GRADINGS OF Bₙ AND Cₙ
OF FINITE REPRESENTATION TYPE

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ABSTRACT. It was shown by Bongartz and Gabriel that the classification of simply-connected algebras (i.e., finite-dimensional, basic, of finite representation type and with a simply-connected Auslander-Reiten graph) can be reduced to the study of certain numerical functions, called gradings, operating on a tree. Here, we classify in terms of their bounden species the simply-connected algebras arising from gradings of the Dynkin trees Bₙ and Cₙ, and show that these are exactly the tilted algebras of types Bₙ and Cₙ, respectively.

In [5] Bongartz and Gabriel defined the notion of grading of a tree and proved that there exists a bijection between the isomorphism classes of graded trees of finite representation type and the isomorphism classes of finite-dimensional basic connected algebras of finite representation type with a simply-connected Auslander-Reiten graph.² They also describe the gradings of finite representation type for the Dynkin graph Aₙ. Here, using the methods and results of [2], we describe these gradings for the Dynkin graphs Bₙ and Cₙ. We obtain the following two theorems:

THEOREM (1). The following assertions are equivalent:
(I) The finite-dimensional algebra A arises from a grading of finite representation type of the tree Bₙ.
(II) The bounden species of the algebra A satisfies the properties (β) of [2] (see also (1.10) below) and, moreover, does not contain a full connected subspecies of one of the forms:

(a) $\cdots \overset{0}{E} + E + E \cdots \overset{0}{E} + F + E$
(b) $\cdots \overset{0}{E} + F + E \cdots \overset{0}{E} + E + E$
(c) $\cdots \overset{0}{E} + E + E \cdots \overset{0}{E} + E + F$

¹This article was written while the first author was a post-doctoral fellow at Carleton University, and he wishes to acknowledge support from NRSC grant No. A-7257
²Although Bongartz and Gabriel were dealing with algebras over a fixed algebraically closed field, their results and proofs remain valid in general, as will be seen in (1.7).
where $E$, $F$, $\pi$ are as in [2], and we assume that there is no other relation between the relations shown.

(III) $A$ is a tilted algebra of type $B_n$.

**Theorem (2).** The following assertions are equivalent:

(I) The finite-dimensional algebra $A$ arises from a grading of finite representation type of the tree $C_n$.

(II) The bounden species of the algebra $A$ satisfies the properties $(\gamma)$ of [2] (see also (1.10) below) and, moreover, does not contain any full connected subspecies of one of the forms:

(a) $F + F + F \rightarrow \cdots F + E + F$

(b) $F + E + F \rightarrow \cdots F + F + F$

(c) $F + F + F \rightarrow \cdots F + F + E$

(d) $E + F + F \rightarrow \cdots F + F + F$

(e) $F + F + F \rightarrow \cdots F + F + F$

(f) $F + F + F \rightarrow \cdots F + E + F \rightarrow \cdots F + F + F$

(g) $E \rightarrow F \rightarrow \cdots F + F \rightarrow \cdots F + F + F$

(h) $E \rightarrow F \rightarrow \cdots F + F \rightarrow \cdots F + F + F$

where $E$, $F$, $\pi$ are as in [2] and we assume that there is no other relation between the relations shown.

(III) $A$ is a tilted algebra of type $C_n$. 
Observe that the existence of complete slices in tilted algebras [10] gives directly in both cases that (III) implies (I).

We apply our results to show that the algebras whose trivial extensions are of finite representation type and Dynkin class \( B_n \) (respectively, \( C_n \)) are precisely the iterated tilted algebras of type \( B_n \) (respectively, \( C_n \)). A similar result has been obtained for \( A_n \) by Hughes and Waschbüch [15].

For the convenience of the reader, the main results of [2 and 5] will be recalled in §1. In §2, we prove Theorem (1), and §3 will consist of Theorem (2), together with the stated application.

1. Preliminaries.

1.1. Let \( k \) be a commutative field, and \( A \) a finite-dimensional \( k \)-algebra. By a module is meant a finite-dimensional right \( A \)-module. The simple \( A \)-modules will be denoted by \( S(i) \), where \( i \) runs through a fixed index set and we shall let \( P(i) \) and \( I(i) \) be, respectively, the projective cover and the injective hull of \( S(i) \). The support \( \text{Supp} \ M \) of the module \( M \) is the set of all \( i \) such that \( S(i) \) appears as a composition factor of \( M \).

Let \( \Sigma = (F_i, M_j)_{i,j \in I} \) be a \( k \)-species [9]. We shall denote by \( T(\Sigma) \) its tensor algebra, and by \( G_{\Sigma} \) the associated (oriented) valued graph [7]. An ideal \( R \subseteq \text{rad}^2 T(\Sigma) \) is called relation ideal, and the algebra \( A = T(\Sigma)/R \) is then said to be given by the bounded \( k \)-species \( (\Sigma, R) \) [2]. For each pair \( (i, j) \) of elements of \( I \), the \( F_i - F_j \) bimodule \( _iF_j \) is a relation on \( \Sigma \). A representation of \( \Sigma \) [7] is bound by \( R \) if the associated \( T(\Sigma) \)-module is annihilated by the ideal \( R \).

1.2. We shall use here, without further reference, properties of the Auslander-Reiten sequences and irreducible maps (cf. [4]). The Auslander-Reiten graph \( \Gamma_A \) of the algebra \( A \) has as a set of vertices \( \Gamma_0 \), the set of isomorphism classes of indecomposable \( A \)-modules, and there is an arrow (oriented edge) \( \alpha: [M] \to [N] \) whenever there is an irreducible map from \( M \) to \( N \), this arrow being endowed with a valuation \( (d_\alpha, d'_\alpha) \) defined as follows: let \( \text{Irr}(M, N) \) denote the bimodule of irreducible maps [12], then \( d_\alpha = \dim_{\text{End} N} \text{Irr}(M, N) \) and \( d'_\alpha = \dim_{\text{End} M} \text{Irr}(M, N) \). We shall let \( \Gamma_1 \) denote the set of all arrows. Note that, if \( \tau = D\text{Tr} \) is the Auslander-Reiten translation, each arrow \( \alpha: [M] \to [N] \) with \( N \) nonprojective is paired with an arrow \( \sigma\alpha: [\tau N] \to [M] \). A topology is defined on \( \Gamma_A \) by considering it as a two-dimensional cell complex [11].

\( \Gamma_A \) becomes a modulated graph [8] if to each vertex \( i = [M] \) is associated the skew field \( F_i = \text{End} M/\text{rad} \text{End} M \), and to each arrow \( \alpha: i \to j \) where \( i = [M] \) and \( j = [N] \), we associate the bimodules \( _iM_j \) and \( _jM_i \). Finally, we let the bilinear forms \( e_i: _iM_j \otimes_j M_i \to F_i \) be the evaluation maps.

1.3. Following [14], a translation species \( (\Gamma_0, \Gamma_1, F, N, \tau, \chi) \) is defined by

(1) a translation quiver \( (\Gamma_0, \Gamma_1, \tau) \) [5];
(2) a map \( F \) associating to each vertex \( i \in \Gamma_0 \) a skewfield \( F_i \);
(3) a map \( N \) associating to each arrow \( \alpha: i \to j \) an \( F_i - F_j \) bimodule \( N(\alpha) \), finite dimensional on both sides;
(4) a map associating to each nonprojective vertex \( i \) an isomorphism \( F_i \sim F_{\tau i} \).
(5) a map associating to each arrow \( \alpha: i \to j \), with \( j \) nonprojective, a nondegenerate bilinear form

\[
\chi_{\alpha}: F_j N(\sigma \alpha)_F \otimes_F N(\alpha)_F \to F_i F_j
\]

(where \( N(\sigma \alpha) \) is considered as a left \( F \)-module by means of the isomorphism \( F_j \cong F_j \)).

The bilinear form \( \chi_{\alpha} \) determines an element \( c_{ij} \in N(\sigma \alpha) \otimes_F N(\alpha) \) as follows: let \( x_1, x_2, \ldots, x_m \) be a basis of \( F_j N(\alpha) \), and \( \xi_1, \xi_2, \ldots, \xi_m \) the dual basis of \( N(\sigma \alpha)_F \) with respect to the bilinear form \( \chi_{\alpha} \); then \( c_{ij} = \sum_{i=1}^m \xi_i \otimes x_i \) is called the canonical element [8].

