MEAN VALUES OF SUBSOLUTIONS
OF ELLIPTIC AND PARABOLIC EQUATIONS

BY

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ABSTRACT. Integral averages of weak subsolutions (and supersolutions) in R^n of quasilinear elliptic and parabolic equations are investigated. The important feature is that these integral averages are defined in terms of measures that reflect interesting geometric phenomena. Harnack type inequalities are established in terms of these integral averages.

1. Introduction. In the theory of regularity of weak solutions that arises in partial differential equations and the calculus of variations, the analysis of the set of Lebesgue points frequently plays an important role. A point $x_0 \in R^n$ is said to be a Lebesgue point of a function u if there is a number $l = l(x_0)$ such that

$$\lim_{r \downarrow 0} r^{-n} \int_{B(x_0, r)} |u(y) - l(x_0)| \, dy = 0.$$

Here $B(x_0, r)$ denotes the open ball of radius r centered at x_0. If u is a weak solution of a partial differential equation, certain regularity properties of u often hold at its Lebesgue points. In many instances it is necessary to investigate the Lebesgue set and the associated integral averages of weak subsolutions and supersolutions. Perhaps the best known results in this connection are those that pertain to subharmonic functions. These results state that if u is subharmonic in R^n, then u has the following sub-mean-value properties:

(1)
$$u(x_0) \leq (\alpha(n)r^n)^{-1} \int_{B(x_0, r)} u(y) \, dy,$$

$$u(x_0) \leq (n\alpha(n)r^n)^{-1} \int_{\partial B(x_0, r)} u(y) \, d\sigma(y),$$

where $\alpha(n)$ denotes the volume of the unit ball in R^n. Other results in this direction are the well-known Harnack inequalities for weak subsolutions and supersolutions of elliptic and parabolic equations [M, T1, T2].

In this paper we investigate integral averages of weak subsolutions of elliptic and parabolic quasilinear equations. This work was motivated by [Z1] in which the

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analysis of Lebesgue points of weak subsolutions of parabolic equations played a vital role in determining the regularity properties of weak solutions of a wide class of degenerate parabolic equations.

In §4 we consider weak subsolutions of equations of the form

$$(2) \qquad\qquad \operatorname{div} A(x, u, u_x) = B(x, u, u_x),$$

where A and B are measurable functions subject to certain structural inequalities; see (12) below. Because of this structure, it is natural to require that weak subsolutions of (2) lie in the Sobolev space $W^{1,p}$. Our results for weak subsolutions u of (2) are somewhat analogous to (1) in that we relate $\sup u$ in $B(x_0, r)$ to the integral average of u over an $(n-k)$-sphere of radius r and center x_0 where $1 \leq k < p$. This is a special case of the main result which allows consideration of integral averages of subsolutions that are defined in terms of certain nonnegative measures (Theorem 4.6). Of course, similar results are valid for weak supersolutions. In §3 we consider $u \in W^{1,p}$ without assuming that u is a subsolution of (2) and show that the integral averages, that are defined relative to certain measures, converge everywhere to u except possibly on a set whose dimension is $n - p$.

In §5 we consider weak subsolutions of parabolic equations of the form

$$(3) \qquad\qquad u_t = \operatorname{div} A(x, t, u, u_x) + B(x, t, u, u_x).$$

We establish results which are analogous to those obtained in the elliptic case. In particular we show that the supremum of a weak subsolution of (3) in a cylinder of radius r in R^{n+1} can be estimated by its integral average over the lateral boundary of a suitable subcylinder plus an error term that tends to 0 as $r \to 0$. Other results of a similar nature can be obtained in which the lateral boundary of the subcylinder can be replaced by other geometric objects of dimension n.

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2. Notation and preliminaries. Points in Euclidean n-space, R^n, will be denoted by $x = (x_1, x_2, \ldots, x_n)$. In the case of a context for parabolic equations, points in R^{n+1} will generally be denoted by $z = (x, t)$ where $x \in R^n$ and $t \in R^1$. We will denote k-dimensional Hausdorff measure by H^k. Thus, H^1 is linear measure and H^n defined on subsets of R^n is Lebesgue measure (cf. [F]). The Lebesgue measure of a set $E \subset R^n$ is denoted by $|E|$ and $\int_E u(y)\, d\mu(y)$ will stand for the integral average $\mu(E)^{-1}\int_E u(y)\, d\mu(y)$, where μ is a nonnegative Radon measure with $\mu(E) \neq 0$. For Ω an open subset of R^n, $W^{1,p}(\Omega)$, $\infty > p \geq 1$, will denote the Sobolev space of functions whose distributional first derivatives are functions that belong to the Lebesgue space $L^p(\Omega)$. $W_0^{1,p}(\Omega)$ will denote the closure in the Sobolev norm of smooth functions with supports contained in Ω. Throughout, $p^* = np/(n-p)$ will denote the Sobolev exponent, and $p' = p/(p-1)$ for $1 \leq p < n$. Following common practice, the letter C will denote a constant that may change from line to line in the same proof.

3. Mean values of Sobolev functions. Whenever $E \subset R^n$ we define, for $1 \leq p < n$,

$$\gamma_p(E) = \inf \left\{ \int |\nabla u|^p \right\}$$

where the infimum is taken over all nonnegative functions $u \in W^{1,p}(R^n)$ for which $E \subset \text{interior}\{x: u(x) \geqslant 1\}$. In the case $p \geqslant n$, the definition must be modified to require that the functions $u \in W^{1,p}(R^n)$ have supports contained in some fixed compact set. It is known (cf. [FZ]) that if $1 < p \leqslant n$, then

$$\gamma_p(E) = 0 \text{ implies } H^{n-p+\varepsilon}(E) = 0 \text{ for each } \varepsilon > 0$$

and

(4) $$H^{n-p}(E) < \infty \text{ implies } \gamma_p(E) = 0.$$

In the event that $p = 1$, then $\gamma_1(E) = 0$ if and only if $H^{n-1}(E) = 0$ [Fl].

