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GRADINGS OF $\mathbf{B}_n$ AND $\mathbf{C}_n$ OF FINITE REPRESENTATION TYPE

BY

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ABSTRACT. It was shown by Bongartz and Gabriel that the classification of simply-connected algebras (i.e. finite-dimensional, basic, of finite representation type and with a simply-connected Auslander-Reiten graph) can be reduced to the study of certain numerical functions, called gradings, operating on a tree. Here, we classify in terms of their bounden species the simply-connected algebras arising from gradings of the Dynkin trees $\mathbf{B}_n$ and $\mathbf{C}_n$, and show that these are exactly the tilted algebras of types $\mathbf{B}_n$ and $\mathbf{C}_n$, respectively.

In [5] Bongartz and Gabriel defined the notion of grading of a tree and proved that there exists a bijection between the isomorphism classes of graded trees of finite representation type and the isomorphism classes of finite-dimensional basic connected algebras of finite representation type with a simply-connected Auslander-Reiten graph.[2] They also describe the gradings of finite representation type for the Dynkin graph $\mathbf{A}_n$. Here, using the methods and results of [2], we describe these gradings for the Dynkin graphs $\mathbf{B}_n$ and $\mathbf{C}_n$. We obtain the following two theorems:

THEOREM (1). *The following assertions are equivalant*:

(I) *The finite-dimensional algebra A arises from a grading of finite representation type of the tree $\mathbf{B}_n$.*

(II) *The bounden species of the algebra A satisfies the properties (β) of* [2] (*see also* (1.10) *below*) *and, moreover, does not contain a full connected subspecies of one of the forms*:

(a) $\overset{0}{\overset{\frown}{E \leftarrow E \leftarrow E}} - E - \dots - \overset{\mu}{\overset{\frown}{E \leftarrow F \leftarrow E}}$

(b) $\overset{\mu}{\overset{\frown}{E \leftarrow F \leftarrow E}} - E - \dots - \overset{0}{\overset{\frown}{E \leftarrow E \leftarrow E}}$

(c) $\overset{0}{\overset{\frown}{E \leftarrow E \leftarrow E}} - E - \dots - \overset{0}{\overset{\frown}{E \leftarrow E \leftarrow F}}$

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[2]Although Bongartz and Gabriel were dealing with algebras over a fixed algebraically closed field, their results and proofs remain valid in general, as will be seen in (1.7).

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(d) $\overset{0}{\overset{\frown}{F \leftarrow E \leftarrow E}} - E - \dots - \overset{0}{\overset{\frown}{E \leftarrow E \leftarrow E}}$

(e) $\overset{0}{\overset{\frown}{E \leftarrow E \leftarrow E}} - E - \dots - \overset{0}{\overset{\frown}{E \leftarrow E \leftarrow E}}$

(f) $\overset{0}{\overset{\frown}{E \leftarrow E \leftarrow E}} - E \dots - \overset{\mu}{\overset{\frown}{E \rightarrow F \rightarrow E}} - \overset{0}{\overset{\frown}{E \leftarrow E \leftarrow E}}$

(g) $F - E - \dots - \begin{array}{c} E \\ \uparrow \\ \overset{0}{\overset{\frown}{E}} \rightarrow E \end{array} - \dots \overset{0}{\overset{\frown}{E \leftarrow E \leftarrow E}}$

(h) $F - E - \dots - \begin{array}{c} E \\ \downarrow \\ \overset{0}{\overset{\frown}{E}} \leftarrow E \end{array} - \dots \overset{0}{\overset{\frown}{E \rightarrow E \rightarrow E}}$

where E, F, μ are as in [**2**], *and we assume that there is no other relation between the relations shown.*

(III) *A is a tilted algebra of type* $\mathbf{B}_n$.

THEOREM (2). *The following assertions are equivalent*:

(I) *The finite-dimensional algebra A arises from a grading of finite representation type of the tree* $\mathbf{C}_n$.

(II) *The bounden species of the algebra A satisfies the properties* (γ) *of* [**2**] (*see also* (1.10) *below*) *and, moreover, does not contain any full connected subspecies of one of the forms*:

(a) $\overset{0}{\overset{\frown}{F \leftarrow F \leftarrow F}} - F - \dots \overset{\pi}{\overset{\frown}{F \leftarrow E \leftarrow F}}$

(b) $\overset{\pi}{\overset{\frown}{F \leftarrow E \leftarrow F}} - F - \dots \overset{0}{\overset{\frown}{F \leftarrow F \leftarrow F}}$

(c) $\overset{0}{\overset{\frown}{F \leftarrow F \leftarrow F}} - F - \dots \overset{0}{\overset{\frown}{F \leftarrow F \leftarrow E}}$

(d) $\overset{0}{\overset{\frown}{E \leftarrow F \leftarrow F}} - F - \dots \overset{0}{\overset{\frown}{F \leftarrow F \leftarrow F}}$

(e) $\overset{0}{\overset{\frown}{F \leftarrow F \leftarrow F}} - F - \dots \overset{0}{F \leftarrow F \leftarrow F}$

(f) $\overset{0}{\overset{\frown}{F \leftarrow F \leftarrow F}} - F - \dots \overset{\pi}{\overset{\frown}{F \rightarrow E \rightarrow F}} - \dots \overset{0}{\overset{\frown}{F \leftarrow F \leftarrow F}}$

(g) $E - F - F - \dots - \begin{array}{c} F \\ \uparrow \\ \overset{0}{\overset{\frown}{F}} \rightarrow F \end{array} - \dots \overset{0}{\overset{\frown}{F \leftarrow F \leftarrow F}}$

(h) $E - F - F - \dots - \begin{array}{c} F \\ \downarrow \\ \overset{0}{\overset{\frown}{F}} \leftarrow F \end{array} - \dots \overset{0}{\overset{\frown}{F \rightarrow F \rightarrow F}}$

where E, F, π are as in [**2**] *and we assume that there is no other relation between the relations shown.*

(III) *A is a tilted algebra of type* $\mathbf{C}_n$.

Observe that the existence of complete slices in tilted algebras [**10**] gives directly in both cases that (III) implies (I).

We apply our results to show that the algebras whose trivial extensions are of finite representation type and Dynkin class $\mathbf{B}_n$ (respectively, $\mathbf{C}_n$) are precisely the iterated tilted algebras of type $\mathbf{B}_n$ (respectively, $\mathbf{C}_n$). A similar result has been obtained for $\mathbf{A}_n$ by Hughes and Waschbüch [**15**].

For the convenience of the reader, the main results of [**2** and **5**] will be recalled in §1. In §2, we prove Theorem (1), and §3 will consist of Theorem (2), together with the stated application.

1. Preliminaries.

1.1. Let k be a commutative field, and A a finite-dimensional k-algebra. By a module is meant a finite-dimensional right A-module. The simple A-modules will be denoted by $S(i)$, where i runs through a fixed index set and we shall let $P(i)$ and $I(i)$ be, respectively, the projective cover and the injective hull of $S(i)$. The support $\operatorname{Supp} M$ of the module M is the set of all i such that $S(i)$ appears as a composition factor of M.

Let $\Sigma = (F_i, {}_iM_j)_{i,j\in I}$ be a k-species [**9**]. We shall denote by $T(\Sigma)$ its tensor algebra, and by G_Σ the associated (oriented) valued graph [**7**]. An ideal $R \subseteq \operatorname{rad}^2 T(\Sigma)$ is called *relation ideal*, and the algebra $A = T(\Sigma)/R$ is then said to be given by the *bounden k-species* (Σ, R) [**2**]. For each pair (i, j) of elements of I, the $F_i - F_j$ bimodule ${}_iR_j = F_iRF_j$ is a relation on Σ. A representation of Σ [**7**] is bound by R if the associated $T(\Sigma)$-module is annihilated by the ideal R.

1.2. We shall use here, without further reference, properties of the Auslander-Reiten sequences and irreducible maps (cf. [**4**]). The *Auslander-Reiten graph* Γ_A of the algebra A has as a set of vertices Γ_0, the set of isomorphism classes of indecomposable A-modules, and there is an arrow (oriented edge) α: $[M] \to [N]$ whenever there is an irreducible map from M to N, this arrow being endowed with a valuation (d_α, d'_α) defined as follows: let $\operatorname{Irr}(M, N)$ denote the bimodule of irreducible maps [**12**], then $d_\alpha = \dim_{\operatorname{End} N} \operatorname{Irr}(M, N)$ and $d'_\alpha = \dim \operatorname{Irr}(M, N)_{\operatorname{End} M}$. We shall let Γ_1 denote the set of all arrows. Note that, if $\tau = D\operatorname{Tr}$ is the Auslander-Reiten translation, each arrow α: $[M] \to [N]$ with N nonprojective is paired with an arrow $\sigma\alpha$: $[\tau N] \to [M]$. A topology is defined on Γ_A by considering it as a two-dimensional cell complex [**11**].

Γ_A becomes a modulated graph [**8**] if to each vertex $i = [M]$ is associated the skew field $F_i = \operatorname{End} M/\operatorname{rad}\operatorname{End} M$, and to each arrow α: $i \to j$ where $i = [M]$ and $j = [N]$, we associate the bimodules ${}_jM_i = \operatorname{Irr}(M, N)$ and ${}_iM_j = \operatorname{Hom}_{F_i}({}_jM_i, F_i)$. Finally, we let the bilinear forms ε_i^j: ${}_iM_j \otimes {}_jM_i \to F_i$ be the evaluation maps.