Let \( i, j \in \Gamma_0 \), and \( \mathcal{P} = (i \to_{\alpha_1} i_2 \to \cdots \to_{\alpha_m} \alpha_m = j) \) be a path in \( (\Gamma_0, \Gamma_1) \). We have an \( F_i - F_j \) bimodule,

\[
N(\mathcal{P}) = N(\alpha_1) \otimes_{F_{i_1}} N(\alpha_2) \otimes_{F_{i_2}} \cdots \otimes_{F_{i_m}} N(\alpha_m),
\]

and hence an \( F_i - F_j \) bimodule \( N(i, j) = \bigoplus_{\mathcal{P}} N(\mathcal{P}) \), the sum being taken over all paths from \( i \) to \( j \). We shall also agree to set \( N(1) = F_i \) if \( \mathcal{P} \) is the trivial path at the point \( i \). We can thus define the tensor category \( \otimes \Gamma \) of \( \Gamma \) to have \( \Gamma_0 \) as set of objects, and \( N(i, j) \) as set of morphisms from \( i \) to \( j \). The mesh category \( \bowtie \Gamma \) of \( \Gamma \) is the factor category of \( \otimes \Gamma \) modulo the ideal generated by the elements \( \Sigma c_{ij} \), the sum being taken over all arrows \( i \to j \) with nonprojective target \( j \).

Thus, the Auslander-Reiten graph \( \Gamma_{\mathcal{A}} \) of the algebra \( \mathcal{A} \) yields, in an obvious way, a translation species, called the Auslander-Reiten species of \( \mathcal{A} \) [14].

1.4. In what follows, we shall limit ourselves to finite-dimensional, basic, connected algebras of finite representation type with a simply-connected Auslander-Reiten graph. Such algebras will be called simply-connected algebras.

Let \( \mathcal{A} \) be a simply-connected algebra. Then, following Bongartz and Gabriel [5], we can associate to \( \Gamma_{\mathcal{A}} \) a (nonoriented) valued graph \( \text{Gr}(\Gamma_{\mathcal{A}}) \) as follows: the vertices of \( \text{Gr}(\Gamma_{\mathcal{A}}) \) are the \( \sigma \)-orbits of the isomorphism classes of indecomposable \( \mathcal{A} \)-modules, and the valued edges correspond to the \( \sigma \)-orbits of the arrows of \( \Gamma_{\mathcal{A}} \); if \( \alpha: [M] \to [N] \) is an arrow in \( \Gamma_{\mathcal{A}} \), there exists an edge \( \sigma^2 \alpha \) between \( \tau^2[M] \) and \( \tau^2[N] \) in \( \text{Gr}(\Gamma_{\mathcal{A}}) \) endowed with the same valuation \( (d_\alpha, d'_\alpha) \). Since \( d_\alpha = d'_\alpha \), this definition is not ambiguous. Clearly \( \text{Gr}(\Gamma_{\mathcal{A}}) \) is homotopically equivalent to \( \Gamma_{\mathcal{A}} \) and, in particular, is a tree.

The natural modulation on \( \Gamma_{\mathcal{A}} \) induces on \( \text{Gr}(\Gamma_{\mathcal{A}}) \) a modulation \( \mathcal{M}(\Gamma_{\mathcal{A}}) \) as follows: to each vertex \( i = \tau^2[M] \) of \( \text{Gr}(\Gamma_{\mathcal{A}}) \), we associate the skew field \( \overline{F}_i = \text{End}_{\mathcal{M}/\text{rad End } \mathcal{M}} \), and for each edge between \( i = \tau^2[M] \) and \( j = \tau^2[N] \) (where we assume, without loss of generality, that the representatives \( M \) and \( N \) are chosen such that \( N \) is not projective, and there is an arrow \( \alpha: [M] \to [N] \)), we put \( \overline{N} = \text{Irr}(\mathcal{M}, N) \) and \( \overline{M} = \text{Irr}(\mathcal{M}, N) \). Finally, the bilinear form of \( \mathcal{M}(\Gamma_{\mathcal{A}}) \) is again the evaluation map.

1.5. Definition [5]. Let \( (T, d) \) be a valued (nonoriented) tree. A grading of \( T \) is a function \( g: T \to \mathbb{N} \) such that

\begin{enumerate}
\item[(G1)] \( g^{-1}(0) \neq \emptyset \);
\item[(G2)] \( g(i) - g(j) \equiv 1 \pmod{2} \), whenever \( i, j \) are neighbours in \( T \).
\end{enumerate}
We shall define a natural grading \( g_A \) on the tree \( \text{Gr}(\Gamma_A) \) associated to the algebra \( A \): let \((\Gamma_0, \Gamma_1, \tau)\) be the underlying translation quiver of \( \Gamma_A \), then there exists a unique morphism of translation quivers \( \text{pr}_A: (\Gamma_0, \Gamma_1, \tau) \to \mathbb{Z}_A \) such that \( \text{Min}_x \in \Gamma_0 \text{pr}_A(x) = 0 \) \([5]\). Let us put \( g_A(x) = \text{pr}_A(P(i)) \), where \( P(i) \) is the unique indecomposable projective such that \( i = \tau^Z[P(i)] \). Clearly, \( g_A \) is a grading.

1.6. To the valued tree \((T, d)\), graded by \( g \) and modulated by \( \mathfrak{M} = (F_i, M_j, e^i_j)_{i,j \in T} \), we can associate a translation species \( \Omega = \Omega_{(T, \mathfrak{M}, g)} = (\Omega_0, \Omega_1, K, N, \tau, \epsilon) \). Let

\[
\Omega_0 = \left\{(n, i) \in \mathbb{N} \times T \mid \frac{1}{2}(n - g(i)) \in \mathbb{N}\right\},
\]

and for each valued edge \( i \to j \), define families of valued arrows
\[
(n, i) \rightarrow (n + 1, j) \quad \text{and} \quad (n, j) \rightarrow (n + 1, j),
\]

with \( n \in \mathbb{N} \), whenever both endpoints lie in \( \Omega_0 \).

Thus, we obtain an infinite-valued graph endowed with a translation: the projective vertices are the pairs \((g(i), i)\), with \( i \in T \), the translate of a nonprojective is defined by \( \tau(n, i) = (n - 2, i) \). The mapping \( K \) is defined by \( K(n, i) = F_i \), the mapping \( N \) by \( N((n, i), (n + 1, j)) = M_j \) and \( N((n, j), (n + 1, i)) = J_i M_j \), while the bilinear forms \( \epsilon \) are given by the \( e^i_j \).

Let \((n, i) \in \Omega_0\), and define \( \nu(n, i) \) to be the set \( \{j \in T \mid d_{ij} \neq 0 \text{ and } g(j) < n\} \). It is obviously possible to define inductively on \( \Omega \) a unique mapping \( \alpha: \Omega_0 \to \mathbb{N}^{\text{Card}T} \) such that:

(a) For every projective vertex \((g(i), i)\) such that all \( j \in \nu(g(i), i) \) satisfy \( d(g(i) - 1, j) > 0 \), we have
\[
d(g(i), i) = \delta_i + \sum_{j \in \nu(g(i), i)} d_{ij}d(g(i) - 1, j).
\]

Here, \( \delta_i \) is the Kronecker delta-function.

(b) For every nonprojective vertex \((n, i)\) such that all \( j \in \nu(n, i) \) satisfy \( d(n - 1, j) > 0 \) and, moreover, \( \sum_{j \in \nu(n, i)} d_{ij}d(n - 1, j) - d(n - 2, i) > 0 \), we have
\[
d(n, i) = \sum_{j \in \nu(n, i)} d_{ij}d(n - 1, j) - d(n - 2, i).
\]

(c) For every other \((n, i) \in \Omega_0\), \( d(n, i) = 0 \).

