ANALYTICITY ON ROTATION IN Variant
FAMILIES OF CURVES

BY

JOSIP GLOBEVIČNIK

Dedicated to Professor Ivan Vidav on the occasion
of his sixty-fifth birthday, January 17, 1983

ABSTRACT. Let \( \mathcal{G} \) be a rotation invariant family of smooth Jordan curves contained
in \( \Delta \), the open unit disc in \( \mathbb{C} \). For each \( \Gamma \in \mathcal{G} \) let \( D_\Gamma \) be the simply connected
domain bounded by \( \Gamma \). We present various conditions which imply that if \( f \) is a
continuous function on \( \Delta \) such that for every \( \Gamma \in \mathcal{G} \) the function \( f|_\Gamma \) has a
continuous extension to \( D_\Gamma \) which is analytic in \( D_\Gamma \), then \( f \) is analytic in \( \Delta \).

1. Introduction. Denote by \( \Delta \) the open unit disc in \( \mathbb{C} \) and by \( G \) the group of
conformal automorphisms of \( \Delta \). Let \( \mathcal{G} \) be a family of smooth (i.e. continuously
differentiable) Jordan curves contained in \( \Delta \) which is Moebius invariant, i.e. \( \omega(\Gamma) \in \mathcal{G} \)
whenever \( \Gamma \in \mathcal{G} \) and \( \omega \in G \). For each \( \Gamma \in \mathcal{G} \) denote by \( D_\Gamma \) the simply connected
domain bounded by \( \Gamma \) and write \( \Gamma^* = \{ \tilde{z} : z \in \Gamma \} \). Agranovski and Valski [2] (see
also [1]) proved that

\[
\text{(1)} \quad \begin{cases} 
\text{if } f \text{ is a continuous function on } \Delta \text{ such that for every } \Gamma \in \mathcal{G} \text{ the} \\
\text{function } f|_\Gamma \text{ has a continuous extension to } D_\Gamma, \text{ which is analytic} \\
in D_\Gamma, \text{ then } f \text{ is analytic in } \Delta.
\end{cases}
\]

Agranovski [1] sharpened this result by proving that if \( f \) is continuous on \( \Delta \) and
satisfies \( \int\int_{\Gamma} f(z) \, dz = 0 \) for every \( \Gamma \in \mathcal{G} \) then \( f \) is analytic in \( \Delta \). This suggests that one
should be able to prove (1) for families \( \mathcal{G} \) much smaller than the Moebius invariant
ones.

A Moebius invariant family \( \mathcal{G} \) is always rotation invariant (i.e. \( s \Gamma \in \mathcal{G} \) whenever
\( \Gamma \in \mathcal{G} \) and \( s \in \mathbb{C}, |s| = 1 \)) and sometimes (e.g. if it consists of circles) it is also
symmetric (i.e. \( \Gamma^* \in \mathcal{G} \) whenever \( \Gamma \in \mathcal{G} \)).

In the present paper we study the rotation invariant, symmetric families \( \mathcal{G} \) which
satisfy (1) and the rotation invariant families \( \mathcal{G} \) which satisfy (1) for smooth
functions \( f \). We present various examples of such minimal families.

If \( f \) is continuous then \( f|_\Gamma \) has a continuous extension to \( D_\Gamma \) which is analytic in
\( D_\Gamma \) if and only if

\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{f(\xi)}{\xi - z} \, d\xi = 0 \quad (z \in \mathbb{C} - D_\Gamma)
\]

Received by the editors September 7, 1982.
1980 Mathematics Subject Classification. Primary 30E05, 30E20.

©1983 American Mathematical Society
0002-9947/83 $1.00 + $.25 per page
We will often use the fact that if \( f \) is continuous on \( \Delta \) and holomorphic in \( \Delta - \{0\} \) then \( f \) is holomorphic in \( \Delta \).

