TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 280, Number 1, November 1983

LOCAL SPECTRA OF SEMINORMAL OPERATORS

BY

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ABSTRACT. The local spectral theory of seminormal operators is studied by examining the connection between two naturally occurring contractive operator functions. These results are used to control the local spectra of cohyponormal operators. An invariant subspace result for seminormal operators whose real part has thin spectra is provided.

Local spectral theory of a bounded operator T on a Hilbert space $\mathcal{H}$ is the theory of the resolvent operator function $(T-\lambda)^{-1}$ "localized" to vectors in $\mathcal{H}$. More specifically, for x in $\mathcal{H}$ the local resolvent is the (multiple-valued) function $x_T(\lambda)$ which consists of all possible (weak) analytic continuations of $(T-\lambda)^{-1}x$ in the complex plane from the resolvent set $\rho(T)$. The operator T is said to have the single valued extension property (s.v.e.p.) in case for every x in $\mathcal{H}$, $x_T(\lambda)$ is single valued.

In case T has the s.v.e.p. the maximal domain $\rho(T: x)$ of $x_T(\lambda)$ is called the local resolvent set of x and $\sigma(T: x) = \mathbf{C}\backslash\rho(T: x)$ the local spectrum of x. For δ a closed set in $\mathbf{C}$

$$M(T: \delta) = \{x \in \mathcal{H}: \sigma(T: x) \subset \delta\}$$

is a T invariant linear manifold. The principal goal of local spectral theory is to use $M(T: \delta)$ to decompose $\mathcal{H}$ and a fortiori T into simpler components. The difficulties with the theory usually are in showing $M = M(T: \delta)$ is closed and nontrivial ($M \neq (0)$ and $M \neq \mathcal{H}$).

An interesting class of nonnormal operators where local spectral theory has been investigated is the class of seminormal operators. An operator T on a Hilbert space $\mathcal{H}$ is called seminormal in case the self-commutator $D = [T, T^*] = TT^* - T^*T$ is semidefinite. In the local spectral theory of seminormal operators there is an interesting distinction between the hyponormal case ($D \leqslant 0$) and the cohyponormal case ($D \geqslant 0$). In fact, if T is hyponormal, then it has the s.v.e.p. and the spectral subspace $M(T: \delta)$ is a (perhaps trivial) closed T invariant subspace (see Stampfli [**21**]). On the other hand when T is cohyponormal (not a scalar multiple of the identity) and the s.v.e.p. is assumed, then it is always possible to find a closed set δ with $(0) \subsetneqq M(T: \delta) \subsetneqq \mathcal{H}$. This ability to "control" the local spectrum of cohyponormal operators follows from a result of Putnam [**18**] (see, for example, [**7**, p. 29]). It will be shown by an example below that local spectral spaces associated

Received by the editors May 31, 1982 and, in revised form, December 1, 1982.
1980 *Mathematics Subject Classification*. Primary 47B20, 47A15, 47B40.
Key words and phrases. Seminormal operators, subnormal operators, local spectrum, Bergman shift.
[1] Work supported by a grant from the National Science Foundation.

with cohyponormal operators may fail to be closed. For a fuller discussion of local spectral theory of seminormal operators, see the references and Chapter 1 of the monograph [7].

In this paper local spectral theory of seminormal operators is studied by examining the connections between two naturally occurring contractive operator functions. More specifically, let T be cohyponormal and $D = TT^* - T^*T$ ($\geqslant 0$). For every complex λ there are unique contractions $C(\lambda)$ and $K(\lambda)$ satisfying

$$(0.1) \qquad (T-\lambda)C(\lambda) = \sqrt{D}\,; \quad T-\lambda = K(\lambda)(T-\lambda)^*$$

with $C^*(\lambda)$ and $K(\lambda)$ zero on the kernel of $T-\lambda$. It is clear that $d(\lambda) = C(\lambda)f$ acts as a local resolvent with respect to T for the vector $d = \sqrt{D}f$ in the range of $\sqrt{D}$. Similarly, when $K(\lambda)$ is an invertible operator from the closure of the range of $(T-\lambda)^*$ to the closure of the range of $T-\lambda$, then $\tilde{d}(\lambda) = [[K(\bar{\lambda})]^*]^{-1}C(\lambda)f$ is a "local resolvent" with respect to T for the vector $d = \sqrt{D}f$ in the range of $\sqrt{D}$.

This paper is structured as follows. §1 describes the basic relation between the operator functions $C = C(\lambda)$ and $K = K(\lambda)$ defined in (0.1). The operator function $C = C(\lambda)$ is also connected to important known unitary invariants of seminormal operators. The results of §1 should be of independent interest. §2 is concerned with applications of the results of §1 to control the local spectra of cohyponormal operators. In particular, it is shown that if T is cohyponormal and Δ is a closed disc whose interior intersects the boundary of the spectrum of T, then there is a nonzero vector x with $\sigma(T: x) \subset \Delta$. §3 concerns the local spectra of cosubnormal operators and examples. In this section an example of Sinanjan [**20**] is shown to lead to an example of a seminormal operator possessing the s.v.e.p. where the local spectral spaces are not closed. In §4 an invariant subspace result for seminormal operators whose real part has thin spectra is provided. This result is based on the material in §1 and extends earlier results in [**1**] and Putnam [**17**].

