SUBCONTINUA WITH DEGENERATE TRANCHES IN HEREDITARILY DECOMPOSABLE CONTINUA

BY
LEX G. OVERSTEEGEN AND E. D. TYMCHATYN

Abstract. A hereditarily decomposable, irreducible, metric continuum M admits a mapping f onto [0, 1] such that each f⁻¹(t) is a nowhere dense subcontinuum. The sets f⁻¹(t) are the tranches of M and f⁻¹(t) is a tranche of cohesion if t ∈ {0, 1} or f⁻¹(t) = Cl(f⁻¹([0, t])) ∩ Cl(f⁻¹((t, 1])). The following answers a question of Mahavier and of E. S. Thomas, Jr.

Theorem. Every hereditarily decomposable continuum contains a subcontinuum with a degenerate tranche.

Corollary. If in an irreducible hereditarily decomposable continuum each tranche is nondegenerate then some tranche is not a tranche of cohesion.

The theorem answers a question of Nadler concerning arcwise accessibility in hyperspaces.

1. Introduction. A continuum is a compact connected metric space. A continuum M is said to be irreducible between two points p and q if no proper subcontinuum of M contains both p and q.

A continuum M is said to be of type λ (see [8, p. 200]) if there exists a map φ of M onto [0, 1] such that each point inverse under φ is a nowhere dense subcontinuum of M. The sets φ⁻¹(t) are called the tranches of M. The sets φ⁻¹(0) and φ⁻¹(1) are called end-tranches of M. The tranche φ⁻¹(t) is said to be a tranche of cohesion if t ∈ {0, 1} or if

φ⁻¹(t) = Cl(φ⁻¹([0, t])) ∩ Cl(φ⁻¹((t, 1))).

We denote the closure of a set A by Cl(A) and the boundary of A by Bd(A).

Irreducible continua have been extensively studied, in particular, under the topic of continuous collections. For example, an irreducible continuum which admits a monotone open mapping onto [0, 1] is a continuum of type λ and has the additional property that each tranche is a tranche of cohesion. Also, irreducible, hereditarily decomposable continua are of type λ.

Thomas in [14] and Mahavier in [9] proved that each hereditarily decomposable arc-like continuum contains a subcontinuum with a degenerate tranche. In the main result of this paper we extend the Thomas and Mahavier result to arbitrary hereditarily decomposable continua. This answers, in the affirmative, Problem 121 in

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the *University of Houston problem book* (due to Mahavier). Our methods are patterned on those used by both Thomas and Mahavier. These methods are abstracted from a proof of Henderson [4].

To prove the existence of an indecomposable continuum, one constructs a sequence $O_i$ of open covers such that $O_{i+1}$ “folds” in $O_i$. The notion of folding in chain covers is intuitively clear. A large part of this paper is devoted to a definition of folding in covers whose nerves are arbitrary polyhedra.

In 1935 Knaster [6] constructed a monotone, open mapping of a certain irreducible continuum onto $[0,1]$ such that each point inverse is nondegenerate. Dyer proved in [2] (see [7] for a simple proof) that each such mapping has a dense $G_δ$ of indecomposable point inverses. As a corollary to our main theorem we complement Dyer’s theorem by proving that if $M$ is a continuum of type $λ$, such that each tranche is nondegenerate and is a tranche of cohesion, then $M$ contains indecomposable subcontinua of arbitrarily small diameters. Also, as a corollary to our main result we obtain an affirmative solution to a question of Nadler [12] concerning arcwise accessibility in hyperspaces.

2. Definitions and preliminaries. We let $M$ be a continuum with a fixed but arbitrary metric $d$. If $𝒰$ is a collection of subsets of $M$ and $A ⊂ M$ we set

$$S^1(A, 𝒰) = S(A, 𝒰) = ∪ \{ U ∈ 𝒰 | U ∩ A ≠ ∅ \}$$

and, inductively,

$$S^n(A, 𝒰) = S(S^{n-1}(A, 𝒰), 𝒰) .$$

We let

$$𝒰^* = \{ S(U, 𝒰) | U ∈ 𝒰 \} \quad \text{and} \quad 𝒰^{**} = \{ S^2(U, 𝒰) | U ∈ 𝒰 \} .$$

If $𝒰$ and $𝑽$ are two collections of subsets of $X$ we say $𝒰$ refines $𝑽$ if for each $U ∈ 𝒰$ there exists $V ∈ 𝒫$ with $U ⊂ V$. If $𝒰 = \{ U_γ | γ ∈ Γ \}$ and $𝑽 = \{ V_γ | γ ∈ Γ \}$ and $U_γ ⊂ V_γ$ for each $γ ∈ Γ$, then $𝒰$ is said to be a precise refinement of $𝑽$.