1.3. Following [**14**], a *translation species* $(\Gamma_0, \Gamma_1, F, N, \tau, \chi)$ is defined by

(1) a translation quiver $(\Gamma_0, \Gamma_1, \tau)$ [**5**];

(2) a map F associating to each vertex $i \in \Gamma_0$ a skewfield F_i;

(3) a map N associating to each arrow α: $i \to j$ an $F_i - F_j$ bimodule $N(\alpha)$, finite dimensional on both sides;

(4) a map associating to each nonprojective vertex i an isomorphism $F_i \xrightarrow{\sim} F_{\tau i}$;

(5) a map associating to each arrow $\alpha\colon i \to j$, with j nonprojective, a nondegenerate bilinear form

$$\chi_\alpha\colon {}_{F_j}N(\sigma\alpha)_{F_i} \otimes_{F_i} N(\alpha)_{F_j} \to {}_{F_j}F_{jF_j}$$

(where $N(\sigma\alpha)$ is considered as a left F_j-module by means of the isomorphism $F_j \xrightarrow{\sim} F_{\tau j}$).

The bilinear form χ_α determines an element $c_i^j \in N(\sigma\alpha) \otimes_{F_i} N(\alpha)$ as follows: let $x_1, x_2, \ldots, x_m$ be a basis of ${}_{F_i}N(\alpha)$, and $\xi_1, \xi_2, \ldots, \xi_m$ the dual basis of $N(\sigma\alpha)_{F_i}$ with respect to the bilinear form χ_α; then $c_i^j = \sum_{t=1}^m \xi_t \otimes x_t$ is called the *canonical element* [**8**].

Let $i, j \in \Gamma_0$, and $\mathcal{P} = (i \xrightarrow{\alpha_1} i_1 \xrightarrow{\alpha_2} i_2 \to \cdots \xrightarrow{\alpha_m} i_m = j)$ be a path in (Γ_0, Γ_1). We have an $F_i - F_j$ bimodule,

$$N(\mathcal{P}) = N(\alpha_1) \otimes_{F_{i_1}} N(\alpha_2) \otimes_{F_{i_2}} \cdots \otimes_{F_{i_{m-1}}} N(\alpha_m),$$

and hence an $F_i - F_j$ bimodule $N(i, j) = \bigoplus_{\mathcal{P}} N(\mathcal{P})$, the sum being taken over all paths from i to j. We shall also agree to set $N(\mathcal{P}_i) = F_i$ if $\mathcal{P}_i$ is the trivial path at the point i. We can thus define the *tensor category* $\otimes\,\Gamma$ of Γ to have Γ_0 as set of objects, and $N(i, j)$ as set of morphisms from i to j. The *mesh category* $\diamond\Gamma$ of Γ is the factor category of $\otimes\,\Gamma$ modulo the ideal generated by the elements Σc_i^j, the sum being taken over all arrows $i \to j$ with nonprojective target j.

Thus, the Auslander-Reiten graph Γ_A of the algebra A yields, in an obvious way, a translation species, called the *Auslander-Reiten species* of A [**14**].

1.4. In what follows, we shall limit ourselves to finite-dimensional, basic, connected algebras of finite representation type with a simply-connected Auslander-Reiten graph. Such algebras will be called *simply-connected algebras*.

Let A be a simply-connected algebra. Then, following Bongartz and Gabriel [**5**], we can associate to Γ_A a (nonoriented) valued graph $\mathrm{Gr}(\Gamma_A)$ as follows: the vertices of $\mathrm{Gr}(\Gamma_A)$ are the τ-orbits of the isomorphism classes of indecomposable A-modules, and the valued edges correspond to the σ-orbits of the arrows of Γ_A: if $\alpha\colon [M] \to [N]$ is an arrow in Γ_A, there exists an edge $\sigma^{\mathbf{Z}}\alpha$ between $\tau^{\mathbf{Z}}[M]$ and $\tau^{\mathbf{Z}}[N]$ in $\mathrm{Gr}(\Gamma_A)$ endowed with the same valuation (d_α, d'_α). Since $d_\alpha = d'_{\sigma a}$, this definition is not ambiguous. Clearly $\mathrm{Gr}(\Gamma_A)$ is homotopically equivalent to Γ_A and, in particular, is a tree.

The natural modulation on Γ_A induces on $\mathrm{Gr}(\Gamma_A)$ a modulation $\mathfrak{M}(\Gamma_A)$ as follows: to each vertex $i = \tau^{\mathbf{Z}}[M]$ of $\mathrm{Gr}(\Gamma_A)$, we associate the skew field $\overline{F}_i = \mathrm{End}\, M/\mathrm{rad}\,\mathrm{End}\, M$, and for each edge between $i = \tau^{\mathbf{Z}}[M]$ and $j = \tau^{\mathbf{Z}}[N]$ (where we assume, without loss of generality, that the representatives M and N are chosen such that N is not projective, and there is an arrow $\alpha\colon [M] \to [N]$), we put ${}_j\overline{M}_i = \mathrm{Irr}(M, N)$ and ${}_i\overline{M}_j = \mathrm{Irr}(\tau N, M)$. Finally, the bilinear form of $\mathfrak{M}(\Gamma_A)$ is again the evaluation map.

1.5. Definition [**5**]. Let $(T, \underline{d})$ be a valued (nonoriented) tree. A *grading* of T is a function $g\colon T \to \mathbf{N}$ such that

(G1) $g^{-1}(0) \neq \varnothing$;

(G2) $g(i) - g(j) \equiv 1 \pmod 2$, whenever i, j are neighbours in T.

We shall define a natural grading g_A on the tree $\mathrm{Gr}(\Gamma_A)$ associated to the algebra A: let $(\Gamma_0, \Gamma_1, \tau)$ be the underlying translation quiver of Γ_A, then there exists a unique morphism of translation quivers pr_A: $(\Gamma_0, \Gamma_1, \tau) \to \mathbf{ZA}_2$ such that $\mathrm{Min}_{x \in \Gamma_0} \mathrm{pr}_A(x) = 0$ [5]. Let us put $g_A(x) = \mathrm{pr}_A(P(i))$, where $P(i)$ is the unique indecomposable projective such that $i = \tau^{\mathbf{Z}}[P(i)]$. Clearly, g_A is a grading.

1.6. To the valued tree $(T, \underline{d})$, graded by g and modulated by $\mathfrak{M} = (F_i, {}_iM_j, \varepsilon_j^i)_{i,j \in T}$, we can associate a translation species $\Omega = \Omega_{(T, \mathfrak{M}, g)} = (\Omega_0, \Omega_1, K, N, \tau, \varepsilon)$. Let

$$\Omega_0 = \{(n, i) \in \mathbf{N} \times T \mid \tfrac{1}{2}(n - g(i)) \in \mathbf{N}\},$$

and for each valued edge $i \overset{(d_{ij}, d_{ji})}{\to} j$, define families of valued arrows

$$(n, i) \overset{(d_{ij}, d_{ji})}{\to} (n+1, j) \quad \text{and} \quad (n, j) \overset{(d_{ji}, d_{ij})}{\to} (n+1, j),$$

with $n \in \mathbf{N}$, whenever both endpoints lie in Ω_0.

Thus, we obtain an infinite-valued graph endowed with a translation: the projective vertices are the pairs $(g(i), i)$, with $i \in T$, the translate of a nonprojective is defined by $\tau(n, i) = (n-2, i)$. The mapping K is defined by $K(n, i) = F_i$, the mapping N by $N((n, i), (n+1, j)) = {}_iM_j$ and $N((n, j), (n+1, i)) = {}_jM_i$, while the bilinear forms ε are given by the ε_j^i.

Let $(n, i) \in \Omega_0$, and define $\nu(n, i)$ to be the set $\{j \in T \mid d_{ij} \neq 0 \text{ and } g(j) < n\}$. It is obviously possible to define inductively on Ω a unique mapping α: $\Omega_0 \to \mathbf{N}^{(\mathrm{Card}\, T)}$ such that:

(a) For every projective vertex $(g(i), i)$ such that all $j \in \nu(g(i), i)$ satisfy $d(g(i) - 1, j) > 0$, we have

$$d(g(i), i) = \delta_i + \sum_{j \in \nu(g(i), i)} d_{ij} d(g(i) - 1, j).$$

Here, δ_i is the Kronecker delta-function.

(b) For every nonprojective vertex (n, i) such that all $j \in \nu(n, i)$ satisfy $d(n-1, j) > 0$ and, moreover, $\sum_{j \in \nu(n,i)} d_{ij} d(n-1, j) - d(n-2, i) > 0$, we have

$$d(n, i) = \sum_{j \in \nu(n, i)} d_{ij} d(n-1, j) - d(n-2, i).$$

(c) For every other $(n, i) \in \Omega_0$, $d(n, i) = 0$.

If $(T, \mathfrak{M}, g) = (\mathrm{Gr}(\Gamma_A), \mathfrak{M}(\Gamma_A), g_A)$, the uniqueness of d implies that $d = \underline{\dim}$, where $\underline{\dim}$ is the mapping associating to each vertex $[M]$ of Γ_A its dimension vector $\underline{\dim}\ M$. By analogy, d is called the *dimension map* of Ω. Now let $\Gamma = \Gamma_{(T, \mathfrak{M}, g)}$ be the full subspecies of Ω defined by

$$\Gamma_0 = \{(n, i) \in \Omega_0 \mid d(n, i) > 0\}.$$

DEFINITION (1). The grading g is *admissible* if Γ contains all the projective vertices $(g(i), i)$ of Ω.