ACKNOWLEDGEMENT. The work on this paper was initiated while both authors were visitors at Indiana University. The authors would like to express their gratitude to the Department of Mathematics for making this collaboration a possibility.

1. Contractive operator functions associated with seminormal operators. (1) The following conventions and notations connected with the bounded operator A on the Hilbert space $\mathcal{H}$ will be employed. The spectrum and resolvent set of A will be denoted by $\sigma(A)$ and $\rho(A)$, respectively, and $\pi_0(A)$ will be used for the set of eigenvalues. The notations $\mathrm{Ran}(A)$ and $\mathrm{Ker}(A)$ will denote the range of A and the kernel of A. If λ is a complex number, then A_λ denotes $A-\lambda$ with $A_\lambda^* = (A-\lambda)^* = A^* - \bar{\lambda}$. The operator A is said to be pure (or completely nonnormal) in case zero is the only subspace reducing A on which A is a normal operator.

Let T be a cohyponormal operator on the Hilbert space $\mathcal{H}$. Hence $D = TT^* - T^*T$ is nonnegative semidefinite. For every complex λ, $D = D^{1/2}D^{1/2} \leqslant T_\lambda T_\lambda^*$ and, therefore, there is a unique contraction $C(\lambda)$ satisfying

$$(1.1) \qquad T_\lambda C(\lambda) = \sqrt{D}\,, \quad C^*(\lambda)f = [C(\lambda)]^*f = 0 \qquad (f \in \mathrm{Ker}(T_\lambda))$$

(see Douglas [**12**]). It follows in a similar manner from the inequality $T_\lambda^* T_\lambda \leqslant T_\lambda T_\lambda^*$ that there is a unique contraction $K(\lambda)$ which satisfies

$$T_\lambda = K(\lambda)T_\lambda^*, \quad K(\lambda)f = 0 \qquad (f \in \operatorname{Ker}(T_\lambda)). \tag{1.2}$$

The elementary identities

$$D = D^{1/2}D^{1/2} = T_\lambda C(\lambda)C^*(\lambda)T_\lambda^*$$

and

$$D = T_\lambda T_\lambda^* - T_\lambda^* T_\lambda = T_\lambda[I - K^*(\lambda)K(\lambda)]T_\lambda^*$$

imply that the contractive operator functions $C = C(\lambda)$ and $K = K(\lambda)$ are related by the fundamental identity

$$I = C(\lambda)C^*(\lambda) + K^*(\lambda)K(\lambda) + P(\lambda), \tag{1.3}$$

where $P(\lambda)$ denotes the orthogonal projection onto $\operatorname{Ker}(T_\lambda)$.

PROPOSITION 1.1. *Let T be a cohyponormal operator on $\mathcal{H}$ and for λ fixed in* $\mathbf{C}$, *let $C(\lambda)$ be the unique contraction satisfying* (1.1). *The following statements are equivalent*:

(a) $\|C(\lambda)\| < 1$.

(b) $\operatorname{Ran}(T_\lambda) = \operatorname{Ran}(T_\lambda^*)$.

PROOF. The identity (1.3) shows that (a) is equivalent to

(a′) $K(\lambda)$: $[\operatorname{Ran}(T_\lambda^*)]^- \to [\operatorname{Ran}(T_\lambda)]^-$ is invertible, where the bar denotes closure.

As a consequence, if (a) is assumed there is an operator $L(\lambda)$ satisfying $T_\lambda^* = L(\lambda)T_\lambda$. Hence,

$$T_\lambda^* = T_\lambda K^*(\lambda), \quad T_\lambda = T_\lambda^* L^*(\lambda)$$

and it follows that (a) implies (b).

Conversely, if (b) is assumed, then by the result of Douglas [**12**] there is a unique operator $L(\lambda)$ which is zero on $\operatorname{Ker}(T_\lambda^*)$ satisfying $T_\lambda^* = L(\lambda)T_\lambda$. The identities $T_\lambda = K(\lambda)T_\lambda^*$ and $T_\lambda^* = L(\lambda)T_\lambda$ imply that $L(\lambda)K(\lambda)$ is the identity on $\operatorname{Ran}(T_\lambda^*)$ and $K(\lambda)L(\lambda)$ is the identity on $\operatorname{Ran}(T_\lambda)$. Thus (b) implies (a′) and (a). This completes the proof.

Later in the final paragraph of §4 we will present examples of pure cohyponormal operators which satisfy (a) and (b) at points λ in $\sigma(T)$. It is easily verified that $C = C(\lambda)$ is weakly continuous on $\mathbf{C}\backslash\pi_0(T)$; however, as the following result shows, the operator function $C(\lambda)$ must be discontinuous in the operator norm at such points.