3.1. DEFINITION. The *Lebesgue set of degree s* for a Lebesgue measurable function u is the set of all points $x \in R^n$ such that

$$\lim_{r \downarrow 0} r^{-n} \int_{B(x, r)} |u(y) - u(x)|^s \, dy = 0.$$

Here $B(x, r)$ denotes the open ball of radius r and center x. The *exceptional set of degree s* for u is the complement in R^n of the Lebesgue set of degree s for u.

It follows from classical differentiation theory that in case $u \in L^s(R^n)$, the exceptional set of degree s for u has Lebesgue measure 0. However, in case $u \in W^{1,p}(R^n)$ the exceptional set is considerably smaller. Indeed, we have the following (vide [FZ, §9]):

3.2. THEOREM. *For every $u \in W^{1,p}(R^n)$, $1 \leqslant p < n$, there exists a function $v \in W^{1,p}(R^n)$ such that $u = v$ a.e. in R^n and the exceptional set of degree p^* for v has γ_p capacity 0, where $p^* = np/(n - p)$.*

In particular, this results implies that the integral averages of u converge at all points of R^n except perhaps those that belong to a set of γ_p capacity 0; that is,

(5) $$\lim_{r \downarrow 0} \int_{B(x, r)} u(y) \, dy = u(x)$$

for all $x \in R^n - E$, where $\gamma_p(E) = 0$. The purpose of this section is to show that the integral average taken with respect to Lebesgue measure in (5) can be replaced by an average relative to a more general measure.

For this purpose we introduce the space of Bessel potentials $g_1 * f$, $f \in L^p(R^n)$, $p > 1$, where the Fourier transform of g_1 is given by $\hat{g}_1(x) = (1 + 4\pi^2 |x|^2)^{-1/2}$. The space of Bessel potentials $g_1 * f$, $f \in L^p(R^n)$, is identical to $W^{1,p}(R^n)$, $1 < p < \infty$. The Bessel kernel g_1 is comparable to the Riesz kernel $R_1(x) = |x|^{1-n}$ in a neighborhood of $|x| = 0$ and has exponential decrease as $|x| \to \infty$. Moreover, $g_1(x) \leqslant c|x|^{1-n}$ for some constant C.

3.3. DEFINITION. For each real number $1 \leqslant \alpha < n$, let M_α^+ denote the space of nonnegative Radon measures m on R^n with compact support which, for some constant M, satisfy the growth condition $m(B_r) \leqslant Mr^{n-\alpha}$ on open n-balls of radius r.

If $p > 1$ and m is a nonnegative Radon measure such that $\|g_1 * m\|_{L^{p'}(R^n)} < \infty$, where $p' = p/(p - 1)$, then clearly $m \in [W^{1,p}(R^n)]^*$, the dual of $W^{1,p}(R^n)$. To see this, let $u \in W^{1,p}(R^n)$ be written as $u = g_1 * f$, $f \in L^p(R^n)$. Then, by changing the

order of integration, we have

$$\int_{R^n} u \, dm = \int g_1 * f \, dm = \int g_1 * mf \, dx \leq \| g_1 * m \|_{p'} \| f \|_p < \infty,$$

and thus, $\int u \, dm$ defines a bounded linear form on $W^{1,p}(R^n)$. Again, by changing the order of integration, we have

$$\int (g_1 * m)^{p'} \, dx = \int g_1 * (g_1 * m)^{1/p-1} \, dm.$$

Therefore, by appealing to Theorems 3.3 and 3.5 of [AM], it follows that if $p > 1$, $1 \leq \alpha < p$ and $m \in M_\alpha^+$, then $m \in [W^{1,p}(R^n)]^*$. In case $p = 1$, it follows from Theorem 4.7 of [MZ] that $m \in [W^{1,1}(R^n)]^*$ if $m \in M_1^+$. The main results of [M] and [MZ] thus yield the following

3.4. THEOREM. *Let $\Omega \subset R^n$ be a bounded Lipschitz domain and suppose $1 \leq p < n$. Let m be a nonnegative Radon measure supported on $\overline{\Omega}$ with $m \in M_1^+$ if $p = 1$ and $m \in M_\alpha^+$ if $1 \leq \alpha < p$. There is a constant $C = C(\Omega, n, p, M)$ such that if $u \in W^{1,p}(\Omega)$, then*

$$\left(\int_\Omega \left| u(y) - \int u \, dm \right|^{p^*} dy \right)^{1/p^*} \leq C \left(\int_\Omega | \nabla u |^p \right)^{1/p}.$$

This result will be crucial in establishing (5) where Lebesgue measure in the integral average is replaced by $m \in M_\alpha^+$.

If $u \in W^{1,p}(R^n)$, define a measure γ_u by

$$\gamma_u(E) = \int_E | \nabla u |^p \, dx$$

whenever E is a measurable subset of R^n. Let

$$P_{n-p} = R^n \cap \left\{ x : \limsup_{r \downarrow 0} r^{p-n} \gamma_u[B(x, r)] > 0 \right\}.$$

3.5. LEMMA. *If $u \in W^{1,p}(R^n)$, $1 \leq p < n$, then $H^{n-p}(P_{n-p}) = 0$.*

PROOF. Let $A_k = R^n \cap \{ x : \limsup_{r \downarrow 0} r^{p-n} \gamma_u[B(x, r)] > k^{-1} \}$ for each positive integer k. It follows from [F, §2.10.19] that there is a constant C such that $H^{n-p}(A_k) \leq C k \gamma_u(A_k)$. Note that γ_u is absolutely continuous with respect to Lebesgue measure. Now $\gamma_u[A_k] < \infty$, $H^{n-p}(A_k) < \infty$, $H^n(A_k) = 0$, $\gamma_u(A_k) = 0$ and $H^{n-p}(A_k) = 0$. Hence $H^{n-p}(P_{n-p}) = 0$ because $P_{n-p} = \bigcup A_k$.