If \( f \) is a continuous function on a circle \( |z| = r, r > 0 \), and if \( n \in \mathbb{Z} \) then define

\[
A_n(f, r) = r^{-n} \frac{1}{2\pi} \int_0^{2\pi} e^{-in\varphi} f(re^{i\varphi}) \, d\varphi.
\]

Note that if \( f \) is analytic in a neighbourhood of \( |z| = r \) then \( A_n(f, r) \) is the \( n \)th coefficient in the Laurent series of \( f \).

**Lemma 1.** Let \( 0 < r_1 < r_2 \) and let \( f \) be a continuous function on \( \Omega = \{ z : r_1 \leq |z| \leq r_2 \} \). Suppose that for each \( n \in \mathbb{Z} \), the function \( r \mapsto A_n(r) \) is constant on \( \{ r : r_1 \leq r \leq r_2 \} \). Then \( f \) is analytic in the interior of \( \Omega \).

**Proof.** By the assumption there are numbers \( a_n, n \in \mathbb{Z} \), such that for each \( r, r_1 \leq r \leq r_2 \), \( \sum a_n r^n e^{in\theta} \) is the Fourier series of the function \( \theta \mapsto f(re^{i\theta}) \). For \( m \in \mathbb{N} \) let

\[
\sigma_m(f, r, e^{i\theta}) = \frac{1}{m} \left[ a_0 + \sum_{k=1}^{m-1} a_k r^k e^{ik\theta} + \cdots + \sum_{k=-(m-1)}^{m-1} a_k r^k e^{ik\theta} \right]
\]

be its \( m \)th Cezàro mean. By the uniform continuity of \( f \) the family \( \{ \theta \mapsto f(re^{i\theta}); r_1 \leq r \leq r_2 \} \) is uniformly equicontinuous. The usual proof of Fejér’s theorem \([5]\) applied to the series \( \sum_{n=-\infty}^{\infty} a_n r^ne^{in\theta} \) shows that \( \sigma_m(f, r, e^{i\theta}) \) converges to \( f(re^{i\theta}) \) uniformly for \( r \) and \( \theta, r_1 \leq r \leq r_2, 0 \leq \theta \leq 2\pi \). Consequently, on \( \Omega \), \( f(z) \) is the uniform limit of the sequence

\[
\frac{1}{m} \left[ a_0 + \sum_{k=1}^{m-1} a_k z^k + \cdots + \sum_{k=-(m-1)}^{m-1} a_k z^k \right]
\]

so \( f \) is analytic in the interior of \( \Omega \). This completes the proof.

**2. Analyticity on a family obtained by rotating a single curve.**

**Lemma 2.** Let \( \Gamma \) be a smooth Jordan curve in \( \mathbb{C} \) and let \( D \) be the simply connected domain bounded by \( \Gamma \). Suppose that \( f \) is a continuous function on \( \Omega = \{ sz : z \in \Gamma, |s| = 1 \} \) such that for every \( s, |s| = 1 \), the function \( f|(s\Gamma) \) has a continuous extension to \( sD \) which is analytic in \( sD \). Then for every \( n \in \mathbb{Z} \) the function \( z \mapsto z^n A_n(|z|) \) has a continuous extension from \( \Gamma - \{0\} \) to \( D \) which is analytic in \( D \).

**Proof.** Fix \( n \in \mathbb{Z} \). Let \( z \in \mathbb{C} - \bar{D} \). By the continuity of \( f \) the function \( (\xi, \varphi) \mapsto e^{-in\varphi} f(e^{i\varphi}\xi)/(\xi - z) \) is continuous on \( \Gamma \times [0, 2\pi] \). By our assumption,

\[
\frac{1}{2\pi i} \int_\Gamma \frac{f(e^{i\varphi}\xi)}{\xi - z} \, d\xi = 0 \quad (0 \leq \varphi \leq 2\pi),
\]