PROPOSITION 1.2. *Let T be a pure cohyponormal operator and $C = C(\lambda)$ ($\lambda \in \mathbf{C}$) the contractive operator function defined by* (1.1). *If $\|C(\lambda)\|$ is continuous at $\lambda_0 \in \sigma(T)$, then $\|C(\lambda_0)\| = 1$.*

PROOF. Suppose by way of contradiction that $\|C(\lambda)\|$ is continuous at $\lambda_0 \in \sigma(T)$ and $\|C(\lambda_0)\| < 1$. Then $\|C(\lambda)\| < 1$ on some open disc Δ centered at λ_0. By

Proposition 1.1, $\operatorname{Ran}(T_\lambda) = \operatorname{Ran}(T_\lambda^*)$, for all λ in Δ. Consequently,

$$\mathfrak{M} = \bigcap_{\lambda \in \Delta} \operatorname{Ran}(T_\lambda) = \bigcap_{\lambda \in \Delta} \operatorname{Ran}(T_\lambda^*)$$

is a linear manifold which is both T and T^* invariant. The identity $T_\lambda C(\lambda) = \sqrt{D}$ shows $\mathfrak{M}$ also contains $\operatorname{Ran}(\sqrt{D})$. However, by Theorem 1 of **[6]**, the linear manifold $\mathfrak{M} = \bigcap_{\lambda \in \Delta} \operatorname{Ran}(T_\lambda^*)$ is closed. In fact, the result in Theorem 1 of **[6]** identifies $\mathfrak{M}$ with the local spectral subspace $\mathfrak{M}(T^* : \delta)$, where $\delta = \{\bar{\lambda} : \lambda \notin \Delta\}$ and by the result of Stampfli **[21]**)the space $\mathfrak{M}(T^* : \delta)$ is closed. Since T is pure we conclude $\mathfrak{M} = \mathcal{H}$ and this leads to the contradiction $\sigma(T) \cap \Delta = \varnothing$. The proof is complete.

(2) In the remainder of this section we relate the operator function $C = C(\lambda)$ ($\lambda \in \mathbf{C}$) defined by (1.1) to several known unitary invariants for nonnormal operators.

It is interesting to first observe that when T is a pure cohyponormal operator the operator function $C = C(\lambda)$ ($\lambda \in \mathbf{C}$) defined by (1.1) is a complete unitary invariant for T. To establish this observation we need to recall some results from **[14]**.

Let A be a pure operator on the Hilbert space $\mathcal{H}$. Following **[14]**, we denote by $\mathfrak{M}_1$ and $H_1(A)$ the subspaces

$$\mathfrak{M}_1 = \bigcap_{j=1}^{\infty} \operatorname{Ker}[A^*, A^j] \quad \text{and} \quad H_1(A) = \mathfrak{M}_1^{\perp}.$$

The following result is not explicitly stated in **[14]**. We include a proof for completeness.

LEMMA 1.1. *$H_1(A)$ is the smallest closed subspace containing the range of $D = AA^* - A^*A$ which is invariant under A^*.*

PROOF. Let Δ denote the derivation $\Delta(X) = XA^* - A^*X$ acting on the algebra of operators on $\mathcal{H}$. Then $\mathfrak{M}_1 = \bigcap_{j=1}^{\infty} \operatorname{Ker} \Delta(A^j)$. Thus $f \in \mathfrak{M}_1$ if and only if

$$\begin{aligned}
&\Delta(A\ )f = Df = 0,\\
&\Delta(A^2)f = A\Delta(A)f + \Delta(A)Af = ADf + DAf = 0,\\
&\quad\vdots \qquad\qquad\qquad \vdots\\
&\Delta(A^j)f = A\Delta(A^{j-1})f + \Delta(A)A^{j-1}f = A\Delta(A^{j-1})f + DA^{j-1}f = 0, \ldots.
\end{aligned}$$

This makes it clear that $f \in \mathfrak{M}_1$ if and only if $DA^j f = 0$ ($j = 0, 1, \ldots$). Equivalently, f is in $\mathfrak{M}_1$ if and only if f is orthogonal to the smallest space invariant under A^* containing $\operatorname{Ran}(D)$. This completes the proof.

The main result of **[14]** is the following. Let A_i be pure operators acting on the Hilbert spaces $\mathcal{H}_i$ and $D_i = A_i A_i^* - A_i^* A_i$ ($i = 1, 2$). The operators A_1 and A_2 are unitarily equivalent if and only if there is a unitary operator U_0: $H_1(A_1) \to H_1(A_2)$ such that $U_0 A_1^* f = A_2^* U_0 f$, $U_0 D_1 f = D_2 U_0 f$, $f \in H_1(A)$. In other words the pair consisting of the operators A^* and D restricted to $H_1(A)$ determine the pure operator A up to unitary equivalence.

We apply the above result to cohyponormal operators as follows.

PROPOSITION 1.3. *Let T_i be pure cohyponormal operators acting on the Hilbert spaces $\mathcal{H}_i$, $D_i = [T_i, T_i^*]$ and $C_i = C_i(\lambda)$ ($\lambda \in \mathbf{C}$) the unique contractive operator functions satisfying*

$$(T_i - \lambda)C_i(\lambda) = \sqrt{D_i}, \quad C_i^*(\lambda)\mathrm{Ker}(T_i - \lambda) = (0)$$

for $i = 1, 2$. The operators T_1 and T_2 are unitarily equivalent if and only if there is a unitary operator U: $\mathcal{H}_1 \to \mathcal{H}_2$ satisfying

$$UC_1(\lambda) = C_2(\lambda)U \tag{1.4}$$

in a neighborhood of ∞.