DEFINITION (2). The grading g is of *finite representation type* if it is admissible and Γ is finite.

For instance, g_A is of finite representation type and $\Gamma = \Gamma_{(\mathrm{Gr}(\Gamma_A), \mathfrak{M}(\Gamma_A), g_A)}$ can be identified to Γ_A.

1.7. Let $(T, \mathfrak{M}, g)$ be a modulated tree endowed with a grading of finite representation type, and consider the algebra

$$A = A_{(T,\mathfrak{M},g)} = \bigoplus_{i,j \in T} \operatorname{Hom}_{\diamond\Gamma}((g(i), i), (g(j), j)).$$

We have the following theorem.

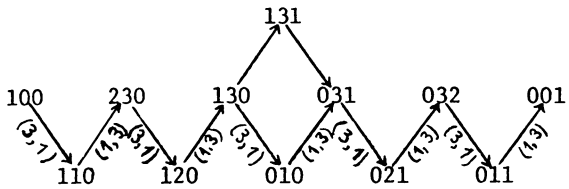
THEOREM (BONGARTZ-GABRIEL). *The map $(T, \mathfrak{M}, g) \to A_{(T,\mathfrak{M},g)}$ induces a bijection between the isomorphism classes of modulated trees equipped with a grading of finite representation type and the isomorphism classes of simply-connected algebras.*

Indeed, the proof of [**5**], done under the assumption that k is algebraically closed, carries over to the general case with only the obvious changes.

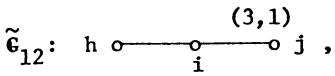
1.8. EXAMPLE. Let F, G be two skew fields, finite-dimensional over the common central subfield k, and such that, moreover, $\dim F_G = 3$. The lower triangular matrix algebra

$$A = \begin{bmatrix} F & 0 & 0 \\ F & G & 0 \\ F & F & F \end{bmatrix}$$

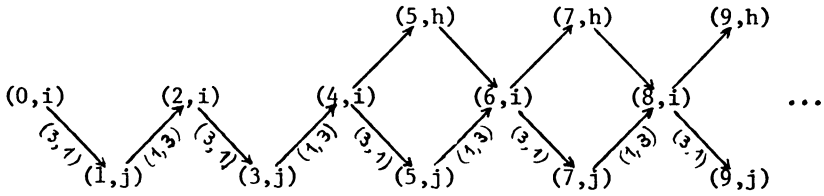
has the following Auslander-Reiten graph:



(where modules are represented by their dimension vectors). Thus $\mathrm{Gr}(\Gamma_A)$ is here the Euclidean graph



and the grading g_A is given by $g_A(h) = 5$, $g_A(i) = 0$, $g_A(j) = 1$. $\Omega = \Omega_{(\mathrm{Gr}(\Gamma_A), \mathfrak{M}(\Gamma)_A, g_A)}$ is here given by



and $\Gamma = \Gamma_A$. In fact, A is tilted of type $\tilde{\mathbf{G}}_{12}$.

1.9. Let A be a finite-dimensional k-algebra. A module T_A is called a *tilting module* [**10**] if

(T1) $\operatorname{pd} T_A \leqslant 1$;

(T2) $\mathrm{Ext}^1_A(T, T) = 0$;

(T3) there is a short exact sequence $0 \to A_A \to T'_A \to T''_A \to 0$ with T' and T'' direct sums of summands of T_A.

A tilting module T_A is *splitting* if every indecomposable B-module N_B, where $B = \mathrm{End}\, T_A$, is such that either $N \otimes_B T = 0$, or $\mathrm{Tor}_1^B(N, T) = 0$. A finite-dimensional k-algebra B is *iterated tilted* (called "generalized tilted" in [**3**]) if:

(1) There exists a sequence of algebras $A_0, A_1, \ldots, A_m = B$ with A_0 hereditary.

(2) There exists a sequence of splitting tilting modules $T^{(i)}_{A_i}$ $(0 \leqslant i \leqslant m-1)$ such that $\mathrm{End}\, T^{(i)}_{A_i} = A_{i+1}$.

If $m \leqslant 1$, B is called *tilted* [**10**]. B is said to be of type Δ for a (nonoriented) valued graph Δ if A_0 is the tensor algebra of an oriented valued graph with nonoriented underlying graph Δ [**6**].

For example, it is not hard to check that the iterated tilted algebras of type $\mathbf{F}_4$ are precisely the simply-connected algebras given by admissible gradings of $\mathbf{F}_4$.

1.10. Let us now recall briefly the main results of [**2**]: Let A be a finite-dimensional k-algebra, and Γ_A its Auslander-Reiter graph. A point x of Γ_A will be called a *border point* if:

(1) There exists at most one arrow α of source x.

(2) There exists at most one arrow β of target x.

(3) If α and β both exist, then $\beta = \sigma\alpha$.

DEFINITION. A k-species $\Sigma = (F_i, {}_iM_j)_{i,j \in I}$ with relation ideal R is said to satisfy the *properties* (β) if it satifies:

$(\beta 1)$ The bounden graph G of (Σ, R) is a tree.

$(\beta 2)$ There is a vertex i_0 such that $F_{i_0} = F$, and for all $i \neq i_0$, $F_i = E$, where E and F are two skew fields, finite-dimensional over the common central subfield k, and such that $\dim E_F = 2$. Also, if ${}_iM_j \neq 0$, then ${}_iM_j = {}_FE_E$ if $i = i_0$, ${}_iM_j = {}_EE_F$ if $j = i_0$, and ${}_iM_j = {}_EE_E$ otherwise.

$(\beta 3)$ i_0 has at most two neighbours, and, if it is so, then $i \to i_0 \to j$ and there is a relation (μ) on the subspecies

$$E \xrightarrow{{}_EE_F} F \xrightarrow{{}_FE_E} E$$

given by an epimorphism μ: ${}_EE_F \otimes {}_FE_E \to {}_EE_E$.

$(\beta 4)$ All relations are of length two, and the only relations besides (μ) are the zero relations.

$(\beta 5)$ Each vertex of G has at most four neighbours.

$(\beta 6)$ If a vertex l has four neighbours, then G contains a full connected subgraph of the form

$$\begin{tikzcd}
i_1 \arrow[dr] & & i_2 \arrow[dl] \\
& l & \\
i_3 \arrow[ur] & & i_4 \arrow[ul]
\end{tikzcd}$$

with the zero relations ${}_{i_4}M_l \otimes {}_lM_{i_3}$ and ${}_{i_2}M_l \otimes {}_lM_{i_1}$.

($\beta 7$) If a vertex l has three neighbours, then G contains a full connected subgraph of one of the forms

$$\begin{array}{ccccccccc} & & i_2 & & & & i_2 & & \\ & & \uparrow & & & & \downarrow & & \\ i_1 & \leftarrow & l & \leftarrow & i_3 & \quad i_1 & \leftarrow & l & \leftarrow & i_3 \end{array}$$

with the zero-relation ${}_{i_3}M_l \otimes {}_lM_{i_1}$.

Then we have

THEOREM (1). *For a finite-dimensional k-algebra A, the following statements are equivalent*:

(a) *A is iterated tilted of type $\mathbf{B}_n$.*

(b) *A is given by a bounden species satisfying the properties (β).*

(c) *A is simply connected and Γ_A satisfies*:

(Γ_1) *There are at most two arrows in Γ_A with a prescribed source or target. For any arrow α: $[M] \to [N]$ in Γ_A, we have $d_\alpha \leqslant 2$ and $d'_\alpha \leqslant 2$. Moreover, there exists a unique τ-orbit $\mathfrak{B}$ of Γ_A, entirely consisting of border points, such that $d_\alpha = 2$ if and only if $[N] \in \mathfrak{B}$ and $d'_2 = 2$ if and only if $[M] \in \mathfrak{B}$.*

(Γ_2) *If P_A is an indecomposable projective A-module and $[R] \to [P]$ is an arrow of Γ_A, there is at most one arrow of target $[R]$. Dually, if I_A is an indecomposable injective module and $[I] \to [J]$ is an arrow of Γ_A, then there is at most one arrow of source $[J]$.*

DEFINITION. A k-species $\Sigma = (F_i, {}_iM_i)_{i,j \in I}$ with relation ideal R is said to satisfy the *properties* (γ) if it satisfies the properties ($\beta 1$), ($\beta 4$), ($\beta 5$), ($\beta 6$) and ($\beta 7$) (now renamed respectively ($\gamma 1$), ($\gamma 4$), ($\gamma 5$), ($\gamma 6$) and ($\gamma 7$)) and:

($\gamma 2$) There is a vertex i_0 such that $F_{i_0} = E$, and for all $i \neq i_0$, $F_i = F$, where E and F are two skew fields, finite-dimensional over k, and such that $\dim E_F = 2$. Moreover, if ${}_iM_j \neq 0$, ${}_iM_j = {}_FE_E$ for $j = i_0$, ${}_iM_j = {}_EE_F$ for $i = i_0$, and ${}_iM_j = {}_FF_F$ otherwise.

($\gamma 3$) The vertex i_0 has at most two neighbours i and j and, if it is so, then $i \to i_0 \to j$ and there is a relation (π) on the subspecies

$$F \xrightarrow{{}_FE_E} E \xrightarrow{{}_EE_F} F$$

defined by an epimorphism π: ${}_FE_E \otimes {}_EE_F \to {}_FF_F$.