so by Fubini,

\[
\frac{1}{2\pi i} \int_\Gamma \left[ \int_0^{2\pi} e^{-in\varphi} f(e^{i\varphi}\xi) \, d\varphi \right] \frac{d\xi}{\xi - z} = \int_0^{2\pi} e^{-in\varphi} \left[ \frac{1}{2\pi i} \int_\Gamma \frac{f(e^{i\varphi}\xi)}{\xi - z} \, d\xi \right] \, d\varphi = 0.
\]
This proves that
\[ \xi \mapsto \frac{1}{2\pi} \int_0^{2\pi} e^{in\varphi} f(e^{i\varphi}\xi) \, d\varphi \]
has a continuous extension from \( \Gamma \) to \( \overline{D} \) which is analytic in \( D \). If \( \xi \neq 0 \) then
\[ \frac{1}{2\pi i} \int_0^{2\pi} e^{-in\varphi} f(e^{i\varphi}\xi) \, d\varphi = e^{-in\arg \xi} \frac{1}{2\pi} \int_0^{2\pi} e^{-in\varphi} f(e^{i\varphi}|\xi|) \, d\varphi = \xi^n A_n(|\xi|). \]
This completes the proof.

For any domain \( D \subset \mathbb{C} \) we write \( D^* = \{ \bar{z} : z \in D \} \).

**Theorem 1.** Let \( \Gamma \) be a smooth Jordan curve in \( \mathbb{C} \). Denote by \( D \) the simply connected domain bounded by \( \Gamma \) and assume that \( 0 \in \overline{D} \). If \( f \) is a continuous function on \( \Omega = \{ sz : z \in \Gamma, |s| = 1 \} \) such that for every \( s, |s| = 1 \):

(i) the function \( f|\{ s\Gamma \} \) has a continuous extension to \( sD \) which is analytic in \( sD \) and

(ii) the function \( f|\{ s\Gamma^* \} \) has a continuous extension to \( sD^* \) which is analytic in \( sD^* \),

then \( f \) is analytic in the interior of \( \Omega \). If \( f \) is smooth (i.e. of class \( C^1 \)) on \( \Omega \) then (i) alone implies that \( f \) is analytic in the interior of \( \Omega \).

**Proof.** Let \( f \) be continuous on \( \Omega \) and suppose that (i) holds. Since \( 0 \notin \overline{D} \) it follows by Lemma 2 that for each \( n \in \mathbb{Z} \) the function \( z \mapsto A_n(|z|) \) has a continuous extension \( F_n \) from \( \Gamma \) to \( \overline{D} \) which is analytic in \( D \).

If (ii) also holds, then by Lemma 2 for each \( n \in \mathbb{Z} \), the function \( z \mapsto A_n(|z|) \) has a continuous extension \( G_n \) from \( \Gamma^* \) to \( \overline{D}^* \) which is analytic in \( D \). Fix \( n \in \mathbb{Z} \). Note that the function \( z \mapsto G_n(z) \) is continuous on \( \overline{D} \) and antianalytic in \( D \). For every \( z \in \Gamma \) we have \( F_n(z) = A_n(|z|) = G_n(z) \) which implies that the functions \( z \mapsto F_n(z), z \mapsto G_n(z) \), continuous on \( \overline{D} \) and harmonic in \( D \), coincide in \( D \). It follows that \( F_n \) is a constant, i.e. \( z \mapsto A_n(|z|) \) is constant on \( \Gamma \). By Lemma 1 it follows that \( f \) is analytic in the interior of \( \Omega \).

Suppose now that \( f \) is smooth on \( \Omega \) and that (i) holds. It is easy to see that for each \( n \in \mathbb{Z} \) the function \( z \mapsto A_n(|z|) \) is smooth on \( \Gamma \). Thus the proof will be complete once we have shown that if \( z \mapsto \Phi(z) \) is a smooth function on \( \Gamma \) that depends only on \( |z| \) and if \( \Phi \) has a continuous extension \( \overline{\Phi} \) to \( \overline{D} \) which is analytic in \( D \), then \( \Phi \) is a constant. We follow Agranovski and Valski [2] and use an argument used by Browder and Wermer [3]. Suppose that \( \Phi \) is not a constant. Then \( \overline{\Phi}(D) \) is open, and so by the smoothness of \( \Phi, \overline{\Phi}(D) \not\subset \Phi(\Gamma) \). Let \( c \in \Lambda(D) - \Phi(\Gamma) \). Then the function \( z \mapsto \overline{\Phi}(z) - c \) is continuous on \( \overline{D} \), analytic in \( D \), has a zero in \( D \), and has no zero on \( \Gamma \). On \( \Gamma \) it depends only on \( |z| \) so the variation of its argument along \( \Gamma \) around \( D \) is zero, a contradiction. This completes the proof.