PROOF. In a neighborhood of ∞, $C_i(\lambda) = (T_i - \lambda)^{-1}\sqrt{D_i}$. Consequently, any unitary U satisfying (1.4) will satisfy $UT_1^j\sqrt{D_1}f = T_2^j\sqrt{D_2}\,Uf$, $f \in \mathcal{H}_1$. It follows that U is a unitary map from $H_1(T_1^*)$ to $H_1(T_2^*)$ implementing a unitary equivalence between the pairs of operators (T_1, D_1) and (T_2, D_2) restricted to these invariant subspaces. Thus by the result of Morrel [**14**] as outlined above, the operators T_1 and T_2 are unitarily equivalent. This completes the proof.

REMARKS. 1°. It would be interesting to replace the condition in Proposition 1.3 that (1.4) hold in a neighborhood of ∞ with the condition that $UC_1(\lambda) = C_2(\lambda)U$ hold on the spectrum of T_1 and T_2. In a nice case, for example, when $\partial\sigma(T_1) = \partial\sigma(T_2) = \Gamma_0 = \{\lambda: |\lambda| = 1\}$ and (1.4) holds with some unitary U on Γ_0, then (1.4) will continue to hold for λ in a neighborhood of ∞. In particular, in such a case the operator function $C_1(\lambda)$ ($\lambda \in \Gamma_0$) is a complete unitary invariant for the pure operator T_1.

2°. Suppose that T and A are pure operators on the Hilbert space $\mathcal{H}$ such that $TT^* - T^*T = \phi \otimes \phi$ and $AA^* - A^*A = \psi \otimes \psi$, where $\|\phi\| = \|\psi\| = 1$. The result in Proposition 1.3 establishes that A and T are unitarily equivalent if and only if there is a unitary U such that in a neighborhood of ∞

$$UT_\lambda^{-1}\phi = A_\lambda^{-1}\psi. \tag{1.5}$$

The beautiful results of Cowen and Douglas [**11**] can be used to find necessary and sufficient conditions for the existence of a unitary U which satisfies (1.5). If one considers the hermitian holomorphic line bundles $\lambda \to \vee\{T_\lambda^{-1}\phi\}$, $\lambda \to \vee\{A_\lambda^{-1}\psi\}$ ($\vee$ denotes span), then the results of [**11**] show there is a unitary U satisfying (1.5) (in a neighborhood of ∞) if and only if the scalar curvatures

$$\kappa_T(\lambda) = \frac{-\partial^2}{\partial\lambda\partial\bar{\lambda}}\log\|T_\lambda^{-1}\phi\|^2 \quad \text{and} \quad \kappa_A(\lambda) = \frac{-\partial^2}{\partial\lambda\partial\bar{\lambda}}\log\|A_\lambda^{-1}\psi\|^2$$

agree in a neighborhood of ∞. It follows that two pure operators T and A for which $[T, T^*] = \phi \otimes \phi$ and $[A, A^*] = \psi \otimes \psi$ with $\|\phi\| = \|\psi\| = 1$ are unitarily equivalent if and only if $\|T_\lambda^{-1}\phi\| = \|A_\lambda^{-1}\psi\|$ in a neighborhood of infinity. We will have more to say about the invariant $\|C(\lambda)\| = \|T_\lambda^{-1}\phi\|$ below.

(3) In the final paragraph of this section we will relate the operator function $C(\lambda)$ ($\lambda \in \mathbf{C}$) to the principal function associated with a seminormal operator having a trace class self-commutator. We briefly recall the definition of this important unitary invariant.

Let $T^* = X + iY$ be the Cartesian form ($X = \frac{1}{2}(T^* + T)$, $Y = (T^* - T)/2i$) of a hyponormal operator T^* such that $D = TT^* - T^*T$ is trace class. There is a nonnegative integrable function $g = g_{T^*}$ with compact support on $\mathbf{R}^2$ such that

$$(1.6)\quad i\,\mathrm{tr}[p(X,Y), q(X,Y)] = \frac{1}{2\pi}\iint_{\mathbf{R}^2}\left\{\frac{\partial p}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial q}{\partial x}\frac{\partial p}{\partial y}\right\} g(x,y)\,dx\,dy,$$

where $p = p(x, y)$, $q = q(x, y)$ are polynomials in the variables x, y. Note that in (1.6) the possible ambiguity caused by the substitution of the noncommuting X, Y into p, q is mitigated in the tracial bilinear form

$$i\,\mathrm{tr}[p(X,Y), q(X,Y)] = i\,\mathrm{tr}[p(X,Y)q(X,Y) - q(X,Y)p(X,Y)],$$

so that both sides of (1.6) define a bilinear form on the algebra $\mathcal{P}(\mathbf{R}^2)$ of polynomials in the variables x, y. The function g is called the principal function associated with T^* and was first studied by Pincus [**15**] in connection with the diagonalization of certain selfadjoint singular integral operators on the real line. The tracial bilinear form connected with g first appears in [**13**] (see also [**5**] and, for an expository report, [**7**, Chapter 5]).

