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HIGHLY CONNECTED EMBEDDINGS IN CODIMENSION TWO

BY

SUSAN SZCZEPANSKI[1]

ABSTRACT. In this paper we study semilocal knots over f into ξ, that is, embeddings of a manifold N into $E(\xi)$, the total space of a 2-disk bundle over a manifold M, such that the restriction of the bundle projection $p\colon E(\xi) \to M$ to the submanifold N is homotopic to a normal map of degree one, $f\colon N \to N$.

We develop a new homology surgery theory which does not require homology equivalences on boundaries and, in terms of these obstruction groups, we obtain a classification (up to cobordism) of semilocal knots over f into ξ.

In the simply connected case, the following geometric consequence follows from our classification. Every semilocal knot of a simply connected manifold $M\#K$ in a bundle over M is cobordant to the connected sum of the zero section of this bundle with a semilocal knot of the highly connected manifold K into the trivial bundle over a sphere.

Introduction. It is a well-known fact that complex hypersurfaces with isolated critical points give rise to embeddings of highly connected manifolds in spheres, that is, (real) codimension two submanifolds K of S^{2k+1} with $\pi_i(K) = 0$ for $i < k - 1$ [**M**]. To study such embeddings, it is natural to consider the techniques employed in the classification of classical knot theory and the higher-dimensional analogues—embeddings of S^n into S^{n+2}. One approach is that of surgery theory. As one application of Γ-homology surgery theory, Cappell and Shaneson obtained a calculation of the knot cobordism groups [**CS**]. The more general problem classifying highly connected codimension two embeddings up to cobordism cannot be answered in terms of known surgery theories. One intent of this paper is the development and application of a new surgery theory, Б-surgery, which is useful when studying geometric problems in which the dominant maps are *not* homotopy or homology equivalences. The maps derived from these situations will have the property that the restriction to the boundary does not induce a homology equivalence, not even over the local coefficients prescribed by the situation. The notion of factorizations of forms, the key new element of Б-surgery theory, is directly related to this property. As an application we will obtain a calculation of the set of cobordism classes of certain highly connected codimension two embeddings in terms of some of these new surgery groups.

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The highly connected link of an isolated singularity of a complex hypersurface motivates the interest in a previously inaccessible geometric situation. The objects of study may be defined more precisely as follows. Let $f\colon N^n \to M^n$ be a degree one normal map of closed, connected smooth (respectively, piecewise linear, topological) manifolds such that f is highly connected, that is, f induces an isomorphism $\pi_j(N) \to \pi_j(M)$ for $j < [n/2]$. If $E(\xi)$ is the total space of a linear 2-plane bundle over M, we define a semilocal knot over f into ξ to be an embedding $\iota\colon N \to E(\xi)$ such that ι is homotopic to f. The techniques of surgery theory were used by Cappell and Shaneson to study the case f is the identity map [**CS**] and by S. Ocken in the case f is a homotopy equivalence [**O**].

In the more general case, it is the development of Б-surgery which leads to answers to the two natural questions:

(1) Which maps $f\colon N \to M$ admit semilocal knots into ξ?

(2) Given two semilocal knots over f into ξ, are they cobordant?

In §2 we concern ourselves with obtaining a complete answer to the second question when $\dim M = 2k - 1$. In the process we obtain a necessary condition for f to admit a semilocal knot into ξ. In a future paper, using similar methods, we obtain a complete answer to the first question in the simply connected case.

THEOREM. *If $f\colon N \to M$ admits a semilocal knot into ξ, then f is normally cobordant to* id_M.

It is not difficult to see that the set of cobordism classes of semilocal knots over a degree one (highly connected) map $K \to S^{2k-1}$ into the trivial bundle is isomorphic to the set of cobordism classes of highly connected knots, that is, embeddings of K into S^{2k+1}. Hence, application of our results to a highly connected map into a trivial bundle over a sphere gives a classification of cobordism classes of highly connected knots. In the case of a manifold with *free homology*, this classification was also obtained by R. Vogt [**V**], using methods similar to those of M. Kervaire and J. Levine.

Let $M\#K$, the connected sum of M and K, denote the manifold obtained by removing the interior of a neighborhood of a point of M and K, and then gluing along the boundary spheres by a diffeomorphism. We may also talk of the connected sums of two maps, e.g., embeddings.

As a consequence of our classification we have, in the simply connected case, the geometric result that every semilocal knot over $\mathrm{id}_M\#f\colon M\#K \to M$, where $f\colon K \to S^{2k-1}$ is a highly connected map, is cobordant to a semilocal knot obtained from the zero-section $\iota_0\colon M \to E(\xi)$ by taking the connected sum with an embedding $K \to S^{2k+1}$. That is, every such knot can be deformed to one that is trivial outside the restriction to $K - \mathrm{int}(D^{2k-1}) \to E(\xi)$.

More precisely, denote by $C(f, \xi)$ the set of cobordism classes of semilocal knots over f into ξ. Essentially by taking the connected sum with id_M, we can define a map $\alpha\colon C(f, \text{trivial}) \to C(\mathrm{id}_M \#f, \xi)$. The geometric result may be stated as follows:

THEOREM (2.10). *Let M be a simply connected n-manifold, $n = 2k - 1 > 4$, and let ξ be a 2-plane bundle over M. Then $\alpha\colon C(f, \text{trivial}) \to C(f\#\mathrm{id}_M, \xi)$ is onto in the P.L. and topological cases.*

As a consequence of their classification, Cappell and Shaneson obtain this result in the case $f = \mathrm{id}\colon S^{2k-1} \to S^{2k-1}$.

In §1 we develop Б-surgery theory, a homology surgery for maps $f\colon (M, \partial M) \to (X, Y)$ in which we do not require that, when restricted to the boundary, the maps induce homology equivalences. The content of this section is summarized in the statement of the next theorem.

Let $\mathbf{Z}[\pi]$ be the integral group ring of the group π, with involution determined by a homomorphism $\omega\colon \pi \to \{\pm 1\}$, and let $\mathfrak{F}\colon \mathbf{Z}[\pi] \to \Lambda$ be a homomorphism of rings with unit and involution. Let $(X^n, \partial X)$ be a Poincaré pair over Λ with

$$\left(\pi_1 X, \omega^1(X)\right) = (\pi, \omega), \qquad n = 2k > 4.$$

Let $(f, b)\colon (W, \partial W) \to (X, Y)$ be a normal map of degree one such that $\partial W = M_0 \cup M_1$ and $f|\partial W, f|M_0$ and $f|M_1$ are Λ-homology highly connected, that is, the maps induced on homology with Λ coefficients are isomorphisms in dimensions less than $k-1$ and epimorphisms in dimension $k-1$. Assume that $Y = Y_0 \cup Y_1$, $(X; Y_0, Y_1)$ is an h-cobordism and $f|\partial W$ is the restriction of a map of h-cobordisms. We set

$$A = \ker\left(H_{k-1}(M_i; \Lambda) \to H_{k-1}(Y_i; \Lambda)\right)$$

and

$$A \oplus B = \ker\left(H_{k-1}(\partial W; \Lambda) \to H_{k-1}(Y; \Lambda)\right).$$

THEOREM. *The normal map (f, b) determines an element $\sigma(f, b)$ of an algebraically defined group $Б_{2k}(\mathfrak{F}; A \oplus B)$. The element $\sigma(f, b)$ vanishes if and only if (f, b) is normally cobordant, relative boundary, to a map of Λ-homology h-cobordisms. Every element of $Б_{2k}(\mathfrak{F}; A \oplus B)$ has the form $\sigma(f, b)$ for some normal map (f, b).*

The key new element in our surgery theory is the notion of factorization of forms. The operation of addition in $Б_{2k}(\mathfrak{F}; A \oplus B)$ corresponds to taking the union of two maps along boundary components. Recall that the restriction of the maps to these components is *not* a Λ-homology equivalence in our setting. Algebraically, this fact is reflected in the use of factorizations of forms in defining the group structure of $Б_{2k}(\mathfrak{F}; A \oplus B)$.

We consider the set of $(-1)^k$ symmetric Hermitian forms over $\mathbf{Z}\pi$ that become forms on stably free modules when tensored with Λ, and such that the cokernel of the adjoint map of the form when tensored with Λ is $A \oplus B$. A form is said to have a unimodular factorization (see Theorem 1.2) if there exists a suitable form-preserving map from it to another form over $\mathbf{Z}\pi$ which becomes a unimodular form on stably free modules when tensored with Λ. Thus, the objects under consideration may be described by specifying a homomorphism of $\mathbf{Z}\pi$-modules and a form over $\mathbf{Z}\pi$ on the target module such that when tensored with Λ the homorphism is a map of stably free modules and the form is unimodular. Our surgery obstruction group is defined as a Grothendieck group on this set of pairs.

The relationships between Б-surgery theory and other surgery theories will be the subject of a future paper.

Our classification of $C(f, \xi)$ can now be stated in terms of these obstruction groups. Let $\mathfrak{F}$: $\mathbf{Z}\pi_1 S(\xi) \to \mathbf{Z}\pi_1 M$ be induced by p: $S(\xi) \to M$; let

$$A = \ker(\pi_{k-1}(N) \to \pi_{k-1}(M)), \quad \dim M = n = 2k - 1.$$

THEOREM (2.11). *There is an exact sequence*

$$Б_{n+3}(\mathfrak{F}; A \oplus A) \to C(f, \xi) \to \text{image } s_H.$$

That is, $Б_{n+3}(\mathfrak{F}; A \oplus A)$ *acts on* $C(f, \xi)$ *and the orbit space is the image of the surgery obstruction map* s_H: $[\Sigma M, G/H] \to L_{n+1}(\pi_1 M)$, $H = 0$, PL *or* TOP.

Finally, much of this material first appeared in my doctoral thesis and so for the many rewarding discussions with Julius L. Shaneson, I am indeed grateful.

1. Б-surgery theory. Let π be a finitely presented group and ω: $\pi \to \mathbf{Z}_2$ a homomorphism. There is a conjugation defined on the integral group ring $\mathbf{Z}\pi$ by the formula $\bar{g} = \omega(g)g^{-1}$ for $g \in \pi$. Denote by $\mathfrak{F}$: $\mathbf{Z}\pi \to \Lambda$ an epimorphism of rings with involution.

