ON THE DISTRIBUTION OF THE PRINCIPAL SERIES  
IN $L^2(\Gamma \backslash G)$  

BY  
ROBERTO J. MIATELLO AND JORGE A. VARGAS$^1$

Abstract. Let $G$ be a semisimple Lie group of split rank one with finite center. If $\Gamma \subset G$ is a discrete cocompact subgroup, then $L^2(\Gamma \backslash G) = \sum_{\omega \in \hat{G}} \eta_\Gamma(\omega) \cdot \omega$. For fixed $\sigma \in \hat{G}$, let $P(\sigma)$ denote the classes of irreducible unitary principal series $\varpi_{\sigma,i}$ ($\sigma \in \mathfrak{A}^*$. Let, for $s > 0$, $\psi_\sigma(s) = \sum_{\omega \in P(\sigma)} \eta_\Gamma(\omega) \cdot e^{i \lambda_\omega}$, where $\lambda_\omega$ is the eigenvalue of $V$ (the Casimir element of $G$) on the class $\omega$. In this paper, we determine the singular part of the asymptotic expansion of $\psi_\sigma(s)$ as $s \to 0^+$ if $\Gamma$ is torsion free, and the first term of the expansion for arbitrary $\Gamma$. As a consequence, if $N_\sigma(r) = \sum_{\omega \in P(\sigma), \delta \subset \omega} \frac{1}{\delta} \cdot \eta_\Gamma(\omega)$ and $G$ is without connected compact normal subgroups, then  
$$N_\sigma(r) \sim c \cdot |Z(G) \cap \Gamma| \cdot \text{vol}(\Gamma \backslash G) \cdot \frac{\dim(\sigma)}{\delta} \cdot r^{e} \quad (e = \frac{1}{2} \dim G/K),$$
as $r \to +\infty$. In the course of the proof, we determine the image and kernel of the restriction homomorphism $i^*: R(K) \to R(M)$ between representation rings.

Introduction. Let $G$ be a connected, real semisimple Lie group with Lie algebra $\mathfrak{g}$. Let $G = K.A.N.$ (respectively $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$) be an Iwasawa decomposition of $G$ (respectively $\mathfrak{g}$) and let $M$ be the centralizer of $A$ in $K$. We assume throughout this paper that $G$ has finite center and split rank one. We do not assume that $G$ is linear. Let $\hat{G}$ denote the set of equivalence classes of irreducible unitary representations of $G$. If $\sigma \in \hat{G}$, $\nu \in \mathfrak{g}^*$ let $\varpi_{\sigma,\nu}$ be the principal series representation of $G$, parametrized as in [DW, §3]. In this parametrization $\varpi_{\sigma,\nu}$ is unitary if $\nu \in i\mathfrak{a}^*$. If $\omega \in \hat{G}$ let $\lambda_\omega$ and $\theta_\omega$ denote, respectively, the eigenvalue of the Casimir element of $G$ on the class $\omega$ and the distributional character of $\omega$. We will abbreviate by writing $\lambda_{\sigma,\nu} = \lambda_{\sigma,\nu^\circ}$. If $\omega \in \hat{G}(G)$, the discrete series of $G$, let $d(\omega)$ denote the formal degree of $\omega$.

For fixed $\sigma \in \hat{G}$ set  
P($\sigma$) = $\{\varpi_{\sigma,i} | \nu \in \mathfrak{g}^* \text{ and } \varpi_{\sigma,i,\nu} \text{ is irreducible} \}.

Recall [KS, §12] that $\varpi_{\sigma,i,\nu}$ is reducible only if rank $G = \text{rank } K$, $\nu = 0$ and in this case $\varpi_{\sigma,0} = \varpi_{\sigma,0}^+ + \varpi_{\sigma,0}^-$ where $\varpi_{\sigma,0}^\pm$ are inequivalent irreducible representations. Let $R(\sigma) = \{\varpi_{\sigma,0}^\pm\}$ if $\varpi_{\sigma,0}$ is reducible and $R(\sigma) = \emptyset$, otherwise. Finally let $C(\sigma)$ denote the subset of $\hat{G}(G)$ of classes $\omega$ such that, if $(\pi_\omega, H_\omega) \subset \omega$, then $H_\omega$ is infinitesimally equivalent to $J_{\sigma,\nu}$ for some $\nu$ s.t. $Re(\nu, \lambda) > 0$ ($J_{\sigma,\nu}$ is as in [DW, Theorem 4.1], and $\lambda$ is the long positive restricted root).
Let $\Gamma$ be a discrete, cocompact subgroup of $G$. The right regular representation $\pi_\Gamma$ of $G$ in $L^2(\Gamma \backslash G)$ decomposes $\pi_\Gamma = \sum n_\Gamma(\omega) \cdot \omega$ and $n_\Gamma(\omega) < \infty$, for any $\omega \in \hat{G}(G)$. If $\tau \in \hat{G}(K)$, 
$$\phi_\tau(s) = \sum_{\omega \in \hat{G}(G)} n_\Gamma(\omega) \cdot [\tau: \omega] \cdot e^{s\lambda_\omega}$$
defines a $C^\infty$ function on $\mathbb{R}^+$, the series converging uniformly on compacta with all derivatives [W]. Hence, if $\sigma \in \hat{G}(M)$ is fixed, the series $\psi_\sigma(s) = \sum_{\omega \in P(\sigma)} n_\Gamma(\omega) \cdot e^{s\lambda_\omega}$ defines a $C^\infty$ function for $s > 0$. The purpose of this paper is to study the asymptotic behavior of $\psi_\sigma(s)$, as $s \to 0^+$. By using the technique in [MI] we determine the singular part of the asymptotic expansion of $\psi_\sigma(s)$, as $s \to 0^+$, when $\Gamma$ is torsion free.

**Theorem 1.** Let $\Gamma \subset G$ be a discrete, cocompact, torsion-free subgroup. Then

$$\psi_\sigma(s) = \text{vol}(\Gamma \backslash G) \cdot e^{-s(\rho_\sigma^2 + \omega_\lambda_\sigma)} \left( \sum_{i=0}^{c + d - 1} b_{2(i-d)+1}(\sigma) \cdot \Gamma(i + 1 - d) \cdot (4s)^{-i+1} \right) - g_\sigma(s)$$

where $g_\sigma(s)$ extends to $\tilde{g}_\sigma(s)$, a $C^\infty$ function on $\mathbb{R}$, such that

(i) if rank $G = \text{rank} K$

$$\tilde{g}_\sigma(0) = \text{vol}(\Gamma \backslash G) \left( \sum_{i=0}^{c-1} b_{2i+1}(\sigma) \left( \frac{(\varepsilon + 1) \cdot 2^{2i+1} - 1}{i+1} \right) \cdot B_{2(i+1)} \right)$$

$$+ \sum_{\omega \in R(\sigma) \cup C(\sigma)} s(\omega) \cdot n_\Gamma(\omega)$$

$$+ \sum_{\omega \in \hat{G}(G)} s(\omega)(n_\Gamma(\omega) - d(\omega) \cdot \text{vol}(\Gamma \backslash G)),$$

(ii) if rank $G > \text{rank} K$

$$\tilde{g}_\sigma(0) = \sum_{\omega \in C(\sigma)} s(\omega) \cdot n_\Gamma(\omega).$$

Here $c = \frac{1}{2} \dim(G/K)$, $d = c - [c]$, $\sigma \in \mathbb{R}^+$, and $\lambda_\sigma$ is the eigenvalue of the Casimir element of $M$ on the class $\sigma$. Furthermore, $b_{2(i-d)+1}(\sigma)$ denotes, for $i = 1, \ldots, c + d - 1$, the $i$th coefficient of the polynomial part of the Plancherel density associated to $\sigma$, $B_{2j}$ is the $j$th Bernoulli number, and $\varepsilon = 1$ or $-1$ depending on $\sigma$. Finally, if $\omega \in R(\sigma) \cup C(\sigma) \cup \hat{G}(G)$, then

$$s(\omega) = [\eta : \omega] = \dim \text{Hom}_K(\eta, \omega) \in \mathbb{Z}$$

where $\eta = \eta_\sigma$ is a virtual representation of $K$ (in particular, $s(\omega)$ depends on $\sigma$ but not on $\Gamma$).