Suppose now that the operator T is pure and let $T_\lambda = W_\lambda\sqrt{P_\lambda}$ be the unique polar factorization of T_λ, where $P_\lambda = T_\lambda^* T_\lambda$ and W_λ is the unique coisometry which is zero on $\mathrm{Ker}(T_\lambda)$ and satisfies $T_\lambda = W_\lambda\sqrt{P_\lambda}$. It is easily verified that $W_\lambda P_\lambda = (P_\lambda + D)W_\lambda$. If $C(\lambda)$ denotes the unique contraction satisfying (1.2), then

$$(1.7)\qquad C(\lambda) = W_\lambda^*(P_\lambda + D)^{-1/2}\sqrt{D},$$

where the range of $\sqrt{D}$ is contained in the domain $\mathcal{D}((P_\lambda + D)^{-1/2})$ of the (possibly unbounded) selfadjoint operator $(P_\lambda + D)^{-1/2}$. Similarly, on $W_\lambda^*\mathcal{D}((P_\lambda + D)^{-1/2})$

$$(1.8)\qquad C^*(\lambda) = \sqrt{D}\,(P_\lambda + D)^{-1/2}W_\lambda.$$

Combining (1.7) and (1.8), we obtain on $W_\lambda^*\mathcal{D}((P_\lambda + D)^{-1/2})$,

$$(1.9)\qquad C(\lambda)C^*(\lambda) = W_\lambda^*(P_\lambda + D)^{-1/2}D(P_\lambda + D)^{-1/2}W_\lambda.$$

It follows from the work of Carey and Pincus [**5**] that for $\mu > 0$

$$(1.10)\qquad \det\left[I - W_\lambda^*(P_\lambda + D + \mu)^{-1/2}D(P_\lambda + D + \mu)^{-1/2}W_\lambda\right] = \exp - \frac{1}{\pi}\iint_{\mathbf{R}^2}\frac{g(x,y)}{|x + iy - \bar{\lambda}|^2 + \mu^2}\,dx\,dy;$$

see, e.g., [**6**]. It is apparent from (1.10) that the condition

$$(1.11)\qquad \nu_g(\lambda) \equiv \frac{1}{\pi}\iint_{\mathbf{R}^2}\frac{g(x,y)}{|x + iy - \bar{\lambda}|^2}\,dx\,dy < \infty$$

implies

$$\sup_{\mu>0}\left\|W_\lambda^*(P_\lambda + D + \mu)^{-1/2}\sqrt{D}\right\| = \alpha_\lambda < 1.$$

From the representation (1.7) we learn that (1.11) implies

$$(1.12)\qquad \|C(\lambda)\| = \alpha_\lambda < 1.$$

In general the assumption that D is trace class does not even force $C(\lambda)$ to be compact (see Remark 2° in §2). On the other hand when D is of finite rank, then $C(\lambda)$ ($\lambda \in \mathbf{C}$) is clearly of finite rank. In this case combining (1.9) and (1.10) we obtain that

$$\det[I - C(\lambda)C^*(\lambda)] = \exp[-\nu_g(\lambda)] \qquad (\lambda \in \mathbf{C}), \tag{1.13}$$

where the possibility that $\nu_g(\lambda) = +\infty$ is admitted. The identity in (1.13) was first noted in **[6]** and used there to study local spectra of cohyponormal operators.

We close this section with the following remark. Suppose $D = \phi \otimes \phi$, where $\|\phi\| = 1$. As we observed earlier $\|C(\lambda)\| = \|T_\lambda^{-1}\phi\|$. Indeed, this equality holds everywhere in the complex plane provided $T_\lambda^{-1}\phi$ denotes the unique solution of $T_\lambda x = \phi$ which is orthogonal to $\operatorname{Ker}(T_\lambda)$. Combining this identity with (1.13) we obtain, for $\lambda \in \mathbf{C}$,

$$\|C(\lambda)\|^2 = \|T_\lambda^{-1}\phi\|^2 = 1 - \exp\left\{-\frac{1}{\pi}\iint_{\mathbf{R}^2} \frac{g(x, y)}{|x - iy - \bar{\lambda}|^2}\,dx\,dy\right\}. \tag{1.14}$$

Equation (1.14) shows the simple relationship between the complete unitary invariants $\|C(\lambda)\|$ (λ in a neighborhood of ∞) and $g = g_{T^*}$. The fact that the principal function is a complete unitary invariant for a pure hyponormal operator with rank one self-commutator is in Pincus **[15]**.

2. Local spectra of cohyponormal operators. (1) In this section we will examine the problem of controlling the local spectra of a cohyponormal operator T. More specifically, suppose Δ is a closed disc with $\Delta \cap \sigma(T) \neq \varnothing$. We consider the problem of whether there is a nonzero vector x with $\sigma(T : x) \subset \Delta$. In general, we do not know whether such a vector x exists. We are able to produce such a vector x in case $D = TT^* - T^*T$ is finite dimensional or, in general, if the disc Δ intersects the boundary $\partial\sigma(T)$ of the spectrum of T. The case where D is finite dimensional was handled in **[6]**, where the argument was based on the identity (1.13). Here our techniques are simpler and based on the study of the analyticity of the operator function $C = C(\lambda)$ defined in (1.1).

(2) The following is our main tool for controlling local spectra of cohyponormal operators.