A collection $𝒰$ of sets is said to be taut if $U, V ∈ 𝒰$ with $C(U) ∩ C(V) ≠ ∅$ implies $U ∩ V ≠ ∅$. The collection $𝒰$ is said to be coherent if $U, V ∈ 𝒰$ implies there exists $U_1 = U, U_2, \ldots, U_n = V$ in $𝒰$ with $U_i ∩ U_{i+1} ≠ ∅$ for each $i = 1, \ldots, n − 1$. If $𝒰$ is a collection of open sets in a set $M$ and $U ∈ 𝒰$ let

$$i(U, 𝒰) = U \setminus C(∪ \{ V | U ≠ V ∈ 𝒰 \}) .$$

If $K ⊂ M$ is a continuum we say a collection $𝒰$ of subsets of $M$ strongly irreducibly covers $K$ if $\{ C(U) | U ∈ 𝒰 \}$ is an irreducible cover of $K$. Notice that if $𝒰$ is an irreducible open cover of $K$ then there exists an open, taut, precise refinement $𝑽$ of $𝒰$ such that $𝑽$ covers $K$ and $𝑽$ is a strongly irreducible cover of $K$.

We shall need the following well-known result (see [8, p. 172]):

**Boundary Bumping Theorem.** If $K$ is a component of a proper open subset $U$ of a continuum $X$ then $Bd(U) ∩ C1(K) ≠ ∅$.

**Remark.** If $M$ is a continuum which contains no indecomposable continuum of diameter less than $ε$ for some $ε > 0$ then $M$ is one dimensional. To see this, let
Let \( p \in M \) and let \( U \) be a closed neighbourhood of \( p \) of diameter less than \( \varepsilon \). Then every component of \( U \) has dimension \( \leq 1 \) by the theorem of Mazurkiewicz [10] that every continuum of dimension \( \geq 2 \) contains an indecomposable continuum. Let \( f: U \to Y \) be the map that identifies components of \( U \) to points. Then \( \dim Y = 0 \) and \( f \) is a closed map. Then \( \dim U = 1 \) by the Hurewicz Theorem [5, VI, 7]. For the sake of geometric intuition the reader may suppose, therefore, that every open cover \( \mathcal{U} \) that will be needed for the proofs of the main results of this paper has nerve \( \mathcal{N}(\mathcal{U}) \) which is a finite graph.

If \( K \subseteq M \) are continua then by a \( M-K \)-cover \( A \) we mean a taut collection of open sets in \( K \) which covers and strongly irreducibly covers \( K \). If \( N \subseteq M \) are continua and \( A \) is a \( N-K \)-cover then \( A \) is a \( M-K \)-cover.

Let \( A \) be an irreducible open cover of a continuum \( M \) and let \( U, V \in A \) such that \( U \not\subseteq S^6(V, A) \). Let \( \langle U, V \rangle \) be a subcollection of \( A \) such that

1. \( U, V \subseteq \langle U, V \rangle \),
2. \( \langle U, V \rangle \) is a cover of some subcontinuum \( K \) of \( M \) such that \( U \cap K \neq \emptyset \neq V \cap K \), and
3. if \( W \subseteq \langle U, V \rangle \) is a cover of a subcontinuum \( L \) of \( M \) such that \( L \cap U \neq \emptyset \neq L \cap V \) and \( U, V \in W \) then \( W = \langle U, V \rangle \).

If \( B = \langle U, V \rangle \) we call \( U \) the first link of \( B \) and write \( U = FB \). Similarly, we call \( V \) the last link of \( B \) and write \( V = LB \). We call a \( M-K \)-cover \( A \) a \( M-K \)-cover from \( FA \) to \( LA \) if \( FA, LA \in A \) and \( A = \langle FA, LA \rangle \).