Let $R(M)$ and $R(K)$ denote the representation rings of $M$ and $K$. We make use of the following.

**Proposition 1.** Let $i^*: R(K) \to R(M)$ be the restriction homomorphism. Then $\text{Im}(i^*) = R(M)^W$, where $W = W(\mathfrak{g}, \mathfrak{h})$. If rank $G = \text{rank} K$, then $R(M)^W = R(M)$ and $i^*$ is surjective.
Let $\Gamma \subset G$ be an arbitrary discrete cocompact subgroup. We assume for simplicity, that $G$ has no nontrivial compact, connected, normal subgroups (see the remark below). Theorem 1.1 in [W], together with Proposition 1, imply

**Corollary 1.** If $Z(\Gamma) = Z(G) \cap \Gamma$ and $\sigma \in \mathcal{E}(M)$ satisfies $\sigma|_{Z(\Gamma)} = 1$, then

$$\lim_{s \to 0^+} s^r \cdot \psi_\sigma(s) = \frac{\dim(\sigma) \cdot |Z(\Gamma)| \cdot \text{vol}(\Gamma \backslash G)}{(4\pi)^r}$$

(if $\sigma|_{Z(\Gamma)} \neq 1$ then $n_{\Gamma}(\pi_{\sigma,r}) = 0$, for all $\nu$, hence $\psi_\sigma(s) = 0$).

Let $N_\sigma(r) = \sum_{\omega \in \rho(\sigma), \lambda_{\omega,r} > r} n_{\Gamma}(\omega)$ $(r > 0)$. Corollary 1 and the Tauberian theorem for the Laplace transform imply

$$\lim_{r \to \infty} r^{-c} \cdot N_\sigma(r) = \frac{\dim(\sigma) \cdot |Z(\Gamma)| \cdot \text{vol}(\Gamma \backslash G)}{\Gamma(c + 1) \cdot (4\pi)^c}.$$

**Remark.** When $G$ has compact normal subgroups, (1) still holds with $\dim V_{\sigma}^{\Gamma \cap N} \cdot |\Gamma \cap N|$ substituting $\dim(\sigma) \cdot |Z(G) \cap \Gamma|$, where $N = \bigcap_{x \in G} xKx^{-1}$. This follows from [W, 1.1], with a correction factor as in [BH, §6], and Proposition 1. Indeed, if $\sigma = i^*(\eta)$, $\eta = \sum_{j \in J} \rho_j \in R(K)$, one can show that $\sum \dim V_{\tau, \eta}^{\Gamma \cap N} = \dim V_{\sigma}^{\Gamma \cap N}$.

The asymptotic formula (1) for the spherical principal series (i.e. $\sigma = 1$) in $L^2(\Gamma \backslash G)$ was proposed by Gelfand ([G, p. 77], see also [GGP, pp. 82, 94]). It was proved by Gangolli for complex $G$, by Eaton for $G$ of split rank one, and by Duistermaat-Kolk-Varadarajan, for general $G$ ([Ga, DKV], see also [GW]). With the aid of Proposition 1, Theorem 1.1 in [W] implies the Gelfand type formula (1) for any $\sigma \in \mathcal{E}(M)$, when $G$ is as above.

The outline of the paper is as follows. In §1, we prove Theorem 1.1 (assuming Proposition 1). The proof of Proposition 1 is given in §2. Finally, we show (Lemma 2.6) that if rank $G = \text{rank } K$, then $J = \ker i^* \neq 0$ and determine $J$ explicitly. We recall that if $\Gamma$ is torsion-free, each $\eta \in \ker i^*$ yields a finite alternating sum formula in the $n_{\Gamma}(\omega)$’s [M3, Theorem 1.2].

1. We first normalize Haar measures conveniently. If $\lambda$ denotes the long positive restricted root of $\mathfrak{g}$ let $H \in \mathfrak{h}$ be so that $\lambda(H) = 2$. Let $a = B(H, H)$, $B$ the Killing form of $\mathfrak{g}$. Let $d\tilde{x}, d\tilde{k}$ denote respectively the invariant Riemannian measures on $G$ and $K$ induced by the inner product on $\mathfrak{g}$, $(X, Y) = a^{-1} \cdot B(X, Y)$. We will use on $G$ and $K$ the measures $dx = \text{vol}(K)^{-1} \cdot d\tilde{x}$, $dk = \text{vol}(K)^{-1} \cdot d\tilde{k}$. As usual, let $dx$ on $\Gamma \backslash G$ be so that

$$\int_{\Gamma \backslash G} \left( \sum_{\gamma} f(\gamma x) \right) dx = \int_G f(x) \ dx, \text{ for } f \in C_c(G).$$

For fixed $\sigma \in \mathcal{E}(M)$, the Plancherel density associated to $\sigma$ can be written $\mu_\sigma(x\lambda) = q_\sigma(x) \cdot \phi_\sigma(x)$, where $q_\sigma(x)$ is a polynomial of degree $2c - 1$ and $\phi_\sigma(x) = 1$, $\tanh \pi x$ or $\coth \pi x$, depending on $\sigma$ [O]. Moreover, $\phi_\sigma = 1$ if and only if rank $G > \text{rank } K$. Let $d = c - \lfloor c \rfloor$, that is, $d = 0$ if rank $G = \text{rank } K$ and $d = \frac{1}{2}$, otherwise. If the Haar
measure on $G$ is normalized as above, then

$$q_\omega(x) = \sum_{i=0}^{c+d-1} b_{2i-1}(\sigma) \cdot x^{2i-1}$$

and $b_{2i-1}(\sigma) = \dim(\sigma)/(\Gamma(c) \cdot \pi^i)$ [M2, §3].

For fixed $\tau$ set, if $x \in G$ and $s > 0$,

$$g_{\tau,s}(x) = \int_{\mathcal{E}(G)} \dim(\tau)^{-1} \cdot \phi_{\tau,\omega}(x^{-1}) \cdot e^{s\lambda_\omega} \cdot d\mu(\omega)$$

where $\phi_{\tau,\omega}$ is the $\tau$-spherical trace function associated to $\omega$ and $\mu(\omega)$ is the Plancherel measure on $\mathcal{E}(G)$. DeGeorge and Wallach (unpublished) have proved a general result which implies that $g_{\tau,s} \in \mathcal{C}^p(G)$ (the $p$-Schwartz space of $G$) for any $p > 0$ ($G$ can be of arbitrary split rank). Using this fact, one shows [M3, 1.1] that $\theta_\omega(g_{\tau,s}) = [\tau : \omega] e^{s\lambda_\omega}$ for any $\omega \in \mathcal{E}(G)$, where $[\tau : \omega] = \dim \text{Hom}_K(\tau, \omega)$.

Let $\Gamma \subset G$ be a discrete, cocompact, torsion-free subgroup. Fix $\sigma \in \mathcal{S}(M)$. We assume first that rank $G = \text{rank } K$. Then, by Proposition 1, there exists $\eta = \sum m_j \tau_j$, $m_j \in \mathbb{Z}$, $\tau_j \in \mathcal{E}(K)$ such that $i^*(\eta) = \sigma$. Set $g_{\eta,s} = \sum m_j g_{\tau_j,s}$. Since $g_{\eta,s} \in \mathcal{C}^p(G)$ for any $p > 0$, and $g_{\eta,s}$ is $K$-finite, the operator $\pi_\Gamma(g_{\eta,s})$ on $L^2(\Gamma \backslash G)$ is trace-class [M1, §2], and

$$\text{tr } \pi_\Gamma(g_{\eta,s}) = \sum_{\omega \in \mathcal{E}(G)} n_\eta(\omega) \cdot [\eta : \omega] \cdot e^{s\lambda_\omega},$$

where $[\eta : \omega] = \sum m_j \cdot [\tau_j : \omega]$.