Let $x_0 \notin P_{n-p}$ and consider $B(x_0, 1)$. If $u \in W^{1,p}(R^n)$, $1 \leq p < n$, and if m is a measure satisfying the hypotheses of Theorem 3.4 where we set $\Omega = B(x_0, 1)$, then

$$(6) \qquad \left(\int_{B(x_0, 1)} \left| u(y) - \int u \, dm \right|^{p^*} dy \right)^{1/p^*} \leq C \left(\int_{B(x_0, 1)} | \nabla u |^p \right)^{1/p}.$$

Let $T_r : B(x_0, 1) \to B(x_0, r)$ be defined by $T_r(x_0 + y) = x_0 + ry$ where $y \in B(0, 1)$. If we define $u_r = u \circ T_r$, then (6) implies that

$$\left(\int_{B(x_0, 1)} \left| u_r(y) - \int u \circ T_r \, dm \right|^{p^*} dy \right)^{1/p^*} \leq C \left(\int_{B(x_0, 1)} | \nabla u_r |^p \right)^{1/p}$$

which is the same as

$$(7) \quad \left(r^{-n} \int_{B(x_0,r)} \left| u(y) - \int u \circ T_r \, dm \right|^{p^*} dy \right)^{1/p^*} \leq C \left(r^{p-n} \int_{B(x_0,r)} |\nabla u|^p \right)^{1/p}.$$

Because $x_0 \notin P_{n-p}$, the right side of (7) tends to 0 as $r \downarrow 0$. From Theorem 2.2 there is a set E with $\gamma_p(E) = 0$ such that

$$(8) \quad \lim_{r \downarrow 0} r^{-n} \int_{B(x_0,r)} |u(y) - u(x_0)|^{p^*} dy = 0$$

whenever $x_0 \notin E$. Note that (4) and Lemma 3.5 imply $\gamma_p(P_{n-p}) = 0$. Now by setting $A = E \cup P_{n-p}$, the next result follows from (7) and (8).

3.6. THEOREM. *Suppose $u \in W^{1,p}(R^n)$, $1 \leq p < n$. There is a set $A \subset R^n$ with $\gamma_p(A) = 0$ such that if $x_0 \notin A$ and m is a nonnegative Radon measure supported on $\overline{B}(x_0, 1)$ with $m \in M_1^+$ if $p = 1$ and $m \in M_\alpha^+$ if $1 \leq \alpha < p$, then*

$$(9) \quad \lim_{r \downarrow 0} \int u \circ T_r \, dm = u(x_0).$$

Of course, the most interesting measures to consider are those that possess some homogeneity properties. For example, if $m = H^{n-1} \lfloor \partial B(x_0, 1)$, then clearly $m \in M_1^+$ and

$$\int u \circ T_r \, dm = \fint_{\partial B(x_0,r)} u(y) \, dH^{n-1}(y).$$

Thus, we have

3.7. COROLLARY. *If $u \in W^{1,1}(R^n)$ there is a set A with $\gamma_1(A) = H^{n-1}(A) = 0$ such that*

$$\lim_{r \downarrow 0} \fint_{\partial B(x_0,r)} r(y) \, dH^{n-1}(y) = u(x_0)$$

whenever $x_0 \notin A$.

A similar result could be obtained by taking $m = H^{n-1} \lfloor \pi^{n-1}(x_0)$ where $\pi^{n-1}(x_0)$ is an $(n-1)$-dimensional plane passing through x_0. In the event that $u \in W^{1,p}(R^n)$, $1 < p < n$, let $m = H^{n-R} \lfloor S^{n-k}(x_0, 1)$ where $1 \leq k < p$ is an integer and $S^{n-k}(x_0, 1)$ is an S^{n-k} sphere with radius 1 centered at x_0. Then we have

$$(10) \quad \lim_{r \downarrow 0} \fint_{S^{n-k}(x_0,r)} u(y) \, dH^{n-k}(y) = u(x_0)$$

for γ_p-a.e. $x_0 \in R^n$. As in the case $p = 1$, a similar result follows by taking $m = H^{n-k} \lfloor \pi^{n-k}(x_0)$.

4. Lebesgue points of subsolutions of elliptic equations. In this section we obtain results for weak subsolutions and supersolutions of quasilinear elliptic equations with measurable coefficients that are analogous to Theorem 3.6 and its corollaries. In the case of weak subsolutions we will show that (9) holds at all points x_0 and, more importantly, that the sup u in $B(x_0, r)$ is bounded above by the left-hand side of (9) plus a term that tends to 0 as $r \downarrow 0$.