Then we have

THEOREM (2). *For a finite-dimensional k-algebra A, the following statements are equivalent*:

(a) *A is iterated tilted of type $\mathbf{C}_n$.*

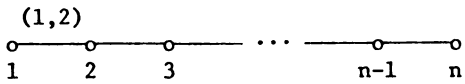
(b) *A is given by a bounden series satisfying the properties (γ).*

(c) *A is simply connected, and Γ_A satisfies property (Γ_2) of Theorem (1) and*:

(Γ'_1) *There are at most two arrows in Γ_A with a prescribed source or target. For any arrow α: $[M] \to [N]$ in Γ_A, we have $d_\alpha \leqslant 2$ and $d'_\alpha \leqslant 2$. Moreover there exists a unique τ-orbit $\mathfrak{B}'$ of Γ_A, entirely consisting of border points and such that $d_\alpha = 2$ if and only if $[M] \in \mathfrak{B}'$, and $d'_\alpha = 2$ if and only if $[N] \in \mathfrak{B}'$.*

2. Gradings of $\mathbf{B}_n$ of finite representation type.

2.1. Our object is to describe the gradings of finite representation type of the tree $\mathbf{B}_n$:

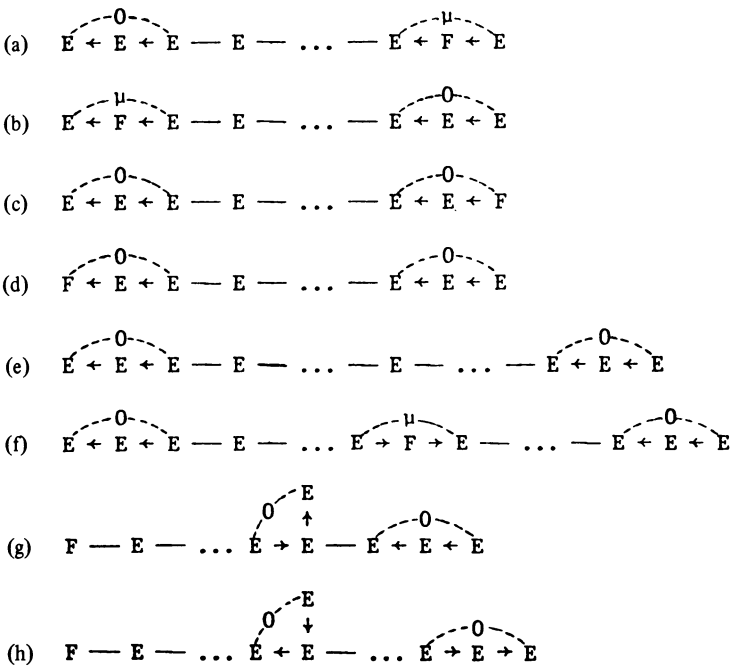


A k-modulation $\mathfrak{M}$ of $\mathbf{B}_n$ is always defined by two skew fields E and F, both finite dimensional over the central subfield k and such that, moreover, $\dim E_F = 2$. We shall prove the following theorem:

THEOREM. *The following assertions are equivalent*:

(I) $A = A_{(\mathbf{B}_n, \mathfrak{M}, g)}$ *for a grading g of finite representation type on the modulated tree* $(\mathbf{B}_n, \mathfrak{M})$.

(II) *The bounden species* (Σ, R) *of A satisfies the properties* (β) *of* (1.10) *and, moreover, does not contain any full connected subspecies of one of the forms*:



where E, F, μ are as in [2], *and we assume that there is no other relation between the relations shown.*

(III) *A is a tilted algebra of type* $\mathbf{B}_n$.

(2.2) LEMMA. *Let the modulated tree* $(\mathbf{B}_n, \mathfrak{M})$ *be graded by a grading g of finite representation type. Then the associated algebra* $A = A_{(\mathbf{B}_n, \mathfrak{M}, g)}$ *is iterated tilted of type* $\mathbf{B}_n$.

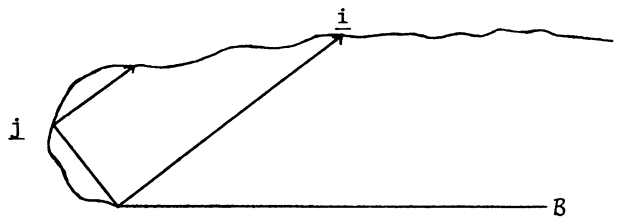
PROOF. The Auslander-Reiten graph Γ_A of A can be identified to the full simply-connected subgraph Γ of $\Omega = \Omega_{(\mathbf{B}_n, \mathfrak{M}, g)}$, which is, moreover, finite since g is of finite representation type. Let $[M] \in \Gamma_0$: there exist at most two arrows of source (or target) $[M]$. This follows from the fact that every point of $\mathbf{B}_n$ has at most two neighbours, and from the definition of Ω. On the other hand, the valuation on an arrow α: $[M] \to [N]$ is inherited from $\mathbf{B}_n$, hence $\dim \operatorname{Irr}(M, N)_{\operatorname{End} M} \leqslant 2$ and $\dim_{\operatorname{End} N} \operatorname{Irr}(M, N) \leqslant 2$. Moreover $\dim \operatorname{Irr}(M, N)_{\operatorname{End} M} = 2$ if and only if $[M]$ belongs to the τ-orbit of the unique projective $(g(1), 1)$ of endomorphism ring F. By definition of Ω, this is a τ-orbit consisting entirely of border points, and we shall denote it by $\mathfrak{B}$. Similarly, $\dim_{\operatorname{End} N} \operatorname{Irr}(M, N) = 2$ if and only if $[N] \in \mathfrak{B}$.

Let now P_A be an indecomposable projective A-module, and $[R] \to [P]$ an arrow in Γ_1. Then there exists at most one arrow of target $[R]$: for, if $[X_1] \to [R]$ and $[X_2] \to [R]$ are two arrows, the four points $[P]$, $[R]$, $[X_1]$ and $[X_2]$ belong to different τ-orbits, hence the vertex $\tau^{\mathbf{Z}}[R]$ of $\operatorname{Gr}(\Gamma_A)$ has three neighbours in $\operatorname{Gr}(\Gamma_A)$, which is impossible, since $\operatorname{Gr}(\Gamma_A) = \mathbf{B}_n$. Similarly, if I_A is an indecomposable injective, and $[I] \to [J]$ an arrow of Γ_1, there exists at most one arrow of source $[J]$. The result then follows from (1.10).

2.3. It follows immediately from this lemma that (Σ, R) satisfies the properties (β). Proposition (4.2) of [2] shows that, to construct (Σ, R), we can use the prefactor and postfactor sets of projective points of Γ. Let $\underline{i} = (g(i), i)$ and $\underline{j} = (g(j), j)$ be two projective points. There is an arrow $\underline{i} \to \underline{j}$ in G_Σ if and only if one of the following two conditions is satisfied:

(1) $\underline{i} = (g(i), i)$ belongs to the maximal sectional path $\{(m, h) \mid h > j,\ m - h = g(j) - j\}$ and, moreover, no other projective $\underline{l} = (g(l), l)$ lies on this path between $\underline{j}$ and $\underline{i}$.

(2) $\underline{i} = (g(i), i)$ belongs to the composite of the two maximal sectional paths $\{(m, h) \mid h < j,\ m + h = g(j) + j\}$ and $\{(m, h) \mid m - h = g(j) + j - 2\}$, and, moreover, no other projective $\underline{l} = (g(l), l)$ lies on this path between $\underline{j}$ and $\underline{i}$:



In the first case, the arrow $\underline{i} \to \underline{j}$ is called a $(+)$-arrow, in the second, a $(-)$-arrow. This allows us to construct G_Σ. The species Σ of A is defined by:

(M1) $F_1 = F$ and $F_i = E$ for $i \neq 1$.

(M2) ${}_iM_j$ equals ${}_EE_E$ if $i, j \neq 1$, ${}_FE_E$ if $i = 1$ and ${}_EE_F$ if $j = 1$.

We now describe the relation ideal R by its minimal generators:

(R1) All relations are of length two.

(R2) If we have $\underline{i} \to \underline{l} \to \underline{j}$ $(l \neq 1)$ and $\underline{i} \to \underline{l}$ is a $(+)$-arrow (respectively, $(-)$-arrow), while $\underline{l} \to \underline{j}$ is a $(-)$-arrow (respectively, $(+)$-arrow), we have a zero-relation ${}_iR_j = {}_iM_l \otimes {}_lM_j$.

(R3) If we have $\underline{i} \to \underline{1} \to \underline{j}$, then necessarily $\underline{i} \to \underline{1}$ is a $(+)$-arrow and $\underline{1} \to \underline{j}$ is a $(-)$-arrow. In this case, ${}_iR_j$ is the kernel of an epimorphism μ: ${}_iM_1 \otimes {}_1M_j = {}_EE_F \otimes {}_FE_E \to {}_EE_E$.

Moreover, the properties (β) imply that G_Σ is a tree such that every vertex is the source (or the target) of at most one $(+)$-arrow and one $(-)$-arrow.