If $\omega \in \mathcal{E}(G)$, by Langlands' classification, either $\omega \in \mathcal{E}_2(G)$ or $\omega \in P(\xi) \cup R(\xi) \cup C(\xi)$, for some $\xi \in \mathcal{E}(M)$. If $\omega \in P(\xi)$, then

$$[\eta : \omega] = [i^*(\eta) : \xi] = \begin{cases} 0, & \xi \neq \sigma, \\ 1, & \xi = \sigma. \end{cases}$$

Hence $\text{tr } \pi_\Gamma(g_{\tau,s}) = \psi_{\omega}(s) + h_{\omega}(s)$, where

$$h_{\omega}(s) = \sum_{\omega \in \mathcal{E}(G) \cup R(\sigma) \cup C(\sigma)} n_\eta(\omega) \cdot [\eta : \omega] \cdot e^{s\lambda_\omega}.$$

Note that the sets $\{\omega \in \mathcal{E}_2(G) | [\eta : \omega] \neq 0\}$, $\{\omega \in C(\sigma) | n_\eta(\omega) \neq 0\}$ are finite [DW, p. 489]. Hence $h_{\omega}(s)$ is analytic.

On the other hand ([M1, 5.1], essentially)

$$\text{tr } \pi_\Gamma(g_{\eta,s}) \sim \text{vol}(\Gamma \backslash G) \cdot g_{\eta,s}(1), \text{ as } s \to 0^+$$

(that is, $\text{tr } \pi_\Gamma(g_{\eta,s}) - \text{vol}(\Gamma \backslash G) \cdot g_{\eta,s}(1) = o(s^n)$ for all $n \in \mathbb{N}$, as $s \to 0^+$). Set $g_{\eta,s}^0 = \sum m_j \cdot g_{\eta,j,s}^0$, where

$$g_{\eta,s}^0 = \sum_{\omega \in \mathcal{E}(G)} d(\omega) \cdot [\eta : \omega] \cdot e^{s\lambda_\omega} \quad \text{(a finite sum)}.$$

By choice of $\eta$, if $h_{\eta,s} = g_{\eta,s} - g_{\eta,s}^0$, then

$$h_{\eta,s}(1) = \int_{-\infty}^{+\infty} e^{s\lambda_{\eta,s}} \cdot \mu_\sigma(x\lambda) \, dx,$$
where \( \lambda_{\sigma, x\lambda} = -4x^2 + \rho_1^2 + a\lambda_\sigma \) [M1, p. 17]. Here if \( X_1, \ldots, X_r \) is a basis of \( \mathcal{V} \) such that \( (X_i, X_j) = -\delta_{ij} \) and \( \Delta_{\mathcal{V}} = -\Sigma X_i^2 \), \( \lambda_\sigma \) is so that \( \sigma(\Delta_{\mathcal{V}}) = \lambda_\sigma \cdot I \). On the other hand \( \mu_\sigma(x\lambda) = q_\sigma(x) \cdot \phi_\sigma(x) \), \( q_\sigma(x) = \Sigma_0^{-1} b_{2i+1}(\sigma) \cdot x^{2i+1} \) and \( \phi_\sigma(x) = \tanh \pi x \) or \( \coth \pi x \). We may write (if \( x \neq 0 \)) \( 1 = \phi_\sigma(x) - 2/((1 + e^{2\pi x}) \cdot x^2) \), where \( \varepsilon = 1 \) if \( \phi_\sigma(x) = \tanh \pi x \) (respectively \( \varepsilon = -1 \), if \( \phi_\sigma(x) = \coth \pi x \)). Hence

\[
h_{\eta, x}(1) = e^{-x(x^2 + a\lambda_\sigma)} \cdot \left[ 2 \int_0^{+\infty} e^{-4sx^2} \cdot q_\sigma(x) \, dx - 4 \int_0^{+\infty} e^{-4sx^2} \cdot q_\sigma(x) \, dx \right]
\]

\[
= e^{-x(x^2 + a\lambda_\sigma)} \cdot \left[ \sum_{i=0}^{c-1} b_{2i+1}(\sigma) \cdot i! (4s)^{-i-1} \right]
\]

\[
- \sum_{i=0}^{c-1} b_{2i+1}(\sigma) \int_0^{+\infty} \frac{4 \cdot e^{-4sx^2} \cdot x^{2i+1}}{1 + e^{2\pi x}} \, dx.
\]

Furthermore [WW, pp. 266–268]

\[
\int_0^{+\infty} \frac{4x^{2i+1}}{1 + e^{2\pi x}} \, dx = \frac{2^{2(i+1)}-1}{i+1} \cdot B_{2(i+1)},
\]

\[
\int_0^{+\infty} \frac{4x^{2i+1}}{1 - e^{2\pi x}} \, dx = -\frac{B_{2(i+1)}}{i+1}.
\]

\( B_{2m} \) the \( m \)th Bernoulli number. Hence

\[
\lim_{s \rightarrow -0^+} \sum_{i=0}^{c-1} b_{2i+1}(\sigma) \int_0^{+\infty} \frac{4 \cdot e^{-4sx^2} \cdot x^{2i+1}}{1 + e^{2\pi x}} \, dx
\]

\[
= \sum_{i=0}^{c-1} b_{2i+1}(\sigma) \left[ \frac{(\varepsilon + 1) \cdot 2^{2i+1}}{i+1} \right] B_{2(i+1)}
\]

(in fact, the full asymptotic expansion

\[
\int_0^{+\infty} \frac{e^{-4sx^2} \cdot x^{2i+1}}{1 + e^{2\pi x}} \, dx \sim \sum_{j=0}^{\infty} a_j \cdot s^j
\]

can be written down explicitly).

Summing up

\[
\psi_\sigma(s) \sim \text{vol}(\Gamma \backslash G) \cdot \left( h_{\eta, x}(1) + g_{\eta, x}^0(1) \right) - h_\sigma(s),
\]

\[
\psi_\sigma(s) \sim \text{vol}(\Gamma \backslash G) e^{-x(x^2 + a\lambda_\sigma)} \cdot \left( \sum_{i=0}^{c-1} b_{2i+1}(\sigma) \cdot i! (4s)^{-i-1} \right) - g_\sigma(s),
\]

where

\[
g_\sigma(s) = \text{vol}(\Gamma \backslash G) \cdot e^{-x(x^2 + a\lambda_\sigma)} \cdot \left( \sum_{i=0}^{c-1} b_{2i+1}(\sigma) \int_0^{+\infty} \frac{4 \cdot e^{-4sx^2} \cdot x^{2i+1}}{1 + e^{2\pi x}} \, dx \right)
\]

\[
- \text{vol}(\Gamma \backslash G) \cdot g_{\eta, x}^0(1) + h_\sigma(s).
\]

This concludes the proof of Theorem 1, in this case.
If rank $G > \text{rank } K$, let $W = W(\mathfrak{g}, \mathfrak{k}) = \{1, u\}$. If $\sigma \in \mathfrak{g}(M)$ is such that $\sigma = \sigma^u$, then by Proposition 1, $\sigma = i^*(\eta)$, $\eta \in R(K)$, and the above proof (with several simplifications) can be repeated. Moreover, in this case $\mu_\sigma(x\lambda) = q_\sigma(x)$, $g^0_{\eta,s} = 0$, hence $g_\sigma(s) = h_\sigma(s)$.