THEOREM 2.1. *Let T be a pure cohyponormal operator and $C = C(\lambda)$ ($\lambda \in \mathbf{C}$) the operator function defined by* (1.1). *Denote by $\Omega(T)$ the maximal domain of analyticity of C and set $\Sigma(T) = \mathbf{C}\backslash\Omega(T)$. The following hold*:

(i) $\partial\sigma(T) \subset \Sigma(T) \subset \sigma(T)$.

(ii) *If $D = TT^* - T^*T$ is a finite rank operator, then $\sigma(T) = \Sigma(T)$.*

PROOF. (i) The inclusion $\Sigma(T) \subset \sigma(T)$ is obvious. Suppose that $\lambda_0 \in \partial\sigma(T)$. If $C = C(\lambda)$ is analytic at λ_0, then $\|C(\lambda)\|$ is continuous at λ_0 and for sufficiently small $s > 0$

$$\|C(\lambda_0)\| \leqslant \frac{1}{2\pi}\int_0^{2\pi} \|C(\lambda_0 + se^{i\theta})\|\,d\theta. \tag{2.1}$$

Since $\lambda_0 \in \partial\sigma(T)$, by applying Proposition 1.2, we know that for some such s, $\|C(\lambda_0 + se^{i\theta})\| < 1$ on an arc of the circle $\{\lambda_0 + se^{i\theta}\colon 0 \leqslant \theta < 2\pi\}$. As a consequence of (2.1), $\|C(\lambda_0)\| < 1$. This would contradict Proposition 1.2 and this completes the proof of (i).

(ii) From (1.1), $C^*(\lambda)T_\lambda^* = \sqrt{D}$ and, consequently, when D has a finite dimensional range, then $C(\lambda)$ ($\lambda \in \mathbf{C}$) has finite rank. Suppose in such a case $C = C(\lambda)$ is analytic at $\lambda_0 \in \sigma(T)$. By Proposition 1.2, $\|C(\lambda_0)\| = 1$. There is a unit vector f in $\mathcal{H}$ such that $\|C(\lambda_0)f\| = 1$. The analyticity of $C(\lambda)f$ at λ_0 forces $C(\lambda)f = C(\lambda_0)f \neq 0$ in a neighborhood of λ_0. For λ in this neighborhood

$$T_\lambda C(\lambda)f = T_\lambda C(\lambda_0)f = \sqrt{D}f = T_{\lambda_0}C(\lambda_0)f$$

and, therefore, $(\lambda - \lambda_0)C(\lambda_0)f = 0$. This is untenable. We conclude that C cannot be analytic at λ_0 and the proof of the theorem is complete.

REMARKS. 1°. The conclusion $\partial\sigma(T) \subset \Sigma(T)$ which appears in the above theorem can be improved. In fact, if we let $\sigma_e(T)$ denote the essential spectrum of T, then

$$\partial\sigma_e(T) \subset \Sigma(T). \tag{2.2}$$

The inclusion (2.2) can be seen as follows. Let π denote the projection of the algebra $\mathcal{L}(\mathcal{H})$ of bounded operators on $\mathcal{H}$ into the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, where $\mathcal{K}(\mathcal{H})$ denotes the ideal of compact operators. Then

$$\pi(T_\lambda)[\pi(I) - \pi(C(\lambda))\pi(C^*(\lambda))]\pi(T_\lambda^*) = \pi(T_\lambda^*)\pi(T_\lambda).$$

Consequently, if $\lambda \notin \sigma_e(T)$ (equivalently, $\pi(T_\lambda)$ is invertible), then $\pi(I) - \pi(C(\lambda))\pi(C^*(\lambda))$ is invertible. Thus $\lambda \notin \sigma_e(T)$ implies $\|\pi(C(\lambda))\| < 1$. Now if $\lambda_0 \in \partial\sigma_e(T)$ and $C(\lambda)$ is analytic at λ_0, we conclude from Proposition 1.1 that $\|C(\lambda_0)\| = 1$, whereas the subharmonicity of $\|\pi(C(\lambda))\|$ implies $\|\pi(C(\lambda_0))\| < 1$. It follows that for some unit vector $\|C(\lambda_0)f\| = 1$, and this leads to the contradiction that $C(\lambda)f = C(\lambda_0)f$ in a neighborhood of λ_0. In this way we are forced to conclude (2.2).

2°. The proof of statement (ii) in Theorem 2.1 does not easily extend to the case where $D = TT^* - T^*T$ is compact. The difficulty is that in such a case $C = C(\lambda)$ may fail to be compact. Such an example can be constructed as follows. Let $T_0 = U_+^* + I$ act on

$$l_2^+ = \left\{\underline{x} = \{x_j\}_{j=0}^\infty\colon \sum_{j=0}^\infty |x_j|^2 < +\infty\right\},$$

where $U_+\{x_j\}_{j=0}^\infty = \{0, x_0, x_1, \dots\}$ is "the" unilateral shift. In this case $D_0 = T_0T_0^* - T_0^*T_0 = \underline{e}_0 \otimes \underline{e}_0$, where $\underline{e}_0 = \{1, 0, 0, \dots\}$ and in the factorization $T_0C_0(0) = \sqrt{D_0} = D_0$, we have $C_0(0) = T_0^{-1}\underline{e}_0 \otimes \underline{e}_0 = \underline{e}_0 \otimes \underline{e}_0$. The operator $T = \sum_{n=0}^\infty \oplus 2^{-n}T_0$ on $\mathcal{H} = \sum_{n=0}^\infty \oplus l_2^+$ satisfies

$$D = TT^* - T^*T = \sum_{n=0}^\infty \oplus (2^{-n}\underline{e}_0 \otimes \underline{e}_0)$$

is compact, however, in the factorization $TC(0) = \sqrt{D}$, the operator $C(0) = \sum_{n=0}^\infty \oplus(\underline{e}_0 \otimes \underline{e}_0)$ is not compact.