2.4. LEMMA. *Let g be a grading of finite representation type on the modulated tree $(\mathbf{B}_n, \mathfrak{M})$. Then the bounden species (Σ, R) of the associated algebra A contains no full connected subspecies of one of the forms:*

(a) $\overset{0}{\overset{\frown}{E \leftarrow E \leftarrow E}} - E - \dots \overset{\mu}{\overset{\frown}{E \leftarrow F \leftarrow E}}$

(b) $\overset{\mu}{\overset{\frown}{E \leftarrow F \leftarrow E}} - E - \dots \overset{0}{\overset{\frown}{E \leftarrow E \leftarrow E}}$

(c) $\overset{0}{\overset{\frown}{E \leftarrow E \leftarrow E}} - E - \dots \overset{0}{\overset{\frown}{E \leftarrow E \leftarrow F}}$

(d) $\overset{0}{\overset{\frown}{F \leftarrow E \leftarrow E}} - E - \dots \overset{0}{\overset{\frown}{E \leftarrow E \leftarrow E}}$

(e) $\overset{0}{\overset{\frown}{E \leftarrow E \leftarrow E}} - E - \dots \overset{0}{\overset{\frown}{E \leftarrow E \leftarrow E}}$

(f) $\overset{0}{\overset{\frown}{E \leftarrow E \leftarrow E}} - E - \dots \overset{\mu}{\overset{\frown}{E \rightarrow F \rightarrow E}} - \dots - \overset{0}{\overset{\frown}{E \leftarrow E \leftarrow E}}$

(g) $F - E - \dots \begin{array}{c} E \\ \uparrow \\ \overset{0}{\overset{\frown}{E \rightarrow E}} \end{array} - \dots \overset{0}{\overset{\frown}{E \leftarrow E \leftarrow E}}$

(h) $F - E - \dots \begin{array}{c} E \\ \downarrow \\ \overset{0}{\overset{\frown}{E \leftarrow E}} \end{array} - \dots \overset{0}{\overset{\frown}{E \rightarrow E \rightarrow E}}$

where it is assumed that there is no other relation between the relations shown.

PROOF. Let us first observe that the cases (b), (d) and (h) can be deduced, respectively, from (a), (c) and (g) by passing to the opposite algebra. We shall thus assume that (Σ, R) contains a full connected subspecies of one of the forms (a), (c), (e), (f) or (g), and show that in each case an injective module lies on the left of a maximal sectional path of Γ_A ending at a projective, from which we shall deduce the contradiction $\mathrm{Gr}(\Gamma_A) \neq \mathbf{B}_n$.

(a) Suppose that (Σ, R) contains a full connected subspecies of the form

$$\overset{0}{\overset{\frown}{E_{i_1} \leftarrow E_{i_2} \leftarrow E_{i_3}}} - E_{i_4} - \dots - \overset{\mu}{\overset{\frown}{E_{i_{t-2}} \leftarrow F_{i_{t-1}} \leftarrow E_{i_t}}}$$

where $t \geqslant 4$, $E_{i_j} = E$ for $j \neq t-1$ and $F_{i_{t-1}} = F$. Let us define $P_{i_{t-1}}(i_t)$ to be the unique submodule of $P(i_t)$ with support $\operatorname{Supp} P_{i_{t-1}}(i_t) = \operatorname{Supp} P(i_t) \cap \operatorname{Supp} P(i_{t-1})$. Obviously, $P_{i_{t-1}}(i_t)$ is indecomposable. Dually, we can define $I^{i_2}(i_1)$ by $\operatorname{Supp} I^{i_2}(i_1) = \operatorname{Supp} I(i_1) \cap \operatorname{Supp} I(i_2)$. This is an indecomposable image of $I(i_1)$. Let us now consider the representation M defined by

$$\begin{aligned}\operatorname{Supp} M = \{i_2, i_3, \ldots, i_{t-1}\} &\cup \{\operatorname{Supp} I(i_1) \cap \operatorname{Supp} I(i_2)\} \\ &\cup \{\operatorname{Supp} P(i_t) \cap \operatorname{Supp} P(i_{t-1})\};\end{aligned}$$

the coordinate vector spaces are $M_{i_{t-1}} = E_F$, $M_{i_j} = E_E \oplus E_E$ if $2 \leqslant j \leqslant t-2$ is such that i_j belongs to $\operatorname{Supp} P(i_{t-1})$, and $M_h = E_E$ if $h \in \operatorname{Supp} M \setminus \operatorname{Supp} P(i_{t-1})$, and the maps between the coordinate vector spaces are the obvious ones. It is clear that M is indecomposable except if $\{i_2, i_3, \ldots, i_{t-1}\} \subseteq \operatorname{Supp} P(i_{t-1})$ and, moreover, the only arrow of target i_2 has source i_3, in which case $M \xrightarrow{\sim} P(i_{t-1}) \oplus P(i_{t-1})$. On the other hand, it is easy to see that $I^{i_2}(i_1)$ is a submodule of M, while $P_{i_{t-1}}(i_t)$ is an image of M. Thus we have an oriented path in Γ_A defined by the sequence of maps

$$I(i_1) \to I^{i_2}(i_1) \to \cdots \to M \to \cdots \to P_{i_{t-1}}(i_t) \to P(i_t)$$

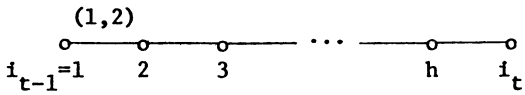
(or

$$I(i_1) \to I^{i_2}(i_1) \to \cdots \to P(i_{t-1}) \to \cdots \to P_{i_{t-1}}(i_t) \to P(i_t)$$

if M is decomposable).

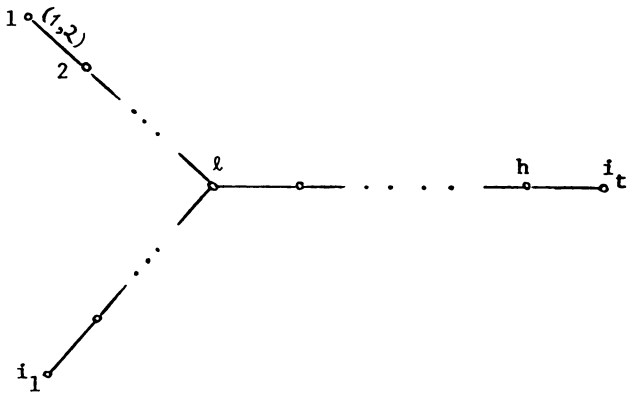
Since $\operatorname{Hom}_A(P(i_{t-1}), P(i_t)) \neq 0$, $P(i_{t-1})$ belongs to the set of prefactors of $P(i_t)$. On the other hand, $i_{t-1} = 1$ implies that $[P(i_{t-1})] \in \mathfrak{B}$, hence there exists a sectional path $\mathfrak{L}$ of source $[P(i_{t-1})]$ and target $[P(i_t)]$. Since $M_{i_{t-1}} \neq 0$, $[M] \in \mathfrak{L}$, and since $I(i_1)_{i_{t-1}} = 0$, $I(i_1)$ lies on its left.

Let us show that this implies $\operatorname{Gr}(\Gamma_A) \neq \mathbf{B}_n$. By hypothesis, $i_{t-1} = 1$. Consequently, $\mathfrak{L}$ induces a full connected subgraph G_0 of $\operatorname{Gr}(\Gamma_A)$ of the form



where $h = \tau^{\mathbf{Z}}[P_{i_{t-1}}(i_t)]$. Since the injective module $I(i_1)$ lies on the left of $\mathfrak{L}$, no module of the form $\tau^s I(i_1)$ $(s \geqslant 0)$ lies on $\mathfrak{L}$, and hence $i_1 \notin G_0$. On the other hand, the oriented path $I(i_1) \to \cdots \to M \to \cdots \to P_{i_{t-1}}(i_t)$ (or $I(i_1) \to \cdots \to P(i_{t-1}) \to \cdots \to P_{i_{t-1}}(i_t)$ if M is decomposable) induces a path of $\operatorname{Gr}(\Gamma_A)$ of the form

Clearly, this path cannot contain i_t, and is not a subgraph of G_0, hence there exists $1 \leqslant l \leqslant h$ such that



is a full connected subgraph of $\mathrm{Gr}(\Gamma_A)$, which gives the wanted contradiction.

(c) Let us now assume that (Σ, R) contains a full connected subspecies of the form

$$\begin{tikzcd} E_{i_1} \arrow[rr, bend left, dashed, "0"] & E_{i_2} \arrow[l] & E_{i_3} \arrow[l] & E_{i_4} \arrow[l, dash] & \cdots \arrow[l, dash] & E_{i_{t-2}} \arrow[rr, bend left, dashed, "0"] & E_{i_{t-1}} \arrow[l] & F_{i_t} \arrow[l] \end{tikzcd}$$

where $F_{i_t} = F$ and $E_{i_j} = E$ for $j \neq t$. Let us define M to be the indecomposable representation of support

$$\begin{aligned} \operatorname{Supp} M = {} & \{i_2, \ldots, i_{t-1}\} \cup \{\operatorname{Supp} I(i_1) \cap \operatorname{Supp} I(i_2)\} \\ & \cup \{\operatorname{Supp} P(i_t) \cap \operatorname{Supp} P(i_{t-1})\}, \end{aligned}$$

and such that $M_h = E_E$ for all $h \in \operatorname{Supp} M$, the only mappings between the coordinate vector spaces being the obvious ones. The rest of the proof of case (a) applies here.