Let $\eta = \pm 1$. A triple $\alpha = (H, \phi, \mu)$ will be called an η-Hermitian form over $\mathfrak{F}$: $\mathbf{Z}\pi \to \Lambda$ if it is an η-Hermitian form over $\mathbf{Z}\pi$ (see (Q1)–(Q5) of [**CS**, p. 286]) with the additional property:

(Q1) $H_\Lambda = H \otimes_{\mathbf{Z}} \pi\Lambda$ is a stably free Λ-module and the adjoint map $A\phi_\Lambda$: $H_\Lambda \to \operatorname{Hom}_\Lambda(H_\Lambda, \Lambda)$ given by $A\phi_\Lambda(x)(y) = \phi_\Lambda(x, y)$ satisfies $\ker A\phi_\Lambda \cong \operatorname{Hom}_\Lambda(\operatorname{coker} A\phi_\Lambda, \Lambda)$.

We call the η-form $\alpha = (H, \phi, \mu)$ strongly equivalent to zero, $\alpha \approx 0$, with respect to a given isomorphism s: $\operatorname{coker} A\phi_\Lambda = A \oplus B$, A and B fixed Λ-modules, if there exists a submodule $K \subset H$ such that:

(P1) $\phi(x, y) = 0$, $\mu(x) = 0$ for all $x, y \in K$.

(P2) K_Λ is a free direct summand of H_Λ and the composition

$$\operatorname{coker} A\phi_{\Lambda|_{K_\Lambda}} \to \operatorname{coker} A\phi_\Lambda \xrightarrow{\pi_2} B$$

is an isomorphism.

Let (X, Y) be a Poincaré pair over $\mathfrak{F}$: $\mathbf{Z}\pi \to \Lambda$ of dimension $2k \geqslant 6$, X connected with $\pi_1 X = \pi$ and ω: $\pi \to \mathbf{Z}_2$ the orientation character of X. Let (f, b): $(M, \partial M) \to (X, Y)$ be a degree one normal map of the manifold pair $(M, \partial M)$ into (X, Y). We assume that for each component Y_j of ∂Y, the map $f|_{M_j}$: $M_j = (f|_{\partial M})^{-1}(Y_j) \to Y_j$ is Λ-homology $(k-1)$-connected, that is, the induced map

$$H_i(M_j; \Lambda) \xrightarrow{(f|_{M_j})_*} H_i(Y_j; \Lambda)$$

is an isomorphism for $i < k - 1$ and surjective for $i = k - 1$.

A map f: $(M, \partial M) \to (X, Y)$ of degree one of Poincaré pairs induces split surjections (injections) of homology (cohomology) groups over B, a $\mathbf{Z}\pi$-module [**W**, 2.2]. We denote the kernels (cokernels) by $K_*(M, \partial M; B)$ (respectively, $K^*(M, \partial M, B)$).

We assume, after perhaps performing surgery on the interior of M, that f is k-connected. Let $\eta = (-1)^k$. As in [**W**, §1; **CS**, §1], $H_{k+1}(f; \mathbf{Z}\pi) \cong K_k(M: \mathbf{Z}\pi)$; intersection and self-intersection forms, ϕ: $K_k(M) \times K_k(M) \to \mathbf{Z}\pi$ and μ: $K_k(M) \to I_{(\eta)}$, respectively, are defined. As for the case of $\mathbf{Z}\pi$ coefficients, $H_{k+1}(f; \Lambda) \cong K_k(M; \Lambda)$ also, and by [**CS**, 1.4], $H_{k+1}(f; \Lambda) \cong H_{k+1}(f; \mathbf{Z}\pi) \otimes \Lambda$. Since f is k-connected and $f|_{\partial M}$ is Λ-homology $(k-1)$-connected, it follows by arguments similar to those in [**CS**, §1] that $K_k(M; \Lambda)$ is stably free and the map

$$K_k(M; \Lambda) \to K_k(M, \partial M; \Lambda) \cong \operatorname{Hom}(K_k(M; \Lambda), \Lambda)$$

corresponds to the adjoint $A\phi_\Lambda$. Clearly, since $\ker A\phi_\Lambda \cong K_k(\partial M; \Lambda)$ and $\operatorname{coker} A\phi_\Lambda \cong K_{k-1}(\partial M; \Lambda)$, (Q1) is satisfied and we have $\alpha(f, b) = (K_k(M); \phi, \mu)$ is an η-form over $\mathcal{F}$: $\mathbf{Z}\pi \to \Lambda$. Let (f, b): $(M, \partial M) \to (X, Y)$ be a map such that $K_i(\partial M; \Lambda) \to K_i(M; \Lambda)$ is an epimorphism for all i and the composition $A \to s^{-1}\iota_2$: $A \to K_{k-1}(\partial M; \Lambda) \to K_{k-1}(M; \Lambda)$ is an isomorphism, s: $K_{k-1}(\partial M; \Lambda) \to A \oplus B$ an isomorphism and ι_1 the canonical injection. Then we say that (f, b) is a map for which the surgery problem is solved over Λ (with respect to s). When the isomorphism is clear from the context, we will usually delete it from the notation; we will write $\operatorname{coker} A\phi_\Lambda = A \oplus B$.

PROPOSITION 1.1. *If $\alpha(f, b) \approx 0$ with respect to s, then (f, b): $(M, \partial M) \to (X, Y)$ is normally cobordant relative to the boundary to a map $(\bar{f}, \bar{b})$: $(\overline{M}, \partial M) \to (X, Y)$ for which the surgery problem is solved over Λ (with respect to s).*

PROOF. Let $H \subset K_k(M; \mathbf{Z}\pi)$ be a trivialization of the form $(K_k(M); \phi, \mu)$. Let $\{x_i\}$ be a set of elements in H whose images in $K_k(M; \Lambda)$ are a basis of $H_\Lambda = H \otimes \Lambda$; we represent $\{x_i\}$ by disjoint, embedded spheres. Performing surgery on these spheres we obtain a map $\bar{f}$: $(\overline{M}, \partial) \to (X, Y)$. We claim that $\bar{f}$ has the desired property. For $i \neq k$, it is quite easy to see that $K_i(\partial M; \Lambda) \to K_i(\overline{M}; \Lambda)$ is surjective. To handle the case $i = k$, let

$$F\colon (W, M \cup \partial M \times I \cup \overline{M}) \to (X \times I, X \times 0 \cup X \times I \cup X \times 1)$$

be the support of these surgeries. Then we can identify $H = K_{k+1}(W, M; \Lambda)$ and we have the commutative diagram *over* Λ:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K_{k+1}(\partial W, M) & \to & K_k(\partial M) & \to & K_k(\overline{M}) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K_{k+1}(W, M) & \to & K_k(M) & \to & K_k(W) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K_{k+1}(W, \partial W) & \to & K_k(M, \partial M) & \to & K_k(W, \overline{M}) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K_k(\partial W, M) & \to & K_{k-1}(\partial M) & \to & K_{k-1}(\overline{M}) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Since $K_k(M) \to K_k(M, \partial M)$ is identified with the map $A\phi_\Lambda$,

$$K_k(\overline{M}, \partial M) \cong K_k(\partial W, M) \to K_{k-1}(\partial M)$$

can be identified with $\operatorname{coker} A\phi_{\Lambda|_H} \to \operatorname{coker} A\phi_\Lambda$. By (P2) this is a monomorphism, and it follows by chasing the diagram that $K_k(\partial M; \Lambda) \to K_k(\overline{M}; \Lambda)$ is an epimorphism. That $(\bar{f}, \bar{b})$ respects the identification s: $K_{k-1}(\partial M; \Lambda) \cong A \oplus B$ follows easily from (P2).

We restrict our attention to forms $(H; \phi, \mu)$ with an additional property.

(Q2) There exists an η-form $(G; \theta, \psi)$ and a homomorphism of $\mathbf{Z}\pi$ modules λ: $H \to G$ such that $(H; \phi, \mu)$ factors through $(G; \theta, \psi)$, that is, $\phi(x, y) = \phi(\lambda(x), \lambda(y))$ and $\mu(x) = \psi(\lambda(x))$, and, over Λ, the sequence $H_\Lambda \to G_\Lambda \to A \to 0$ is exact. The factorization is called a unimodular factorization if the form θ_Λ is unimodular.

Consider the case $X = Y \times I$. Let (h, c): $(P^{2k-1}, \partial P) \to (Y, \partial Y)$ be a map for which the surgery problem is solved over Λ, that is, $K_i(\partial P; \Lambda) \to K_i(P; \Lambda)$ is an epimorphism for all i. Thus

$$h \times \mathrm{id}\colon (P \times I; P \times 0, P \times 1) \to (X; Y \times 0, Y \times I)$$

is a solved problem over Λ with respect to the obvious isomorphism

$$s\colon K_{k-1}(\partial(P \times I); \Lambda) \cong K_{k-1}(P \times 0; \Lambda) \oplus K_{k-1}(\partial(P \times I), P \times 0; \Lambda).$$

Actually, we may replace $P \times I$ by an h-cobordism over Λ and $h \times \mathrm{id}$ by a map of h-cobordisms over Λ.

REMARK 1. Let $(\bar{f}, \bar{b})$: $(\overline{M}, \partial M) \to (Y \times I, \partial(Y \times I))$ be as in the conclusion of Proposition 1.1 with respect to an isomorphism

$$s\colon K_{k-1}(\partial M; \Lambda) \cong K_{k-1}(P_i; \Lambda) \oplus K_{k-1}(\partial M, P_i; \Lambda), \qquad i = 0, 1,$$

compatible with the obvious exact sequences, where $\partial M = P_0 \cup P_1$. Then it is easily seen from this proof of Proposition 1.1 that $(\bar{f}, \bar{b})$: $(\overline{M}; P_0, P_1) \to (X; Y \times 0, Y \times 1)$ is a map of h-cobordisms over Λ. Unless otherwise stated, when mapping to a product, the isomorphism s will always be as just mentioned.