**Remark 1.** Let \( \Gamma \) and \( D \) be as in Theorem 1. Suppose that \( f \) is continuous on \( \Omega \). It is an open question whether (i) alone implies that \( f \) is analytic in the interior of \( \Omega \). Standard arguments, e.g. convolving \( f \) with an approximate identity of a group to get a smooth function [2, 9], do not apply since the rotation group is too small and the process smoothens the function only in the direction perpendicular to the radius. From the proof of Theorem 1 it follows that we could answer the question if we knew that if \( z \mapsto F(z) \) is a continuous function on \( \overline{D} \), analytic in \( D \) and such that its
boundary values depend only on $|z|$, then $F$ is a constant. We are not able to prove this. For certain domains $D$ this problem is related to Question 1 in [3, p. 129].

The following corollary gives a characterization of analytic functions in an annulus in terms of the behavior on certain circles (compare with [11, p. 169; 7, Theorem 12.3.11]).

**Corollary 1.** Let $0$ be in the exterior of a circle $\Gamma \subset \mathbb{C}$ and let $\Omega$ be the annulus obtained by rotating $\Gamma$ around the origin. If $f$ is a continuous function on $\Omega$ such that for every $s$, $|s| = 1$, the function $f|(s\Gamma)$ has a continuous extension to the closed disc bounded by $s\Gamma$ which is analytic in its interior, then $f$ is analytic in the interior of $\Omega$.

**Example 1.** The function

$$z \mapsto f(z) = \begin{cases} z^3/\bar{z}, & z \neq 0, \\ 0, & z = 0, \end{cases}$$

is smooth on $\mathbb{C}$. Let $a, b \in \mathbb{C}$, $b \neq 0$. If $|s| = 1$ and $a + sb \neq 0$ then $f(a + sb) = s(a + sb)^3/(b + sa)$ which shows that if $|b| \geq |a|$ then $s \mapsto f(a + sb)$ has a continuous extension from $\partial \Delta$ to $\Delta$ which is analytic in $\Delta$. Consequently, for any open disc $D \subset \mathbb{C}$ such that $0 \in \overline{D}$, the function $f|\partial D$ has a continuous extension to $\overline{D}$ which is analytic in $D$. This shows that in Theorem 1 one cannot drop the assumption that $0 \in D$.

**Remark 2.** Let $r > 0$ and let $\Theta$ be the family of all circles in $\mathbb{C}$ of radius $r$. Let $f$ be continuous on $\mathbb{C}$ and let $n \in \mathbb{N}$. If

$$\int_{\Gamma} f(z) \, dz = \int_{\Gamma} z^n f(z) \, dz = 0$$

for every $\Gamma \in \Theta$, then by a result of Zalcman [10, 11] $f$ is entire. So, vanishing of only two negative Fourier coefficients implies the analyticity of $f$. One cannot relax the assumptions in Corollary 1 in this direction—here one needs the vanishing of all negative Fourier coefficients. To see this, let $n \in \mathbb{N}$ and define

$$f(e^{i\tau}(2 + e^{i\theta})) = \cos n\theta \quad (0 \leq \tau \leq 2\pi, \ 0 \leq \theta \leq 2\pi).$$

Then $f$ is well defined in $\{z: 1 \leq |z| \leq 3\}$. We have

$$\int_0^{2\pi} e^{im\varphi} f(2e^{i\tau} + e^{i\varphi}) \, d\varphi = \int_0^{2\pi} e^{im\varphi} \cos n(\varphi - \tau) \, d\varphi$$

$$= 2^{-1} \int_0^{2\pi} e^{im\varphi} \left[ e^{-in\varphi} e^{in\tau} + e^{in\varphi} e^{-in\tau} \right] \, d\varphi = 0$$

whenever $0 \leq \tau \leq 2\pi$ and $m \in \mathbb{N}$, $m \neq n$. So, all negative Fourier coefficients vanish, except one, and yet $f$ is not analytic in $\{z: 1 \leq |z| \leq 3\}$.