1. Note. Let \( A \) be an irreducible open cover of a continuum \( M \) and suppose \( U, V \in A \) such that \( U \not\subseteq S^6(V, A) \). Since \( A \) is finite there exists a \( \langle U, V \rangle \) which need not, however, be unique. If \( N \) is a continuum in \( M \) such that \( N \subseteq \bigcup \langle U, V \rangle \) and \( N \cap U \neq \emptyset \neq N \cap V \), let \( \langle \langle U, V \rangle \rangle \) be a \( M-N \)-cover which is a precise refinement of \( \langle U, V \rangle \) (we can show as before that \( \langle \langle U, V \rangle \rangle \) exists). For each \( W \subseteq \langle U, V \rangle \) let \( W' \) be the element of \( \langle \langle U, V \rangle \rangle \) which corresponds to \( W \). Then \( \langle \langle U, V \rangle \rangle \) is a \( M-N \)-cover from \( U' \) to \( V' \). It suffices to show that if \( K \) is a continuum in \( \bigcup \langle \langle U, V \rangle \rangle \) such that \( K \cap U' \neq \emptyset \neq K \cap V' \) and \( O \subseteq \langle \langle U, V \rangle \rangle \backslash \{U, V\} \), then \( K \not\subseteq \bigcup \{\langle \langle U, V \rangle \rangle \backslash \{O\} \} \). If \( K \subseteq \bigcup \{\langle \langle U, V \rangle \rangle \backslash \{O\} \} \), then \( K \subseteq \bigcup \{\langle \langle U, V \rangle \rangle \backslash \{O\} \} \) and \( K \cap U \neq \emptyset \neq K \cap V \), which contradicts the definition of \( \langle U, V \rangle \).

We say that a \( N-K \)-cover \( B \) is embedded in a \( M-N \)-cover \( A \) if \( \{S^3(U, B) \mid U \in B\} \) refines \( A \). If \( A \) is a \( M-N \)-cover from \( FA \) to \( LA \), then we say \( A \) is a \( M-N \)-cover from \( FA \) to \( LA \) if \( B \) is embedded in \( A \), \( B \) is a \( N-K \)-cover from \( FB \) to \( LB \), \( CL(FB) \subseteq i(FA, A) \) and \( CL(U) \subseteq i(LA, A) \) for some \( U \in B \).

Remark. If \( B \) is a \( N-K \)-cover embedded in a \( M-N \)-cover \( A \) from \( FA \) to \( LA \) then for each \( U \in A \) there exists \( W \in B \) such that \( CL(W) \subseteq U \).

Proof. Without loss of generality \( FA \neq U \neq LA \). Let \( V \in B \) such that \( V \subseteq i(LA, A) \). Since \( B \) is an irreducible cover of \( K \subseteq N \), \( FB \subseteq i(FA, A), V \subseteq i(LA, A) \) and \( A \) is a \( M-N \)-cover from \( FA \) to \( LA \), there exists \( x \in K \cap U \backslash \bigcup \{T \subseteq A \mid T \neq U\} \). Let \( W \in B \) such that \( x \in W \). Then \( CL(W) \subseteq U \).

If it follows from the above Remark that if \( B \) is a \( N-K \)-cover embedded in the \( M-N \)-cover \( A \) from \( FA \) to \( LA \) and \( C \) is a \( K-L \)-cover embedded in \( B \) from \( FB \) to \( LB \), then \( C \) is embedded in \( A \) from \( FA \) to \( LA \).
2. Note. If $A$ is a $M$-$N$-cover from $FA$ to $LA$, then there exists for each $\varepsilon > 0$, by Note 1, a $N$-$K$-cover $B$ of mesh less than $\varepsilon$ embedded in $A$ from $FA$ to $LA$ and $C(LB) \subset i(LA, A)$.

If $A$ is a $M$-$N$-cover from $FA$ to $LA$ then by an endpiece $T$ of $A$ we mean a coherent subcollection of $A$ which contains $\{ W \in A \mid W \subset S^3(LA, A) \}$ and such that $FA \cap \bigcup T = \emptyset$. By $fT$ we denote

$$\{ W \in T \mid W \cap Z \neq \emptyset \text{ for some } Z \in A \backslash T \},$$

and we call $fT$ the first links of $T$.