If $\sigma \neq \sigma^u$ then $\sigma + \sigma^u = i^*(\eta)$, $\eta \in R(K)$. Define $g_{\eta,s}$ as before. In this case $g_{\eta,s} = h_{\eta,s}$. Arguing as above, one obtains

$$\sum_{\omega \in \Omega(G)} n_\Gamma(\omega) \cdot [\eta : \omega] \cdot e^{i\lambda_\omega} \sim \text{vol}(\Gamma \setminus G) \cdot h_{\eta,s}(s), \text{ as } s \to 0^+.$$ 

The left-hand side equals

$$2\psi_\sigma(s) + 2 \sum_{\omega \in \Omega(\sigma)} n_\Gamma(\omega) \cdot e^{i\lambda_\omega}$$

since $\mathcal{C}_2(G) = R(\sigma) = \phi$, $\pi_{\sigma,s} = \pi_{\sigma^u,s}$ ($\nu \in \mathfrak{h}^*_C$) and $[\eta : \omega] = 1$ if $\omega \in C(\sigma)$, in this case. Similarly,

$$h_{\eta,s}(1) = 2 \cdot e^{-\frac{i}{2}(\rho^2 + a\lambda_\omega)} \cdot \left( \sum_{\alpha \in \Delta^+} b_{\alpha,\sigma}(\sigma) \int_{-\infty}^{+\infty} e^{-4\pi x^2} \cdot x^{2i} \, dx \right)$$

$$= 2e^{-\frac{i}{2}(\rho^2 + a\lambda_\omega)} \cdot \left( \sum_{\alpha \in \Delta^+} b_{\alpha,\sigma}(\sigma) \cdot \Gamma \left( i + \frac{1}{2} \right) \cdot (4\pi)^{-1/2} \right).$$

This concludes the proof. We observe that, if $\sigma \in \text{Im}(i^*)$, Corollary 1 is an immediate consequence of Theorem 1.1 in [W] and Proposition 1 (with our normalization of measures $C_G = 1$, $C_G$ as in [W, 1.1]). If $\sigma \not\in \text{Im}(i^*)$, then $\sigma + \sigma^u = i^*(\eta)$ and (essentially) the above argument yields the result.

2. This section is mainly devoted to the proof of Proposition 1. Assume first that rank $G > \text{rank } K$. Then rank $K = \text{rank } M$. Let $T_1 \subset M$ be a maximal torus. There is a commutative diagram

$$
\begin{array}{ccc}
R(K) & \xrightarrow{i^*} & R(M) \\
\downarrow j^*_K & & \downarrow j^*_M \\
R(T_1)^{W_K} & & R(T_1)^{W_M}
\end{array}
$$

where $j^*_K$ is an isomorphism onto $R(T_1)^{W_K}$. If $M^* = N_K(A)$ (the normalizer of $A$ in $K$), there is $u \in M^* \cap N_K(T)$, $u \not\in M$. Therefore, $W_K$ is generated by $W_M$ and $u(\frac{1}{2} W_K/W_M) = 2$. Thus $\text{Im}(j^*_K) = R(T)^{W_K} = (R(T)^{W_M})^W$ and Proposition 1 is clear, in this case.

From now on, we assume that rank $G = \text{rank } K$. Fix $\mathfrak{g} \subset \mathfrak{h}$, a Cartan subalgebra, and let $\Delta = \Delta(\mathfrak{g}_C, \mathfrak{h}_C)$. Then $\Delta = \Delta_c \cup \Delta_a$, where $\Delta_c (\Delta_a)$ is the set of compact (noncompact) roots. Fix $\Delta^+ \subset \Delta$ a system of positive roots, $\Delta^+ = \Delta^+_c \cup \Delta^+_a$. Let $\{X_\alpha\}_{\alpha \in \Delta^+_c}$, $\{H_\alpha\}_{\alpha \in \Delta^+_a}$ be a Weyl basis of $\mathfrak{g}_C$ adapted to the compact form $\mathfrak{g}_c = \mathfrak{h} \oplus i\mathfrak{p}_C$ [H, p. 421]. Then, if $\sigma$ denotes the conjugation of $\mathfrak{g}_C$ with respect to $\mathfrak{g}$, $\sigma X_\alpha = -X_{-\alpha}$ ($\alpha \in \Delta_a$) and $\sigma X_\alpha = X_{-\alpha}$ ($\alpha \in \Delta_c$). From now on, we fix $\beta \in \Delta_a^+$ and choose $\mathfrak{h} = R(X_\beta + X_{-\beta})$. The following lemma is not difficult.
2.1. Lemma.

\[ \mathfrak{m}_C = \ker \beta \oplus \sum_{\alpha \in \Delta} C \cdot X_\alpha \oplus \sum_{\alpha \in \Delta, \alpha + \beta \in \Delta} C(X_\alpha + c_\alpha X_{\alpha+2\beta}) \]

where \( c_\alpha = -N_{\alpha, \beta}/N_{\alpha+2\beta, \beta} \) and \( N_{\alpha, \beta} \) is such that \( [X_\alpha, X_\beta] = N_{\alpha, \beta} \cdot X_{\alpha, \beta} \). Furthermore, \( \ker \beta \) is a Cartan subalgebra of \( \mathfrak{m}_C \) and

\[ \Delta_\mathfrak{m} = \Delta(\mathfrak{m}_C, \ker \beta) = \{ \alpha' = \alpha \in \ker \beta \mid \alpha \pm \beta \notin \Delta \} \cup \{ \alpha' = \alpha \in \ker \beta \mid \alpha + 2\beta \in \Delta \} \]

The root spaces are \( \mathfrak{g}_\alpha = C X_\alpha \), if \( \alpha \pm \beta \notin \Delta \) and \( \mathfrak{g}_\alpha = C(X_\alpha + c_\alpha X_{\alpha+2\beta}) \), if \( \alpha + 2\beta \in \Delta \).

Let \( \Delta_+ \subset \Delta_\mathfrak{m} \) be the positive system induced by \( \Delta^+ \). Let also \( T_1 = \exp(\ker \beta \cap \mathfrak{h}) \), a maximal torus of \( M^0 \) (the connected component of 1 in \( M \)).

2.2. Lemma. Let \( G \) be semisimple, of split rank one, and such that rank \( G = \) rank \( K \). Let \( W = W(\mathfrak{g}, \mathfrak{a}) = \{ 1, u \} \). Then \( \sigma = \sigma^u \), for any \( \sigma \in \mathfrak{g}(M) \).

Proof. In [KS, §16] the lemma is verified for \( G = \text{Spin}(2n, 1) \), \( G = \text{SU}(n, 1) \) and \( G = \text{Sp}(n, 1) \). We give a different proof. It is well known that \( M = Z(G) \cdot M_0 \). Moreover, \( W \) is generated by \( u = \exp(\pi i H_\beta/\langle \beta, \beta \rangle) \). If \( \sigma \in \mathfrak{g}(M) \), then \( \chi_\sigma(x) = \chi_\sigma(x) \) for any \( x \in M \), since this holds for \( x \in T_1 \) and \( x \in Z(G) \). Hence \( \sigma^u = \sigma \).

Remark. In [KS, Theorem 12.5], Knapp and Stein prove that if \( G \) is a linear group of split rank one, \( \pi_{\sigma, 0} \) is reducible only if \( \nu = 0 \). Moreover, \( \pi_{\sigma, 0} \) is reducible if and only if (i) \( \sigma = \sigma^u \), (ii) \( \mu_\sigma(0) > 0 \). Lemma 2.2 says that if rank \( G = \) rank \( K \), (i) is automatic. If rank \( G > \) rank \( K \) it is no longer true that \( \sigma = \sigma^u \). In fact, \( \sigma = \sigma^u \) forces \( \mu_\sigma(0) = 0 \), hence \( \pi_{\sigma, 0} \) is irreducible.