**3. Analyticity on general families.** The analogue of Corollary 1 in the case when $\Omega$ is a disc does not hold: If $\Theta$ is the family of circles obtained by rotating the circle $|z - 1/2| = 1/2$ around the origin, then Example 1 shows that if

$$\begin{cases} f \text{ is a continuous function on } \overline{\Delta} \text{ such that for every } \Gamma \in \Theta \text{ the function} \\
 f|\Gamma \text{ has a continuous extension to the closed disc bounded by } \Gamma \text{ which} \\
 \text{is analytic in its interior} \end{cases}$$

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
then it does not necessarily follow that \( f \) is analytic in \( \Delta \). Consequently, for no family \( \mathcal{O} \) obtained by rotating only one circle, (2) implies the analyticity of \( f \) in \( \Delta \) (see the discussion following Remark 3). It is interesting that two (suitable chosen) circles suffice: If \( \Gamma_0 \subset \Delta \) is any circle having 0 in its exterior and if \( \mathcal{O} \) is the family of all circles obtained by rotating the circles \( \Gamma_0 \) and \( |z - 1/2| = 1/2 \) around the origin, then (2) implies that \( f \) is analytic in \( \Delta \). This follows from Theorem 1, Lemma 2, and the following

**Lemma 3.** Let \( \Gamma, D, \Omega \) and \( f \) be as in Lemma 2. Suppose that \( f \) is analytic in an open annulus contained in \( \Omega \). Then \( f \) is analytic in the interior of \( \Omega \).

**Proof.** By the assumption there are a sequence \( a_n, n \in \mathbb{Z} \), and an open annulus \( \Sigma \subset \Omega \) such that \( A_n(|z|) = a_n \) \( (z \in \Sigma, n \in \mathbb{Z}) \). It follows that there is a relatively open set \( U \subset \Gamma \) such that

\[
A_n(|z|) = a_n \quad (z \in U, n \in \mathbb{Z}).
\]

By Lemma 2, for each \( n \in \mathbb{Z} \) the function \( z \mapsto z^nA_n(|z|) \) has a continuous extension from \( \Gamma - \{0\} \) to \( \overline{D} \) which is analytic in \( D \). Let \( n \geq 0 \). By (3) we have \( z^nA_n(|z|) = a_n z^n \) \( (z \in U) \). Since \( z \mapsto a_n z^n \) also has a continuous extension from \( \Gamma - \{0\} \) to \( \overline{D} \), which is analytic in \( D \), it follows that \( z^nA_n(|z|) = a_n z^n \) \( (z \in \Gamma - \{0\}) \) and consequently, \( A_n(|z|) = a_n \) \( (z \in \Gamma - \{0\}) \). Let \( n < 0 \). Then \( z \mapsto z^{-n} \) has a continuous extension from \( \Gamma - \{0\} \) to \( \overline{D} \) which is analytic in \( D \) so the same holds for the function \( z \mapsto z^{-n}[z^nA_n(|z|)] = A_n(|z|) \). By (3) it follows that \( A_n(|z|) = a_n \) \( (z \in \Gamma - \{0\}) \). Thus we have proved that \( A_n(|z|) = a_n \) \( (z \in \Gamma - \{0\}, n \in \mathbb{Z}) \). Now the assertion follows from Lemma 1. This completes the proof.