Let $B$ be a $N$-$K$-cover embedded in a $M$-$N$-cover $A$ from $FA$ to $LA$. Let $S$ be an endpiece of $A$ and let $T$ be an endpiece of $B$. We say that $T$ folds in $S$ if $C(LB) \subset \bigcup fS$, $C(\bigcup fT) \subset \bigcup fS$, $C(\bigcup T) \subset \bigcup S$, and no coherent subcollection of $\{ W \in T \mid W \subset i(LA, A) \}$ contains both $LB$ and an element of $fT$.

3. Lemma. Let $B$ be a $N$-$K$-cover embedded in a $M$-$N$-cover $A$ from $FA$ to $LA$, let $S$ be an endpiece of $A$ and suppose $\{ W \in B \mid C(LW) \subset i(LA, A) \}$ contains at least two maximal distinct coherent subcollections $P$ and $Q$ and elements $U \in P$ and $V \in Q$ such that $C((U \cup V)) \subset \bigcup fS$. Then there exists a $K$-$L$-cover $C$ embedded in $A$ from $FA$ to $LA$ and an endpiece $T$ of $C$ such that $T$ folds in $S$.

Proof. Let $\{ U \in B \mid C(U) \subset i(LA, A) \} = R \cup R'$, where $R$ is the maximal coherent subcollection of $\{ U \in B \mid C(U) \subset i(LA, A) \}$ which contains $FB$ and $R \cap R' = \emptyset$. Let $R'' = \{ U \in R' \mid C(U) \subset \bigcup S \}$.

If $R'' = R'$ let $U \in R''$ such that $C(U) \subset \bigcup fS$. By Note 1 choose $\langle FB, U \rangle \subset B$, a continuum $L \subset K \cap \bigcup \langle FB, U \rangle$ such that $L \cap FB \neq \emptyset \neq L \cap U$, and a $K$-$L$-cover $C = \langle \langle FB, U \rangle \rangle$ from $FC$ to $LC$ with $FC \subset FB$ and $LC \subset U$. Then $C$ is embedded in $A$ from $FA$ to $LA$. Let $T$ be the maximal coherent subcollection of $C$ which contains $LC$ and such that $C(\bigcup T) \subset \bigcup S$. Then $C(\bigcup fT) \subset \bigcup fS \cap \bigcup R$. To see this, let $W \in fT$. There exists $Z \in C \backslash T$ such that $Z \cap W \neq \emptyset$. Let $V \in A$ such that $S^3(Z, C) \subset V$. Since $C(Z) \subset V$, $V \notin S$. Let $V_1, \ldots, V_n \in S$ such that $C(W) \subset V_1 \cup \cdots \cup V_n$. Then $V_i \cap V \neq \emptyset$ for each $i$ and, hence, $V_i \in fS$. Hence, $T$ folds in $S$.

If $R'' \neq R'$, let $x_0 \in K \cap i(FB, B)$. Then by the Boundary Bumping Theorem there exists a continuum $K' \subset K \cap (\bigcup R \cup LA \cup \bigcup R'')$ such that $x_0 \in K'$ and

$$C(S(K', B)) \cap \bigcup R' \subset \bigcup S.$$
4. **Lemma.** Let $M_0$ be a continuum. Suppose $B_1, B_2, \ldots$ is a sequence such that $B_{i+1}$ is a $M_i$-$M_{i+1}$-cover embedded in the $M_{i-1}$-$M_i$-cover $B_i$ from $F B_i$ to $L B_i$, and $T_1, T_2, \ldots$ is a sequence such that $T_i$ is an endpiece of $B_i$ and $T_{i+1}$ folds in $T_i$. Then $\bigcap_{i=1}^{\infty} (\bigcup \{U \in T_i\})$ contains an indecomposable continuum.