The equations that are considered in this section are of the form

$$(11) \qquad \operatorname{div} A(x, u, u_x) = B(x, u, u_x)$$

where A and B are, respectively, vector and scalar valued Baire functions defined on $\Omega \times R^1 \times R^n$. Here Ω is an open subset of R^n. The functions A and B are required to satisfy the following structural inequalities:

$$
(12) \qquad
\begin{aligned}
|A(x, u, w)| &\leq a_0|w|^{p-1} + a_1|u|^{p-1} + a_2, \\
|B(x, u, w)| &\leq b_0|w|^p + b_1|w|^{p-1} + b_2|u|^{p-1} + b_3, \\
A(x, u, w) \cdot w &\geq |w|^p - C_1|u|^p - C_2.
\end{aligned}
$$

We assume $1 < p < n$ and a_0, b_0 are nonnegative constants. The results below are valid if the remaining coefficients are nonnegative measurable functions that are assumed to lie in appropriate $L^q(\Omega)$ spaces (vide [GZ]), but to minimize technical detail, we will assume that the coefficients are bounded by some constant $K > 0$:

$$(13) \qquad a_i(x) \leq K, \quad b_i(x) \leq K, \quad c_i(x) \leq K.$$

A function $u \in W^{1,p}_{\text{loc}}(\Omega)$ is called a *weak subsolution* (*supersolution*) of (11) if

$$(14) \qquad \int_\Omega A(x, u, u_x) \cdot \nabla\phi + B(x, u, u_x)\phi \leq 0 \quad (\geq 0)$$

for all bounded $\phi \geq 0$, $\phi \in W^{1,p}_0(\Omega)$.

The following result is due to Trudinger [T1] whose proof is based on the familiar Moser iteration method [Mo]. Therefore, a complete proof will not be given, but for the convenience of the reader, we will give an outline of the main steps.

Let $u \in W^{1,p}_{\text{loc}}(\Omega)$ be a weak subsolution of (11) that is bounded above by L in Ω. Choose $x_0 \in \Omega$, and for each real number k, let $u_k = (u - k)^+$ and define

$$
(15) \qquad
\begin{aligned}
\mu_k(r) &= \sup\{u_k(x) : x \in B(x_0, r)\}, \\
\mu(r) &= \sup\{u(x) : x \in B(x_0, r)\}.
\end{aligned}
$$

4.1. Theorem. *Let* $u \in W^{1,p}_{\text{loc}}(\Omega)$, $1 < p < n$, *be a weak subsolution of* (11) *such that* $0 \leq u(x) \leq L$ *for each* $x \in \Omega$. *There is a constant* C *depending only on* L, n, p *and the structure* (12) *such that if* $B(x_0, r) \subset \Omega$, *then*

$$\left(\fint_{B(x_0, 7r/8)} |\mu(r) - u|^\gamma \right)^{1/\gamma} \leq C[\mu(r) - \mu(r/2) + a(r)],$$

where $a(r) = r + Kr + (Kr)^{p/(p-1)}$ *and* $\gamma < n(p-1)/(n-p)$.

Proof. Let

$$(16) \qquad \phi = \eta^p e^{-b_0 u}(\mu(r) + a(r) - u)^\alpha,$$

where $\eta \in C^\infty_0[B(x_0, r)]$ and $\alpha < 0$. Then ϕ is a bounded, nonnegative test function that can be employed in (14). If we set $v = \mu(r) + a(r) - u$, the structure (12) and elementary estimates yield

$$(17) \qquad \int \eta^p v^{\alpha-1}|\nabla v|^p \leq C(\alpha) \int (\eta^p + |\nabla\eta|^p)v^{p+\alpha+1},$$

where $C(\alpha)$ is a finite constant when α is bounded away from 0 and $1 - p$. Let $w = v^q$ where $pq = p + \alpha - 1$ and apply Sobolev's inequality to find that (17) yields

$$(18) \qquad \|\eta w\|_{\sigma p} \leq C(\alpha)\|(\eta + |\nabla \eta|)w\|_p$$

where $\sigma = n/(n - p)$. For $0 < s < t \leq r$, let $\eta \in C_0^\infty[B(x_0, t)]$ be such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B(x_0, s)$ and $|\nabla \eta| \leq 2(t - s)^{-1}$. Then (18) implies

$$(19) \qquad \|w\|_{\sigma p;s} \leq C(\alpha)(t - s)^{-p}\|w\|_{p;t}.$$

Let $r_j = r(2^{-1} + 2^{-j-2})$ for $j = 0, 1, \ldots$ and iterate inequality (19) to obtain for any $p_0 > 0$,

$$(20) \qquad \theta(-\infty, r/2) \geq C\theta(-p_0, 3r/4)$$

where

$$\theta(p, r) = \left(\fint_{B(x_0, r)} v^p \right)^{1/p}.$$

By performing a finite iteration of (19) and choosing the r_j in a different way, we have

$$(21) \qquad \theta(\gamma, 7r/8) \leq C\theta(p_0, 3r/4)$$

for any $p_0 > 0$ and $\gamma < n(p - 1)/(n - p)$. Return now to the definition of the test function ϕ and notice that in case $\alpha = -1$, substitution of ϕ into (14) gives

$$\int \eta^p |\nabla \log v|^p \leq C \int |\nabla \eta|^p$$

whenever $\eta \in C_0^\infty[B(x_0, r)]$. Thus, from the John-Nirenberg lemma (cf. [GT, p. 158]), it follows that there exist constants C and $p_0 > 0$ such that

$$\int_{B(x_0, s)} v^{-p_0} \int_{B(x_0, s)} v^{p_0} \leq CS^{2n}$$

for $0 < s \leq 3r/4$. That is,

$$(22) \qquad \theta(p_0, 3r/4) \leq \theta(-p_0, 3r/4).$$

Thus, (20)–(22) imply

$$\theta(\gamma, 7r/8) \leq C\theta(-\infty, r/2)$$

or

$$\left(\fint_{B(x_0, 7r/8)} [\mu(r) + a(r) - u]^\gamma \right)^{1/\gamma} = \left(\fint_{B(x_0, 7r/8)} v^\gamma \right)^{1/\gamma}$$
$$\leq C \min_{B(x_0, r/2)} v = C[\mu(r) - \mu(r/2) + a(r)].$$

4.2. REMARK. Note that the right-hand side of the inequality in Theorem 4.1 tends to 0 with r and therefore

$$\lim_{r \downarrow 0} \fint_{B(x_0, r)} u(y) \, dy = \lim_{r \downarrow 0} \mu(r) = \lim_{x \to x_0} \sup u(x).$$

This implies that *every bounded weak subsolution* of (11) *is upper semicontinuous on* Ω after redefinition on a set of measure 0.