If \((T, \mathfrak{M}, g) = (\text{Gr}(\Gamma_A), \mathfrak{M}(\Gamma_A), g_A)\), the uniqueness of \( d \) implies that \( d = \dim \), where \( \dim \) is the mapping associating to each vertex \( [M] \) of \( \Gamma_A \) its dimension vector \( \dim M \). By analogy, \( d \) is called the \textit{dimension map} of \( \Omega \). Now let \( \Gamma = \Gamma_{(T, \mathfrak{M}, g)} \) be the full subspecies of \( \Omega \) defined by
\[
\Gamma_0 = \left\{(n, i) \in \Omega_0 \mid d(n, i) > 0\right\}.
\]

**Definition (1).** The grading \( g \) is \textit{admissible} if \( \Gamma \) contains all the projective vertices \((g(i), i)\) of \( \Omega \).

**Definition (2).** The grading \( g \) is of \textit{finite representation type} if it is admissible and \( \Gamma \) is finite.

For instance, \( g_A \) is of finite representation type and \( \Gamma = \Gamma_{(\text{Gr}(\Gamma_A), \mathfrak{M}(\Gamma_A), g_A)} \) can be identified to \( \Gamma_A \).
1.7. Let \((T, \mathcal{R}, g)\) be a modulated tree endowed with a grading of finite representation type, and consider the algebra

\[ A = A_{(T, \mathcal{R}, g)} = \bigoplus_{i,j \in T} \text{Hom}(g(i), g(j)). \]

We have the following theorem.

**Theorem (Bongartz-Gabriel).** The map \((T, \mathcal{R}, g) \to A_{(T, \mathcal{R}, g)}\) induces a bijection between the isomorphism classes of modulated trees equipped with a grading of finite representation type and the isomorphism classes of simply-connected algebras.

Indeed, the proof of [5], done under the assumption that \(k\) is algebraically closed, carries over to the general case with only the obvious changes.

1.8. Example. Let \(F, G\) be two skew fields, finite-dimensional over the common central subfield \(k\), and such that, moreover, \(\dim F_c = 3\). The lower triangular matrix algebra

\[
A = \begin{bmatrix}
F & 0 & 0 \\
F & G & 0 \\
F & F & F
\end{bmatrix}
\]

has the following Auslander-Reiten graph:

(100) \rightarrow (130) \rightarrow (131) \rightarrow \cdots

(3,1) \rightarrow (3,2) \rightarrow (3,3) \rightarrow \cdots

Thus \(\text{Gr}(\Gamma_A)\) is here the Euclidean graph

\[ \mathbf{e}_{12} : \quad h \rightarrow (3,1) \rightarrow (3,2) \rightarrow \cdots \]

and the grading \(g_A\) is given by \(g_A(h) = 5, \ g_A(i) = 0, \ g_A(j) = 1\). \(\Omega = \Omega(\text{Gr}(\Gamma_A), \mathcal{R}(\Gamma_A), g_A)\) is here given by

\[ (0,1) \rightarrow (2,1) \rightarrow (4,1) \rightarrow (5,1) \rightarrow \cdots \]

\[ (1,1) \rightarrow (3,1) \rightarrow (5,1) \rightarrow (7,1) \rightarrow \cdots \]

and \(\Gamma = \Gamma_A\). In fact, \(A\) is tilted of type \(\mathbf{e}_{12}\).

1.9. Let \(A\) be a finite-dimensional \(k\)-algebra. A module \(T_A\) is called a tilting module [10] if

\[(T1) \text{ pd } T_A \leq 1;\]
(T2) $\text{Ext}^1_A(T, T) = 0$;

(T3) there is a short exact sequence $0 \to A_A \to T'_A \to T''_A \to 0$ with $T'$ and $T''$ direct sums of summands of $T_A$.

A tilting module $T_A$ is splitting if every indecomposable $B$-module $N_B$, where $B = \text{End} T_A$, is such that either $N \otimes_B T = 0$, or $\text{Tor}_B^1(N, T) = 0$. A finite-dimensional $k$-algebra $B$ is iterated tilted (called "generalized tilted" in [3]) if:

1. There exists a sequence of algebras $A_0, A_1, \ldots, A_m = B$ with $A_0$ hereditary.
2. There exists a sequence of splitting tilting modules $T_{A_i}^{(i)} (0 \leq i \leq m - 1)$ such that $\text{End} T_{A_i}^{(i)} = A_{i+1}$.

If $m = 1$, $B$ is called tilted [10]. $B$ is said to be of type $\Delta$ for a (nonoriented) valued graph $\Delta$ if $A_0$ is the tensor algebra of an oriented valued graph with nonoriented underlying graph $\Delta$ [6].

For example, it is not hard to check that the iterated tilted algebras of type $\mathbb{F}_4$ are precisely the simply-connected algebras given by admissible gradings of $\mathbb{F}_4$.

1.10. Let us now recall briefly the main results of [2]: Let $A$ be a finite-dimensional $k$-algebra, and $\Gamma_A$ its Auslander-Reiter graph. A point $x$ of $\Gamma_A$ will be called a border point if:

1. There exists at most one arrow $a$ of source $x$.
2. There exists at most one arrow $b$ of target $x$.
3. If $a$ and $b$ both exist, then $b = \sigma a$.

Definition. A $k$-species $\Sigma = (F_{i,j}, M_{i,j})_{i,j \in I}$ with relation ideal $R$ is said to satisfy the properties $(\beta)$ if it satisfies:

1. The bounden graph $G$ of $(\Sigma, R)$ is a tree.
2. There is a vertex $i_0$ such that $F_{i_0} = F$, and for all $i \neq i_0$, $F_i = E$, where $E$ and $F$ are two skew fields, finite-dimensional over the common central subfield $k$, and such that $\dim E_F = 2$. Also, if $i, M_j \neq 0$, then $i, M_j = E E_F$ if $i = i_0, j, M_j = E E_F$ if $j = i_0$, and $i, M_j = E E_F$ otherwise.
3. $i_0$ has at most two neighbours, and, if it is so, then $i \to i_0 \to j$ and there is a relation $(\mu)$ on the subspecies

$$E \xrightarrow{\mu} F \xrightarrow{\nu} E$$

given by an epimorphism $\mu: F_E F \otimes_F E_E \to E_E$.

4. All relations are of length two, and the only relations besides $(\mu)$ are the zero relations.
5. Each vertex of $G$ has at most four neighbours.
6. If a vertex $l$ has four neighbours, then $G$ contains a full connected subgraph of the form

$$i_1 \xleftarrow{l} i_2$$

with the zero relations $i_4 M_l \otimes i_1 M_3$ and $i_2 M_l \otimes i_1 M_1$. 

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If a vertex \( l \) has three neighbours, then \( G \) contains a full connected subgraph of one of the forms

\[
\begin{array}{c}
\downarrow \\
i_2 & i_2 \\
\uparrow \\
i_1 & l & i_3 \\
i_1 & l & i_3
\end{array}
\]

with the zero-relation \( i_2 M_i \otimes i_2 M_i \).

Then we have

**Theorem (1).** For a finite-dimensional \( k \)-algebra \( A \), the following statements are equivalent:

(a) \( A \) is iterated tilted of type \( B' \).
(b) \( A \) is given by a bounden species satisfying the properties (\( B' \)).
(c) \( A \) is simply connected and \( \Gamma_A \) satisfies:

(\( \Gamma_1 \)) There are at most two arrows in \( \Gamma_A \) with a prescribed source or target. For any arrow \( \alpha: [M] \to [N] \) in \( \Gamma_A \), we have \( d_\alpha \leq 2 \) and \( d'_\alpha \leq 2 \). Moreover, there exists a unique \( \tau \)-orbit \( \mathcal{O} \) of \( \Gamma_A \), entirely consisting of border points, such that \( d_\alpha = 2 \) if and only if \([N] \in \mathcal{O}\) and \( d'_\alpha = 2 \) if and only if \([M] \in \mathcal{O}\).

(\( \Gamma_2 \)) If \( P_A \) is an indecomposable projective \( A \)-module and \([R] \to [P] \) is an arrow of \( \Gamma_A \), there is at most one arrow of target \([R] \). Dually, if \( I_A \) is an indecomposable injective module and \([I] \to [J] \) is an arrow of \( \Gamma_A \), then there is at most one arrow of source \([J] \).