**Proof.** By the Boundary Bumping Theorem there exists a continuum $K_i$ in $\bigcup T_i$ such that $K_i$ meets $fT_i$ and $L B_i$. Then $K_i$ also meets $fT_j$ and $L B_j$ for $j < i$. Without loss of generality $\text{Lim } K_i = K$. Then $K$ is a continuum,

$$K \subset \bigcap_{i=1}^{\infty} \text{Cl}(\bigcup \{U \in T_i\}) = \bigcap_{i=1}^{\infty} \bigcup \{U \in T_i\}$$

and $K$ meets $\text{Cl}(\bigcup fT_i)$ and $\text{Cl}(L B_i)$ for each $i$. So $K \cap \bigcup fT_i \neq \emptyset$ for each $i$. Let $h: [0, 1] \to [0, 1]$ be defined by

$$h(x) = \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2}], \\ 2 - 2x & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Define $f_1: \text{Cl}(\bigcup T_i) \to [0, 1]$ to be a continuous function such that $f_1^{-1}(0) = \text{Cl}(\bigcup fT_i)$ and $f_1^{-1}(1) = \text{Cl}(L B_i)$. Define $f_2: \text{Cl}(\bigcup T_2) \to [0, 1]$ so that if $R$ is the union of the maximal coherent subcollections of $\{U \in B_2 \mid U \not\subseteq i(L B_1, B_1)\}$ which meets $\bigcup fT_2$, then

$$f_2(x) = \begin{cases} \frac{1}{2} f_1(x) & \text{if } x \in \text{Cl}(\bigcup R) \cap \text{Cl}(\bigcup T_2), \\ \frac{1}{2} & \text{if } x \in f_1^{-1}(1) \cap \text{Cl}(\bigcup T_2), \\ 1 - \frac{1}{2} f_1(x) & \text{if } x \in \text{Cl}(\bigcup T_2) \setminus (\text{Cl}(\bigcup R) \cup f_1^{-1}(1)). \end{cases}$$

Notice that $L B_2 \cap \text{Cl}(\bigcup R) = \emptyset$ and $K \cap \bigcup fT_2 \neq \emptyset \neq K \cap \text{Cl}(L B_2)$, so $f_2(K) = [0, 1]$. Then $f_1(x) = h \circ f_2(x)$ for each $x \in \text{Cl}(\bigcup T_2)$. By induction we define continuous functions $f_i: \text{Cl}(\bigcup T_i) \to [0, 1]$ such that $f_i(x) = h \circ f_{i+1}(x)$ for each positive integer $i$ and for each $x \in \text{Cl}(\bigcup T_{i+1})$ and $f_i(K) = [0, 1]$. Then $f = \lim f_i \mid K$ is a mapping of $K \subset \bigcap_{i=1}^{\infty} \bigcup \{U \in T_i\}$ onto Knaster’s indecomposable continuum $Y = \lim_{i,j} (I_i, h^j)$, where $I_i = [0, 1]$ and $h^j = h$ for all $i$ and $j$ (as in the proof of Bellamy [1 Theorem, p. 305]). Since $f$ maps $K$ onto $Y$, $K$ contains an indecomposable continuum.

5. **Lemma.** Let $M$ be a continuum which contains no indecomposable subcontinuum of diameter less than $\varepsilon$ for some $\varepsilon > 0$. Let $A$ be a $M$-$N$-cover of mesh less than $\varepsilon$ from $F A$ to $L A$. Then there exists a $N$-$N$-cover $B$ embedded in $A$ from $F A$ to $L A$, and an endpiece $T$ of $B$ such that $\text{Cl}(\bigcup T) \subset i(L A, A)$, and such that if $C$ is a $N$-$K$-cover embedded in $B$ from $F B$ to $L B$ then no endpiece of $C$ folds in $T$. Moreover, $B$ can be chosen to have arbitrarily small mesh.

**Proof.** As in Note 2 there is a $N$-$L$-cover $B_1$ of arbitrarily small mesh embedded in $A$ from $F A$ to $L A$ and an endpiece $T_1$ of $B_1$ such that $\text{Cl}(\bigcup T_1) \subset L A$. The lemma now follows by contradiction from Lemma 4.
3. The main results. The first three results in this section were proved by Thomas in [14] and Mahavier in [9] for the special case of arc-like continua.

6. Theorem. If $M$ is a continuum which contains no indecomposable subcontinuum of diameter less than $\epsilon$ for some $\epsilon > 0$ and $x, y \in M$, then there exists a subcontinuum $K$ of $M$ irreducible from $p$ to $q$ such that $K$ is locally connected at $q$, $d(x, p) < \epsilon$ and $d(y, q) < \epsilon$. In particular, if $K$ is irreducible from $p$ to $q'$ then $q' = q$.