If $b_0 = 0$ in the structure (12), then the same conclusion will hold without assuming that u is bounded on Ω. In this case it is possible to show that a weak subsolution $u \in W^{1,p}_{loc}(\Omega)$ is locally bounded above on Ω [**S**]. On the other hand, u is not necessarily bounded below and, therefore, the estimate in Theorem 3.1 may not hold. However, one can easily show that for each $k \in R^1$,

$$\lim_{r \downarrow 0} \fint_{B(x_0, 7r/8)} |\mu_k(r) - u_k|^\gamma = 0.$$

To see this, replace u by u_k and $\mu(r)$ by $\mu_k(r)$ in the definition of ϕ, (16), and observe that the proof of Theorem 3.1 yields

$$(23) \quad \left(\fint_{B(x_0, 7r/8)} |\mu_k(r) - u_k|^\gamma \right)^{1/\gamma} \leq C(k) \big[\mu_k(r) - \mu_k(r/2) + a(r) \big].$$

Let $\lambda(x_0) = \lim_{r \downarrow 0} \mu(r)$ and choose $k < \lambda(x_0)$. Then

$$\int_{B(x_0, 7r/8)} \mu_k(r) - u_k \geq (\lambda(x_0) - k) \, | B(x_0, 7r/8) \cap \{u < k\} |$$

and, thus, it follows from (23) that

$$\lim_{r \downarrow 0} r^{-n} | B(x_0, r) \cap \{u < k\} | = 0.$$

Therefore, for each $\varepsilon > 0$,

$$\lim_{r \downarrow 0} r^{-n} | B(x_0, r) \cap \{x : | u(x) - \lambda(x_0) | > \varepsilon\} | = 0.$$

This states that u is *approximately continuous* at x_0 and its approximate limit at x_0 is $\lambda(x_0)$. However, every measurable function is approximately continuous almost everywhere and, therefore, *every weak subsolution of* (11) *with* $b_0 = 0$ *in* (12) *is upper semicontinuous on* Ω *after redefinition on a set of measure* 0.

We will now show that (9) holds at all $x_0 \in \Omega$ if u is a subsolution of (11).

4.3. THEOREM. *Let* $u \in W^{1,p}_{loc}(\Omega)$, $1 < p < n$, *be a weak subsolution of* (11) *such that* $0 \leq u(x) \leq L$ *for each* $x \in \Omega$. *For each* $x_0 \in \Omega$ *with* $B(x_0, r_0) \subset \Omega$ *for some* $r_0 > 0$, *let* $m \in M^+_\alpha$, $1 \leq \alpha < p$, *be a Radon measure supported on* $\bar{B}(x_0, r_0/2)$. *There is a constant* C *depending only on* M, L, n, p *and the structure* (12) *such that*

(24)

$$\left(r^{-n} \int_{B(x_0, r)} \left| u(y) - \int u \circ T_r \, dm \right|^{p^*} dy \right)^{1/p^*} \leq C [\mu(r) - \mu(r/2) + a(r)]^{p - 1/p}$$

for each $r \leq r_0/2$ *where* $T_r : B(x_0, r_0/2) \to B(x_0, r/2)$ *is defined by* $T_r(x_0 + y) = x_0 + r/r_0 y$.

PROOF. We proceed to obtain an estimate of $\int_{B(x_0, r)} | \nabla u |^p$ as in [**GZ**]. Let $\eta \in C^\infty_0 [B(x_0, 7r/8)]$ be a cut-off function such that $\eta \equiv 1$ on $B(x_0, r/2)$ and let

$\phi = \eta^p e^{b_0 u}$. Hence, by substituting ϕ into (14) and utilizing the structure (12), it follows that there is a constant C such that

(25)

$$\int \eta^p \, | \, \nabla u \, |^p \leq C\bigg[\int \eta^{p-1} \, | \, \nabla \eta \, |^{p-1} \, | \, \nabla \eta \, | + \int \eta^p + \int \eta^{p-1} \, | \, \nabla \eta \, | + \int \eta^p \, | \, \nabla u \, |^{p-1} \bigg].$$

As in the proof of Theorem 4.1, let $v = \mu(r) - u + a(r)$ and use (17) and Theorem 4.1 to obtain, for sufficiently small $\varepsilon > 0$,

$$
\begin{aligned}
(26) \quad \int \eta^{p-1} \, | \, \nabla u \, |^{p-1} \, | \, \nabla \eta \, | &= \int \big(\eta^{p-1} v^{-(1+\varepsilon)/p'} \, | \, \nabla u \, |^{p-1} \big) \big(v^{(1+\varepsilon)/p'} \, | \, \nabla \eta \, | \big) \\
&\leq \bigg(\int \eta^p v^{-(1+\varepsilon)} \, | \, \nabla v \, |^p \bigg)^{p-1/p} \bigg(\int v^{(p-1)(1+\varepsilon)} \, | \, \nabla \eta \, |^p \bigg)^{1/p} \\
&\leq C \bigg(\int (\eta^p + | \, \nabla \eta \, |^p) v^{p-1-\varepsilon} \bigg)^{p-1/p} \\
&\quad \times \big([\mu(r) - \mu(r/2) + a(r)]^{(p-1)(1+\varepsilon)} \cdot r^{n-p} \big)^{1/p} \\
&\leq C \big([\mu(r) - \mu(r/2) + a(r)]^{(p-1-\varepsilon)} r^{n-p} \big)^{p-1/p} \\
&\quad \times \big([\mu(r) - \mu(r/2) - a(r)]^{(p-1)(1+\varepsilon)} r^{n-p} \big)^{1/p} \\
&\leq C [\mu(r) - \mu(r/2) + a(r)]^{p-1} r^{n-p}.
\end{aligned}
$$