(3) In this paragraph we relate the results on nonanalyticity of the operator function $C = C(\lambda)$ to the local spectra of a cohyponormal operator. In order to ensure that the cohyponormal operator has the s.v.e.p. we will often assume that the set $\pi_0(T)$ of eigenvalues is empty. This makes life a lot simpler. In particular, when $\pi_0(T) = \varnothing$, then $C(\lambda)f = T_\lambda^{-1}\sqrt{D}f$, for all $f \in \mathcal{H}$, $\lambda \in \mathbf{C}$, and with the notation of Theorem 2.1

$$\Sigma(T) = \bigcup_{d \in \operatorname{Ran}(\sqrt{D})} \sigma(T: d). \tag{2.3}$$

The following is an immediate consequence of (2.3) and Theorem 2.1.

COROLLARY 2.1. *Let T be a pure cohyponormal operator with $\pi_0(T) = \varnothing$ and let $D = TT^* - T^*T$. Then*

$$\text{(i)} \quad \partial\sigma(T) \subset \bigcup_{d \in \operatorname{Ran}(\sqrt{D})} \sigma(T: d). \tag{2.4}$$

(ii) *If D is of finite rank, then*

$$\sigma(T) = \bigcup_{d \in \operatorname{Ran}(\sqrt{D})} \sigma(T: d). \tag{2.5}$$

The following result was established in [**6**].

COROLLARY 2.2. *Let T be a pure cohyponormal operator with the s.v.e.p. and suppose $D = TT^* - T^*T$ is of finite rank. If Δ is a closed disc such that the interior of Δ intersects $\sigma(T)$, then there is a nonzero vector x in $\mathcal{H}$ such that*

$$\sigma(T: x) \subset \Delta. \tag{2.6}$$

PROOF. If $\Delta \cap \pi_0(T) \neq \varnothing$, then it is easily seen that any eigenvector x associated with $\lambda_0 \in \Delta \cap \pi_0(T)$ satisfies $\sigma(T: x) = \{\lambda_0\}$, hence, (2.6). If $\Delta \cap \pi_0(T) = \varnothing$, then $C(\lambda)f = T_\lambda^{-1}\sqrt{D}f$, $f \in \mathcal{H}$, $\lambda \in \Delta$. By Theorem 2.1 we know C is not analytic on the interior of Δ and, consequently, for some $f \in \mathcal{H}$, $\sigma(T: \sqrt{D}f)$ intersects the interior of Δ. We may, therefore, choose a simple closed rectifiable curve γ (having positive orientation) in the interior of Δ so that

$$x = \frac{1}{2\pi i}\int_\gamma C(\lambda)f\,d\lambda \neq 0.$$

(This last integral exists as a weak integral because of the weak continuity of $C = C(\lambda)$ on $\mathbf{C}\backslash\pi_0(T)$.) The vector function

$$x_T(\lambda) = \frac{-1}{2\pi i}\int_\gamma \frac{C(\mu)f}{\mu - \lambda}\,d\mu$$

defines a local resolvent for x on $\mathbf{C}\backslash\Delta$. It follows that $\sigma(T: x) \subset \Delta$ and this completes the proof.

3. Local spectra of cosubnormal operators. (1) An operator S on a Hilbert space $\mathcal{H}$ is called subnormal in case it is the restriction of a normal operator N acting on a superspace $\mathcal{R} \supset \mathcal{H}$ for which $N\mathcal{H} \subset \mathcal{H}$. If the normal operator N is chosen to be minimal in the sense that $\mathcal{R} = \bigvee\{N^{*j}h: h \in \mathcal{H}, j = 0, 1, 2, \ldots\}$, then the operator

N is uniquely determined up to unitary equivalence which leaves S and $\mathcal{H}$ fixed. The adjoint of a subnormal operator is referred to as a cosubnormal operator. For a recent account of the theory of subnormal operators, see Conway [**10**].

In this section we will examine briefly a simple result connecting the local spectra of a cosubnormal operator and the spectral resolution of its minimal normal extension. This result will be combined with an example of a cosubnormal operator which has the s.v.e.p. to show that the local spectral subspaces associated with cohyponormal operators need not be closed. (See Example 1. The results in Radjabalipour [**19**] could also be used to construct this example.) We also give a short discussion of the peculiar local spectral theory of the adjoint of the dual Bergman operator.

(2) Let S be subnormal on $\mathcal{H}$ and N the minimal normal extension acting on $\mathcal{R} \supset \mathcal{H}$. Relative to the decomposition $\mathcal{R} = \mathcal{H} \oplus \mathcal{H}^{\perp}$ the operator N has the 2×2 matrix form

$$N = \begin{bmatrix} S & X \\ 0 & T^* \end{bmatrix}, \tag{3.1}$$

where the subnormal operator T on $\mathcal{H}^{\perp}$ is called the dual of S.

