \notin S^2(x, 0)$. Let $x \in U \in \emptyset$ and $y \in V \in \emptyset$. By Note 1 choose $\langle U, V \rangle \subseteq \emptyset$, a continuum $N_1 \subseteq \bigcup \langle U, V \rangle$ such that $N_1 \cap U \neq \emptyset \neq N_1 \cap V$ and $B_1 = \langle \langle U, V \rangle \rangle$, a $N_0$-$N_1$-cover from $FB_1 \subseteq U$ to $LB_1 \subseteq V$. By Lemma 5 there exists a $N_1$-$N_2$-cover $B_2$ of mesh less than $\frac{1}{2}$ embedded in $B_1$ from $FB_1$ to $LB_1$ and an endpiece $T_2$ of $B_2$ such that $Cl(\bigcup T_2) \subseteq i(LB_1, B_1)$, and such that if $D$ is a $N_2$-$K$-cover embedded in $B_2$ from $FB_2$ to $LB_2$, then no endpiece of $D$ folds in $T_2$.

By repeated application of Lemma 5 there exist sequences of continua $N_1, N_2, \ldots$, covers $B_1, B_2, \ldots$ and endpieces $T_2, T_3, \ldots$ such that for each $i = 1, 2, \ldots$:

(i) $B_{i+1}$ is a $N_i$-$N_{i+1}$-cover embedded in $B_i$ from $FB_i$ to $LB_i$;

(ii) mesh $B_i < 1/i$;

(iii) $T_{i+1}$ is an endpiece of $B_{i+1}$ with $Cl(\bigcup T_{i+1}) \subseteq i(LB_i, B_i)$;

(iv) if $D$ is a $N_{i+1}$-$K$-cover embedded in $B_{i+1}$ from $FB_{i+1}$ to $LB_{i+1}$, then no endpiece of $D$ folds in $T_{i+1}$.

Let $K = \bigcap N_i$, $\{p\} = \bigcap FB_i$ and $\{q\} = \bigcap LB_i$. Then $K$ is a continuum. Since $B_i = \langle FB_i, LB_i \rangle$ for each $i$, $B_i$ is an irreducible cover of every subcontinuum of $K$ which contains both $p$ and $q$. Since mesh $B_i < 1/i$ it follows that $K$ is an irreducible continuum from $p$ to $q$.

Suppose $K$ is not connected im kleinen at $q$. There exists $\delta > 0$ such that $\delta < \varepsilon$ and such that no subcontinuum of $K$ of diameter less than $\delta$ contains a neighbourhood of $q$ in $K$. Let $Q$ be the component of $q$ in $K \cap Cl(B(q, \delta/4))$, where $B(q, \delta/4)$ denotes the open $\delta/4$ ball centered at $q$. Let $r \in Q \setminus B(q, \delta/4)$. Let $n$ be an integer so that $LB_{n-1} \subseteq B(q, \delta/4)$. If for each sufficiently large integer $i$ the maximal coherent subcollection of $\{U \in B_i \mid Cl(U) \not\subseteq i(LB_n, B_n)\}$ which contains $FB_i$, also contains $r$ in its union, then there exists a component $N$ of $K \setminus LB_{n+1}$ which contains both $p$ and $r$. By the irreducibility of $K$ from $p$ to $q$ this would imply that $K = N \cup Q$ since $N \cup Q$ is a continuum in $K$ which contains $p$ and $q$. Thus, $q$ is in the interior of $Q$ in $K$ which contradicts the choice of $\delta$. Thus, for some sufficiently large $m > n$ the maximal coherent subcollection of $\{U \in B_m \mid Cl(U) \not\subseteq i(LB_n, B_n)\}$ which contains $FB_m$ does not contain $r$ in its union. By Lemma 3 there exists a $N_m$-$K$-cover $D$ embedded in $B_n$ from $FB_n$ to $LB_n$, and an endpiece $T$ of $D$ such that $T$ folds in $T_n$. This is a contradiction and the connectedness im kleinen of $K$ at $q$ is proved.