We next prove a lemma. Let \( K_1 \) be a Lie group with finitely many components, such that \( \text{Ad}(K_1) \) is compact. Let \( K_2 \subset K_1 \) be a closed subgroup. As usual, let \( R(K_i) \) and \( \mathfrak{e}(K_i) \) denote, respectively, the representation ring and the unitary dual of \( K_i \) (\( i = 1, 2 \)). Let \( S \) be a closed subgroup of \( Z(K_1) \) (the center of \( K_1 \)) such that \( S \subset K_2 \). Then \( R(K_1/S) \) can be identified with the subring of \( R(K_1) \) generated by those representations \( \tau \) of \( K_1 \) such that \( S \subset \ker \tau \). Let \( i^*_S: R(K_1/S) \to R(K_2/S) \), \( i^* : R(K_1) \to R(K_2) \) denote the restriction homomorphisms.

2.3. Lemma. \( \text{Im}(i^*_S) = \text{Im}(i^*) \cap R(K_2/S) \).

Proof. Let \( \tau \in \mathfrak{e}(K_1) \). If \( i^*(\tau) = \sum r_j \cdot \xi_j \) (\( r_j \neq 0 \)) we say that \( \xi_j \) is a \( K_2 \)-type of \( \tau \). We note that if \( \tau \) has a \( K_2 \)-type \( \xi \) such that \( \xi \mid_S = 1 \), then \( \tau \mid_S = 1 \). Indeed, since \( S \) is central in \( K_1 \), then \( \text{Ind}^{K_2}_{K_1}(\xi) \mid_S = 1 \). Thus \( \tau \mid_S = 1 \), too. As a consequence, if \( \tau, \gamma \in \mathfrak{e}(K_1) \) have a common \( K_2 \)-type and \( \tau \mid_S = 1 \), then \( \gamma \mid_S = 1 \). We now prove the lemma. Let \( \eta \in R(K_1) \) be such that \( i^*(\eta) \in R(K_2/S) \). If \( \eta = \sum t_j \cdot \tau_j \), \( i^*\eta = \sum \sigma_j \cdot \tau_j \) set \( \mathfrak{e}_\eta(K_1) = \{ \tau_1, \ldots, \tau_k \} \), \( \mathfrak{e}_i^{i^*\eta}(K_2) = \{ \sigma_1, \ldots, \sigma_l \} \). By assumption \( \sigma_j \mid_S = 1, j = 1, \ldots, l \).

Define inductively

\[ S_1 = \{ \gamma \in \mathfrak{e}_\eta(K_1) \mid \gamma \text{ contains a } K_2 \text{-type in } \mathfrak{e}_i^{i^*\eta}(K_2) \} \]

\[ S_{i+1} = \{ \gamma \in \mathfrak{e}_\eta(K_1) \mid \gamma \text{ has a common } K_2 \text{-type with some } \tau \in S_i \} \].
Then $S_1 \subset S_2 \subset \cdots \subset S_n(K_i)$. By the above observation, if $\tau \in S_j$ for some $j$, then $\tau|_S = 1$. Thus, if $S_n = S_n(K_i)$ for some $n \in \mathbb{N}$, then $\eta \in R(K_i/S)$ and the lemma is proved. Otherwise, there exists $n$ such that $S_n = S_{n+1} \neq S_1(K_i)$. It is then easy to see that if $\eta' = \sum_{\tau, \in S_n} m_{\tau} \tau$, then $i^*(\eta') = 0$. Thus $i^*(\eta) = i^*(\eta - \eta')$ and $\eta - \eta' \in R(K_i/S)$. We note that in general it is not true that $\ker i^* \subset R(K_i/S)$, as the example $K_1 = S^1, K_2 = S = \{\pm 1\}$ already shows.

2.4. Lemma. Let $G$ be a simply connected Lie group of split rank one. Assume that rank $G = \text{rank } K$ and $\mathcal{G} \neq \mathcal{G}(l(2, \mathbb{R})$. Then $M$ is simply connected.

Proof. By applying the long exact sequence in homotopy to the fibration $M \to K \to K/M$, one readily obtains $\pi_0(M) = \pi_1(M) = \{1\}$ ($K/M$ is diffeomorphic to the unit sphere in $\mathcal{P}$ and dim $\mathcal{P} \geq 4$).

2.5. Proof of Proposition 1. By Lemma 2.2, in order to prove Proposition 1, we must show that $i^*: R(K) \to R(M)$ is surjective, if rank $G = \text{rank } K$. By Lemma 2.3 (applied to $(K_1, K_2) = (K, A/2)$), it is enough to verify this under the assumption that $K$ (hence $G$) be simply connected. Now, since $G$ has split rank one, we may assume that $G$ is simple and, on the other hand, if $\mathcal{G} = \mathcal{G}(l(2, \mathbb{R})$, it is clear that $i^*$ is surjective. We thus assume that $G$ is simple, simply connected, rank $G = \text{rank } K$ and $\mathcal{G} \neq \mathcal{G}(l(2, \mathbb{R})$.

It will be enough to show, by Lemma 2.4, that the fundamental representations of $\mathfrak{M}_c$ are restrictions of virtual representations of $\mathcal{M}_c$. We give a proof by case-by-case verification. Though a direct proof would be desirable, by this method, one finds explicitly $\eta \in R(K)$ with $i^*(\eta) = \sigma$, for each fundamental representation $\sigma$ of $\mathfrak{M}_c$.

Since, by Theorem 1, the coefficients $a_i (i \geq 0)$ of the asymptotic expansion of $\psi_{\eta}(s)$ involve the numbers $[\eta; \omega]$ ($\omega \in \mathcal{E}(G)$), the knowledge of $\eta$ may be of some use.

From now on we identify, via the Killing form, the imaginary dual of $\mathfrak{g}$ with a convenient subspace of $\mathfrak{r}^\ast$, so that the usual inner product of $\mathfrak{r}^\ast$ corresponds to a multiple of the Killing form. Let $(\epsilon_1, \ldots, \epsilon_n)$ be the canonical basis of $\mathfrak{r}^\ast$. We often denote by $1$ the trivial representation (of any Lie algebra). We will make use of the well-known branching formulas (see [Z, pp. 128–132 or B, 10]).

(i) $\mathcal{G} = \mathcal{G}(l(n, 1) (n \geq 2)$.

$$i_\mathfrak{g}^* = \left\{ \sum_{a=1}^{n+1} t_a e_i | t_1 + \cdots + t_{n+1} = 0 \right\}, \quad \mathfrak{r}_c = \mathfrak{l}(n), \quad \mathfrak{m}_c = \mathfrak{l}(n-1),$$

$$\Delta_c^+ = \{ e_i - e_j | 2 \leq i < j \leq n + 1 \}, \quad \Delta_n^+ = \{ e_i - e_j | 2 \leq i < n + 1 \}, \quad \beta = e_1 - e_2.$$

The centers of $\mathfrak{r}$ and $\mathfrak{m}$ correspond, respectively, to $\mathfrak{r}(e_1, -\frac{1}{n}(e_2 + \cdots + e_{n+1}))$ and $\mathfrak{r}(e_1 + e_2 - 2(e_3 + \cdots + e_{n+1}))/(n-1)$. Any $a \in \mathfrak{r}$ defines a character $\phi_a$ ($\phi'_a$) on $\mathfrak{r}(\mathfrak{g})$ ($\mathfrak{r}(\mathfrak{m})$) by the rule

$$\phi_a \left( e_1 - \frac{1}{n} \left( \sum_{a=2}^{n+1} e_i \right) \right) = ia \left( \phi'_a \left( e_1 + e_2 - \frac{2(\sum_{a=1}^{n+1} e_i)}{n-1} \right) = ia \right).$$

Hence $\phi_a$ ($\phi'_a$) defines a one-dimensional representation of $\mathfrak{r}_c (\mathfrak{m}_c)$ and it is easy to verify that $i^*(\phi_a) = \phi'_a$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
The fundamental representations are \( \lambda_i = \varepsilon_2 + \cdots + \varepsilon_i \) (2 \( \leq i \leq n \)), for \( \mathfrak{R}_C \), and \( \lambda'_i = \varepsilon_3 + \cdots + \varepsilon_i \) (3 \( \leq j \leq n \)), for \( \mathfrak{W}_C \). The branching formulas imply

\[
i^*(\lambda_2) = \phi_1 \oplus \phi_2 \otimes \lambda'_3,
i^*(\lambda_i) = \phi_{2i-3} \otimes \lambda'_i \oplus \phi_{2i-2} \otimes \lambda'_{i+1}, \quad 3 \leq i \leq n - 1,
i^*(\lambda_n) = \phi_{2n-3} \otimes \lambda'_n \oplus \phi_{2n-2},
\]

where \( \phi'_j = \phi_{aj} \) (a, j can be easily computed). Since \( \text{Im}(i^*) \) contains \( \phi'_a \) for any \( a \), this clearly implies that \( \lambda'_j \in \text{Im}(i^*) \) for 3 \( \leq j \leq n \).