**Definition.** A \( k \)-species \( \Sigma = (F, i, M_i)_{i \in I} \) with relation ideal \( R \) is said to satisfy the properties (\( \gamma \)) if it satisfies the properties (\( B' \)), (\( B' \)), (\( B' \)), (\( B' \)) and (\( B' \)) (now renamed respectively (\( \gamma' \)), (\( \gamma' \)), (\( \gamma' \)), (\( \gamma' \)) and (\( \gamma' \)) and:

(\( \gamma_2 \)) There is a vertex \( i_0 \) such that \( F_{i_0} = E \), and for all \( i \neq i_0, F_i = F \), where \( E \) and \( F \) are two skew fields, finite-dimensional over \( k \), and such that \( \dim E_F = 2 \). Moreover, if \( i M_j \neq 0, i M_j = F F_E \) for \( j = i_0, i M_j = F F_E \) for \( i = i_0 \), and \( i M_j = F F_E \) otherwise.

(\( \gamma_3 \)) The vertex \( i_0 \) has at most two neighbours \( i \) and \( j \), and if it is so, then \( i \to i_0 \to j \) and there is a relation (\( \pi \)) on the subspecies

\[
\begin{array}{c}
F \\
F \otimes_E E_F \\
E_F \\
F
\end{array}
\]

defined by an epimorphism \( \pi: F F_E \otimes_E E_F \to F F_E \).

Then we have

**Theorem (2).** For a finite-dimensional \( k \)-algebra \( A \), the following statements are equivalent:

(a) \( A \) is iterated tilted of type \( C' \).
(b) \( A \) is given by a bounden series satisfying the properties (\( \gamma \)).
(c) \( A \) is simply connected, and \( \Gamma_A \) satisfies property (\( \Gamma_2 \)) of Theorem (1) and:

(\( \Gamma_2' \)) There are at most two arrows in \( \Gamma_A \) with a prescribed source or target. For any arrow \( \alpha: [M] \to [N] \) in \( \Gamma_A \), we have \( d_\alpha \leq 2 \) and \( d'_\alpha \leq 2 \). Moreover there exists a unique \( \tau \)-orbit \( \mathcal{O}' \) of \( \Gamma_A \), entirely consisting of border points and such that \( d_\alpha = 2 \) if and only if \([M] \in \mathcal{O}' \), and \( d'_\alpha = 2 \) if and only if \([N] \in \mathcal{O}' \).
2. Gradings of $\mathbf{B}_n$ of finite representation type.

2.1. Our object is to describe the gradings of finite representation type of the tree $\mathbf{B}_n$:

\[
\begin{array}{cccccccc}
1 & 2 & 3 & \cdots & n-1 & n \\
(1,2)
\end{array}
\]

A $k$-modulation $\mathcal{M}$ of $\mathbf{B}_n$ is always defined by two skew fields $E$ and $F$, both finite dimensional over the central subfield $k$ and such that, moreover, $\dim E_F = 2$. We shall prove the following theorem:

**Theorem.** The following assertions are equivalent:

(I) $A = A(\mathbf{B}_n, \mathcal{M}, g)$ for a grading $g$ of finite representation type on the modulated tree $(\mathbf{B}_n, \mathcal{M})$.

(II) The bounden species $(\Sigma, R)$ of $A$ satisfies the properties $(\beta)$ of (1.10) and, moreover, does not contain any full connected subspecies of one of the forms:

(a) $E + E + E \to E \to \cdots \to E + F + E$

(b) $E + F + E \to E \to \cdots \to E + E + E$

(c) $E + E + E \to E \to \cdots \to E + E + F$

(d) $F + E + E \to E \to \cdots \to E + E + E$

(e) $E + E + E \to E \to \cdots \to E \to \cdots \to E + E + E$

(f) $E + E + E \to E \to \cdots \to E \to F \to E \to \cdots \to E + E + E$

(g) $F \to E \to \cdots \to E + E \to \cdots \to E + E + E$

(h) $F \to E \to \cdots \to E + E \to \cdots \to E + E + E$

where $E, F, \mu$ are as in [2], and we assume that there is no other relation between the relations shown.

(III) $A$ is a tilted algebra of type $\mathbf{B}_n$.

(2.2) **Lemma.** Let the modulated tree $(\mathbf{B}_n, \mathcal{M})$ be graded by a grading $g$ of finite representation type. Then the associated algebra $A = A(\mathbf{B}_n, \mathcal{M}, g)$ is iterated tilted of type $\mathbf{B}_n$.
Proof. The Auslander-Reiten graph $\Gamma_A$ of $A$ can be identified to the full simply-connected subgraph $\Gamma = \Omega(B_n,\mathcal{R},g)$, which is, moreover, finite since $g$ is of finite representation type. Let $[M] \in \Gamma_0$: there exist at most two arrows of source (or target) $[M]$. This follows from the fact that every point of $B_n$ has at most two neighbours, and from the definition of $\Omega$. On the other hand, the valuation on an arrow $\alpha$: $[M] \rightarrow [N]$ is inherited from $B_n$, hence $\text{dim} \text{Irr}(M,N)_{\text{End}M} \leq 2$ and $\text{dim} \text{End}M \text{Irr}(M,N) \leq 2$. Moreover $\text{dim} \text{Irr}(M,N)_{\text{End}M} = 2$ if and only if $[M]$ belongs to the $\tau$-orbit of the unique projective $(g(1),1)$ of endomorphism ring $F$. By definition of $\Omega$, this is a $\tau$-orbit consisting entirely of border points, and we shall denote it by $\mathcal{B}$. Similarly, $\text{dim} \text{End}M \text{Irr}(M,N) = 2$ if and only if $[N] \in \mathcal{B}$.

Let now $P_A$ be an indecomposable projective $A$-module, and $[R] \rightarrow [P]$ an arrow in $\Gamma_1$. Then there exists at most one arrow of target $[R]$; for, if $[X_1] \rightarrow [R]$ and $[X_2] \rightarrow [R]$ are two arrows, the four points $[P], [R], [X_1]$ and $[X_2]$ belong to different $\tau$-orbits, hence the vertex $\tau^2 [R]$ of $\text{Gr}(\Gamma_A)$ has three neighbours in $\text{Gr}(\Gamma_A)$, which is impossible, since $\text{Gr}(\Gamma_A) = B_n$. Similarly, if $I_A$ is an indecomposable injective, and $[J] \rightarrow [I]$ an arrow of $\Gamma_1$, there exists at most one arrow of source $[J]$. The result then follows from (1.10).

2.3. It follows immediately from this lemma that $(\Sigma, R)$ satisfies the properties $(\beta)$. Proposition (4.2) of [2] shows that, to construct $(\Sigma, R)$, we can use the prefactor and postfactor sets of projective points of $\Gamma$. Let $i = (g(i),i)$ and $j = (g(j),j)$ be two projective points. There is an arrow $i \rightarrow j$ in $G_\Sigma$ if and only if one of the following two conditions is satisfied:

1. $i = (g(i),i)$ belongs to the maximal sectional path $\{(m,h) | h > j, m - h = g(j) - j\}$ and, moreover, no other projective $l = (g(l),l)$ lies on this path between $j$ and $i$.

2. $i = (g(i),i)$ belongs to the composite of the two maximal sectional paths $\{(m,h) | h < j, m + h = g(j) + j\}$ and $\{(m,h) | m - h = g(j) + j - 2\}$, and, moreover, no other projective $l = (g(l),l)$ lies on this path between $j$ and $i$:

\[\begin{array}{c}
1 \\
\end{array}\]

In the first case, the arrow $i \rightarrow j$ is called a $(\rightarrow)$-arrow, in the second, a $(\leftarrow)$-arrow. This allows us to construct $G_\Sigma$. The species $\Sigma$ of $A$ is defined by:

1. $F_1 = F$ and $F_i = E$ for $i \neq 1$.

2. $M_i$ equals $E_E$ if $i, j \neq 1$, $F_E$ if $i = 1$ and $E_F$ if $j = 1$.