Also, the last term in (25) can be written as

$$\int \eta^p \, | \, \nabla u \, |^{p-1} = \int (\eta \, | \, \nabla u \, |)^{p-1} \eta \leq \frac{1}{2} \int \eta^p \, | \, \nabla u \, |^p + C \int \eta^p.$$

Hence, it follows from (25) and (26) that

$$r^{p-n} \int_{B(x_0, r/2)} | \, \nabla u \, |^p \leq C [\mu(r) - \mu(r/2) + a(r)]^{p-1}.$$

Now apply Theorem 3.4 as in the proof of (7) to establish the conclusion.

Because the right-hand side of (24) tends to 0 as $r \downarrow 0$, the following is immediate.

4.4. COROLLARY. *If $u \in W_{\mathrm{loc}}^{1,p}(\Omega)$ is a bounded subsolution of (11) on Ω, then*

$$\lim_{r \downarrow 0} \int u \circ T_r \, dm = u(x_0)$$

for each $x_0 \in \Omega$.

4.5. REMARK. As in (10), if we let $m = H^{n-k} \, | \, S^{n-k}(x_0, r_0/2)$, we have for each $x_0 \in \Omega$,

$$\lim_{r \downarrow 0} \fint_{S^{n-k}(x_0, r)} u(y) \, dH^{n-k}(y) = u(x_0)$$

whenever k is an integer such that $1 \leq k < p$. A similar result holds if m is taken as the restriction of H^{n-k} to an $(n-k)$-plane passing through x_0.

Finally, we establish an inequality for weak subsolutions of (11) which is reminiscent of the weak Harnack inequalities proved in [**T1**].

4.6. THEOREM. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $1 < p < n$, be a weak subsolution of (11) and assume that $0 \leqslant u \leqslant L$ on Ω. For each $x_0 \in \Omega$ with $B(x_0, r_0) \subset \Omega$ for some $r_0 > 0$, there is a nonnegative function $g(r)$ with $\int_0 g(r)^{p'} \, dr/r < \infty$ such that*

$$\sup_{B(x_0, r)} u \leqslant \fint_{S^{n-k}(x_0, r)} u(y) \, dH^{n-k}(y) + g(r)$$

for $0 < r < r_0/2$. Here k is an integer such that $1 \leqslant k < p$.

PROOF. This follows immediately from Theorem 4.1 and (24), for setting $c(r) = \fint_{S^{n-k}(x_0, r)} u(y) \, dH^{n-k}(y)$, we have

$$\mu(r) - c(r) = \fint_{B(x_0, 7r/8)} \mu(r) - c(r)$$

$$\leqslant \fint_{B(x_0, 7r/8)} |\mu(r) - u| + \fint_{B(x_0, 7r/8)} |u - c(r)|$$

$$\leqslant C[\mu(r) - \mu(r/2) + a(r)]^{p-1/p}.$$

Now set $g(r) = C[\mu(r) - \mu(r/2) + a(r)]^{p-1/p}$ and the conclusion follows.

5. Parabolic equations. In this section we consider parabolic equations of the form

$$(27) \qquad u_t = \mathrm{div} A(x, t, u, u_x) + B(x, t, u, u_x)$$

where A and B are Baire functions defined on $\Omega_T \times R^1 \times R^n$. Here $\Omega_T = \Omega \times (0, T)$ where Ω is an open subset of R^n.

The structure imposed on A and B is similar to that considered in the elliptic case, (12):

$$|A(x, t, u, w)| \leqslant a_0 |w| + a_1 |u| + a_2,$$
$$(28) \qquad |B(x, t, u, w)| \leqslant b_0 |w|^2 + b_1 |w| + b_2 |u| + b_3,$$
$$A(x, t, u, w) \cdot w \geqslant |w|^2 - c_1 |u|^2 - c_2.$$

a_0 and b_0 are nonnegative constants and the remaining coefficients are required to lie in appropriate Lebesgue spaces (cf. [**LSU, T2**]), but as in §4, for simplicity of exposition, we will assume that all coefficients are bounded by some constant $K > 0$. A function $u \in W^{1,2}_{\mathrm{loc}}(\Omega_T)$ is called a *weak subsolution (supersolution)* of (27) if

$$(29) \qquad \int -u\phi_t + A \cdot \nabla\phi - B\phi \leqslant 0 \quad (\geqslant 0)$$

for all bounded $\phi \geqslant 0$, $\phi \in W^{1,2}_0(\Omega_T)$.

The object of this section is to establish results for subsolutions of (27) analogous to those in Theorems 4.3 and 4.6. For this purpose we consider an arbitrary point $Z_0 \in \Omega_T$ and, for convenience, set $Z_0 = (0, 0)$. We shall utilize space-time cylinders of the following form:

$$R(r) = B(r) \times \left(\sigma_1 r^2, \sigma_2 r^2\right), \quad R^-(r) = B(\alpha r) \times \left(\tau_1 r^2, \tau_2 r^2\right),$$
$$R^*(r) = B(\beta r) \times \left(\rho_1 r^2, \rho_2 r^2\right),$$

where $B(r)$ denotes the n-ball of radius r and center $x_0 = 0$ and where

$$(30) \qquad 0 < \beta < \alpha < 1, \qquad \sigma_1 < \rho_1 < \rho_2 < \rho_3 < 0 < \rho_4 < \sigma_2.$$

Note that

$$(31) \qquad R(r/2^N) \subset R^-(r)$$

for some integer N. Finally, for $u \in W_{\text{loc}}^{1,2}(\Omega_T)$ a subsolution of (27), let $\mu(r) = \sup\{u(z): z \in R(r)\}$. The following result is proved in [T2] and its proof runs parallel to that of the elliptic version in Theorem 4.1.