(ii) \( \mathfrak{O} = \mathfrak{S}(2n, 1) \).

\[
i^\mathfrak{O} = \left\{ \sum_{i=1}^n t_i \varepsilon_i | t_i \in \mathbb{R} \right\}, \quad \Delta^+ = \{ \varepsilon_i | 1 \leq i \leq n, \varepsilon_i \pm \varepsilon_j | 1 \leq j < n \},
\]

\[
\Delta_i^+ = \{ \varepsilon_i \pm \varepsilon_j | 1 \leq i < j \leq n \}, \quad \Delta_n^+ = \{ \varepsilon_i | 1 \leq i \leq n \}, \quad \beta = \varepsilon_1, \quad \Delta_{2\mathfrak{W}}^+ = \{ \varepsilon_i | 2 \leq i \leq n, \varepsilon_i \pm \varepsilon_j | 2 \leq j < n \}.
\]

Fundamental weights:

\( \mathfrak{R}_C: \lambda_i = \varepsilon_1 + \cdots + \varepsilon_i \) (i = 1, \ldots, n - 2), \quad \lambda'_\pm = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{n-1} \mp \varepsilon_n),

\( \mathfrak{W}_C: \lambda'_i = \varepsilon_2 + \cdots + \varepsilon_i \) (i = 2, \ldots, n - 1), \quad \lambda'_+ = \frac{1}{2}(\varepsilon_2 + \cdots + \varepsilon_n).

By the branching formulas

\[
i^*(\lambda_i) = \lambda'_i \oplus \lambda'_{i+1}, \quad i = 1, \ldots, n - 2, (\lambda'_i = 1), \quad i^*(\lambda'_\pm) = \lambda'_\mp.
\]

Hence \( \lambda'_{i+1} = i^*(\lambda_i - \lambda_{i-1} + \lambda_{i-2} - \cdots \pm 1) \), \( \lambda'_+ = i^*(\lambda'_\mp) \). We include the case \( \mathfrak{O} = \mathfrak{S}(2n + 1, 1) \), for completeness.

(iii) \( \mathfrak{O} = \mathfrak{S}(2n + 1, 1) \).

\[
i^\mathfrak{O} = \left\{ \sum_{i=1}^{n+1} t_i \varepsilon_i | t_i \in \mathbb{R} \right\}, \quad \Delta^+ = \{ \varepsilon_i \pm \varepsilon_j | 1 \leq j \leq n + 1 \},
\]

\[
\Delta_i^+ = \{ \varepsilon_i \pm \varepsilon_j | 2 \leq i < j \leq n - 1, \varepsilon_i | 2 \leq i < n + 1 \}, \quad \Delta_{2\mathfrak{W}}^+ = \{ \varepsilon_i \pm \varepsilon_j | 2 \leq j < n + 1 \}.
\]

Fundamental weights:

\( \mathfrak{R}_C: \lambda_i = \varepsilon_2 + \cdots + \varepsilon_i \) (2 \( \leq i \leq n \)), \quad \lambda'_+ = \frac{1}{2}(\varepsilon_2 + \cdots + \varepsilon_{n+1}),

\( \mathfrak{W}_C: \lambda'_i = \varepsilon_2 + \cdots + \varepsilon_i \) (2 \( \leq j \leq n + 1 \)), \quad \lambda'_+ = \frac{1}{2}(\varepsilon_2 + \cdots + \varepsilon_n \pm \varepsilon_{n+1}).

Moreover, \( i^*(\lambda_i) = \lambda'_i \oplus \lambda'_{i-1} (2 \leq i \leq n) \), \( i^*(\lambda_+) = \lambda'_+ \oplus \lambda'_- \).

Hence \( \lambda'_i = i^*(\lambda_i - \lambda_{i-1} + \lambda_{i-2} - \cdots \pm 1) \in \text{Im}(i^*) \) (i = 2, \ldots, n).

Recall [Hu, p. 188] that \( \lambda'_+ \oplus \lambda'_- = \lambda'_n \oplus \lambda'_{n-2} \oplus \cdots \).

Thus \( \lambda'_+ \oplus \lambda'_- \) and \( \lambda'_+ \oplus \lambda'_- \in \text{Im}(i^*) \). On the other hand, if \( W = \{1, u\} \) one knows that \( (\lambda'_i)^u = \lambda'_i (i = 2, \ldots, n - 1) \), \( (\lambda'_\pm)^u = \lambda'_\pm \).

Hence \( R(M)^W = \mathbb{Z}[\lambda'_{2}, \ldots, \lambda'_{n-1}][\lambda'_+, \lambda'_-]^W \) is a polynomial ring over \( \mathbb{Z}[\lambda'_{2}, \ldots, \lambda'_{n-1}] \) in the symmetric functions \( \lambda'_+ \oplus \lambda'_- \). Hence, if \( M \) is simply connected (i.e. \( G = \text{Spin}(2n + 1, 1) \)) \( \text{Im}(i^*) = R(M)^W \).
The case \( G = \text{SO}(2n, 1) \) follows from Lemma 2.3.

(iv) \( \mathfrak{g} = \mathfrak{s}\mathfrak{p}(n, 1) \ (n \geq 2) \).

\[ \mathfrak{g} \cong \mathfrak{s}\mathfrak{p}(1) \times \mathfrak{s}\mathfrak{p}(n), \quad i\mathfrak{g}^* = \left\{ \sum_{i=1}^{n+1} t_i e_i | t_i \in \mathbb{R} \right\}. \]

\[ \Delta^+ = \left\{ 2e_i, 1 \leq i \leq n + 1; e_i = e_j = 1, 1 \leq i < j \leq n + 1 \right\}, \]

\[ \Delta^+_e = \left\{ 2e_i, 1 \leq i \leq n + 1; e_i = e_j, 1 \leq i < j \leq n \right\}, \quad \Delta^+_n = \left\{ e_i = e_{n+1}, 1 \leq i \leq n \right\}. \]

\[ \beta = e_1 - e_{n+1}, \quad \ker \beta = C(e_1 + e_{n+1}) + \sum_{2}^{n} C \cdot e_i, \]

\[ \Delta^+_\mathfrak{g} = \left\{ e_1 + e_{n+1} \right\} \cup \left\{ 2e_i \ | \ker \beta \right\} 2 \leq i \leq n \} \cup \left\{ e_i = e_j \ | \ker \beta \right\} 2 \leq i < j \leq n \}. \]

It will be understood from now on that roots and weights of \( \mathfrak{m}_C \) are restricted to \( \ker \beta \).