Proof. Let $N_0 = M$. Let $\emptyset$ be a $M$-$N_0$-cover of mesh less than $\min\{\epsilon, 1\}$ such that $y \not\in S^2(x, 0)$. Let $x \in U \in \emptyset$ and $y \in V \in \emptyset$. By Note 1 choose $\langle U, V \rangle \subseteq \emptyset$, a continuum $N_1 \subseteq \bigcup \langle U, V \rangle$ such that $N_1 \cap U \neq \emptyset \neq N_1 \cap V$ and $B_1 = \langle \langle U, V \rangle \rangle$, a $N_0$-$N_1$-cover from $FB_1 \subseteq U$ to $LB_1 \subseteq V$. By Lemma 5 there exists a $N_1$-$N_2$-cover $B_2$ of mesh less than $\frac{1}{1}$ embedded in $B_1$ from $FB_1$ to $LB_1$ and an endpiece $T_2$ of $B_2$ such that $C1(\bigcup T_2) \subseteq \langle i(LB_1, B_1) \rangle$, and such that if $D$ is a $N_2$-$K$-cover embedded in $B_2$ from $FB_2$ to $LB_2$, then no endpiece of $D$ folds in $T_2$.

By repeated application of Lemma 5 there exist sequences of continua $N_1, N_2, \ldots$, covers $B_1, B_2, \ldots$, and endpieces $T_2, T_3, \ldots$ such that for each $i = 1, 2, \ldots$:

(i) $B_{i+1}$ is a $N_i$-$N_{i+1}$-cover embedded in $B_i$ from $FB_i$ to $LB_i$;
(ii) mesh $B_i < 1/i$;
(iii) $T_{i+1}$ is an endpiece of $B_{i+1}$ with $C1(\bigcup T_{i+1}) \subseteq \langle i(LB_i, B_i) \rangle$;
(iv) if $D$ is a $N_{i+1}$-$K$-cover embedded in $B_{i+1}$ from $FB_{i+1}$ to $LB_{i+1}$, then no endpiece of $D$ folds in $T_{i+1}$.

Let $K = \cap N_i$, $\{p\} = \cap FB_i$ and $\{q\} = \cap LB_i$. Then $K$ is a continuum. Since $B_i = \langle FB_i, LB_i \rangle$ for each $i$, $B_i$ is an irreducible cover of every subcontinuum of $K$ which contains both $p$ and $q$. Since mesh $B_i < 1/i$ it follows that $K$ is an irreducible continuum from $p$ to $q$.

Suppose $K$ is not connected im kleinen at $q$. There exists $\delta > 0$ such that $\delta < \epsilon$ and such that no subcontinuum of $K$ of diameter less than $\delta$ contains a neighbourhood of $q$ in $K$. Let $Q$ be the component of $q$ in $K \cap C1(B(q, \delta/4))$, where $B(q, \delta/4)$ denotes the open $\delta/4$ ball centered at $q$. Let $r \in Q \setminus B(q, \delta/4)$. Let $n$ be an integer so that $LB_{n-1} \subseteq B(q, \delta/4)$. If for each sufficiently large integer $i$ the maximal coherent subcollection of $\{U \in B_i \mid C1(U) \not\subseteq i(LB_n, B_n)\}$ which contains $FB_i$ also contains $r$ in its union, then there exists a component $N$ of $K \setminus LB_{n+1}$ which contains both $p$ and $r$. By the irreducibility of $K$ from $p$ to $q$ this would imply that $K = N \cup Q$ since $N \cup Q$ is a continuum in $K$ which contains $p$ and $q$. Thus, $q$ is in the interior of $Q$ in $K$ which contradicts the choice of $\delta$. Thus, for some sufficiently large $m > n$ the maximal coherent subcollection of $\{U \in B_m \mid C1(U) \not\subseteq i(LB_n, B_n)\}$ which contains $FB_m$ does not contain $r$ in its union. By Lemma 3 there exists a $N_n$-$K$-cover $D$ embedded in $B_n$ from $FB_n$ to $LB_n$, and an endpiece $T$ of $D$ such that $T$ folds in $T_n$. This is a contradiction and the connectedness im kleinen of $K$ at $q$ is proved.