5.1. THEOREM. *Let $u \in W_{\text{loc}}^{1,2}(\Omega_T)$ be a weak subsolution of (27) such that $0 \leqslant u(z) \leqslant L$ for each $z \in \Omega_T$. There is a number $\gamma > 1$ and a constant C depending only on L, n, and the structures (28) and (30) such that if $R(r) \subset \Omega_T$, then*

$$\left(\fint_{R^*(r)} |\mu(r) - u|^\gamma \right)^{1/\gamma} \leqslant C \min_{R^-(r)} \left[\mu(r) - u + a(r) \right]$$

where $a(r) = r + Kr$.

We now will establish an estimate on the growth of the V^2-norm of $\mu(r) - u$ where u is a weak subsolution of (27). This estimate is similar to the one that appears in [Z2, Theorem 4.3] and is the parabolic analogue of the estimate that appears in Theorem 4.3 above.

For this purpose let u be a bounded, nonnegative weak subsolution of (27) and let

$$(32) \qquad \phi = \eta^2 u e^{b_0 u},$$

where η is a smooth cut-off function that will be specified below. Because u is assumed to be bounded, we may assume that the terms involving $a_1 |u|$, $b_2 |u|$, and $c_1 |u|^2$ in (28) can be absorbed in a_2, b_3 and c_2, respectively. Therefore, substitution of (32) into (29) yields

$$(33) \qquad \iint \eta^2 e^{b_0 u} u u_t + \iint \eta^2 e^{b_0 u} (1 + b_0 u)(|\nabla u|^2 - C_2)$$

$$\leqslant \iint 2\eta u e^{b_0 u} [a_0 |\nabla u| + a_2] |\nabla \eta|$$

$$+ \iint \eta^2 e^{b_0 u} u [b_0 |\nabla u|^2 + b_1 |\nabla u| + b_3].$$

Now let $v = \mu(r) - u$ (for simplicity we will write $\mu = \mu(r)$) and define

$$(34) \qquad f(u) = b_0^{-1} e^{b_0 u} (b_0^{-1} - u) - b_0^{-1} e^{b_0 \mu} (b_0^{-1} - \mu).$$

Then

$$(35) \qquad f'(u) = -u e^{b_0 u}, \quad f''(u) = -e^{b_0 u} (1 + b_0 u),$$

$$\frac{1}{2}(\mu^2 - u^2) \leqslant f(u) \leqslant \mu e^{b_0 \mu}(\mu - u) \quad \text{for } 0 \leqslant u \leqslant \mu.$$

There is a constant C such that (33) can be written as

$$(36) \quad \iint \eta^2 \frac{\partial f(u)}{\partial t} + \iint \eta^2 |\nabla u|^2$$

$$\leqslant C \Big[\iint \eta |\nabla u| |\nabla \eta| + \iint \eta^2 + \iint \eta |\nabla \eta| + \iint \eta^2 |\nabla u| \Big].$$

The constant C depends upon the structure (28) and the bound for $|u|$. Now define $R'(r) = R^*(\gamma r)$ where $0 < \gamma < 1$, and choose η so that $\eta \equiv 1$ on $R'(r)$ and spt $\eta \subset R^*(r)$. Refer to (30) and set $t_1 = \rho_1 r^2$ and $t_2 = \rho_2 r^2$. Choose $t^* \in (t_1, t_2)$ so that

$$\int \eta^2(x, t^*) f^2(x, t^*) \, dx > \frac{1}{2} \sup_{(t_1, t_2)} \int \eta^2(x, t) f^2(x, t) \, dx.$$

If we replace t by t^* in (36), we are led to the following estimate:
(37)

$$\sup_{(t_1, t_2)} \int [\eta f(x, t)]^2 \, dx + \iint \eta^2 |\nabla v|^2 \, dx \, dt$$

$$\leqslant C \Big[\iint \eta |\nabla u| |\nabla \eta| + \iint \eta^2 + \iint \eta |\nabla \eta| + \iint \eta^2 |\nabla u| + \iint \eta |\eta_t| f \Big].$$

We now proceed exactly as in the proof of Theorem 4.3 to estimate the rate at which the right side of (37) tends to 0. Indeed, by employing (31) and Theorem 5.1, we find that there is a constant C depending only on the given data such that the first four terms on the right side of (37) are bounded by

$$(38) \qquad\qquad Cr^n \big[\mu(r) - \mu(r/2^N) + a(r) \big].$$

Refer to (35) to find that $|f(u)| \leqslant C_1 |v|$ and, therefore, by using Theorem 5.1 again, we have

$$(39) \qquad\qquad \iint \eta |\eta_t| |f(u)| \leqslant Cr^n \big[\mu(r) - \mu(r/2^N) + a(r) \big].$$

Referring again to (35), the next lemma now follows from (37)–(39).

5.2. LEMMA. *Let $u \in W_{\mathrm{loc}}^{1,2}(\Omega_T)$ be a bounded, nonnegative weak subsolution of* (27). *Then*

$$\sup_{(t_1, t_2)} \int_{B(\beta \gamma r)} [\mu(r) - u]^2 \, dx + \iint_{R'(r)} |\nabla u|^2 \, dx \, dt \leqslant Cr^n \big[\mu(r) - \mu(r/2^N) + a(r) \big]$$

whenever $R(r) \subset \Omega_T$.