Simple roots:

\[ \mathfrak{m}_C: 2e_{n+1}, e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, 2e_n, \]

\[ \mathfrak{m}_C: \lambda = (e_1 + e_{n+1}), e_1 - e_2, \ldots, e_{n-1} - e_n, 2e_n. \]

Fundamental weights:

\[ \mathfrak{m}_C: \lambda' = \frac{1}{2}(e_1 + e_{n+1}), \lambda'_j = e_2 + \cdots + e_j (1 \leq j \leq n) \ (\text{with dimensions} \ d_\lambda = 2, \ d_{\lambda'} = (\frac{n}{2}) - (\frac{n}{2}) \text{ respectively}) \]

\[ \mathfrak{m}_C: \lambda' = \frac{1}{2}(e_1 + e_{n+1}), \lambda'_j = e_2 + \cdots + e_j (2 \leq j \leq n). \]

It can be shown directly or by using the branching formulas in [B, 10.7] that

\[ i^*(\lambda) = \lambda' \otimes 1, \quad i^*(\lambda_1) = \lambda' \otimes 1 \otimes \lambda_2, \quad i^*(\lambda_2) = \lambda' \otimes \lambda_2' \otimes 1 \otimes \lambda_3 \otimes 1, \]

\[ i^*(\lambda_j) = \lambda' \otimes \lambda_j' \otimes 1 \otimes \lambda_j'_{+1} \otimes 1 \otimes \lambda_j'_{-1} \text{ if } 3 \leq j \leq n - 1, \]

\[ i^*(\lambda_n) = \lambda' \otimes \lambda_n' \otimes 1 \otimes \lambda_{n-1}' - 1. \]

It then follows by induction that \( i^* \) is surjective. Note that making some conventions these formulas can be written in a closed form.

(v) \( \mathfrak{g} = \mathfrak{s}\mathfrak{o}(-20) \).

\[ \mathfrak{g}^* = \mathbb{R}^4, \quad \Delta^+ = \left\{ \frac{1}{2}(e_1 + e_2 \pm e_3 \pm e_4); e_i = e_j = 1 \leq i < j \leq 4; e_i, 1 \leq i \leq 4 \right\}. \]

\[ \Delta^+_e = \left\{ e_i = e_j, 1 \leq i < j \leq 4; e_i, 1 \leq i \leq 4 \right\}, \quad \Delta^+_n = \left\{ \frac{1}{2}(e_1 = e_2 \pm e_3 \pm e_4) \right\}. \]

\[ \beta = \frac{1}{2}(e_1 - e_2 - e_3 - e_4), \]

\[ \Delta^+_\mathfrak{g} = \left\{ \alpha \ | \ker \beta \right\} \alpha = e_i \pm e_j, 2 \leq i < j \leq 4 \text{ or } \alpha = e_i, 2 \leq i \leq 4 \}

\[ = \left\{ \alpha \ | \ker \beta \right\} \alpha = e_i \pm e_j \text{ (even number of \( \pm \))}, \alpha = e_i \pm e_j, 2 \leq i < j \leq 4 \}

\[ = e_i + e_j, 2 \leq i \leq 4 \text{ or } \alpha = e_i - e_j, 2 \leq i < j \leq 4 \}

We have: \( \mathfrak{g}_C = \mathfrak{s}\mathfrak{o}(9), \mathfrak{m}_C = \mathfrak{s}\mathfrak{o}(7) \). The simple roots for \( \Delta^+_\mathfrak{g} \) are \( e_3 - e_4, e_2 - e_3, \frac{1}{2}(e_1 - e_2 + e_3 + e_4) \). The fundamental representations for \( \mathfrak{m}_C: \lambda_1 = e_1, \lambda_2 = e_1 + e_2, \lambda_3 = e_1 + e_2 + e_3, \lambda_+ = \frac{1}{2}(e_1 + e_2 + e_3 + e_4) \) of dimensions 9, 36, 84 and 16, respectively,

\[ \mathfrak{m}_C: \lambda'_1 = \frac{1}{2}(e_1 + e_2 + e_3 - e_4) \ | \ker \beta, \]

\[ \lambda'_2 = (e_1 + e_2) \ | \ker \beta, \lambda'_+ = \frac{1}{2}(3e_1 + e_2 + e_3 + e_4) \ | \ker \beta. \]
DISTRIBUTION OF THE PRINCIPAL SERIES IN \( L^2(\Gamma \backslash G) \)

(dimensions 7, 21, and 8). The branching formulas are

\[ i^*(\lambda_1) = \lambda'_1 + \lambda'_2 + \lambda'_3, \quad i^*(\lambda_2) = \lambda'_1 + \lambda'_2 + \lambda'_3, \quad i^*(\lambda_3) = \lambda'_1 + \lambda'_2 + \lambda'_3 \]

and

\[ i^*(\lambda_4) = (\lambda'_1 + \lambda'_2 + \lambda'_3), \quad \lambda'_1 + \lambda'_2 + \lambda'_3. \]

Therefore, \( \lambda'_1 = i^*(\lambda_1 - 1) \), \( \lambda'_2 = i^*(\lambda_2 - \lambda_1) \), \( \lambda'_2 = i^*(\lambda_2 - \lambda_1 + 1) \) and \( i^* \) is surjective. We sketch the proof of the branching formulas.

A basis for the unipotent radical of the Borel subalgebra of \( \mathfrak{m}_C \) defined by \( A^1 \) is \( \mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \mathfrak{m}_4 \), \( \mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \mathfrak{m}_4 \), \( \mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \mathfrak{m}_4 \), and \( \mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \mathfrak{m}_4 \), where the constants \( c_i \) are as in Lemma 2.1.

Since \( \lambda_1 = \frac{1}{2} \beta + \lambda'_3 \), the restriction of \( \lambda_1 \) contains \( \lambda'_3 \). Since \( \dim(\lambda_1) = 9 \), \( \dim(\lambda'_3) = 8 \), the first identity is clear.

Now \( \lambda_2 = (-\frac{1}{2}) \beta + \lambda'_4 \). Hence \( i^*(\lambda_2) \) contains \( \lambda'_4 \). Since any weight of \( \lambda_4 \) is of the form \( \frac{1}{2}(\beta + \epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4) \), one checks that any vector of weight \( \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4) \) is \( \mathfrak{m}_C \)-dominant. Thus, \( \lambda_4 \) restricted to \( \mathfrak{m}_C \) contains \( \lambda'_4 \). Since \( \dim(\lambda'_4) = 7 \), the third identity follows. Now we study \( \lambda_2 \) and \( \lambda_1 \), restricted to \( \mathfrak{m}_C \). This is the adjoint representation of \( \mathfrak{g}_C \) with weights \( \pm \epsilon_i \pm \epsilon_j, 1 \leq i < j \leq 4 \), \( \pm \epsilon_i, 1 \leq i \leq 4 \), and 0, with multiplicity 4. Clearly, \( i^*(\lambda_2) \) contains the \( \mathfrak{m}_C \)-module with highest weight \( \lambda'_2 \). On the other hand, it is easily checked that any vector of weight \( \epsilon_i \) is \( \mathfrak{m}_C \)-dominant. Since \( \epsilon_i \in \ker(\tau) \), then \( i^*(\lambda_2) = \lambda'_2 + \lambda'_4 + \mu \), a representation of dimension 7. Now if \( \nu_1 \neq 0 \) is of weight \( \epsilon_2 + \epsilon_3 \) and \( \nu_2 \neq 0 \) is of weight \( \epsilon_1 - \epsilon_4 \), then \( X_{\epsilon_1 - \epsilon_4}(v_1) \) and \( X_{\epsilon_1 + \epsilon_4}(v_2) \) are nonzero vectors of weight \( \epsilon_1 + \epsilon_2 \). Hence, we can choose \( v_1 \) and \( v_2 \) so that \( \epsilon_1 X_{\epsilon_1 - \epsilon_4}(v_1) + X_{\epsilon_1 + \epsilon_4}(v_2) = 0 \). It is easy to verify that with this choice \( v_1 + v_2 \) is \( \mathfrak{m}_C \)-dominant. Since

\[ \epsilon_2 + \epsilon_3 = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4) - \beta, \quad \epsilon_1 - \epsilon_4 = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4) + \beta, \]

the \( \mathfrak{m}_C \)-submodule spanned by \( v_1 + v_2 \) has highest weight \( \lambda'_1 \). This proves the third identity, since \( \dim(\lambda'_1) = 7 \). We omit the proof of the last one, since from the first three one already concludes that \( i^* \) is surjective.