We now describe the relation ideal $R$ by its minimal generators:

1. All relations are of length two.

2. If we have $i \rightarrow l \rightarrow j (l \neq 1)$ and $i \rightarrow l$ is a $(\rightarrow)$-arrow (respectively, $(\leftarrow)$-arrow), while $l \rightarrow j$ is a $(\leftarrow)$-arrow (respectively, $(\rightarrow)$-arrow), we have a zero-relation $R_j = \langle iM_l \otimes jM_l \rangle$. 

(R3) If we have $i \rightarrow 1 \rightarrow j$, then necessarily $i \rightarrow 1$ is a (+)-arrow and $1 \rightarrow j$ is a (-)-arrow. In this case, $iR_j$ is the kernel of an epimorphism $\mu: iM_i \otimes jM_j \rightarrow E_E \otimes E_F \rightarrow E_E$.

Moreover, the properties ($\beta$) imply that $G_\Sigma$ is a tree such that every vertex is the source (or the target) of at most one (+)-arrow and one (-)-arrow.

2.4. Lemma. Let $\mathfrak{g}$ be a grading of finite representation type on the modulated tree $(B_n, \equiv)$. Then the bounden species $(\Sigma, R)$ of the associated algebra $A$ contains no full connected subspecies of one of the forms:

(a) $\begin{array}{c}
\xrightarrow{0} E + E + E \rightarrow E \rightarrow \ldots \rightarrow E + F + E
\end{array}$

(b) $\begin{array}{c}
\xrightarrow{0} E + F + E \rightarrow E \rightarrow \ldots \rightarrow E + E + E
\end{array}$

(c) $\begin{array}{c}
\xrightarrow{0} E + E + E \rightarrow E \rightarrow \ldots \rightarrow E + F
\end{array}$

(d) $\begin{array}{c}
\xrightarrow{0} F + E + E \rightarrow E \rightarrow \ldots \rightarrow E + E + E
\end{array}$

(e) $\begin{array}{c}
\xrightarrow{0} E + E + E \rightarrow E \rightarrow \ldots \rightarrow E + E + E
\end{array}$

(f) $\begin{array}{c}
\xrightarrow{0} E + E + E \rightarrow E \rightarrow \ldots \rightarrow E + F \rightarrow E \rightarrow \ldots \rightarrow E + E + E
\end{array}$

(g) $\begin{array}{c}
\xrightarrow{0} F \rightarrow E \rightarrow \ldots \rightarrow E + E \rightarrow \ldots \rightarrow E + E + E
\end{array}$

(h) $\begin{array}{c}
\xrightarrow{0} F \rightarrow E \rightarrow \ldots \rightarrow E + E \rightarrow \ldots \rightarrow E + E + E
\end{array}$

where it is assumed that there is no other relation between the relations shown.

Proof. Let us first observe that the cases (b), (d) and (h) can be deduced, respectively, from (a), (c) and (g) by passing to the opposite algebra. We shall thus assume that $(\Sigma, R)$ contains a full connected subspecies of one of the forms (a), (c), (e), (f) or (g), and show that in each case an injective module lies on the left of a maximal sectional path of $\Gamma_A$ ending at a projective, from which we shall deduce the contradiction $Gr(\Gamma_A) \neq B_n$.

(a) Suppose that $(\Sigma, R)$ contains a full connected subspecies of the form

\begin{align*}
E_1 + E_2 + E_3 \rightarrow E_4 \rightarrow \ldots \rightarrow E_t \rightarrow E_{t-2} + F_{t-1} + E_t
\end{align*}
where \( t \geq 4, E_j = E \) for \( j \neq t - 1 \) and \( F_{i_{t-1}} = F \). Let us define \( P_{i_{t-1}}(i_t) \) to be the unique submodule of \( P(i_t) \) with support \( \text{Supp} \, P_{i_{t-1}}(i_t) = \text{Supp} \, P(i_t) \cap \text{Supp} \, P(i_{t-1}) \). Obviously, \( P_{i_{t-1}}(i_t) \) is indecomposable. Dually, we can define \( I^2(i_1) \) by \( \text{Supp} \, I^2(i_1) = \text{Supp} \, I(i_1) \cap \text{Supp} \, I(i_2) \). This is an indecomposable image of \( I(i_1) \). Let us now consider the representation \( M \) defined by

\[
\text{Supp} \, M = \{i_2, i_3, \ldots, i_{t-1}\} \cup \{\text{Supp} \, I(i_1) \cap \text{Supp} \, I(i_2)\}
\]

\[
\cup \{\text{Supp} \, P(i_t) \cap \text{Supp} \, P(i_{t-1})\};
\]

the coordinate vector spaces are \( M_{i_j} = E_F, M_{i_t} = E_E \oplus E_E \) if \( 2 < j < t - 2 \) is such that \( i_j \) belongs to \( \text{Supp} \, P(i_{t-1}) \), and \( M_h = E_E \) if \( h \in \text{Supp} \, M \setminus \text{Supp} \, P(i_{t-1}) \), and the maps between the coordinate vector spaces are the obvious ones. It is clear that \( M \) is indecomposable except if \( \{i_2, i_3, \ldots, i_{t-1}\} \subseteq \text{Supp} \, P(i_{t-1}) \) and, moreover, the only arrow of target \( i_2 \) has source \( i_3 \), in which case \( M \cong P(i_{t-1}) \oplus P(i_{t-1}) \). On the other hand, it is easy to see that \( I^2(i_1) \) is a submodule of \( M \), while \( P_{i_{t-1}}(i_t) \) is an image of \( M \). Thus we have an oriented path in \( \Gamma_A \) defined by the sequence of maps

\[
I(i_1) \rightarrow I^2(i_1) \rightarrow \cdots \rightarrow M \rightarrow \cdots \rightarrow P_{i_{t-1}}(i_t) \rightarrow P(i_t)
\]

(or

\[
I(i_1) \rightarrow I^2(i_1) \rightarrow \cdots \rightarrow P(i_{t-1}) \rightarrow \cdots \rightarrow P_{i_{t-1}}(i_t) \rightarrow P(i_t)
\]

if \( M \) is decomposable).

Since \( \text{Hom}_A(P(i_{t-1}), P(i_t)) \neq 0 \), \( P(i_{t-1}) \) belongs to the set of prefactors of \( P(i_t) \). On the other hand, \( i_{t-1} = 1 \) implies that \( [P(i_{t-1})] \in \mathcal{B} \), hence there exists a sectional path \( \mathcal{E} \) of source \( [P(i_{t-1})] \) and target \( [P(i_t)] \). Since \( M_{i_{t-1}} \neq 0, [M] \in \mathcal{E} \), and since \( I(i_1)_{i_{t-1}} = 0, I(i_1) \) lies on its left.

Let us show that this implies \( \text{Gr}(\Gamma_A) \neq \mathcal{B}_n \). By hypothesis, \( i_{t-1} = 1 \). Consequently, \( \mathcal{E} \) induces a full connected subgraph \( G_0 \) of \( \text{Gr}(\Gamma_A) \) of the form

\[
\begin{array}{cccccc}
(1,2) & \cdots & h & 1_t \\
1_{t-1} = 1 & 2 & 3 & \cdots & h & 1_t
\end{array}
\]

where \( h = \tau^Z[P_{i_{t-1}}(i_t)] \). Since the injective module \( I(i_1) \) lies on the left of \( \mathcal{E} \), no module of the form \( \tau^s I(i_1) \) \( (s \geq 0) \) lies on \( \mathcal{E} \), and hence \( i_1 \notin G_0 \). On the other hand, the oriented path \( I(i_1) \rightarrow \cdots \rightarrow M \rightarrow \cdots \rightarrow P_{i_{t-1}}(i_t) \) (or \( I(i_1) \rightarrow \cdots \rightarrow P(i_{t-1}) \rightarrow \cdots \rightarrow P_{i_{t-1}}(i_t) \) if \( M \) is decomposable) induces a path of \( \text{Gr}(\Gamma_A) \) of the form

\[
i_1 \cdots \cdots h \]

Clearly, this path cannot contain \( i_t \), and is not a subgraph of \( G_0 \), hence there exists \( 1 \leq l \leq h \) such that
is a full connected subgraph of $\text{Gr}(\Gamma_A)$, which gives the wanted contradiction.