For the sake of exposition, we assume that $\ker A\phi_\Lambda \cong \operatorname{Hom}_\Lambda(\operatorname{coker} A\phi_\Lambda, \Lambda) = 0$, in particular, $A^* = B^* = 0$. The more general result follows from a similar, but more technical argument, which will not be given here.

THEOREM 1.2. *Let* (f, b): $(W, \partial W) \to (Y \times I, \partial(Y \times I))$ *be a highly connected cobordism, relative boundary, of* (h_0, c_0): $(P_0, \partial P) \to (Y, \partial Y)$, *to* (h_1, c_1): $(P_1, \partial P) \to (Y, Y)$ *such that*:

(i) (f_i, c_i) *is a solved problem over* Λ *for* $i = 0, 1$, *and* $A = K_{k-1}(P_0; \Lambda) = K_{k-1}(P_1; \Lambda)$;

(ii) $f|\partial W$ *and* $f|P_i$ *are* Λ*-homology highly connected*;

(iii) *there exists a set of* $(k-1)$ *spheres and/or disjoint tori* $S^{k-2} \times S^1$ *imbedded in* ∂P *whose images in* P_0 *generate* $K_{k-1}(P_0; \Lambda)$.

Suppose $\ker A\phi_\Lambda = 0$. *Then there exists a unimodular factorization of* $\alpha(f, b)$.

Note. The applications to be considered in this paper satisfy the condition that the toral generators can be disjointly embedded in ∂P. The theorem remains true under much less restrictive and purely algebraic hypotheses.

PROOF. Let $\{x_j\}_{j=1}^r$ be the set of toral classes and $\{y_j\}_{j=1}^s$ the set of spherical classes whose images generate $K_{k-1}(P_0; \Lambda)$. We view these as disjoint and unlinked by taking each to be imbedded in its own copy of $\partial P \times I$. We now replace the toral classes by spherical ones which are still disjoint. To do this let w_i: $S^{k-2} \to \partial P \times I$ be an imbedding obtained by pushing off the torus in the normal direction. Since W is k-connected, each of these spheres bounds; by general position, these disks can be chosen to be embedded and disjoint. Thus, we can view W as the union along M of W' and U, where U is the elementary cobordism determined by handles having these disks as cores; that is, U is an elementary cobordism from P_0 to M where M is obtained from P_0 by attaching handles to P_0 along the w_i's. Thus, since the toral classes in P_0 actually live in ∂P_0, they correspond to spherical classes $\{\bar{x}_i\}$ in M, and these classes, together with the classes $\{y_i\}$, are disjoint. Furthermore, these spherical classes are unlinked in the sense that if a linear combination of the spheres $\{\bar{x}_i, y_i\}$ bounds a chain z, then this chain may be chosen so that its intersection with any of the spheres $\{\bar{x}_i, y_i\}$ is empty. With regards to P_0, since the classes $\{x_i, y_i\}$ live in ∂P_0 the analogous statement is immediate as we may "slide" the torus sphere past the chain towards ∂P_0, that is, we may view the toral or spherical class as contained in a collar of the boundary, $\partial P_0 \times I$ and the cycle z in the complement of this collar. With regards to M and the spherical classes $\bar{x}_i$, the collar $\partial P_0 \times I$ is to be replaced by V_i, the manifold obtained from $\partial P_0 \times I$ by attaching the handle corresponding to x_i. As the sphere $\bar{x}_i$ is not contained in ∂P_0 it does not follow in general that we may "slide" the sphere past the cycle z into V_i. However, if the tori are imbedded disjointly in ∂P_0, as we slide the torus x_i, the handle attached along the $(k-2)$ sphere may be slid past the other surgeries as well. Hence, we may view M as the union of V_i and some manifold M' along a copy of ∂P_0 with z contained in M'. And so we see that the images in $K_{k-1}(U; \Lambda)$ of the toral and spherical classes in $K_{k-1}(P_0; \Lambda)$ are given by spherical classes $K_{k-1}(M; \Lambda)$ with unlinked representatives. Now, since

$$K_{k-1}(W'; \mathbf{Z}\pi) \cong K_{k-1}(W; \mathbf{Z}\pi) = 0,$$

these spheres bound in W'. Let $G \subset K_k(W', M) = K_k(W, U)$ be the submodule generated by the classes represented by chains, each of whose boundary is its intersection with M and is one of the spheres $\{\bar{x}_i, y_i\}$. Let Q be the submodule of $K_{k-1}(U)$ generated by the spheres $\{\bar{x}_i, y_i\}$. We have the exact sequence $K_k(W; \mathbf{Z}\pi) \to G \to Q \to 0$. Observe that over the coefficients Λ, the spheres $\bar{x}_i$, y_i generate $K_{k-1}(P_0; \Lambda)$ and the above sequence becomes

$$K_k(W; \Lambda) \to K_k(W, P_0; \Lambda) \to K_{k-1}(P_0; \Lambda) \to 0.$$

On the module G, there is a well-defined form $\beta = (G; \theta, \chi)$, where θ and χ are defined, respectively, by the intersection and self-intersection numbers of the above chains. By choosing the spherical classes to be disjoint and unlinked we have assured that the restriction of β to $K_k(W)$ coincides with the form on $K_k(W)$ given by intersection and self-intersection of *closed* classes $\alpha(f, b)$.

To prove the unimodularity of this factorization, we consider the following diagram over Λ:

$$\begin{array}{ccccccccc}
 & & & & & & K_{k-1}(\partial P) & & \\
 & & & & & & \downarrow & & \\
 & & K_k(W) & \xrightarrow{\delta} & K_k(W, P_1) & \to & K_{k-1}(P_1) & \to & 0 \\
 & & \lambda\downarrow & A\theta_\Lambda\nearrow & \downarrow\lambda^* & & \downarrow 0 & & \\
 & & K_k(W, P_0) & \xrightarrow[\delta^*]{} & K_k(W, \partial W) & \to & K_{k-1}(P_1, \partial) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
K_{k-1}(\partial P) & \to & K_{k-1}(P_0) & \xrightarrow{0} & K_{k-1}(P_0, \partial P) & \to & K_{k-2}(\partial P) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We identify $K_k(W, P_1; \Lambda) \cong (K_k(W, P_0; \Lambda))^*$ using Poincaré duality and the connectivity of P_i over Λ. Since

$$\delta^* - (\lambda^* \circ A\theta_\Lambda)\colon K_k(W, P_0; \Lambda) \to K_k(W, \partial W; \Lambda)$$

is the zero map on image λ, it will factor through a map $\operatorname{coker}\lambda = K_{k-1}(P_0; \Lambda) \to K_k(W, \partial W; \Lambda)$. But, $K_{k-1}(P_0; \Lambda) = A$ and $K_k(W, \partial W; \Lambda)$ is stably free. Thus, $\delta^* = \lambda^* \circ A\theta_\Lambda$. We observe that $K_k(P_i, \partial P; \Lambda) \cong A^*$. This implies that δ^* is a monomorphism, so $A\theta_\Lambda$ is also. It remains to show that $(\operatorname{coker} A\theta_\Lambda) = 0$. Since $A^* = B^* = 0$, the homomorphisms λ, λ^*, δ and δ^* are all monomorphisms. For the composition of monomorphisms $A\theta_\Lambda \circ \lambda = \delta$, there is an exact sequence of cokernels

$$0 \to K_{k-1}(P_0; \Lambda) \xrightarrow{\alpha} K_{k-1}(P_1; \Lambda) \to \operatorname{coker} A\theta_\Lambda \to 0$$

and similarly,

$$0 \to K_{k-1}(P_1; \Lambda) \xrightarrow{\iota} K_{k-1}(\partial W; \Lambda) \to K_{k-1}(\partial W, P_1; \Lambda) \to 0.$$

The maps α and ι are induced by $A\theta_\Lambda$ and λ^*, respectively. The composition $\iota \circ \alpha$ is induced by δ^* and is just the map $K_{k-1}(P_0; \Lambda) \to K_{k-1}(\partial W; \Lambda)$. It follows that α is an isomorphism; hence θ_Λ is unimodular.

A $(-1)^k$ form $\alpha = (H; \phi, \mu)$ over $\mathfrak{F}\colon \mathbf{Z}\pi \to \Lambda$ with a given decomposition $\operatorname{coker} A\phi_\Lambda = A \oplus B$ and satisfying (Q1) and (Q2) is completely described by specifying the map $\lambda\colon H \to G$ and the form on $G, (G; \theta, \psi)$. In what follows, α will denote a $(-1)^k$ form with a unimodular factorization. We define an addition $\oplus$ on the set of isomorphism classes of such forms.

$$(\alpha) \oplus (\beta) = \left(\lambda_{\alpha\oplus\beta}\colon H_{\alpha\oplus\beta} \to G_\alpha \oplus G_\beta, \left(G_\alpha \oplus G_\beta; \theta_\alpha \perp \theta_\alpha, \mu_\alpha \perp \mu_\beta\right)\right)$$

where $H_{\alpha\oplus\beta} = \ker(G_\alpha \oplus G_\beta \to A \oplus A/\Delta A \to 0)$. This operation is commutative and associative. If $\alpha = (\lambda\colon H \to G, (G; \theta, \psi))$, then we define

$$-\alpha = (\lambda\colon H \to G, (G; -\theta, -\psi)).$$

From now on we write $\alpha \approx 0$ to mean that the orthogonal sum of α and a (Wall) kernel over $\mathbf{Z}\pi$ is strongly equivalent to zero in the sense previously defined. In other words, we ignore the addition of kernels. This is of no geometric consequence, since the addition of a kernel can be realized by surgeries on trivial spheres.

LEMMA 1.3. $(\alpha) \oplus (-\alpha) \approx 0$.

PROOF. After taking the orthogonal direct sum with a kernel over $\mathbf{Z}\pi$, we may assume that H_Λ is free. Let $K \subset G \oplus G$ be the diagonal submodule. It is easy to see that $K \subset H_{\alpha\oplus(-\alpha)}$ satisfies (P1). An argument similar to [**W**, L. 5.4] establishes (P2).