We conclude the paper by computing \( \ker(\tau^*: R(K) \to R(M)) \) explicitly. Recall that each \( \eta \in \ker(\tau^*) \) yields an alternating sum formula in the multiplicities \( \eta(\omega) \), if \( \Gamma \) is torsion-free [M3, 1.2]. We assume from now on that \( G \) is a connected, semisimple Lie group of split rank one, with finite center. If \( K \) is compact and \( \tilde{K} \to K \), a finite covering, we identify \( \tilde{\mathcal{O}}(K) \) with \( \{ \tau \in \tilde{\mathcal{O}}(\tilde{K}) | \ker p \subset \ker \tau \} \) and \( R(K) \) with the corresponding subring of \( R(\tilde{K}) \). Let \( T \) be a maximal torus of \( K \) and \( \tilde{T} = p^{-1}(T) \).

2.6. Lemma. (i) If rank \( G > \text{rank } K \), then \( \ker i^* = 0 \).

(ii) If rank \( G = \text{rank } K \), let \( \tilde{G} \to G \) be a finite covering so that \( \delta_n = \frac{1}{2}(\Sigma_{\alpha} \alpha) \) is a weight of \( \tilde{T} = p^{-1}(T) \). Then \( \ker i^* = R(K) \cap R(\tilde{K}) \cdot \eta_1 \), where \( \eta_1 \in R(\tilde{K}) \) is such that \( \eta_1(t) = t^{\delta_n} \cdot \prod_{\Delta_n}(t^\gamma - 1), t \in \tilde{T} \).

Proof. As noted at the beginning of the section, if rank \( G > \text{rank } K \), \( i^*: R(K) \to R(M)^w \) is an isomorphism.
We thus assume that rank $G = \text{rank } K$. We also assume that $\delta_n$ is a weight of $T$.
The lemma is obvious once it is proved in this case.
Let $\beta \in \Delta_n^+$ and $\mathfrak{H} = \mathfrak{R}(X_\beta + X_{-\beta})$, as above.
If $\eta \in \ker i^*$, then $\eta(t) = 0$ for $t \in T_\beta$, since $T_\beta \subseteq M$. Therefore ([A, 6.4], essentially), there is $\eta' \in R(T)$ so that

$$\eta(t) = (t^\beta - 1) \cdot \eta'(t), \quad t \in T.$$  

Since $\eta' = \eta$ ($s \in W_K$), then $\eta(t) = 0$, for $t \in sT_\beta = T_\beta'$. If $s\beta \neq \pm \beta$, then $\dim T_\beta \cap T_\beta' < \dim T_\beta$. Thus, by continuity, $\eta'(t) = 0$, $t \in T_\beta$. Hence, $\eta'(t) = (t^{s\beta} - 1) \cdot \eta''(t)$, for some $\eta'' \in R(T)$.

We may thus write

$$(*) \quad \eta = \prod_{\gamma \in \Psi} (t^\gamma - 1) \cdot \eta' \quad (\eta' = \eta'(\Psi) \in R(T)),$$

where $\Psi$ is any subset of $W_K \cdot \beta$ such that $\Psi \cap -\Psi = \emptyset$.

Since $\emptyset$ is of split rank one, then either $\mathfrak{R}_C$ acts irreducibly on $\mathfrak{g}_C$, or $\mathfrak{g}_C = \mathfrak{g}_+^+ \oplus \mathfrak{g}_-^+$, where $\mathfrak{g}_+^+ = \Sigma_{\Delta_n^+} \mathfrak{H}_a$, $\mathfrak{g}_-^+ = \Sigma_{\Delta_n^-} \mathfrak{H}_a$ and $\mathfrak{g}_{+/-}$ are irreducible subspaces. Furthermore, all noncompact roots have the same length [KW, 12.1]. Thus $W_K \cdot \beta = \Delta_n^+$ or $W_K \cdot \beta = \Delta_n^-$, since $W_K$ acts transitively on weights of a fixed length.

Then, if $\Psi = \Delta_n^+$ in $(*)$, we may write

$$\eta = \eta_0 \cdot \eta'' \quad \text{with } \eta_0(t) = \prod_{\gamma \in \Delta_n^+} (t^\gamma - 1), \eta'' \in R(T),$$
or

$$\eta = \eta_1 \cdot \eta', \quad \text{where } \eta_1(t) = t^{-\delta_n} \cdot \eta_0(t) \in R(T)^{W_K} \text{ and } \eta' \in R(T)^{W_K}.$$  

On the other hand, $M = Z(G) \cdot M^0$ ($M^0$, the connected component of 1 in $M$) and $T_\beta = Z(G) \cdot T^0_\beta$ ($T^0_\beta = \exp(ker/\beta \cap \emptyset)$), a maximal torus of $M^0$). Hence, $M = \cup \{x \cdot T^0_\beta \cdot x^{-1} | x \in M\}$ and $\eta_1 \in \ker i^*$, since $\eta_1(t) = 0$ for $t \in T_\beta$. Thus ker $i^* = R(K) \cdot \eta_1$, as asserted.

**Examples.** (i) $G$ simply connected. Then ker $i^* = R(K) \cdot \eta_1$.

(ii) $G = SL(2, \mathbb{R})$.

Then

$$K = T = \left\{ k(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \right\},$$

$\mathfrak{O}(K) = \{ \tau_n | \tau_n(k(\theta)) = e^{in\theta} \}$, $\Delta = \Delta_n = \{ \pm \alpha \}$, $k(\theta)^n = e^{2i\theta} = \tau_2(k(\theta))$. Hence ker $i^* = R(K) \cdot (\tau_1 - \tau_{-1})$, as in [M3, Lemma 2.1].

(iii) $\emptyset = \mathfrak{O}(n, 1)$.

Then $W_K \cdot \beta = \Delta_n^+$ and $\eta_0(t) = \prod_{\Delta_n^+} (t^\gamma - 1) \in R(T)^{W_K}$. Hence ker $i^* = R(K) \cdot \eta_0$ (if $\delta_n$ is a weight of $K$, $\eta_0$ and $\eta_1$ differ by a unit in $R(T)^{W_K} \cong R(K)$).

(iv) $G = SO(2n, 1)$.

In the notation of 2.5(ii), by Lemma 2.6,

$$\ker i^* = \{ \eta = \eta' \otimes \eta_1 | \eta' \in R\text{Spin}(2n), \eta \in RSO(2n) \},$$

where $\eta_1 = \lambda_+ - \lambda_\cdot \in R\text{Spin}(2n)$. Now

$$R\text{Spin}(2n) = \mathbb{Z}[\lambda_1, \ldots, \lambda_n][\lambda_+, \lambda_\cdot] \subset RSO(2n)[\lambda_+, \lambda_\cdot]$$  

[Hu, Chapter 13].
DISTRIBUTION OF THE PRINCIPAL SERIES IN $L^2(\Gamma \backslash G)$

It is then easy to check that $\eta' \otimes \eta_1 \in RSO(2n)$ if and only if $\eta' = \eta^+ \otimes \lambda_+ + \eta^- \otimes \lambda_- \in RSO(2n)$. That is, 

$$\ker i^* = \{(\eta^+ \otimes \lambda_+ + \eta^- \otimes \lambda_-) \otimes (\lambda_+ - \lambda_-) | \eta^+ \in RSO(2n)\}.$$ 

REFERENCES

[M2], On the Plancherel measure for linear Lie groups of rank one, Manuscripta Math. 29 (1979), 249–276.

SCHOOL OF MATHEMATICS, INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY 08540

INSTITUTO DE MATEMÁTICA, ASTRONOMÍA Y FÍSICA, UNIVERSIDAD NACIONAL DE CÓRDOBA, VALPARAISO Y ROGELIO MARTINEZ, CIUDAD UNIVERSITARIA, CÓRDOBA 5000, ARGENTINA (Current address of both authors)