**Theorem 2.** Let \( \mathcal{O} \) be a rotation invariant, symmetric family of smooth Jordan curves contained in \( \Delta \). For each \( \Gamma \in \mathcal{O} \) let \( D_\Gamma \) be the simply connected domain bounded by \( \Gamma \), and let \( B_\Gamma \) be the interior of the set \( \{sz: z \in \Gamma, |s| = 1\} \). Let \( B = \bigcup_{\Gamma \in \mathcal{O}} B_\Gamma \). Suppose that

(i) every continuous function on \( \Delta \), which is analytic in \( B \), is analytic in \( \Delta \),

(ii) every connected component of \( B \) contains a domain \( D_\Gamma \), \( \Gamma \in \mathcal{O} \), such that \( 0 \not\in D_\Gamma \).

Then

(b) if \( f \) is a continuous function on \( \Delta \) such that for every \( \Gamma \in \mathcal{O} \) the function \( f|\Gamma \) has a continuous extension to \( \overline{D_\Gamma} \) which is analytic in \( D_\Gamma \), then \( f \) is analytic in \( \Delta \).

If we assume (a)(i) and if every \( \Gamma \in \mathcal{O} \) is a circle then (a)(ii) and (b) are equivalent.

If we assume that 0 is in the exterior of every \( \Gamma \in \mathcal{O} \) then (a)(i) and (b) are equivalent.

**Proof.** Note first that a connected component of \( B \) is either a disc \( \{z: |z| < r\} \) or an annulus \( \{z: r_1 < |z| < r_2\} \).

The last statement is an easy consequence of Theorem 1.

Suppose that (a) holds. Assume that \( f \) is a continuous function on \( \Delta \) such that for every \( \Gamma \in \mathcal{O} \) the function \( f|\Gamma \) has a continuous extension to \( \overline{D_\Gamma} \) which is analytic in \( D_\Gamma \). Let \( B_0 \) be a connected component of \( B \). By (a)(ii) there is a \( \Gamma_0 \in \mathcal{O} \) such that \( D_{\Gamma_0} \subset B_0 \) and \( 0 \not\in \overline{D_{\Gamma_0}} \). By Theorem 1, \( f \) is analytic in \( B_{\Gamma_0} \). Let \( K \) be a closed annulus contained in \( B_0 \) such that \( K \cap B_{\Gamma_0} \neq \emptyset \). There are \( \Gamma_1, \Gamma_2, \ldots, \Gamma_n \in \mathcal{O} \) such that
$K \subset \bigcup_{i=1}^{n} B_{r_i} \subset B_0$ and such that $B_{r_i} \cap B_{r_{i+1}} \neq \emptyset$ ($1 \leq i \leq n - 1$). Further, there is some $j$, $1 \leq j \leq n$, such that $B_{r_j} \cap B_{r_0} \neq \emptyset$. By Lemma 3, $f$ is analytic in $B_{r_j}$.

Using Lemma 3 step by step we prove that $f$ is analytic in $\bigcup_{j=1}^{n} B_{r_j}$. As $K$ was arbitrary it follows that $f$ is analytic in $B_0$ and as $B_0$ was an arbitrary connected component of $B$ it follows by (a)(i) that $f$ is analytic in $\Delta$. This proves that (a) implies (b).

Assume that every $\Gamma \in \mathfrak{G}$ is a circle and that (a)(i) holds. Suppose that (b) holds. We have to prove that this implies (a)(ii). Assume, contrarily to (a)(ii), that there is a connected component $B_0$ of $B$ such that $0 \not\in D_\Gamma$ whenever $\Gamma \in \mathfrak{G}$ satisfies $B_\Gamma \subset B_0$.

If $B_0 = \{z: |z| < 1\}$ then define

$$f(z) = \begin{cases} 0, & z = 0, \\ z^2/\bar{z}, & 0 < |z| \leq r, \\ z^3/r^2, & r < |z| < 1, \end{cases}$$

and if $B_0 = \{z: r_1 < |z| < r_2\}$ then define

$$f(z) = \begin{cases} z^3/r_1^2, & |z| \leq r_1, \\ z^2/\bar{z}, & r_1 < |z| \leq r_2, \\ z^3/r_2^2, & r_2 < |z| \leq 1. \end{cases}$$

Then $f$ is continuous on $\Delta$ and (see Example 1) for every $\Gamma \in \mathfrak{G}$ the function $f|\Gamma$ has a continuous extension to $\overline{D}_\Gamma$ which is analytic in $D_\Gamma$ yet $f$ is not analytic in $\Delta$, a contradiction. This complete the proof.