We say that $\alpha \sim \beta$ (α is equivalent to β) if and only if $\alpha + (-\beta) \approx 0$. Let $\text{Б}_\eta(\mathcal{F}; A \oplus B)$ be the set of equivalence classes of η-forms with unimodular factorizations under the equivalence relation generated by $\sim$; addition $\oplus$ induces the structure of an abelian group. Note that forms α and β represent the same element in the group if and only if there exist forms δ_i with $\delta_i \approx 0$ and $\alpha \oplus (-\beta) \oplus \delta_1 \oplus \cdots \oplus \delta_k \approx 0$. As in [**CS**, 1.2] each form $\alpha = (\lambda\colon H \to G, (G; \theta, \psi))$ is equivalent to a form $\alpha_0 = (\lambda_0\colon H_0 \to G_0, (G_0; \theta_0, \psi_0))$ with $\lambda_0\colon H_0 \to G_0$ a map of free $\mathbf{Z}\pi$-modules.

LEMMA 1.4. *If α represents zero in $\text{Б}_\eta(\mathcal{F}; A \oplus B)$ then there exist forms $\bar{\delta}_i$ strongly equivalent to zero over $\mathrm{id}_{\mathbf{Z}\pi}$ with respect to the identifications with $A \oplus B$ (as a $\mathbf{Z}\pi$-module) and with $\alpha \oplus \bar{\delta}_1 \oplus \bar{\delta}_2 \oplus \cdots \oplus \bar{\delta}_j \approx 0$.*

PROOF. If α represents zero, then there exists $\delta_i \approx 0$, $i = 1, \ldots, j$, with $\alpha \oplus \delta_1 \oplus \cdots \oplus \delta_j \approx 0$. Let $\delta_i = (\lambda_i\colon H_i \to G_i, (G_i; \theta_i, \psi_i))$ with K_i a submodule satisfying (P1) and (P2), with $(H_i)_\Lambda = (K_i)_\Lambda \oplus (L_i)_\Lambda$. Take L_i to be the submodule of H_i generated by lifts of the basis of $(L_i)_\Lambda$. Composing the obvious map $h\colon K_i \oplus L_i \to H_i$ with λ_i, we have a form

$$\bar{\delta}_i = (\lambda_i \circ h\colon K_i \oplus L_i \to G_i, (G; \theta_i, \psi_i)).$$

Clearly, $\bar{\delta}_i \approx 0$ over $\mathrm{id}_{\mathbf{Z}\pi}$ and $h_\Lambda\colon (K_i)_\Lambda \oplus (L_i)_\Lambda \to (H_i)_\Lambda$ is an isomorphism. Since the form $\bar{\delta}_i$ factors through δ_i, we have

$$H_{\alpha\oplus\Sigma\delta_i} = H_{\alpha\oplus\Sigma\bar{\delta}_i} \subset G_\alpha \oplus \Sigma G_i \quad \text{and} \quad \alpha \oplus \Sigma\bar{\delta}_i \approx 0.$$

In the case $A \oplus B = 0$, the group $\text{Б}_\eta(\mathcal{F}; A \oplus B)$ is just $\Gamma_\eta(\mathcal{F})$, and each element of the group can be represented by $\alpha = (\mathrm{id}\colon H \to H, (H; \theta, \psi))$. In fact, it can be shown that the equivalence class of a form on H is independent of the factorization, i.e.,

$$(\mathrm{id}\colon H \to H, (H; \phi, \mu)) \sim (\lambda\colon H \to G, (G; \theta, \psi)) \sim (\mathrm{id}\colon G \to G, (G; \theta, \psi)).$$

This is essentially a consequence of the fact that over the coefficients the factorizations are isomorphic. In the case $A \neq 0$, we have the following.

LEMMA 1.5. *Let $\alpha = (\lambda_\alpha\colon H_\alpha \to G_\alpha, (G_\alpha; \theta_\alpha, \psi_\alpha))$ and $\beta = (\lambda_\beta\colon H_\beta \to G_\beta, (G_\beta; \theta_\beta, \psi_\beta))$, and let $\tau\colon G_\alpha \to G_\beta$ be a map of forms such that τ_Λ is an isomorphism. Then $\alpha \oplus (-\beta) \approx 0$, that is, $\alpha \sim \beta$.*

PROOF. Let $\Delta(\tau) = \{(g, \tau(g))/g \in G_\alpha\} \subset G_\alpha \oplus G_\beta$. It is not difficult to see that $\Delta(\tau) \subset H_{\alpha\oplus -\beta} \subseteq G_\alpha \oplus G_\beta$ satisfies (P1) and (P2).

The forms α and β in the above lemma are said to have a common factorization over Λ.

LEMMA 1.6. *For any* $\gamma \in \text{Б}_\eta(\mathfrak{F}; A \oplus B)$, *there is a representative* $(\lambda\colon H \to G, (G; \theta, \psi))$ *with* $H \to G \to A \to 0$ *an exact sequence of* $\mathbf{Z}\pi$ *modules.*

PROOF. Assuming $A^* = 0$, let $\alpha = (\lambda'\colon H' \to G, (G, \theta, \psi))$ be any representative of γ. Then

$$H = \ker\left(G \to \operatorname{coker} \lambda' \xrightarrow{\otimes\Lambda} A \to 0\right)$$

contains H' and $(H')_\Lambda = (H)_\Lambda$. Certainly $\alpha = (\lambda\colon H' \to G, (G; \theta, \psi))$ is a form over $\mathfrak{F}$ with a unimodular factorization. By the previous lemma $\alpha \sim \alpha'$.

A factorization as in the preceding lemma will be called normalized. The following, which will not be used here, can be proven.

LEMMA 1.7. *Assume* $A^* = 0$. *If* $\alpha_i = (\lambda_i\colon H \to (G_i; \theta_i, \psi_i))$, $i = 1, 2$, *are two normalized factorizations of a form on* H, *then* $\alpha_1 \sim \alpha_2$.

Let $\sigma(f, b)$ denote the element in $\text{Б}_\eta(\mathfrak{F}; A \oplus B)$ represented by $\alpha(f, b)$.

LEMMA 1.8. *If* $(f, b)\colon (M, \partial M) \to (Y \times I, (Y \times 0 \cup Y \times 1))$ *is normally cobordant relative boundary to* $f_0\colon (N, \partial M) \to (Y \times I, Y \times 0 \cup Y \times 1)$, *a solved problem over* Λ, *then* $\sigma(f, b) = 0$.

PROOF. Let $(F, B)\colon (W, M, N) \to (Y \times I \times I, Y \times I \times 0, Y \times I \times 1)$ be a normal cobordism. We may assume F is highly connected and, after some handle subtractions, $K_k(W, M) = 0$. As in [**W**, L. 5.5], handle subtractions have the effect of adding a kernel over $\mathbf{Z}\pi$. We would like to show that the form $(K_k(M); \theta, \mu)$ is strongly equivalent to zero. The image of $\partial\colon K_{k+1}(W, M) \to K_k(M)$ satisfies (P1), and since $f|_{\partial M}$ is Λ-homology highly connected and f is highly connected, it follows easily that $K_i(W; \Lambda) = 0$ for $i \neq k$ and, hence,

$$0 \to K_{k+1}(W, M; \Lambda) \to K_k(M; \Lambda) \to K_k(W; \Lambda) \to 0$$

is a split exact sequence of stably free modules [**W**, Corollary to 2.4]. The map of cokernels of (P2) may be identified with $K_k(N, \partial M; \Lambda) \to K_{k-1}(\partial M; \Lambda)$; since f_0 is a solved problem over Λ, (P2) holds.

Let $f_i\colon (M_i; N_i, N_{i+1}) \to (Y \times I; Y \times 0, Y \times 1)$, $i = 1, 2$, be highly connected degree one normal maps, $\partial M_i = N_i \cup N_{i+1}$ and $f_1|_{N_2}, = f_2|_{N_2}$. To obtain a form from the union $f_1 \cup f_2$, we must make it highly connected. To do this, we perform low-dimensional surgeries, relative boundary, to obtain $(h, c)\colon (W; N_1, N_3) \to (Y \times I, Y \times 0, Y \times 1)$. Let

$$\alpha_i = \alpha(f_i, b_i) = (\lambda_i\colon K_k(M_i) \to G_i, (G_k; \theta_i, \psi_i))$$

and

$$\alpha(h, b) = (\lambda\colon K_k(W) \to G, (G; \theta, \psi))$$

be the forms with factorizations determined as in Theorem 1.2. By general position the preliminary surgeries may be performed so that we have a geometrically induced map $G_1 \oplus G_2 \to G$. Since f_1 and f_2 are Λ-homology k-connected and $f_1|_{N_2}$ is Λ-homology $(k-1)$-connected, these surgeries may be performed so that the above map induces an isomorphism $(G_1 \oplus G_2)_\Lambda \cong G_\Lambda$. This last statement follows from an argument similar to that to be used in the proof of Lemma 1.10 below; the details are left to the reader. By Lemma 1.5, we have $\alpha(h, c) \sim \alpha_1 \oplus \alpha_2$ and, hence, the additivity principle,

$$\sigma(h, c) = \sigma(f_1, b_1) + \sigma(f_2, b_2).$$

If $f_1|N_2\colon N_2 \to Y$ is $(k-1)$-connected, then $f_1 \cup f_2$ is k-connected and there is no need to perform the preliminary surgeries. The additivity principle in this case states that

$$(*) \qquad \sigma(f_1 \cup f_2) = \sigma(f_1) + \sigma(f_2).$$

Given any normal map

$$(f, b)\colon (M, N_0, N_1) \to (Y \times I, Y \times 0, Y \times 1)$$

of degree one, $\partial M = N_0 \cup N_1$ with $f|_{\partial M}$ satisfying the conditions of Theorem 1.2, we set $\sigma(f, b) = \sigma(f_0, b_0)$, where (f_0, b_0) is a k-connected normal map normally cobordant, relative boundary to (f, b). That this is well defined will follow from the next lemma. In particular, we will have $\sigma(h, b) = \sigma(f_1 \cup f_2, b_1 \cup b_2)$ and the general statement of the additivity principle will take the form of $(*)$.