(e) and (f) We now consider the case where (Σ, R) contains the full connected subspecies

$$\begin{tikzcd} E_{i_1} \arrow[rr, bend left, dashed, "0"] & E_{i_2} \arrow[l] & E_{i_3} \arrow[l] & \cdots \arrow[l, dash] & E_{i_{t-2}} \arrow[rr, bend left, dashed, "0"] & E_{i_{t-1}} \arrow[l] & E_{i_t} \arrow[l] \end{tikzcd}$$

(where, if $t \geqslant 7$, we may have a $4 \leqslant j \leqslant t-3$ such that $i_j = 1$ and then we are in case (f)). We shall construct the module M as follows:

$$\begin{aligned} \operatorname{Supp} M = {} & \{i_2, \ldots, i_{t-1}\} \cup \{\operatorname{Supp} I(i_1) \cap \operatorname{Supp} I(i_2)\} \\ & \cup \{\operatorname{Supp} P(i_t) \cap \operatorname{Supp} P(i_{t-1})\}. \end{aligned}$$

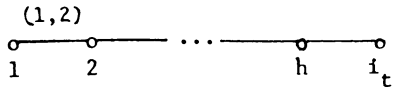
(i) If $1 \notin \operatorname{Supp} M$, we put $M_h = E_E$ for all $h \in \operatorname{Supp} M$, with the obvious maps.

(ii) If we have a $4 \leqslant j \leqslant t-3$ such that $i_j = 1$, we put $M_{i_j} = E_F$ and $M_h = E_E$ if $h \in \operatorname{Supp} M$, $h \neq 1$. The maps between the coordinate vector spaces are the identity, the zero maps, and μ: $M_1 \otimes_F E_E \to E_E$.

(iii) If $1 \in \operatorname{Supp} I(i_1) \cap \operatorname{Supp} I(i_2)$, we put $M_1 = E_F$ and $M_h = E_E$ if $h \in \operatorname{Supp} M$, $h \neq 1$, the maps between the coordinate vector spaces being the obvious ones.

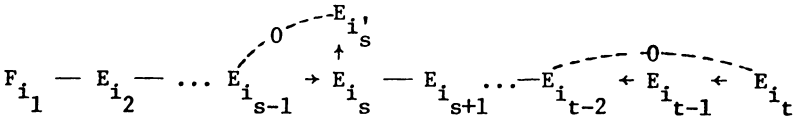
In all the previous cases, M is an indecomposable module lying on the prefactor set of $P(i_t)$, while $I(i_1)$ lies on its left. Let us observe that this prefactor set consists in general of three sectional paths. In all cases, however, we obtain a contradiction.

If there exists a sectional path joining $[P(i_{t-1})]$ to $[P(i_j)]$, this contradiction is obtained as in (a). Otherwise there is a path from $[P(i_{t-1})]$ to $[P(i_t)]$ consisting of two sectional paths factoring over $[N] \in \mathfrak{B}$, and the sectional path $\mathfrak{L}$ from $[N]$ to $[P(i_t)]$ induces a full connected subgraph G_0 of $\mathrm{Gr}(\Gamma_A)$ of the form



where $h = \tau^{\mathbf{Z}}[P_{i_{t-1}}(i_t)]$, the latter submodule being defined as in (a). Assume that $i_1 \in G_0$. Then there exists an $s \geqslant 0$ such that $[\tau^s I(i_1)] \in \mathfrak{L}$. This, however, is impossible, since the existence of a chain of irreducible maps $\tau^s I(i_1) \to \cdots \to I(i_1) \to \cdots \to M \to \cdots \to P(i_t)$ implies that $[I(i_1)] \in \mathfrak{L}$, in particular $I(i_1)_{i_{t-1}} \neq 0$, a contradiction. Thus $i_1 \notin G_0$ and the proof proceeds as in (a).

(g) Finally, assume that we have a full connected subspecies of the form



where $t \geqslant 4$, $F_{i_1} = F$ and $E_h = E$ for $h \neq i_1$. We define M by

$$\begin{aligned}\operatorname{Supp} M = \{i_1, i_2, \ldots, i_{t-1}\} &\cup \{\operatorname{Supp} I(i'_s) \cap \operatorname{Supp} I(i_s)\} \\ &\cup \{\operatorname{Supp} P(i_{t-1}) \cap \operatorname{Supp} P(i_t)\}\end{aligned}$$

and put $M_1 = F$, $M_h = E_E$ for $h \in \operatorname{Supp} M$, $h \neq 1$, together with the obvious maps between the coordinate vector spaces. As in (a), we have that $I(i'_s)$ lies on the left of a sectional path ending at $P(i_t)$, hence the contradiction $\mathrm{Gr}(\Gamma_A) \neq \mathbf{B}_n$.

2.5. Lemma. *Let A be a tilted algebra of finite representation type of type Δ, where Δ is a tree. Then the natural grading g_A of $\mathrm{Gr}(\Gamma_A)$ is a grading of Δ of finite representation type.*

Proof. This follows at once from the existence of complete slices [**10**].

2.6. Proof of the theorem. We have already shown that (III) implies (I) and that (I) implies (II). In order to complete the proof of the theorem, it suffices to show that (II) implies (III).

2.6.1. Let A be an algebra whose bounden species (Σ, R) satisfies the conditions of (II). In order the show that A is tilted, we shall construct a complete slice in Γ_A. Let $s_1, \ldots, s_m$ be the set of sources of G_Σ, and $P(s_1), \ldots, P(s_m)$ be the corresponding indecomposable projective A-modules. Define s to be the full connected subgraph of Γ_A consisting of those points $[M]$ such that if there exists an oriented path from $[M]$ to $[P(s_i)]$, for some i, then this path is sectional.

Since Γ_A has no oriented cycles, and no indecomposable projective A-module lies on the right of the subsection S, it suffices to show that no indecomposable injective

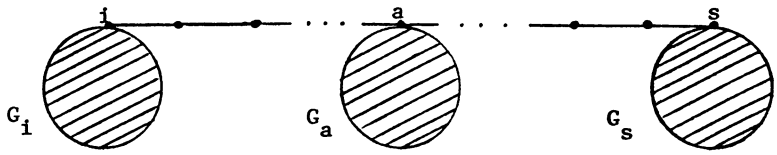
A-module lies on the left of S. Indeed, this will imply that S intersects each τ-orbit, and therefore, that S is the required complete slice [**1**].

Thus let $I(i)$ and $P(s)$ be, respectively, an indecomposable injective and an indecomposable projective A-module, with s a source, such that there exists an oriented path in Γ_A: $[I(i)] \to [M_1] \to [M_2] \to \cdots \to [M_m] \to [P(s)]$. Our aim is to show that such a path is sectional (thus, $[I(i)] \in S$). We may obviously assume that i is a sink.

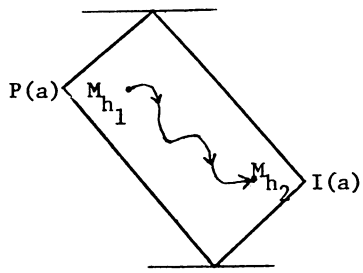
2.6.2. Let w be the (nonoriented) path of G_Σ joining i and s, and A_w the algebra of the subspecies with graph w, together with the inherited relations. We shall also let $I_w(i)$ and $P_w(s)$ be, respectively, the restrictions of $I(i)$ and $P(s)$ to w, considered as A-modules via the natural embedding [**13**]. Thus we have homomorphisms of A-modules

$$I_w(i) \to I(i) \to M_1 \to \cdots \to M_m \to P(s) \to P_w(s).$$

We claim that it might be assumed that $\operatorname{Supp} M_j \cap w \neq \varnothing$ for all $1 \leqslant j \leqslant m$. Let us denote by G_a the branch of the tree G_Σ attached at the vertex a of w:

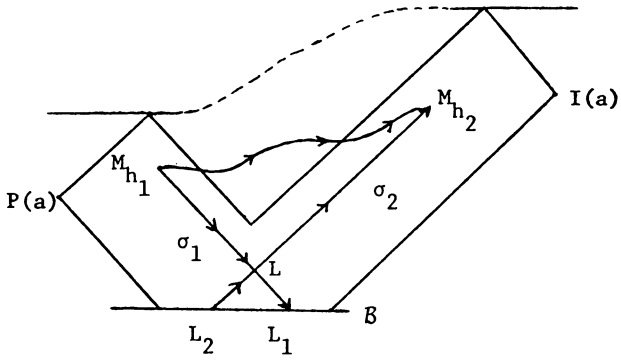


If $\operatorname{Supp} M_j \cap w = \varnothing$ for some $1 \leqslant j \leqslant m$ then there exist $h_1 < h_2$ such that all M_h $(h_1 < h < h_2)$ have their supports not intersecting w while the supports of M_{h_1} and M_{h_2} intersect w. Since the M_h are indecomposable, there exists a vertex $a \in w$ such that $\operatorname{Supp} M_{h_1+1} \subseteq G_a$. For the same reason, $\operatorname{Supp} M_h \subseteq G_a$ for all $h_1 < h < h_2$. However $\operatorname{Hom}_A(M_{h_1}, M_{h_1+1}) \neq 0$ and $\operatorname{Hom}_A(M_{h_2-1}, M_{h_2}) \neq 0$ imply that a belongs to the supports of M_{h_1} and M_{h_2}, and thus $\operatorname{Hom}_A(P(a), M_{h_1}) \neq 0$, $\operatorname{Hom}_A(P(a), M_{h_2}) \neq 0$. Now, either there exists no indecomposable A-module N such that both $N_1 \neq 0$ and $N_a \neq 0$, and the set of those indecomposable A-modules M with $M_a \neq 0$ is given by a rectangle in Γ_A:



(then, clearly, if M_{h_1} and M_{h_2} lie in this rectangle, so do all the M_h, for $h_1 < h < h_2$, a contradiction), or else there exists such an N, and the set of those indecomposable

A-modules M with $M_a \neq 0$ is given by the region Γ_a of Γ_A shown below:



Then there exist sectional paths σ_1 and σ_2, lying entirely within Γ_a, or source $[M_{h_1}]$ (respectively, target $[M_{h_2}]$) and target $[L_1]$ on β (respectively, source $[L_2]$ on $\mathfrak{B}$). We have two cases:

(i) L_2 lies on the left of L_1. Then necessarily the sectional paths σ_1 and σ_2 intersect at a point L, say, and the oriented path of Γ_A given by $[M_{h_1}] \to \cdots \to [L] \to \cdots \to [M_{h_2}]$ lies entirely within Γ_a (and thus the supports of all its modules intersect w).