**Remark 3.** It is not known whether in Theorem 2 one can drop the assumption that $\mathfrak{G}$ is symmetric. In our proof we used the symmetry when we applied Theorem 1. So this question is related to Remark 1. However, for smooth functions one can prove a theorem, analogous to Theorem 1, without assuming that $\mathfrak{G}$ is symmetric. In the proof, analogous to the one above, one uses the second half of Theorem 1.

Now we present two examples of minimal rotation invariant families that satisfy (b) in Theorem 2. Note that if $\mathfrak{G}$ satisfies (b) then $\bigcup_{\Gamma \in \mathfrak{G}} \Gamma$ must be dense in $\Delta$. Were this not so, there would be a continuous function $f$ on $\Delta$, vanishing on an open subset of $\Delta$ containing $\bigcup_{\Gamma \in \mathfrak{G}} \Gamma$ and not vanishing identically. By (b), $f$ would be analytic in $\Delta$, a contradiction.

**Example 2.** Let $r_n, n \in \mathbb{Z}$, be a sequence satisfying $0 < r_n < r_{n+1} < 1$ ($n \in \mathbb{Z}$), $\lim_{n \to \infty} r_n = 1$, $\lim_{n \to -\infty} r_n = 0$. For each $n \in \mathbb{Z}$ let $\Gamma_n$ be the circle of radius $(r_{n+1} - r_n)/2$ with center $(r_{n+1} + r_n)/2$. By Morera’s theorem, $\mathfrak{G} = \{s\Gamma_n: |s| = 1, n \in \mathbb{Z}\}$ satisfies (a)(i) so, by Theorem 2, $\mathfrak{G}$ is a minimal rotation invariant family satisfying (b).

**Example 3.** Let $r_n, n \geq 2$, be a strictly increasing sequence, $r_2 = 1/2$, $\lim_{n \to \infty} r_n = 1$. Let $\Gamma_1 = \{z: |z - 1/4| = 1/4\}$, $\Gamma_2 = \{z: |z - 1/4| = 1/8\}$, and for each $n \geq 2$, let $\Gamma_n$ be the circle of radius $(r_n - r_{n-1})/2$ with center $(r_n + r_{n-1})/2$. Again, by Theorem 2, $\mathfrak{G} = \{s\Gamma_n: |s| = 1, n \in \mathbb{N}\}$ is a minimal rotation invariant family satisfying (b).
Our next example shows that in general (a) and (b) in Theorem 2 are not equivalent.

**Example 4.** Let $D$ be the convex hull of

$$\{z : |4z - 3e^{i\pi/4}| < 1\} \cup \{z : |4z - 3e^{-i\pi/4}| < 1\} \cup \{re^{i\varphi} : 0 \leq r < 1, \pi/4 \leq |\varphi| \leq \pi\}.$$

Then $\Gamma = \partial D$ is smooth and satisfies $\Gamma = \Gamma^*$. Let $\Omega = \{sz : z \in \Gamma, |s| = 1\}$ and let $f$ be a continuous function on $\Omega$ such that for every $s, |s| = 1$, the function $f|(s\Gamma)$ has a continuous extension to $s\overline{D}$ which is analytic in $sD$. Fix $s, |s| = 1$. We show that $f$ is analytic in $(sD) \cap \text{Int} \Omega$ by showing that, in $(sD) \cap \text{Int} \Omega$, $f$ coincides with the continuous extension $\Phi$ of $f|(s\Gamma)$ to $s\overline{D}$ which is analytic in $sD$. Fix a point $z \in (sD) \cap \text{Int} \Omega$. By the definition of $\Gamma$ there is an $s_1, |s_1| = 1$, such that $z \in s_1\Gamma$ and such that $(s_1\Gamma) \cap (s\Gamma)$ contains an arc. Consequently, if $\Phi_1$ is the continuous extension of $f|(s_1\Gamma)$ to $s_1\overline{D}$ which is analytic in $s_1D$ then $\Phi_1 = \Phi$ on $(sD) \cap (s_1\overline{D})$ so $f(z) = \Phi_1(z) = \Phi(z)$. This proves that for any $s, |s| = 1$, $f$ is analytic in $(sD) \cap \text{Int} \Omega$, so $f$ is analytic in $\text{Int} \Omega$. This shows that in Theorem 2, (b) does not imply (a)(ii).