LEMMA 1.9. *If (f_1, b_1) and (f_2, b_2) are normally cobordant relative the boundary, then $\sigma(f_1, b_1) = \sigma(f_2, b_2)$.*

PROOF. Let (F, B), $F\colon W \to X \times I \times I$ be the normal cobordism, relative boundary, from f_1 and f_2. By reorganizing the boundary $F|_{\partial W}\colon \partial W \to \partial(Y \times I \times I)$, we may view F as a cobordism from $f_1 \cup -f_2$ to the product $f_1|N_1 \times I$. By the discussion of the additivity principle $f_1 \cup -f_2$ is normally cobordant relative boundary to a highly connected map (h, c) and

$$\sigma(h, c) = \sigma(f_1, b_1) + \sigma(-f_2, b_2).$$

By Lemma 1.8, $\sigma(h, c) = 0$ and the result follows.

Let $(h, c)\colon (P, \partial P) \to (Y, \partial Y)$ be a Λ-homology highly connected map such that P is a manifold of dimension $2k - 1$ and h is a solved problem over Λ.

LEMMA 1.10. *(h, c) is normally cobordant relative boundary to a highly connected map, $(h_0, c_0)\colon (P_0, \partial P) \to (Y, \partial Y)$, which is a solved problem over Λ.*

PROOF. It suffices to show that if (h, c) is i-connected, $i < k - 1$, then it is normally cobordant relative boundary to an $(i+1)$-connected map (h_0, c_0) which is a solved problem over Λ. We surger spherical classes generating $K_i(P)$ to obtain a normal map $(h', c')\colon (P', \partial P) \to (Y, \partial Y)$ which is $(i+1)$-connected. Since P is Λ-homology highly connected, the image of $K_{i+1}(P'; \Lambda) \to K_{i+1}(P', \partial P; \Lambda)$ is a free module with generators represented by $(i+1)$ spheres in P'. The result of attaching

handles to kill these classes is an $(i+1)$-connected map $(h_0, c_0)\colon (P_0, \partial P) \to (Y, \partial Y)$, which is a solved problem.

Assume that $K_i(\partial P; \Lambda) = 0$, $i \leqslant k-3$, and that there exist tori $S^{k-2} \times S^1$ imbedded in ∂P whose images in P generate $K_{k-1}(P; \Lambda)$.

THEOREM 1.11. *Let $\gamma \in \text{Б}_{2k}(\mathfrak{F}; A \oplus B)$. Then there exists a normal cobordism (f, b) of (h, c), relative boundary, to a solved problem over Λ with $\sigma(f, b) = \gamma$.*

PROOF. By Lemma 1.10, there is a normal cobordism

$$(F, B)\colon (Z, \partial Z) \to (Y \times I, \partial(Y \times I))$$

from (h, c) to a highly connected solved problem $(\bar{h}, \bar{c})$. Since

$$K_i(\partial P; \Lambda) = 0, \quad i \leqslant k-3, \qquad K_i(\partial Z; \Lambda) = 0, \quad i \leqslant k-2,$$

and $\delta = \sigma(F, B) \in \text{Б}_{2k}(\mathfrak{F}; A \oplus B)$ is defined. By the additivity principle, it would be enough to realize $\gamma - \delta$. Hence, we assume that (h, c) is highly connected.

A cobordism, $(f, b)\colon W_0 \to Y \times I$, relative boundary, from (h, c) to $(h_0, c_0)\colon (P_0, \partial P) \to (Y, \partial Y)$ will be obtained in three stages, that is, $W_0 = U \cup W \cup V$. The first stage, $F|_U\colon U \to Y \times [0, 1]$, is an elementary cobordism from P to M obtained by attaching $(k-1)$ handles along disjoint, unlinked $(k-2)$ spheres in $(\partial P \times I)$; these spheres are chosen by pushing off a set of toral generators, to be specified, in the normal direction. Each of the toral generators may be imbedded in a separate copy of $\partial P \times I$, and then we perform surgery on the disjoint $(k-2)$ spheres $S^{k-2} \times \text{pt} \subset S^{k-2} \times S^1$ of each torus. Now we have arranged that these toral classes of P are represented by spherical classes in M' which are given by disjoint and unlinked spheres. (Compare Theorem 1.2.) More precisely, the images in $K_{k-1}(U; \Lambda)$ of the toral classes in $K_{k-1}(P; \Lambda)$ are the images of spherical classes in $K_{k-1}(M')$. As in [**W**, p. 53], these spheres can be subjected to regular homotopies with prescribed intersection and self-intersection numbers. Then W is the cobordism from M to M' resulting from attaching k-disks to the $(k-1)$ spheres at the final stage of the regular homotopies. Finally, it is easy to see that the restrictions to $W' = W \cup U_0$ and M' are $(k-1)$-connected and $K_{k-1}(M'; \Lambda) \to K_{k-1}(\overline{W}; \Lambda)$ is an epimorphism. Having arranged that

$$\begin{aligned} K_k(W, U; \Lambda) &\cong K_k(W, M; \Lambda) \to K_{k-1}(M; \Lambda) \\ &\cong (K_{k-1}(P; \Lambda) \oplus K_{k-1}(U, P; \Lambda)) \end{aligned}$$

is onto $K_{k-1}(P; \Lambda)$, we may identify

$$K_{k-1}(\overline{W}; \Lambda) \cong K_{k-1}(\overline{W}, P; \Lambda) \cong K_{k-1}(U, P; \Lambda).$$

We may choose a set of $(k-1)$ spheres in M to represent generators of the free module $K_{k-1}(W; \Lambda)$. Perform surgery to kill these spheres; the support of these surgeries is the final stage of our construction. The result of these surgeries will be denoted by $(h_0, c_0)\colon (P_0, \partial P) \to (Y, \partial Y)$.

Let γ be represented by $(\lambda\colon H \to G, (G; \theta, \psi))$ with G a free module of rank n. We choose n-generators in the first stage of the construction to be images of basis elements of G under the composite epimorphism $G \to G_\Lambda \to A = K_{k-1}(P; \Lambda)$ (see

(Q2) above), and subject the $(k-1)$ spheres of the second stage to regular homotopies with intersection numbers and self-intersection numbers given by θ and ψ, respectively. Then $K_k(W, M)$ and G are isomorphic as forms.

To obtain the normal map (h_0, c_0) only surgery on $(k-2)$ and $(k-1)$ spheres was performed; consequently, (h_0, c_0) is $(k-1)$-connected. We will show that (h_0, c_0) is a solved problem over Λ; that is, the image of ρ_i: $(K_i(P_0; \Lambda) \to K_i(P_0, \partial P; \Lambda))$ is trivial for $i = k-1, k$. The case $i = k$ follows immediately from the assumption $A^* = 0$. Over the coefficients Λ we may identify

$$K_k(W_0, P; \Lambda) \cong K_k(W_0, U; \Lambda) \cong K_k(W, M; \Lambda)$$

We have the following exact diagram over Λ:

$$\begin{array}{ccccccc}
K_k(W_0) & \overset{\delta}{\to} & K_k(W_0, P_0) & \to & K_{k-1}(P_0) & \to & 0 \\
\lambda\downarrow & A\theta_\Lambda\nearrow & \downarrow\lambda^* & & \downarrow\rho_{k-1} & & \\
K_k(W_0, P) & \overset{\delta^*}{\to} & K_k(W_0, \partial W_0) & \to & K_{k-1}(P_0, \partial P) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
K_{k-1}(P) & \to & K_{k-1}(P, \partial P) & \cong & K_{k-1}(\partial P) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

$\lambda^* \circ A\theta_\Lambda = \delta^*$ follows as in Theorem 1.2. As $A\theta_\Lambda$ is unimodular, the result, $\rho_{k-1} = 0$, follows from an elementary diagram chase.

Since $K_i(\partial P; \Lambda) = 0$ for $i < k-2$, $f|_{\partial W_0}$: $\partial W_0 = P \cup P_0 \to (Y \times I)$ is a Λ-homology highly connected. Thus, $\sigma(f, b)$ is defined. To analyze (f, b) we must first make it highly connected. In the third stage we performed surgeries on $(k-1)$ spheres to kill the free module $K_{k-1}(U, P; \Lambda) = K_{k-1}(U, P; \mathbf{Z}\pi) \otimes \Lambda$. Since these same spheres represent generators over $\mathbf{Z}\pi$, we have actually arranged that

$$K_k(W_0, U) \cong K_k(V \cup W, M) \to K_{k-1}(U, P)$$

is onto, that is, $K_{k-1}(W_0, P; \mathbf{Z}\pi) = 0$. Thus by the same argument as in [**CS**, 1.8], we may take

$$(f_1, b_1) = (g, d) \cup (f, b)\colon W_1 = Z \cup W_0 \to Y \times I$$

where (g, d) is a highly connected map normally cobordant, relative boundary, to $(h, c) \times \mathrm{id}$: $P \times I \to Y \times I$. We note (f_1, b_1) is a cobordism satisfying the conditions of Theorem 1.2 and, as such, $\sigma(f_1, b_1)$ is essentially determined by the unimodular factorization. If $\sigma(g, d)$ is represented by $\alpha(g, b) = (\lambda_g\colon H \to G', (G', \theta_g, \psi_g))$, where G' is a geometrically realized factorization as in Theorem 1.2, then $(G' \oplus K_k(W, M), \theta_g \perp \theta, \psi_g \perp \psi)$ is a unimodular factorization for a representative of $\sigma(f_1, b_1)$. In fact, since $\alpha(g, d) \oplus \gamma$ has the same factorization, it follows from Lemma 1.5 that

$$(**) \qquad \alpha(f_1, b_1) \sim \alpha(g, d) \oplus (\lambda\colon H \to G, (G; \theta, \psi)),$$

that is, $\sigma(f_1, b_1) = \gamma$.

We note that for any highly connected normal cobordism (g, d) from a solved problem over Λ to (h, c), $(f_1, b_1) = (g, d) \cup (f, b)$ is highly connected, and from the argument in the above theorem we may conclude that $(**)$ holds, that is, $\sigma(f_1, b_1) = \sigma(g, d) + \gamma$.