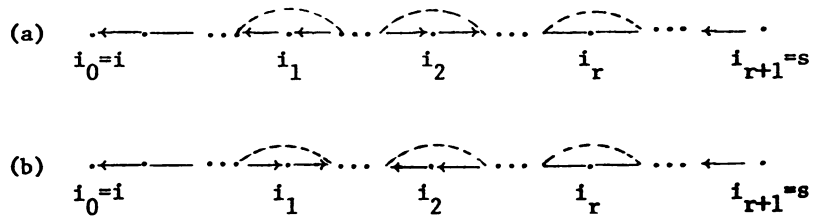
(ii) L_2 lies on the right of L_1, say $L_2 = \tau^{-p}L_1$ with $p \geqslant 0$. Then the oriented path of Γ_A given by

$$[M_{h_1}] \to \cdots \to [L_1] \to * \to [\tau^{-1}L_1] \to \cdots \to [\tau^{-p}L_1] = [L_2] \to \cdots \to [M_{h_2}]$$

lies entirely within Γ_a.

This completes the proof of our claim.

2.6.3. We may thus consider the induced path of Γ_{A_w} given by $[I_w(i)] \to [M_1'] \to [M_2'] \to \cdots \to [M_m'] \to [P_w(s)]$. The path w has necessarily one of the following two forms:



where only the zero-relations are represented by dotted lines. Then, by the conditions (II):

(i) There is no relation between i_j and i_{j+1} for all $1 \leqslant j \leqslant r-1$.

(ii) There is no zero-relation between i_0, i_1 and between i_r, i_{r+1}, but there may be the relation (μ) and in the direction opposite to the adjacent zero-relation.

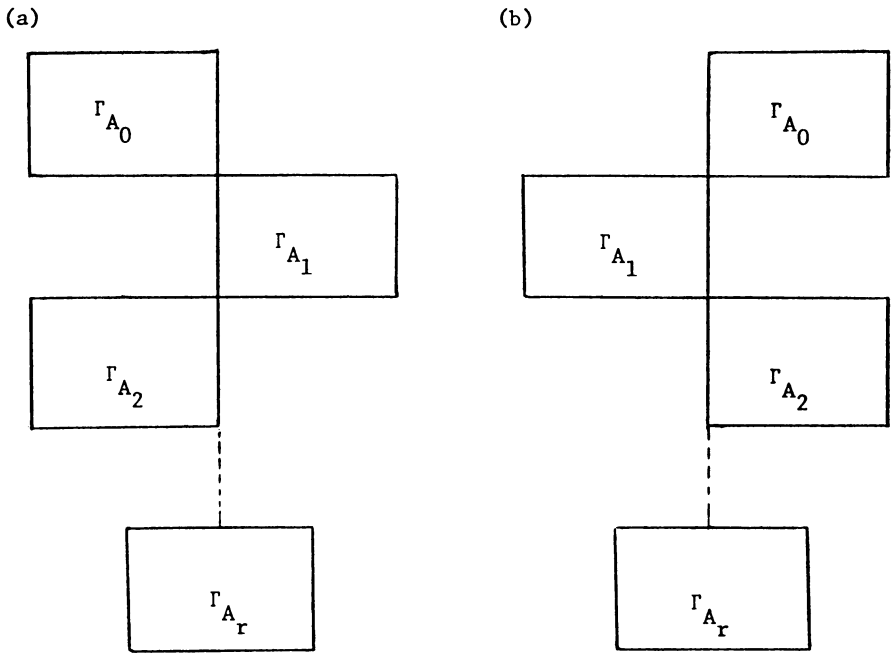
(iii) Two consecutive zero-relations are oriented in opposite directions.

(iv) Nonoriented edges can be oriented arbitrarily.

We may thus write $w = \bigcup_{j=0}^{r} w_j$ where w_j is the full connected subpath

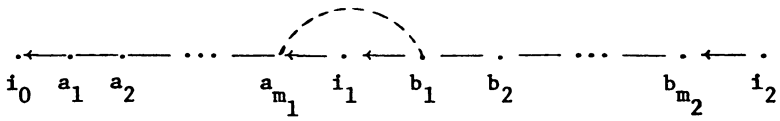
$$i_j \cdot \text{——} \cdot \text{——} \cdots \text{——} \cdot \text{——} \cdot i_{j+1}.$$

Let A_j be the algebra given by the bounden species of w_j. For $j \neq 0$, r, A_j is hereditary of type $\mathbf{A}_n$ and A_0, A_r are either hereditary of type $\mathbf{A}_n$ or tilted of type $\mathbf{B}_n$ (if and only if 1 belongs to w_0 or w_r, for then, A_0 or A_r has an indecomposable sincere representation [**10**]). Every indecomposable A_w-module is in fact an indecomposable A_j-module for some $0 \leqslant j \leqslant r$, and Γ_{A_w} has the following form:



where $\Gamma_{A_j} \cap \Gamma_{A_{j+1}} = \{[S(i_{j+1})]\}$ [**1**].

The existence of an oriented path of Γ_{A_w} from $I_w(i_0)$ to $P_w(i_{r+1})$ implies that $r \geqslant 1$, for, otherwise A_w has an indecomposable sincere representation M_w, and hence we have nonzero mappings $P_w(i_{r+1}) \to M_w$ and $M_w \to I_w(i_0)$, which gives a contradiction. Also, w cannot be of type (b), or of type (a) with $r \geqslant 1$. Consequently, w has the following form:



Moreover, the oriented path from $I_w(i_0)$ to $P_w(i_2)$ in Γ_{A_w} factors necessarily through the simple module $S_w(i_1)$ which is injective when considered as an A_0-module, and projective when considered as an A_1-module. We claim that no a_t $(1 \leqslant t \leqslant m_1)$ is a source, and dually, no b_t $(1 \leqslant t \leqslant m_2)$ is a sink. This implies that all arrows are oriented to the left, and also, that if $1 \in w$, then either $i_0 = 1$ or $i_2 = 1$.

Indeed, assume inductively that no a_s is a source for $s < t$, while a_t is a source:

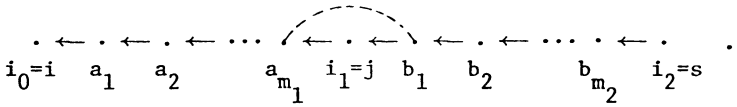
$$i_0 \leftarrow a_1 \leftarrow a_2 \leftarrow \cdots \leftarrow a_t \rightarrow a_{t+1} \text{——} \cdots \text{——} i_1;$$

we then have a sectional path in Γ_{A_w} given by a sequence of irreducible maps

$$I_w(i_0) \to I_w(a_1) \to I_w(a_2) \to \cdots \to I_w(a_t) = S_w(a_t),$$

and thus we have no oriented path from $I_w(i_0)$ to $S_w(i_1)$. The proof of the second assertion is dual.

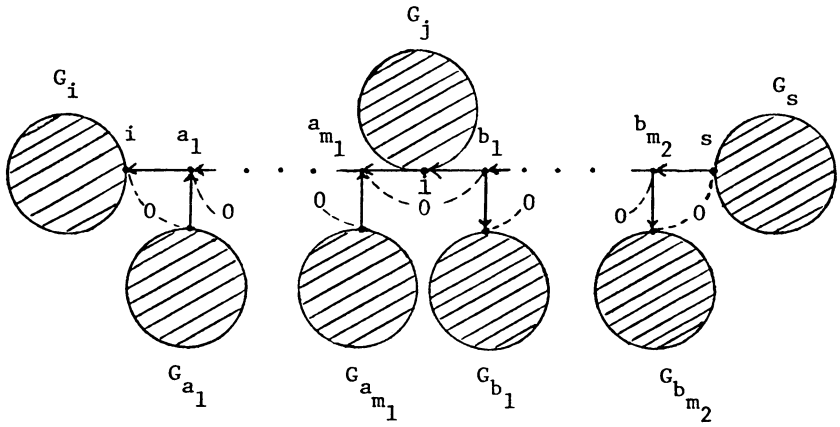
2.6.4. We have proved that the path w has the form



In particular, $\mathrm{Hom}_A(I(i), I(j)) \neq 0$ and $\mathrm{Hom}_A(P(j), P(s)) \neq 0$, whence $I(j)$ belongs to the postfactor set of $I(i)$, and $P(j)$ to the prefactor set of $P(s)$.