(c) Let us now assume that $(\Sigma, R)$ contains a full connected subspecies of the form

\[ E_{i_1}' + E_{i_2}' + E_{i_3}' \rightarrow E_{i_4}' \rightarrow \cdots \rightarrow E_{i_{t-2}}' + E_{i_{t-1}}' + E_{i_t}' \]

where $F_{i_j} = F$ and $E_{i_j} = E$ for $j \neq t$. Let us define $M$ to be the indecomposable representation of support

\[ \text{Supp} M = \{i_2, \ldots, i_{t-1}\} \cup \{\text{Supp} I(i_1) \cap \text{Supp} I(i_2)\} \]
\[ \cup \{\text{Supp} P(i_t) \cap \text{Supp} P(i_{t-1})\}, \]

and such that $M_h = E_E$ for all $h \in \text{Supp} M$, the only mappings between the coordinate vector spaces being the obvious ones. The rest of the proof of case (a) applies here.

(e) and (f) We now consider the case where $(\Sigma, R)$ contains the full connected subspecies

\[ E_{i_1}' + E_{i_2}' + E_{i_3}' \rightarrow \cdots \rightarrow E_{i_{t-2}}' + E_{i_{t-1}}' + E_{i_t}' \]

(where, if $t \geq 7$, we may have a $4 \leq j \leq t - 3$ such that $i_j = 1$ and then we are in case (f)). We shall construct the module $M$ as follows:

\[ \text{Supp} M = \{i_2, \ldots, i_{t-1}\} \cup \{\text{Supp} I(i_1) \cap \text{Supp} I(i_2)\} \]
\[ \cup \{\text{Supp} P(i_t) \cap \text{Supp} P(i_{t-1})\}. \]

(i) If $1 \notin \text{Supp} M$, we put $M_h = E_E$ for all $h \in \text{Supp} M$, with the obvious maps.

(ii) If we have a $4 \leq j \leq t - 3$ such that $i_j = 1$, we put $M_{i_j} = E_E$ and $M_h = E_E$ if $h \in \text{Supp} M$, $h \neq 1$. The maps between the coordinate vector spaces are the identity, the zero maps, and $\mu: M_1 \otimes_F E_E \rightarrow E_E$.

(iii) If $1 \in \text{Supp} I(i_1) \cap \text{Supp} I(i_2)$, we put $M_1 = E_E$ and $M_h = E_E$ if $h \in \text{Supp} M$, $h \neq 1$, the maps between the coordinate vector spaces being the obvious ones.

In all the previous cases, $M$ is an indecomposable module lying on the prefactor set of $P(i_j)$, while $I(i_1)$ lies on its left. Let us observe that this prefactor set consists in general of three sectional paths. In all cases, however, we obtain a contradiction.
If there exists a sectional path joining \([P(i_{r-1})]\) to \([P(i_j)]\), this contradiction is obtained as in (a). Otherwise there is a path from \([P(i_{r-1})]\) to \([P(i_j)]\) consisting of two sectional paths factoring over \([N]\) \(\in\mathfrak{B}\), and the sectional path \(\mathcal{E}\) from \([N]\) to \([P(i_j)]\) induces a full connected subgraph \(G_0\) of \(\text{Gr}(\Gamma_A)\) of the form

\[
\begin{array}{cccccccc}
1 & 2 & \cdots & h & f & t \\
\end{array}
\]

where \(h = \tau^k[P(i_{r-1})]\), the latter submodule being defined as in (a). Assume that \(i_t \in G_0\). Then there exists an \(s \geq 0\) such that \(\tau^s[I(i_1)] \in \mathcal{E}\). This, however, is impossible, since the existence of a chain of irreducible maps \(\tau^s[I(i_1)] \to \cdots \to I(i_j) \to \cdots \to M \to \cdots \to P(i_j)\) implies that \([I(i_1)] \in \mathcal{E}\), in particular \(I(i_j)_{i_{r-1}} \neq 0\), a contradiction. Thus \(i_t \notin G_0\) and the proof proceeds as in (a).

(g) Finally, assume that we have a full connected subspecies of the form

\[
\begin{array}{cccccccc}
F_1 & E_1 & \cdots & E_{s-1} & E_s & \cdots & E_{s+1} & \cdots & E_{t-1} & E_t \\
\end{array}
\]

where \(t \geq 4\), \(F_i = F\) and \(E_h = E\) for \(h \neq i_t\). We define \(M\) by

\[
\text{Supp } M = \{i_1, i_2, \ldots, i_{r-1}\} \cup \{\text{Supp } I(i') \cap \text{Supp } I(i_j)\} \\
\cup \{\text{Supp } P(i_{i-1}) \cap \text{Supp } P(i_j)\}
\]

and put \(M_1 = F\), \(M_h = E_E\) for \(h \in \text{Supp } M\), \(h \neq 1\), together with the obvious maps between the coordinate vector spaces. As in (a), we have that \(I(i_t')\) lies on the left of a sectional path ending at \(P(i_j)\), hence the contradiction \(\text{Gr}(\Gamma_A) \neq \mathcal{B}_n\).

2.5. Lemma. Let \(A\) be a tilted algebra of finite representation type of type \(\Delta\), where \(\Delta\) is a tree. Then the natural grading \(g_A\) of \(\text{Gr}(\Gamma_A)\) is a grading of \(\Delta\) of finite representation type.

PROOF. This follows at once from the existence of complete slices [10].

2.6. PROOF OF THE THEOREM. We have already shown that (III) implies (I) and that (I) implies (II). In order to complete the proof of the theorem, it suffices to show that (II) implies (III).

2.6.1. Let \(A\) be an algebra whose bounden species \((\Sigma, R)\) satisfies the conditions of (II). In order the show that \(A\) is tilted, we shall construct a complete slice in \(\Gamma_A\). Let \(s_1, \ldots, s_m\) be the set of sources of \(G_2\), and \(P(s_1), \ldots, P(s_m)\) be the corresponding indecomposable projective \(A\)-modules. Define \(s\) to be the full connected subgraph of \(\Gamma_A\) consisting of those points \([M]\) such that if there exists an oriented path from \([M]\) to \([P(s_i)]\), for some \(i\), then this path is sectional.

Since \(\Gamma_A\) has no oriented cycles, and no indecomposable projective \(A\)-module lies on the right of the subsection \(S\), it suffices to show that no indecomposable injective
A-module lies on the left of S. Indeed, this will imply that S intersects each \( r \)-orbit, and therefore, that S is the required complete slice [1].

Thus let \( I(i) \) and \( P(s) \) be, respectively, an indecomposable injective and an indecomposable projective \( A \)-module, with \( s \) a source, such that there exists an oriented path in \( \Gamma_4 \): \( [I(i)] \to [M_1] \to [M_2] \to \cdots \to [M_m] \to [P(s)] \). Our aim is to show that such a path is sectional (thus, \([I(i)] \subseteq S \)). We may obviously assume that \( i \) is a sink.