We are now in a position to establish the parabolic analogue of Theorem 4.3. This result is concerned with the behavior of a weak subsolution in a neighborhood of an arbitrary point $z_0 \in \Omega_T$. We will continue to assume that $z_0 = (0, 0)$ and the geometric configuration imposed by (30).

5.3. THEOREM. *Let* $u \in W^{1,2}_{\text{loc}}(\Omega_T)$ *be a bounded, nonnegative weak subsolution of* (27). *For each* $z_0 \in \Omega_T$ *with* $R_{z_0}(r_0) \subset \Omega_T$ *for some* $r_0 > 0$, *let* $m_t \in M^+_\alpha$, $1 \leq \alpha < 2$, *be a Radon measure supported on* $\overline{B}(\beta\gamma r_0) \times \{t\}$. *There is a constant C depending only on the bound for u and the given data such that*

$$\oint_{\gamma t_1}^{\gamma t_2} \int_{B(\beta\gamma r)} \left| u(x,t) - \int u \circ T_r(y,t) \, dm_t(y) \right|^2 dx \, dt \leq C\left[\mu(r) - \mu(r/2^N) + a(r) \right]$$

for each $r \leq r_0$ *where* $T_r \colon B(\beta\gamma r_0) \times \{t\} \to B(\beta\gamma r) \times \{t\}$ *is defined by* $T_r(x,t) = (r/r_0 x, t)$.

PROOF. Because $u \in W^{1,2}_{\text{loc}}(\Omega_T)$, it is an elementary fact concerning Sobolev functions that $u(\cdot, t) \in W^{1,2}_{\text{loc}}(\Omega \times \{t\})$ for a.e. t. For all such t apply Theorem 3.4 to obtain

$$\int_{B(\beta\gamma r)} \left| u(x,t) - \int u \circ T_r(y,t) \, dm_t(y) \right|^2 dx \leq Cr^{2-n} \int_{B(\beta\gamma r)} |\nabla u(x,t)|^2 \, dx.$$

Now integrating with respect to t from $\gamma t_1 = \gamma\rho_1 r^2$ to $\gamma t_2 = \gamma\rho_2 r^2$ and applying Lemma 5.2 yields the desired result.

5.4. COROLLARY. *For each* $z_0 \in \Omega_T$ *with* $R_{z_0}(r_0) \subset \Omega_T$ *for some* $r_0 > 0$, *there is a nonnegative function* $g(r)$ *with* $\int_0 g(r)^2 \, dr/r < \infty$ *such that*

$$\sup_{R(r)} u \leq \oint_{\gamma t_1}^{\gamma t_2} \int u \circ T_r(y,t) \, dm_t(y) \, dt + g(r)$$

for $0 < r \leq r_0$.

PROOF. From Theorems 5.1 and 5.3 it follows that

$$\oint_{\gamma t_1}^{\gamma t_2} \left[\mu(r) - \int u \circ T_r(y,t) \, dm_t(y) \right]^2 dt$$

$$= \oint_{\gamma t_1}^{\gamma t_2} \oint_{B(\beta\gamma r)} \left[\mu(r) - \int u \circ T_r(y,t) \, dm_t(y) \right]^2 dx \, dt$$

$$\leq C \iint_{R'(r)} \left[\mu(r) - u \right]^2 + C\oint_{\gamma t_1}^{\gamma t_2} \int_{B(\beta\gamma r)} \left| u(x,t) - \int u \circ T_r(y,t) \, dm_t(y) \right|^2 dx \, dt$$

$$\leq C\left[\mu(r) - \mu(r/2^N) + a(r) \right].$$

Thus,

$$\oint_{\gamma t_1}^{\gamma t_2} \left[\mu(r) - \int u \circ T_r(y,t) \, dm_t(y) \right] dt \leq C\left[\mu(r) - \mu(r/2^N) + a(r) \right]^{1/2}.$$

The result now follows if we set $g(r) = C[\mu(r) - \mu(r/2^N) + a(r)]^{1/2}$.

As in §4, the most interesting case to consider is when the measure m is taken as a geometric measure with certain homogeneity properties. For example, let

$$m_t = H^{n-1} | S^{n-1}(x_0, \beta\gamma r_0) \times \{t\}.$$

Then

$$\int u \circ T_r(y, t)\, dm_t(y) = \fint_{S^{n-1}(x_0, \beta\gamma r) \times \{t\}} u(y, t)\, dH^{n-1}(y)$$

and, therefore,

$$\fint_{\gamma t_1}^{\gamma t_2} \int u \circ T_r(y, t)\, dm_t(y)\, dt = \fint_{\gamma t_1}^{\gamma t_2} \fint_{S^{n-1}(x_0, \beta\gamma r) \times \{t\}} u(y, t)\, dH^{n-1}(y)$$

$$= \fint_{\partial^* R'(r)} u(z)\, dH^n(z),$$

where $\partial^* R'(r)$ denotes the lateral boundary of the cylinder $R'(r)$.

We have thus proved

5.5. COROLLARY. *For each $z_0 \in \Omega_T$ with $R_{z_0}(r_0) \subset \Omega_T$ for some $r_0 > 0$, there is a nonnegative function $g(r)$ with $\int_0 g(r)^2\, dr/r < \infty$ such that*

$$\sup_{R(r)} u \leq \int_{\partial^* R'(r)} u(z)\, dH^n(z) + g(r)$$

for $0 < r \leq r_0$.

As in §§3 and 4, one could establish similar results by making different selections for the measure m. For example, in addition to the choice made in the above corollary, one could also take $m_t = H^{n-1} \,|\, \Pi^{n-1}(x_0) \times \{t\}$, where Π^{n-1} is an $(n - 1)$-plane that is orthogonal to the t-axis.

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