Finally, we show that $K$ is locally connected at $q$. Let $V$ be any closed connected neighbourhood of $q$ of diameter less than $\epsilon$ such that $p \not\in V$. Let $L$ be the closure of the component of $K \setminus V$ which contains $p$. Then $U = K \setminus L$ is an open set containing $q$. By the irreducibility of $K$ from $p$ to $q$, $U \subseteq V$ since $V$ is connected. Let $N$ be the
component of $U$ which contains $q$. Then $Cl(N) \cap Bd(U) \neq \emptyset$ and, since $Bd(U) \subset L$, $Cl(N) \cap L \neq \emptyset$. Moreover, $Cl(N) \setminus N \subset L$ and, by the irreducibility of $K$, $K = N \cup L$. Hence $N = K \setminus L = U \subset V$ is a connected open set containing $q$. This completes the proof of the theorem.

The next corollary gives an affirmative answer to a question of Mahavier (cf. Problem 121, University of Houston problem book).

7. Corollary. Let $M$ be a hereditarily decomposable continuum and $x, y \in M$. Then for each $\epsilon > 0$, there exists a subcontinuum $K'$ of $M$ such that $K'$ is irreducible from $x$ to $q$ for some $q$ with $d(y, q) < \epsilon$ and such that $\{q\}$ is an end-tranche of $K$.

Proof. Assume $M$ is irreducible from $x$ to $y$. Let $f: M \to [0, 1]$ be a finest monotone map with $f(x) = 0$ and $f(y) = 1$. Choose $K$ and $p \in f^{-1}([0, \frac{1}{2}])$ and $q \in B(y, \epsilon) \cap f(\left(\frac{1}{2}, 1\right]$ as in Theorem 6. Let $K' = K \cup f^{-1}([0, f(p)])$.

In [14 and 3] are given examples of hereditarily decomposable arc-like continua $X$ so that if $A'$ is a subcontinuum of $X$ with a degenerate tranche $L$ then $L$ is an end-tranche of $K$.

8. Corollary. If $M$ is a continuum of type $\lambda$ which contains no indecomposable continuum of diameter $< \epsilon$ for some $\epsilon > 0$ and such that each tranche of $M$ is a tranche of cohesion, then a dense $G_\delta$-set of tranches of $M$ are degenerate.

Proof. Let $\Phi: M \to [0, 1]$ be a map such that for each $t \in [0, 1]$, $\Phi^{-1}(t)$ is a nowhere dense subcontinuum of $M$. Let $[a, b] \subset [0, 1]$ such that $a < b$. By Theorem 6, there exists a continuum $K \subset \Phi^{-1}([a, b])$ irreducible from $p$ to $q$ such that $\Phi(p) < \Phi(q)$ and $\{q\}$ is a degenerate end-tranche of $M$. It also follows that if $t \in D = \{s \in [0, 1] | \Phi^{-1}(s) is a tranche of continuity\}$, then $\Phi^{-1}(t)$ is degenerate, and it is known (see [8, p. 202]) that $D$ is a dense $G_\delta$ in $[0, 1]$.

9. Corollary. If $M$ is an irreducible hereditarily decomposable continuum such that each tranche of $M$ is a tranche of cohesion, then a dense $G_\delta$-set of tranches of $M$ is degenerate.

For any compact metric space $M$ we denote by $2^M$ (respectively, $C(M)$) the space of all nonempty, compact subsets (respectively, subcontinua) of $M$ with the topology induced by the Hausdorff metric.

Let $M$ be a continuum and let $x \in M$. Then $\{x\}$ is said to be arcwise accessible from $2^M \setminus C(M)$ (see [11 and 12]) provided there exists an arc $A$ in $2^M$ such that $A \cap C(M) = \{x\}$. The next corollary follows from Theorem 6 and Theorem 4.1 of [3]. It gives a positive solution to a question of Nadler (see [11, 12.19 and 12, 8.1]).

10. Corollary. Let $M$ be a hereditarily decomposable continuum. There exists a point $x \in M$ such that $\{x\}$ is arcwise accessible from $2^M \setminus C(M)$.

In view of Corollary 8 the following question is interesting:

11. Question. If $X$ is an irreducible continuum which admits a continuous monotone decomposition onto an arc, does $X$ contain hereditarily indecomposable tranches? In particular, does Knaster's continuum in [6] contain tranches which are pseudoarcs?
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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA, BIRMINGHAM, ALABAMA 35294
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SASKATEWAN, SASKATOON, CANADA S7N 0W0