2.6.2. Let \( w \) be the (nonoriented) path of \( G_2 \) joining \( i \) and \( s \), and \( A_w \) the algebra of the subspecies with graph \( w \), together with the inherited relations. We shall also let \( I_w(i) \) and \( P_w(s) \) be, respectively, the restrictions of \( I(i) \) and \( P(s) \) to \( w \), considered as \( A \)-modules via the natural embedding [13]. Thus we have homomorphisms of \( A \)-modules

\[
I_w(i) \to I(i) \to M_1 \to \cdots \to M_m \to P(s) \to P_w(s).
\]

We claim that it might be assumed that \( \text{Supp } M_j \cap w \neq \emptyset \) for all \( 1 \leq j \leq m \). Let us denote by \( G_a \) the branch of the tree \( G_2 \) attached at the vertex \( a \) of \( w \):

If \( \text{Supp } M_j \cap w = \emptyset \) for some \( 1 \leq j \leq m \) then there exist \( h_1, h_2 \) such that all \( M_h \) \((h_1 < h < h_2)\) have their supports not intersecting \( w \) while the supports of \( M_{h_1} \) and \( M_{h_2} \) intersect \( w \). Since the \( M_h \) are indecomposable, there exists a vertex \( a \in w \) such that \( \text{Supp } M_{h_1+1} \subseteq G_a \). For the same reason, \( \text{Supp } M_h \subseteq G_a \) for all \( h_1 < h < h_2 \). However \( \text{Hom}_A(M_{h_1}, M_{h_1+1}) \neq 0 \) and \( \text{Hom}_A(M_{h_2-1}, M_{h_2}) \neq 0 \) imply that \( a \) belongs to the supports of \( M_{h_1} \) and \( M_{h_2} \), and thus \( \text{Hom}_A(P(a), M_{h_1}) \neq 0 \), \( \text{Hom}_A(P(a), M_{h_2}) \neq 0 \). Now, either there exists no indecomposable \( A \)-module \( N \) such that both \( N_1 \neq 0 \) and \( N_a \neq 0 \), and the set of those indecomposable \( A \)-modules \( M \) with \( M_a \neq 0 \) is given by a rectangle in \( \Gamma_4 \):

(then, clearly, if \( M_{h_1} \) and \( M_{h_2} \) lie in this rectangle, so do all the \( M_h \), for \( h_1 < h < h_2 \), a contradiction), or else there exists such an \( N \), and the set of those indecomposable
A-modules \( M \) with \( M_a \neq 0 \) is given by the region \( \Gamma_a \) of \( \Gamma_A \) shown below:

Then there exist sectional paths \( \sigma_1 \) and \( \sigma_2 \), lying entirely within \( \Gamma_a \), or source \([M_{h_1}]\) (respectively, target \([M_{h_2}]\)) and target \([L_1]\) on \( B \) (respectively, source \([L_2]\) on \( B \)). We have two cases:

(i) \( L_2 \) lies on the left of \( L_1 \). Then necessarily the sectional paths \( \sigma_1 \) and \( \sigma_2 \) intersect at a point \( L \), say, and the oriented path of \( \Gamma_A \) given by \([M_{h_1}] \rightarrow \cdots \rightarrow [L] \rightarrow \cdots \rightarrow [M_{h_2}]\) lies entirely within \( \Gamma_a \) (and thus the supports of all its modules intersect \( w \)).

(ii) \( L_2 \) lies on the right of \( L_1 \), say \( L_2 = \tau^{-p}L_1 \) with \( p \geq 0 \). Then the oriented path of \( \Gamma_A \) given by

\[
[M_{h_1}] \rightarrow \cdots \rightarrow [L_1] \rightarrow \ast \rightarrow [\tau^{-1}L_1] \rightarrow \cdots \rightarrow [\tau^{-p}L_1] = [L_2] \rightarrow \cdots \rightarrow [M_{h_2}]
\]

lies entirely within \( \Gamma_a \).

This completes the proof of our claim.

2.6.3. We may thus consider the induced path of \( \Gamma_{A_w} \) given by \([I_w(i)] \rightarrow [M'_1] \rightarrow [M'_2] \rightarrow \cdots \rightarrow [M'_n] \rightarrow [P_w(s)]\). The path \( w \) has necessarily one of the following two forms:

(a) \( i_0 = i \rightarrow \cdots \rightarrow i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_r \rightarrow \ast \rightarrow i_{r+1} = s \)

(b) \( i_0 = i \rightarrow \cdots \rightarrow i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_r \rightarrow \ast \rightarrow i_{r+1} = s \)

where only the zero-relations are represented by dotted lines. Then, by the conditions (II):

(i) There is no relation between \( i_j \) and \( i_{j+1} \) for all \( 1 \leq j \leq r - 1 \).

(ii) There is no zero-relation between \( i_0, i_1 \) and between \( i_r, i_{r+1} \), but there may be the relation (\( \mu \)) and in the direction opposite to the adjacent zero-relation.

(iii) Two consecutive zero-relations are oriented in opposite directions.

(iv) Nonoriented edges can be oriented arbitrarily.

We may thus write \( w = \bigcup_{j=0}^{r-1} w_j \) where \( w_j \) is the full connected subpath

\[
i_j \rightarrow \cdots \rightarrow i_{j+1}
\]
Let $A_j$ be the algebra given by the bounden species of $w_j$. For $j \neq 0$, $r$, $A_j$ is hereditary of type $A_n$ and $A_0$, $A_r$ are either hereditary of type $A_n$ or tilted of type $B_n$ (if and only if 1 belongs to $w_0$ or $w_r$, for then, $A_0$ or $A_r$ has an indecomposable sincere representation [10]). Every indecomposable $A_w$-module is in fact an indecomposable $A_j$-module for some $0 \leq j \leq r$, and $\Gamma_{A_w}$ has the following form:

(a) $\Gamma_{A_0}$ \hspace{1cm} (b) $\Gamma_{A_0}$

\begin{align*}
\Gamma_{A_1} & \quad \Gamma_{A_1} \\
\Gamma_{A_2} & \quad \Gamma_{A_2} \\
\Gamma_{A_r} & \quad \Gamma_{A_r}
\end{align*}

where $\Gamma_{A_j} \cap \Gamma_{A_{j+1}} = \{[S(j+1)]\}$ [1].

The existence of an oriented path of $\Gamma_{A_w}$ from $I_w(i_0)$ to $P_w(i_{r+1})$ implies that $r \geq 1$, for, otherwise $A_w$ has an indecomposable sincere representation $M_w$, and hence we have nonzero mappings $P_w(i_{r+1}) \to M_w$ and $M_w \to I_w(i_0)$, which gives a contradiction. Also, $w$ cannot be of type (b), or of type (a) with $r > 1$. Consequently, $w$ has the following form:

$$i_0 \leftarrow a_1 \leftarrow a_2 \leftarrow \cdots \leftarrow a_t \rightarrow a_{t+1} \rightarrow \cdots \rightarrow i_1$$

Moreover, the oriented path from $I_w(i_0)$ to $P_w(i_2)$ in $\Gamma_{A_w}$ factors necessarily through the simple module $S_w(i)$ which is injective when considered as an $A_0$-module, and projective when considered as an $A_r$-module. We claim that no $a_t$ ($1 \leq t \leq m_1$) is a source, and dually, no $b_t$ ($1 \leq t \leq m_2$) is a sink. This implies that all arrows are oriented to the left, and also, that if $1 \in w$, then either $i_0 = 1$ or $i_2 = 1$.

Indeed, assume inductively that no $a_s$ is a source for $s < t$, while $a_t$ is a source:

$$i_0 \leftarrow a_1 \leftarrow a_2 \leftarrow \cdots \leftarrow a_t \rightarrow a_{t+1} \rightarrow \cdots \rightarrow i_1$$
we then have a sectional path in $\Gamma_A$ given by a sequence of irreducible maps
$$I_w(i_0) \to I_w(a_1) \to I_w(a_2) \to \cdots \to I_w(a_i) = S_w(a_i),$$
and thus we have no oriented path from $I_w(i_0)$ to $S_w(i_1)$. The proof of the second assertion is dual.

2.6.4. We have proved that the path $w$ has the form

$$i_0 = i \leftarrow \ldots \leftarrow a_1 \leftarrow a_2 \leftarrow \ldots \leftarrow a_{m_1} \leftarrow i_1 = j \leftarrow \ldots \leftarrow \ldots \leftarrow \leftarrow b_1 \leftarrow b_2 \leftarrow \ldots \leftarrow b_{m_2} \leftarrow i_2 = s.$$ 

In particular, $\text{Hom}_A(I(i), I(j)) \neq 0$ and $\text{Hom}_A(P(j), P(s)) \neq 0$, whence $I(j)$ belongs to the postfactor set of $I(i)$, and $P(j)$ to the prefactor set of $P(s)$.

We claim that either there exists a sectional path from $I(i)$ to $I(j)$ or from $P(j)$ to $P(s)$. In the first case the existence of oriented paths from $I(i)$ to $P(s)$ and from $P(s)$ to $I(j)$ implies that $P(s)$ lies on this path, and hence the path from $I(i)$ to $P(s)$ is sectional. The second case is dual.