Let (f, b): $(N, \partial M) \to (Y \times I, \partial(Y \times I))$ be a degree one normal map, $\partial M = N_0 \cup N_1$, satisfying the conditions of Theorem 1.2. We have defined the invariant $\sigma(f, b)$. To establish that this invariant exactly measures the obstruction to obtaining a solved problem over Λ by performing surgeries relative boundary, it remains only to establish the following.

LEMMA 1.12. *If $\sigma(f, b) = 0 \in \text{Б}_{2k}(\mathfrak{F}; A \oplus B)$, then (f, b) is normally cobordant, relative boundary to a solved problem over Λ. (Recall that by Remark 1, a solved problem is a map of h-cobordisms over Λ.)*

PROOF. By surgeries we may assume that f is k-connected, so $\alpha(f, b)$ is defined. By Lemma 1.4, there exist forms $\delta_i \sim 0$ over $\mathrm{id}_{\mathbf{Z}\pi}$, $i = 1, k$, with $\alpha(f, b) \oplus \Sigma\delta_i \approx 0$. Assume $k = 1$, that is, $\alpha(f, b) \oplus \delta \approx 0$, $\delta \approx 0$, over $\mathrm{id}_{\mathbf{Z}\pi}$. The general case will follow similarly.

Let $g_\delta \cup g_\varepsilon$: $W_0 \cup Z \to Y \times [1, 2]$ be the map constructed in realizing $[\delta] \in \text{Б}_{2k}(\mathfrak{F}; A \oplus B)$ as the obstruction of a cobordism form $f|_{N_1}$: $N_1 \to Y \times 1$. Thus, g is the normal map (f, b) of Theorem 1.11 and g_ε is (g, d). By the observation following Theorem 1.11, $f \cup g_\delta$ is k-connected and $\alpha(f \cup g_\delta) = \alpha(f, b) \oplus \delta$. By Proposition 1.1, $f \cup g_\delta$ is normally cobordant, relative boundary, to a solved problem. Also, $g_\varepsilon \cup -g_\varepsilon$: $Z \cup -Z \to Y \times [2, 4]$ is normally cobordant to the product $g_\varepsilon|_{Z \cap W_0} \times \mathrm{id}_{[2,4]}$. Let $\alpha(g_\varepsilon, b)$ be denoted by ε.

Finally, we claim that $g_\delta \cup g_\varepsilon \cup -g_\varepsilon$ is normally cobordant to a solved problem. Recalling the notation used in defining the addition of forms, we see without difficulty that $K = K_\delta \oplus \Delta(G_\varepsilon \oplus G_\varepsilon)$ is a submodule of $H_{\delta \oplus \varepsilon \oplus -\varepsilon}$ satisfying (Q1) and (Q2), where $K_\delta \subset H_\delta$ satisfies (Q1) and (Q2). Hence,

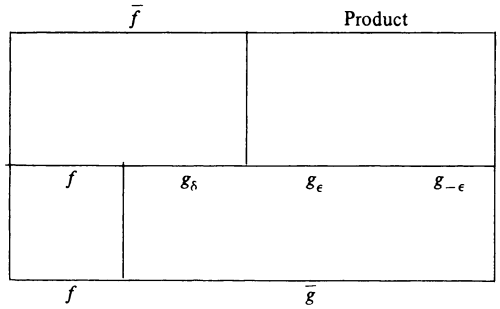
$$\delta \oplus \varepsilon \oplus -\varepsilon = \alpha(g_\delta \cup g_\varepsilon \cup g_{-\varepsilon}) \approx 0,$$

and

$$g_\delta \cup g_\varepsilon \cup g_{-\varepsilon}: W_0 \cup Z \cup -Z \to Y \times [1, 4]$$

is normally cobordant to a solved problem.

We have shown that $f \cup g_\delta \cup g_\varepsilon \cup g_{-\varepsilon}$ is normally cobordant to a solved problem $\bar{f}$ and to $f \cup \bar{g}$ where $\bar{g}$ is a solved problem. By Remark 1, $\bar{f}$ and $\bar{g}$ will be maps of h-cobordisms over Λ. Schematically, we have the following:



Thus, by simple rearranging, we see that f is normally cobordant to $\bar{f} \cup -\bar{g}$, which is a solved problem over Λ.

2. Local knots over highly connected maps. Let M^n be a closed, connected, smooth (piecewise linear, topological) manifold, ξ a 2-plane bundle over M with disk bundle $E = E(\xi)$ and boundary sphere bundle $S = S(\xi) = \partial E$. Let $\pi' = \pi_1(M) = \pi_1(E)$, $\pi = \pi_1(S)$, and $\mathfrak{F}$: $\mathbf{Z}\pi \to \mathbf{Z}\pi' = \Lambda$ the map induced by $S \subset E$. The involution on $\mathbf{Z}\pi$ is given by $\bar{g} = \omega(g)g^{-1}$, $g \in \pi$, where ω: $\pi \to \{\pm 1\}$ is the orientation character of S.

Let f: $N^n \to M^n$ be a highly connected map of closed, connected smooth (resp., P.L., topological) manifolds. By a smooth (resp., P.L., topological) local knot in ξ over f is meant a smooth (resp., P.L. locally flat, topologically locally flat) embedding ι: $N \to E - S$ homotopic to f, that is, up to homotopy

$$\begin{array}{ccc} & & E \\ & \iota \nearrow & \downarrow p \\ N & \underset{f}{\to} & M \end{array}$$

commutes. Two local knots ι and ι_1 are said to be cobordant if there exists an h-cobordism V of N to itself and a smooth (resp., P.L. locally flat, topologically locally flat) embedding of (V, N, N) into an h-cobordism (W, E, E) restricting to ι and ι_1.

Cobordism defines an equivalence relation; the cobordism classes of smooth (resp., P.L., topological) local knots in ξ over f will be denoted by $C_0(f, \xi)$ (resp., $C_{\mathrm{PL}}(f, \xi)$, $C_{\mathrm{TOP}}(f, \xi)$). It can be shown that $C_H(\mathrm{id}_M, \xi) = C_H(M, \xi)$ (as in **[CS, §4]**) if $Wh(\pi_1 M) = \{e\}$. In what follows, the assertions about "local knots" are meant to cover all three categories; the proofs of such assertions will usually apply as written in the smooth and PL categories and with modifications using **[LR, KS]** in the topological category.

LEMMA 2.1. *Let ι: $N \to E(\xi)$ be a local knot. Then the inclusion $S \subset E(\xi) - \iota(N)$ is Λ-homology highly connected, $\Lambda = \mathbf{Z}\pi'$. That is, $H_i(E - \iota(N), S; \Lambda) = 0$ for $i \leqslant [n/2]$.*

PROOF. By Poincaré duality over Λ and the fact that ι is Λ-homology highly connected.

LEMMA 2.2. *Let ι: $N^n \to E$ be a local knot with tubular neighborhood $T \subset E - S$. Let $W = \mathrm{cl}(E - T)$. If $n \geqslant 4$ then there exists a map g: $W \to S$, with $g|S$ the identity and $g|\partial T$: $\partial T \to S$ a bundle map over f so that pg is homotopic to $p|W$. Further, g is unique up to a homotopy, relative S, that is an isotopy of bundle maps on ∂T. In particular, ι has normal bundle equivalent to $f^*(\xi)$.*

PROOF. Consider the lifting problem:

$$\begin{array}{ccc} S & \overset{\mathrm{id}}{\to} & S \\ \cap & \nearrow & \downarrow p \\ W & \overset{p|W}{\to} & M \end{array}$$

The obstructions to lifting $p \mid W$ to S rel id_S, all lie in cohomology groups of (W, S) with local coefficients in $\pi_i(S^1)$ viewed as a module over $\mathbf{Z}\pi'$ by the action determined by the circle fibration $S(\xi)$. By Lemma 2.1, if $n \geqslant 4$, these groups must vanish. The lemma follows by the same argument as in [**CS**, 4.2].

Let $T, W, f\colon W \to S$ be as in Lemma 2.2 and $\gamma\colon (W; \partial T, S) \to (I; 0, 1)$ a Morse function. The map

$$G = (g, \gamma)\colon (W, \partial T, S) \to (S \times I, S \times 0, S \times 1)$$

is called the *complementary map of* ι. Let $f\colon T \to E$ be the canonical extension of $g \mid \partial T\colon \partial T \to S$ by a bundle map. The map

$$h = (f \cup G)\colon (T \cup W, S) \to (E \cup S \times I, S \times 1) \cong (E, S),$$

is called the *characteristic map of* ι.

LEMMA 2.3. *Let $\iota\colon N \to E(\xi)$ be a smooth (resp., P.L., topological) local knot. Then $h\colon (T \cup W, S) \to (E, S)$, the characteristic map of ι, is homotopic as a map of pairs to a diffeomorphism (resp., PL-homeomorphism, homeomorphism).*

PROOF. Since $\iota \sim f$, it follows that $ph \mid \iota(N) \sim p \mid \iota(N)$. Since f has degree one, it induces monomorphisms in cohomology with local coefficients by Poincaré duality. Hence, the same holds for ι. By obstruction theory, we see that $ph \sim p$ since the compositions with ι are homotopic. The same argument as in [**CS**, 4.6] then shows that h is homotopic to a bundle map.

As a consequence of the above lemma, we observe that the complementary map of ι is a degree one normal map, and the induced map on homology with local coefficients enables us to identify $H_i(W, S; \Lambda) \cong K_i(W, S; \Lambda) \cong K_i(W; \Lambda)$. Hence, by Lemma 2.1, the complementary map is Λ-homology highly connected.

LEMMA 2.4. *Let $h = (\bar{f} \cup G)\colon (T \cup W, S) \to (E, S)$ be the characteristic map for the local knot $\iota\colon N \to E$. Then for all j, the sequences*

(a) $$0 \to K_{j+1}(W, \partial T; \Lambda) \to K_j(\partial T; \Lambda) \to K_j(W; \Lambda) \to 0,$$

(b) $$0 \to K_{j+1}(T, \partial T; \Lambda) \to K_j(\partial T; \Lambda) \to K_j(T; \Lambda) \to 0$$

are exact and

$$K_j(\partial T; \Lambda) = K_j(T; \Lambda) \oplus K_j(W; \Lambda).$$

PROOF. This follows immediately from 2.3 and the Mayer-Vietoris sequence.