Finally, we show that $K$ is locally connected at $q$. Let $V$ be any closed connected neighbourhood of $q$ of diameter less than $\varepsilon$ such that $p \notin V$. Let $L$ be the closure of the component of $K \setminus V$ which contains $p$. Then $U = K \setminus L$ is an open set containing $q$. By the irreducibility of $K$ from $p$ to $q$, $U \subseteq V$ since $V$ is connected. Let $N$ be the
The next corollary gives an affirmative answer to a question of Mahavier (cf. Problem 121, University of Houston problem book).

7. Corollary. Let $M$ be a hereditarily decomposable continuum and $x, y \in M$. Then for each $\varepsilon > 0$, there exists a subcontinuum $K'$ of $M$ such that $K'$ is irreducible from $x$ to $q$ for some $q$ with $d(y, q) < \varepsilon$ and such that \{q\} is an end-tranche of $K$.

Proof. Assume $M$ is irreducible from $x$ to $y$. Let $f : M \rightarrow [0, 1]$ be a finest monotone map with $f(x) = 0$ and $f(y) = 1$. Choose $K$ and $p \in f^{-1}([0, \frac{1}{2}])$ and $q \in B(y, \varepsilon) \cap f((\frac{1}{2}, 1])$ as in Theorem 6. Let $K' = K \cup f^{-1}([0, f(p)])$.

In [14 and 3] are given examples of hereditarily decomposable arc-like continua $X$ so that if $A''$ is a subcontinuum of $X$ with a degenerate tranche $L$ then $L$ is an end-tranche of $A''$.

8. Corollary. If $M$ is a continuum of type $\lambda$ which contains no indecomposable continuum of diameter $< \varepsilon$ for some $\varepsilon > 0$ and such that each tranche of $M$ is a tranche of cohesion, then a dense $G_\delta$-set of tranches of $M$ are degenerate.

Proof. Let $\Phi : M \rightarrow [0, 1]$ be a map such that for each $t \in [0, 1]$, $\Phi^{-1}(t)$ is a nowhere dense subcontinuum of $M$. Let $[a, b] \subset [0, 1]$ such that $a < b$. By Theorem 6, there exists a continuum $K \subset \Phi^{-1}([a, b])$ irreducible from $p$ to $q$ such that $\Phi(p) < \Phi(q)$ and \{q\} is a degenerate end-tranche of $M$. It also follows that if $t \in D = \{s \in [0, 1] \mid \Phi^{-1}(s)$ is a tranche of continuity\}, then $\Phi^{-1}(t)$ is degenerate, and it is known (see [8, p. 202]) that $D$ is a dense $G_\delta$ in $[0, 1]$.

9. Corollary. If $M$ is an irreducible hereditarily decomposable continuum such that each tranche of $M$ is a tranche of cohesion, then a dense $G_\delta$-set of tranches of $M$ is degenerate.

For any compact metric space $M$ we denote by $2^M$ (respectively, $C(M)$) the space of all nonempty, compact subsets (respectively, subcontinua) of $M$ with the topology induced by the Hausdorff metric.

Let $M$ be a continuum and let $x \in M$. Then \{x\} is said to be arcwise accessible from $2^M \setminus C(M)$ (see [11 and 12]) provided there exists an arc $A$ in $2^M$ such that $A \cap C(M) = \{x\}$. The next corollary follows from Theorem 6 and Theorem 4.1 of [3]. It gives a positive solution to a question of Nadler (see [11, 12.19 and 12, 8.1]).

10. Corollary. Let $M$ be a hereditarily decomposable continuum. There exists a point $x \in M$ such that \{x\} is arcwise accessible from $2^M \setminus C(M)$.

In view of Corollary 8 the following question is interesting:

11. Question. If $X$ is an irreducible continuum which admits a continuous monotone decomposition onto an arc, does $X$ contain hereditarily indecomposable tranches? In particular, does Knaster's continuum in [6] contain tranches which are pseudoarcs?
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