Example 4 also shows that, for some curves, $\Gamma$ one can drop the assumption that $0 \not\in \overline{D}$ in Theorem 1. By Example 1 one cannot do this for circles so it is natural question whether the circles are the only curves $\Gamma$ for which one cannot drop the assumption that $0 \not\in \overline{D}$ in Theorem 1. There are other curves having this property:

**Example 5.** Let $\Gamma$ be a smooth curve which can be parametrized as $z = r(\varphi)e^{i\varphi}$ ($-\pi \leq \varphi \leq \pi$) where $\varphi \mapsto r(\varphi)$ is strictly increasing and positive on $[0, \pi]$ and satisfies $r(\varphi) = r(-\varphi)$ ($0 \leq \varphi \leq \pi$). Clearly $\Gamma = \Gamma^*$ and if $z \in \Gamma$ then $\bar{z} \in \Gamma$ and there are no other points $w \in \Gamma$ satisfying $|w| = |z|$. Denote by $D$ the simply connected domain bounded by $\Gamma$ and let $\psi : \overline{D} \to \Delta$ be a continuous map which maps $D$ bianalytically onto $\Delta$ and which satisfies $\psi(0) = 0, \psi'(0) = 1$. Then by symmetry, $\psi(\bar{z}) = \overline{\psi(z)}$ ($z \in \Gamma$). Define the function $\Phi$ on $\Gamma$ by $\Phi(z) = \psi(z) + 1/\psi(z)$. Since

$$\Phi(z) = \psi(z) + 1/\psi(z) = \psi(z) + 1/\psi(z) = 1/\psi(z) + \psi(z) = \Phi(z) \quad (z \in \Gamma),$$

it follows by the properties of $\Gamma$ that one can define a continuous function $F$ on $\Omega = \{sz : z \in \Gamma, |s| = 1\}$ by $F(z) = \Phi(z)$ ($z \in \Gamma$). Define $f$ on $\Omega$ by $f(z) = zF(|z|)$ ($z \in \Omega$). Fix $s, |s| = 1$. We have $f(sz) = szF(|z|) = sz[\psi(z) + 1/\psi(z)]$ ($z \in \Gamma$). Since $\psi$ has a single zero at 0 the function $z \mapsto sz[\psi(z) + 1/\psi(z)]$ has a continuous extension from $\Gamma$ to $\overline{D}$ which is holomorphic in $D$. Consequently, $f$ is a continuous function on $\Omega$ such that for every $s, |s| = 1$, the function $f|(s\Gamma)$ has a continuous extension to $s\overline{D}$ which is analytic in $sD$, yet $f$ is not analytic in the interior of $\Omega$. Examples 1, 4 and 5 indicate that there may be no simple characterization of the curves $\Gamma$ for which one can drop the assumption that $0 \not\in \overline{D}$ in Theorem 1 and therefore that there may be no simple characterization of rotation invariant, symmetric families $\mathcal{O}$ which satisfy (b) in Theorem 2.

This work was supported in part by the Republic of Slovenia Science Foundation.
REFERENCES


INSTITUTE OF MATHEMATICS, PHYSICS AND MECHANICS, E. K. UNIVERSITY OF LJUBLJANA, LJUBLJANA, YUGOSLAVIA