Assume $n = 2k - 1$. Given two local knots ι_0 and ι_1 into ξ over $f\colon N \to M$, we wish to determine whether they represent the same class in $C_H(f, \xi)$. Let $h_j\colon (T \cup W_j, S) \to (E \cup S \times I, S \times I)$ be the characteristic map for ι_j, $j = 0, 1$. By Lemma 2.3 each is homotopic to a diffeomorphism; by gluing, using these diffeomorphisms, we obtain a normal cobordism H from h_0 to h_1 with underlying manifold an s-cobordism. After making H transverse to $M \times I \to E \times I$, we have a normal cobordism $H \mid V\colon V = H^{-1}(M \times I) \to M \times I$ from $H \mid \iota_0(N) = f$ to $H \mid \iota_1(N) = f$; since f is normally cobordant to id_M, this cobordism determines an element $\sigma^\epsilon(\iota_0, \iota_1)$ in the image of

$$[\Sigma M, G/H] \underset{s_H}{\to} L^\epsilon_{n+1}(\pi_1 M) \underset{p^!}{\to} L^\epsilon_{n+2}(\pi_1 S),$$

$p^!$ induced by the projection p: $S(\xi) \to M$, $\varepsilon = s$ or h. That f is normally cobordant to id_M is a consequence of Lemma 2.3. Using this fact, a more illuminating description of $\sigma^\varepsilon(\iota_0, \iota_1)$ is obtained by identifying $\iota_0(N)$ and $\iota_1(N)$ to obtain a normal map $\tilde{H}$: $\tilde{V} \to M \times S^1$. Since $\sigma^h(f) \in L_n^h(\pi')$ is trivial,

$$\sigma^\varepsilon(\tilde{H}) \in L_{n+1}^\varepsilon(\pi') \subseteq L_{n+1}^\varepsilon(\pi' \times \mathbf{Z})$$

by [**Sh**, 5.1]. Clearly, $\sigma^\varepsilon(\iota_0, \iota_1) = p^!(\sigma^\varepsilon(H))$. A simple argument using the fact that s_H is a homomorphism allows us to conclude the additivity

$$\sigma^\varepsilon(\iota_0, \iota_1) + \sigma^\varepsilon(\iota_1, \iota_2) = \sigma^\varepsilon(\iota_0, \iota_2).$$

To establish that σ^ε is a well-defined invariant of the pair of local knots, it now suffices to show the following.

LEMMA 2.5. *If ι_0 and ι_1 are cobordant, then $\sigma^h(\iota_0, \iota_1) = 0$. For the trivial bundle, $\sigma^s(\iota_0, \iota_1) = 0$.*

PROOF. Let f': $(Y, N, N) \to (E \times I, E \times 0, E \times 1)$ be the cobordism from ι_0 to ι_1. The relative version of Lemma 2.2 provides a characteristic map of this cobordism which we will denote by G: $(T(Y) \cup Z, S \times I) \to (E \times I, S \times I)$ and which has the property $G \mid G^{-1}(E \times j) = h_j$, the characteristic map of ι_j, $j = 0, 1$. Let H: $(T(V) \cup W, S \times I) \to (E \times I, S \times I)$ be the map of h-cobordisms in the description of $\sigma^\varepsilon(\iota_0, \iota_1)$, $T(V)$ the tubular neighborhood of V with associated S^1-bundle $S(V)$. Taking the union $\overline{H} = H|_W \cup G|_Z$ along the complementary maps of the local knots we obtain a normal cobordism

$$\begin{aligned}\overline{H}\colon &\big(W \cup Z; S(V) \cup S(Y), S(\xi) \times S^1\big)\\ &\to \big(S(\xi) \times I \times S^1; S(\xi) \times 0 \times S^1, S(\xi) \times 1 \times S^1\big),\end{aligned}$$

with $H \mid S(\xi) \times S^1$ the identity. By the same argument used to establish the additivity of σ^ε, it follows that $\sigma^h(\iota_0, \iota_1) = \sigma^h(\overline{H} \mid S(V) \cup S(Y))$. The normal cobordism $\overline{H}$ shows that the latter vanishes.

In the case of the trivial bundle, $\overline{H} \mid S(Y) = f' \times \mathrm{id}_{S^1}$. Taking the product with S^1 always kills the torsion [**KSz**]; hence, $\overline{H} \mid S(Y)$ is an s-cobordism and the second statement of the lemma then follows as above.

Assume ξ is the trivial bundle and write $\sigma = \sigma^s$. Actually in this case, we may view $\sigma(\iota_0, \iota_1)$ as an element of $L_{n+1}^h(\pi_1 M)$ as $p^!$ is a monomorphism.

LEMMA 2.6. *If $\sigma(\iota_0, \iota_1) = 0$ then $H \mid V$ is normally cobordant relative boundary to a map of h-cobordisms.*

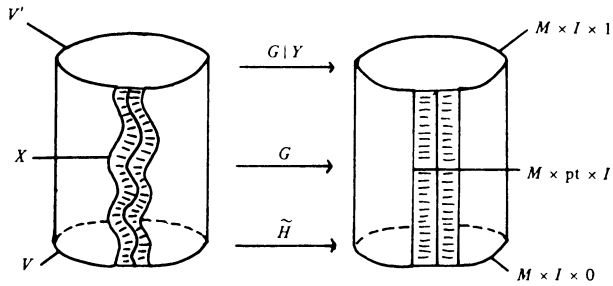
PROOF. Let G: $(Z, \tilde{V}, Y) \to (M \times S^1 \times I, M \times S^1 \times 0, M \times S^1 \times 1)$ be a normal cobordism from $\tilde{H}$: $\tilde{V} \to M \times S^1 \times 0$ obtained by applying the cobordism extension theorem to a cobordism from $f = H \mid \iota_0(N) = H \mid \iota_1(N)$ to a diffeomorphism. We may view the map on Y as having the decomposition

$$G \mid Y\colon \big(Y = (M \times I) \cup V'\big) \to \big((M \times I) \cup (M \times I)\big)$$

and

$$p^!\big(\sigma^h(G \mid V')\big) = \sigma(\iota_0, \iota_1) \in L_{n+2}\big(\pi_1(S(\xi))\big).$$

Since ξ is the trivial bundle, $p^!$ is a monomorphism [**Sh**] and we may assume that V' is an h-cobordism. Removing the interior of a neighborhood of the cobordism $G|X$, $X = G^{-1}(M \times \mathrm{pt} \times I)$ from f to the diffeomorphism, we obtain a normal cobordism from $G|V$ to $G|X \cup V' \cup -X$. Since f is highly connected, $G|X$ determines a $(-1)^k$-Hermitian form over $\mathrm{id}_{\mathbf{Z}\pi'}$ satisfying property (Q1). As we are taking the union of $G|X$ and $G|-X$ along an h-cobordism α, the $(-1)^k$-Hermitian form over $\mathrm{id}_{\mathbf{Z}\pi'}$ determined by $G|X \cup V' \cup X$ is just the orthogonal sum of the form determined by $G|X$ and its negative. The diagonal submodule clearly satisfies (P1) and (P2) so we may conclude $\alpha \approx 0$. It now follows from Proposition 1.1 that $G|X \cup V' \cup -X$ is normally cobordant relative boundary to an h-cobordism.



Let ι_x be a local knot over $f\colon N \to M$ with $[\iota_x] = x \in C_H(f, \xi)$. Let

$$A = K_{k-1}(N; \Lambda) \cong K_{k-1}(\partial T; \Lambda).$$

By Poincaré duality and Lemma 2.3, $K_k(W_x; \Lambda) \cong A$ where $G_x\colon (W_x, \partial T, S) \to (S \times I, S \times 0, S \times 1)$ is the complementary map of ι_x. Let $(\bar{H}, \bar{B})$ be a normal cobordism relative boundary between the complementary maps of ι_x and ι_y with underlying manifold W. An identification

$$K_k(\partial W; \Lambda) = A \oplus A \xrightarrow{\pi_1} A = K_k(W_x; \Lambda)$$

follows from the Mayer-Vietoris sequence, as in Remark 1 of §1.

An element $\gamma \in \mathbf{Б}_{n+3}(\mathcal{F}; A \oplus A)$ is said to act on the local knot if there exists a normal cobordism H from the characteristic map of ι_x to that of ι_y, and the induced cobordism between the complementary maps has surgery obstruction $\gamma \in \mathbf{Б}_{n+3}(\mathcal{F}; A \oplus A)$. We denote this action by writing $\iota_y = \gamma \cdot \iota_x$. Given a knot ι_x and $\gamma \in \mathbf{Б}_{n+3}(\mathcal{F}; A \oplus A)$, a local knot $\iota_y = \gamma \cdot \iota_x$ is determined by the following construction. By Theorem 1.11, starting from the complementary map of ι_x, we may realize γ as the surgery obstruction of $(\bar{H}, \bar{B})$, a normal cobordism, relative boundary, to another solved problem over Λ, $G'\colon W' \to S \times I$. It is not difficult to see that the conclusion of Lemma 2.4 holds for G'; it follows that $\bar{f} \cup G'\colon E' = T \cup W' \to E \cup S \times I$ is a homotopy equivalence. Obviously it is normally cobordant relative boundary to the identity. Since ξ is trivial, the induced map $P^*\colon L_{n+3}(\pi_1 S) \to L_{n+3}(\pi_1 M)$ is onto; $\bar{f} \cup G'$ represents zero in $\mathcal{S}_H^h(E, S)$, that is, $\bar{f} \cup G'$ is h-cobordant as a map of pairs to id_E. Such an h-cobordism can be constructed by attaching handles entirely in W' since, by general position, the natural map $\pi_1 W' \to \pi_1 E$ is

onto [**M1**, §11]. Hence, there is a proper embedding of $N \times I$ into this h-cobordism. Take $\iota_y = \gamma \cdot \iota_x$ to be $N \times 1 \hookrightarrow E$.