A simple combinatorial argument shows that the graph $G_\Sigma$ has the following form:

$$G_1 \quad \cdots \quad G_{m_1} \quad \cdots \quad G_2 \quad \cdots \quad G_{m_2} \quad \cdots \quad G_s,$$

where $1 \not\in G_a$, $G_b$, for $1 \leq t \leq m_1$, $1 \leq r \leq m_2$. We have three cases to consider according as $1$ belongs to $G_i$, $G_j$ or $G_s$. The cases $1 \in G_i$ and $1 \in G_s$ are dual.

Let us assume that $1 \in G_j$, and that there is no sectional path from $I(i)$ to $I(j)$ (in particular, $i \neq 1$). Then there exists an indecomposable $A$-module $M_A$ such that $[M] \in \mathfrak{B}$, $\text{Hom}_A(I(i), M) \neq 0$ and $\text{Hom}_A(M, I(j)) \neq 0$. Obviously, $\text{Hom}_A(I(i), M) \neq 0$ implies $\text{Hom}_A(M, I(i)) = 0$. Thus $M_i = 0$, $M_j \neq 0$ and also $M_1 \neq 0$ (because $[M] \in \mathfrak{B}$). But this contradicts the fact that the support of an indecomposable module must be connected. Thus, in this case, there is a sectional path from $I(i)$ to $I(j)$.

Finally, suppose that $1 \in G_j$ and let $w'$ be the (nonoriented) path of $G_\Sigma$ joining $1$ and $j$. If there is no sectional path from $I(i)$ to $I(j)$, there exists an indecomposable module $M_A$ such that $[M] \in \mathfrak{B}$, $\text{Hom}_A(I(i), M) \neq 0$ and $\text{Hom}_A(M, I(j)) \neq 0$. 

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Now it is easy to see that $[M] \in \mathcal{B}$ implies that $\dim M_a \leq 1$ for any $a \in G_\Sigma$, and also $M_1 \neq 0$. Thus $\dim M_1 = 1$, $\dim M_j = 1$, whence $\dim M_a = 1$ for all $a \in w'$. But this means in particular that there is no zero-relation on the path $w'$. However, in this case, we can construct an indecomposable module $N_\lambda$ such that $\text{Hom}(P(s), N) \neq 0$ and $\text{Hom}(N, I(i)) \neq 0$, which will give an oriented path from $P(s)$ to $I(i)$, a contradiction. Let us define $N$ as follows:

$$N_a = \begin{cases} 
E_E & \text{if } a \in w, a \neq j, \\
E_E \oplus E_E & \text{if } a \in w', a \neq 1, \\
E_F & \text{if } a = 1, \\
0 & \text{otherwise.}
\end{cases}$$

The maps between the coordinate vector spaces are the canonical inclusion $[1, 0]$: $E_E \to E_E \oplus E_E$, the projection $[0, 1]$: $E_E \oplus E_E \to E_E$, the map

$$\phi: E_F \otimes _F E_E \to E_E \oplus E_E \to E_E \oplus E_E,$$

or its adjoint $\phi$ (cf. [6 or 7]). It is easy to see, using the methods of [6], that $N$ is indecomposable, and it is clear that $\text{Hom}(P(s), N) \neq 0$ and $\text{Hom}(N, I(i)) \neq 0$.

We have thus completed the proof of our claim and, hence, of the theorem.


3.1. For the tree $C_n$:

a modulation $\mathfrak{M}$ is again given by two skew fields $E, F$, finite-dimensional over the central subfield $k$ and such that $\dim EF = 2$. We have the following theorem.

**Theorem.** The following assertions are equivalent:

(I) $A = A(C_n, \mathfrak{M}, g)$ for a grading $g$ of finite representation type on the modulated tree $(C_n, \mathfrak{M})$.

(II) The bounden species $(\Sigma, R)$ of $A$ satisfies the properties $(\gamma)$ of (1.10) and, moreover, does not contain any full connected subspecies of one of the forms:

(a) $F + F + F \to F \to \ldots \to F + E + F$

(b) $F + E + F \to F \to \ldots \to F + F + F$

(c) $F + F + F \to F \to \ldots \to F + F + E$

(d) $E + F + F \to F \to \ldots \to F + F + F$

(e) $F + F + F \to F \to \ldots \to F + F + F$
(f) \[ F + \cdots + F \rightarrow F \rightarrow \cdots \rightarrow F + \cdots + F \]

(g) \[ E \rightarrow F \rightarrow F \rightarrow \cdots \rightarrow F + \cdots + F \]

(h) \[ E \rightarrow F \rightarrow F \rightarrow \cdots \rightarrow F + \cdots + F \]

where \( E, F, \pi \) are as in [2], and we assume that there is no relation between the relations shown.

(III) \( A \) is a tilted algebra of type \( C_n \).

The same method can be used as in the case \( B_n \). An alternate proof would use the (easily seen) fact that there is a one-to-one correspondence between tilted algebras of types \( B_n \) and \( C_n \), given by simply interchanging the skew fields \( E \) and \( F \).

3.2. The above results can be used to find the finite-dimensional algebras whose trivial extensions are of finite representation type and Dynkin class \( A = B_n \). This problem has already been considered for \( A = A_n \) by D. Hughes and J. Waschbäuschen [15] who showed that these are exactly the iterated tilted algebras of type \( A_n \). We shall here imitate their proof (and use their notations).

**Corollary.** The finite-dimensional algebra \( A \) is iterated tilted of type \( B_n \) (respectively, \( C_n \)) if and only if its trivial extension algebra \( A \otimes DA \) is of finite representation type and Dynkin class \( B_n \) (respectively, \( C_n \)).

**Proof.** By the main result of [15], the trivial extension algebras of finite representation type and Dynkin class \( B_n \) (respectively, \( C_n \)) are exactly the trivial extensions of tilted algebras of the corresponding type. Thus, let \( B \) be an arbitrary tilted algebra of type \( B_n \) (respectively, \( C_n \)), we construct the algebra \( \tilde{B} \) following the method explained in §3 of [15]. Namely, we take countably many copies \( \{(G, n) \mid n \in \mathbb{Z}\} \) of the graph \( G_S \) of the bounden species \( (\Sigma, R) \) of \( B \) together with the following additional arrows: for each maximal nonzero path in a copy of \( G \),

\[
(i, m) \rightarrow \cdots \rightarrow (i, m) \quad (i, m) \in G_S, \ 1 \leq s \leq t, \ m \in \mathbb{Z},
\]

add a new arrow \( (i, m + 1) \rightarrow (i, m) \) together with the following relations:

(a) For \( l \leq s \leq t \),

\[
(i, m) \otimes (i, m) \otimes \cdots \otimes (i, m) = (i, m) \otimes (i, m) \otimes \cdots \otimes (i, m)
\]

and

\[
(i, m) \otimes (i, m) \otimes \cdots \otimes (i, m) = (i, m) \otimes (i, m) \otimes \cdots \otimes (i, m)
\]

(b) If there is a \( j \in G_S \), \( j \neq i, \) such that there is an arrow \( (j, m) \rightarrow (i, m) \), then

\[
(j, m) \otimes (j, m) \otimes \cdots \otimes (j, m) = (j, m) \otimes (j, m) \otimes \cdots \otimes (j, m)
\]
and, dually, if there is a $j \in G_2, j \neq i_2$, such that there is an arrow $(i_1, m) \rightarrow (j, m)$, then
\[(i_1, m + 1) R_{(j, m)} = (i_1, m + 1) M_{(i_1, m)} \otimes (i_1, m) M_{(j, m)},\]
and all possible $\mu, \pi$ and commutativity relations.

Then, since application of the "reflection operator" $S_i$ of [15] to an iterated tilted algebra of type $B_n$ (respectively, $C_n$) again yields an iterated tilted algebra of the same type, the algebras of complete $\nu$-slices through $\hat{B}$ are precisely the iterated tilted algebras of type $B_n$ (respectively, $C_n$).

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