We claim that either there exists a sectional path from $I(i)$ to $I(j)$ or from $P(j)$ to $P(s)$. In the first case the existence of oriented paths from $I(i)$ to $P(s)$ and from $P(s)$ to $I(j)$ implies that $P(s)$ lies on this path, and hence the path from $I(i)$ to $P(s)$ is sectional. The second case is dual.

A simple combinatorial argument shows that the graph G_Σ has the following form:



where $1 \notin G_{a_t}, G_{b_r}$ for $1 \leq t \leq m_1$, $1 \leq r \leq m_2$. We have three cases to consider according as 1 belongs to G_i, G_j or G_s. The cases $1 \in G_i$ and $1 \in G_s$ are dual.

Let us assume that $1 \in G_i$, and that there is no sectional path from $I(i)$ to $I(j)$ (in particular, $i \neq 1$). Then there exists an indecomposable A-module M_A such that $[M] \in \mathfrak{B}$, $\mathrm{Hom}_A(I(i), M) \neq 0$ and $\mathrm{Hom}_A(M, I(j)) \neq 0$. Obviously, $\mathrm{Hom}_A(I(i), M) \neq 0$ implies $\mathrm{Hom}_A(M, I(i)) = 0$. Thus $M_i = 0$, $M_j \neq 0$ and also $M_1 \neq 0$ (because $[M] \in \mathfrak{B}$). But this contradicts the fact that the support of an indecomposable module must be connected. Thus, in this case, there is a sectional path from $I(i)$ to $I(j)$.

Finally, suppose that $1 \in G_j$ and let w' be the (nonoriented) path of G_Σ joining 1 and j. If there is no sectional path from $I(i)$ to $I(j)$, there exists an indecomposable module M_A such that $[M] \in \mathfrak{B}$, $\mathrm{Hom}_A(I(i), M) \neq 0$ and $\mathrm{Hom}_A(M, I(j)) \neq 0$.

Now it is easy to see that $[M] \in \mathfrak{B}$ implies that $\dim M_a \leqslant 1$ for any $a \in G_\Sigma$, and also $M_1 \neq 0$. Thus $\dim M_1 = 1$, $\dim M_j = 1$, whence $\dim M_a = 1$ for all $a \in w'$. But this means in particular that there is no zero-relation on the path w'. However, in this case, we can construct an indecomposable module N_A such that $\mathrm{Hom}_A(P(s), N) \neq 0$ and $\mathrm{Hom}_A(N, I(i)) \neq 0$, which will give an oriented path from $P(s)$ to $I(i)$, a contradiction. Let us define N as follows:

$$N_a = \begin{cases} E_E & \text{if } a \in w,\ a \neq j, \\ E_E \oplus E_E & \text{if } a \in w',\ a \neq 1, \\ E_F & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The maps between the coordinate vector spaces are the canonical inclusion $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$: $E_E \to E_E \oplus E_E$, the projection $[0\ 1]$: $E_E \oplus E_E \to E_E$, the map

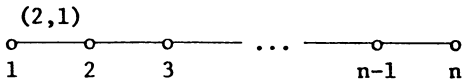
$$\phi\colon E_F \otimes_F E_E \xrightarrow{\sim} E_E \oplus E_E \xrightarrow{\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]} E_E \oplus E_E,$$

or its adjoint $\bar{\phi}$ (cf. [**6** or **7**]). It is easy to see, using the methods of [**6**], that N is indecomposable, and it is clear that $\mathrm{Hom}_A(P(s), N) \neq 0$ and $\mathrm{Hom}_A(N, I(i)) \neq 0$.

We have thus completed the proof of our claim and, hence, of the theorem.

3. Gradings of $\mathbf{C}_n$ of finite representation type.

3.1. For the tree $\mathbf{C}_n$:

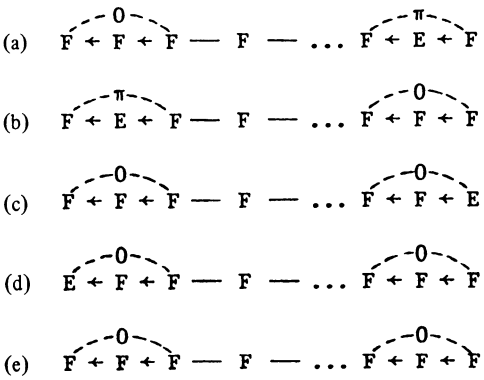


a modulation $\mathfrak{M}$ is again given by two skew fields E, F, finite-dimensional over the central subfield k and such that $\dim E_F = 2$. We have the following theorem.

THEOREM. *The following assertions are equivalent*:

(I) $A = A_{(\mathbf{C}_n, \mathfrak{M}, g)}$ *for a grading g of finite representation type on the modulated tree* $(\mathbf{C}_n, \mathfrak{M})$.

(II) *The bounden species* (Σ, R) *of A satisfies the properties* (γ) *of* (1.10) *and, moreover, does not contain any full connected subspecies of one of the forms*:



(f) $\overset{0}{\overset{\frown}{F \leftarrow F \leftarrow F}} - F - \dots \overset{\pi}{\overset{\frown}{F \rightarrow E \rightarrow F}} - \dots \overset{0}{\overset{\frown}{F \leftarrow F \leftarrow F}}$

(g) $E - F - F - \dots - \overset{F}{\overset{\uparrow}{\overset{0}{\overset{\frown}{F \rightarrow F}}}} - \dots \overset{0}{\overset{\frown}{F \leftarrow F \leftarrow F}}$

(h) $E - F - F - \dots - \overset{F}{\overset{\downarrow}{\overset{0}{\overset{\frown}{F \leftarrow F}}}} - \dots \overset{0}{\overset{\frown}{F \rightarrow F \rightarrow F}}$

where E, F, π are as in [**2**], *and we assume that there is no relation between the relations shown.*

(III) *A is a tilted algebra of type $\mathbf{C}_n$.*

The same method can be used as in the case $\mathbf{B}_n$. An alternate proof would use the (easily seen) fact that there is a one-to-one correspondence between tilted algebras of types $\mathbf{B}_n$ and $\mathbf{C}_n$, given by simply interchanging the skew fields E and F.

3.2. The above results can be used to find the finite-dimensional algebras whose trivial extensions are of finite representation type and Dynkin class $\Delta = \mathbf{B}_n$ or $\mathbf{C}_n$. This problem has already been considered for $\Delta = \mathbf{A}_n$ by D. Hughes and J. Waschbüsch [**15**] who showed that these are exactly the iterated tilted algebras of type $\mathbf{A}_n$. We shall here imitate their proof (and use their notations).

COROLLARY. *The finite-dimensional algebra A is iterated tilted of type $\mathbf{B}_n$ (respectively, $\mathbf{C}_n$) if and only if its trivial extension algebra $A \ltimes DA$ is of finite representation type and Dynkin class $\mathbf{B}_n$ (respectively, $\mathbf{C}_n$).*

PROOF. By the main result of [**15**], the trivial extension algebras of finite representation type and Dynkin class $\mathbf{B}_n$ (respectively, $\mathbf{C}_n$) are exactly the trivial extensions of tilted algebras of the corresponding type. Thus, let B be an arbitrary tilted algebra of type $\mathbf{B}_n$ (respectively, $\mathbf{C}_n$), we construct the algebra $\hat{B}$ following the method explained in §3 of [**15**]. Namely, we take countably many copies $\{(G_\Sigma, m) \mid m \in \mathbf{Z}\}$ of the graph G_Σ of the bounden species (Σ, R) of B together with the following additional arrows: for each maximal nonzero path in a copy of G_Σ,

$$(i_1, m) \rightarrow \cdots \rightarrow (i_t, m) \qquad (i_s \in G_\Sigma,\ 1 \leqslant s \leqslant t,\ m \in \mathbf{Z}),$$

add a new arrow $(i_t, m+1) \rightarrow (i_1, m)$ together with the following relations:

(a) For $1 \leqslant s \leqslant t$,

$${}_{(i_s,m)}R_{(i_{s+1},m-1)} = {}_{(i_s,m)}M_{(i_{s+1},m)} \otimes \cdots \otimes {}_{(i_s,m-1)}M_{(i_{s+1},m-1)}$$

and

$${}_{(i_t,m)}R_{(i_1,m-2)} = {}_{(i_t,m)}M_{(i_1,m-1)} \otimes \cdots \otimes {}_{(i_t,m-1)}M_{(i_1,m-2)}.$$

(b) If there is a $j \in G_\Sigma, j \neq i_{t-1}$, such that there is an arrow $(j, m) \rightarrow (i_t, m)$, then

$${}_{(j,m)}R_{(i_1,m-1)} = {}_{(j,m)}M_{(i_t,m)} \otimes {}_{(i_t,m)}M_{(i_1,m-1)},$$

and, dually, if there is a $j \in G_\Sigma, j \neq i_2$, such that there is an arrow $(i_1, m) \to (j, m)$, then

$$_{(i_t, m+1)}R_{(j,m)} = {}_{(i_t, m+1)}M_{(i_1, m)} \otimes {}_{(i_1, m)}M_{(j,m)},$$

and all possible μ, π and commutativity relations.

Then, since application of the "reflection operator" S_i of [**15**] to an iterated tilted algebra of type $\mathbf{B}_n$ (respectively, $\mathbf{C}_n$) again yields an iterated tilted algebra of the same type, the algebras of complete ν-slices through $\hat{B}$ are precisely the iterated tilted algebras of type $\mathbf{B}_n$ (respectively, $\mathbf{C}_n$).

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