THEOREM 2.7. *The group* $\text{Б}_{n+3}(\mathfrak{F}; A \oplus A)$ *acts on the set* $C_H(f, \xi)$. *Two classes* x *and* y *are in the same orbit if and only if* $\sigma(x, y) = 0$.

PROOF. To show that the action described in the preceding paragraph is well defined on the set of cobordism classes, we must show that $[\iota_y]$ is independent of our choice of representative of $\gamma \in \text{Б}_{n+3}(\mathfrak{F}; A \oplus A)$ and $[\iota_x]$.

Let ι_x be a local knot representing $x \in C_H(f, \xi)$. Let (G_0, B_0) and (G_1, B_1) be two normal cobordisms relative $h|\partial T$ from h, the complementary map of ι_x, to complementary maps h_0 and h_1, respectively, such that $\sigma(G_0, B_0) = \sigma(G_1, B_1) = \gamma \in \text{Б}_{n+3}(\mathfrak{F}; A \oplus A)$. Taking the union of these two cobordisms along h, we obtain a normal cobordism (H, C) from h_0 to h_1; by the additivity of surgery obstructions, $\sigma(H, C) = 0$. By Lemma 1.12, (H, C) is normally cobordant relative boundary to (H', C'), a map of h-cobordisms over Λ. Since the construction is achieved relative $h|\partial T$, we may extend this to a cobordism between characteristic maps, which is actually a map of h-cobordisms, by gluing in $T \times I$ along $\partial T \times I$. Hence, the local knot defined above, using (G_0, B_0), is cobordant to that obtained by using (G_1, B_1).

Let ι_0 and ι_1 be two representatives of the class $x \in C_H(f, \xi)$. Using the relative version of Lemma 2.2, there is a map H, a cobordism of the complementary maps, h_0 and h_1, which is a map of h-cobordisms. If G is a cobordism betwen h_0 and h', the complementary maps of ι_0 and $\gamma \cdot \iota_0$, respectively, we obtain a cobordism between h_1 and h' by taking the union $G \cup H$ along h_0. Since $\sigma(G \cup H) = \sigma(G) = \gamma$, by the previous paragraph $[\gamma \cdot \iota_0] = [\gamma \cdot \iota_1] \in C_H(f, \xi)$. As usual, additivity of surgery obstructions implies that $(\gamma + \gamma') \cdot (x) = \gamma \cdot (\gamma' \cdot (x))$. That $0 \cdot x = x$ follows from the argument of the preceding paragraph and the fact that the vanishing of the surgery obstruction enables us to replace the given normal cobordism with a map of h-cobordisms over Λ. The second statement of the theorem is obvious.

A local knot ι: $N \to E$ is called 1-simple if the inclusion $S \subset E - \iota(N)$ induces an isomorphism of fundamental groups.

LEMMA 2.8. *Every local knot* ι: $N \to E$ *over* f *is cobordant to a* 1-*simple local knot*.

PROOF. By Lemma 2.4 the complementary map of a local knot is a solved problem over Λ. By Lemma 1.10 and Theorem 1.11 a representative of the zero element of $\text{Б}_{n+3}(\mathfrak{F}; A \oplus A)$ may be realized as a cobordism relative boundary from the complementary map to a highly connected map which is a solved problem over Λ; by Proposition 1.1 and Remark 1 this cobordism is then normally cobordant to a map of h-cobordisms. Using this representative of $0 \in \text{Б}_{n+3}(\mathfrak{F}; A \oplus A)$, we obtain a 1-simple knot $0 \cdot \iota$ which is cobordant to ι.

Let ι: $N \to E$ be a local knot and $B^n \subset N$, $B_0^n \subset M$ be closed balls. Then ι is said to be in normal form (with respect to B) if $\iota|B = \iota_0|B_0$, ι_0: $M \to E$ the zero-section and $\iota(N) \cap E(\xi|B_0) = \iota(B)$. If a local knot is in normal form with respect to B, it can be placed in normal form with respect to any given ball by an isotopy.

LEMMA 2.9. *Every* 1-*simple local knot* $\iota\colon N \to E$ *is equivalent to a local knot in normal form provided* $n \geqslant 3$.

PROOF. The argument is essentially the same as that of [**CS**, 6.1].

Let $f\colon K^n \to S^n$ be a degree one normal map, K a highly connected manifold, i.e., $\pi_j K = 0$ for $j < [n/2]$. Assume that f admits a local knot $\iota\colon K \to S^n \times D^2$. It is easy to see that $C_H(f, \text{trivial})$ is isomorphic to the usual set of cobordism classes of embeddings $K \hookrightarrow S^{n+2}$. Hence, using connected sum we define a map

$$\alpha = \alpha(f\#\mathrm{id}, \xi)\colon C_H(f, \text{trivial}) \to C_H(f\#\mathrm{id}_M, \xi)$$

by $\alpha(x) = \iota_x\#\iota_0$ where $\iota_0\colon M \to E(\xi)$ is the zero-section.

THEOREM 2.10. *Let* M^n, $n = 2k - 1 \geqslant 5$, *be simply connected. Let* $H = \mathrm{PL}, \mathrm{TOP}$. *Then* $\alpha_H(f\#\mathrm{id}_M, \xi)$ *is onto.*

PROOF. Assume ξ is the trivial bundle. Let ι be a local knot over $f\#\mathrm{id}_M$, and ι_x a local knot over f which is in normal form with respect to $B \subset K$. Then $\iota_K\#\iota_0$ represents an element in the image of α;

$$\sigma(\iota, \iota_k\#\iota_0) = \gamma \in L_{n+2}(\mathbf{Z}) = L_{n+1}(e)$$

is defined. By [**CS**, 13.10] $\rho\colon C_N^H = C_H(S^n, \text{trivial}) \to L_{n+1}(e)$ is onto. Let $\iota_S\colon S^n \to S^{n+2}$ be a knot with $\rho[\iota_S] = -\gamma$. It is not hard to see that $\sigma(\iota_K, \iota_K\#\iota_S) = -\gamma$. Using the fact that ι_K is in normal form, and the idea of the argument used in [**CS**] to prove the commutativity of the square in 6.4, we see that

$$\sigma(\iota_K, \bar{\iota}_K) = \sigma(\iota_K\#\iota_0, \bar{\iota}_K\#\iota_0) \in L_{n+1}(e).$$

Hence, we may assume that $\sigma(\iota, \iota_K\#\iota_0) = 0$.

Now suppose $\delta \cdot \iota = \iota_K\#\iota_0$. Again by the argument used in [**CS**, 6.4], $(-\delta \cdot \iota_k)\#\iota_0 = -\delta \cdot (\iota_K\#\iota_0)$. By additivity of surgery obstructions, ι is h-cobordant to $(-\delta \cdot \kappa)\#\iota_0$. This proves that α is onto.

For the case of the general bundle ξ, we have not shown that $\sigma(\iota, \iota_0\#\iota_K)$ is a cobordism invariant. Nonetheless, this obstruction may be killed by the procedure described above, and the remainder of the proof is the same.

The group $L^h_{n+3}(\pi)$ acts on $\mathfrak{B}_{n+3}(\mathfrak{F}; A \oplus A)$ by orthogonal direct sum, that is, if $\delta = (\mathrm{id}\colon F \to F, (F; \phi, \mu))$ and $\alpha = (\lambda\colon H \to G, (F; \theta, \psi))$ represent elements in $L^h_{n+3}(\pi)$ and $\mathfrak{B}_{n+3}(\mathfrak{F}; A \oplus A)$, respectively, then we set

$$\delta \cdot \alpha = \left(\begin{pmatrix} \lambda & 0 \\ 0 & \mathrm{id}_F \end{pmatrix}\colon H \oplus F \to G \oplus F, (G \oplus F; \theta \perp \phi, \psi \perp \mu)\right).$$

It is clear that this is well defined.

Let $\tilde{\mathfrak{B}}_{n+3}(\mathfrak{F}; A \oplus A)$ denote the orbit space. We claim that there is an induced action of $\tilde{\mathfrak{B}}_{n+3}(\mathfrak{F}; A \oplus A)$ on $C_H(f, \xi)$. Each element of $L^h_{n+3}(\pi_1 S)$ may be realized as the surgery obstruction of a normal cobordism, relative boundary, from a given map of an h-cobordism to $S \times I$ to another such map. Let

$$(G, B)\colon (W, W_x, W_y) \to (S \times I \times I, S \times I \times 0, S \times I \times 1)$$

be a normal cobordism between complementary maps of ι_x and $\iota_y = \gamma \cdot \iota_x$ as in the description of the action of $\mathfrak{B}_{n+3}(\mathfrak{F}; A \oplus A)$. Let (H, C) be a normal cobordism

from $G|G^{-1}(S \times 0 \times I)$ to another h-cobordism with $\sigma(H, C) = \delta \in L_{n+3}(\pi)$. Taking the union along $G|G^{-1}(S \times 0 \times I)$, we obtain another cobordism between the complementary maps of ι_x and ι_y with $\sigma(G \cup H) = \delta \cdot \gamma$.

From the above discussion and Theorem 2.7 we obtain our final result.

THEOREM 2.11. *If* $\dim M = 2k - 1 = n \geqslant 4$, *the following sequence is exact*:

$$\tilde{\mathbf{B}}_{n+3}(\mathcal{F}; A \oplus A) \to C_H(f, \xi) \xrightarrow{\sigma^h} L^h_{n+1}(\pi').$$

The image of σ^h is the image of the surgery obstruction map s_H: $[\Sigma M; G/H] \to L^h_{n+1}(\pi')$.

As usual, the "exact sequence" means that the left acts on $C_H(f, \xi)$ and σ^h faithfully represents the orbit space. It is clear from the definition that Image σ^h is contained in Image s_H, and the opposite inclusion follows by arguments similar to those of [**CS**, §7]. In the simply-connected case, s_H is surjective.

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DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139

Current address: Department of Mathematics, Lehigh University, Bethlehem, Pennsylvania 18015