PROPOSITION 3.1. *Let S be subnormal on $\mathcal{H}$ and N the minimal normal extension acting on the space $\mathcal{R}$. Suppose the cosubnormal operator S^* has the s.v.e.p. and denote by P the orthogonal projection of $\mathcal{R}$ onto $\mathcal{H}$. If x is in $\mathcal{R}$, then*

$$\sigma(S^* : Px) \subset \sigma(N^* : x). \tag{3.2}$$

PROOF. From (3.1), we have

$$N^* - \lambda = \begin{bmatrix} S^* - \lambda & 0 \\ X^* & T - \lambda \end{bmatrix}$$

and, as a consequence, the identity $(N^* - \lambda)x_{N^*}(\lambda) = x$ implies $(S^* - \lambda)Px_{N^*}(\lambda) = Px$. This completes the proof.

Suppose now that $N^* = \int_{\mathbf{C}} z\, dG(z)$ is the spectral resolution of N^* on $\mathcal{R}$, where N is the minimal normal extension of the subnormal operator S acting on $\mathcal{H}$ and P: $\mathcal{R} \to \mathcal{H}$ is the orthogonal projection. If δ is any closed subset of $\mathbf{C}$, then $G(\delta)\mathcal{R}$ equals the spectral subspace $M(N^* : \delta)$ of N^* associated with δ. It is easily verified that $PG(\delta)\mathcal{R}$ is invariant under S^*. Moreover, if S^* has the s.v.e.p., then (3.2) implies

$$PG(\delta)\mathcal{R} \subseteq M(S^* : \delta). \tag{3.3}$$

If S is pure, then it is plain that $G(\delta) \neq 0$ implies $PG(\delta)\mathcal{R} \neq (0)$. The inclusion (3.3) can be used to control the local spectra of a cosubnormal operator as is shown by the following:

COROLLARY 3.1. *Let S be a subnormal operator on $\mathcal{H}$ such that S^* has the s.v.e.p. If Δ is a closed disc whose interior intersects $\sigma(S^*)$, then there is a nonzero vector x in $\mathcal{H}$ such that*

$$\sigma(S^* : x) \subset \Delta. \tag{3.4}$$

PROOF. Without loss of generality it can be assumed that S is pure. If $\pi_0(S^*) \cap \Delta \neq \varnothing$, then any eigenvector of S^* associated with an eigenvalue λ_0 in Δ satisfies $\sigma(S^* : x) = \{\lambda_0\}$, hence (3.4). If $\pi_0(S^*) \cap \Delta = \varnothing$, then from the familiar spectral inclusion theorem $\sigma(N) \subset \sigma(S)$, we conclude $\sigma(N^*)$ intersects the interior of Δ. If δ is a closed disc in the interior of Δ such that the spectral resolution $N^* = \int_{\mathbf{C}} z\, dG(z)$ satisfies $G(\delta) \neq 0$ and x is a nonzero vector in $PG(\delta)\mathfrak{R}$, then (3.3) implies $\sigma(S^* : x) \subset \delta \subset \Delta$. This completes the proof.

(3) The remainder of this section is concerned with examples. If K is a compact subset of the plane having positive two-dimensional measure, then $L^2(K)$ will denote the usual Lebesgue space of functions on K which are square integrable with respect to planar Lebesgue measure. The closure in $L^2(K)$ of the functions analytic in a neighborhood of K is denoted by $L_a^2(K)$. The operator N defined on $L^2(K)$ by $Nf(z) = zf(z)$ leaves $L_a^2(K)$ invariant, and the subnormal operator obtained by restricting N to $L_a^2(K)$ will be denoted by S. It is easily verified that N is the minimal normal extension of S. The operator S may not be pure, for example, if K has no interior with connected complement, then $L_a^2(K) = L^2(K)$. The spectral resolution $N^* = \int_{\mathbf{C}} z\, dG(z)$ is easily described. If δ is a Borel set, then $G(\delta)f(z) = \chi_{\bar{\delta}} f(z)$, where $\chi_{\bar{\delta}}$ denotes the characteristic function of $\bar{\delta} = \{\bar{z} : z \in \delta\}$.

EXAMPLE 1. Let E_0 be the compact nowhere dense set constructed by Sinanjan [**20**]. The space $L_a^2(E_0)$ has the following remarkable uniqueness property. If Δ is an open disc such that $\Delta \cap E_0 \neq \varnothing$, then the only f in $L_a^2(E_0)$ vanishing on $\Delta \cap E_0$ is the zero function.

The subnormal operator $Sf(z) = zf(z)$ acting on $L_a^2(E_0)$ has the property that S^* has the single valued extension property, however, if δ is any closed disc such that the interior of δ intersects E_0 with $\delta \cap E_0 \subsetneq E_0$, then the spectral space $M(S^* : \bar{\delta})$ is not closed.

These last remarks can be seen as follows. In general, for S on $L_a^2(E)$, $\sigma(S) = \sigma(N) = \{\lambda : \Delta \cap E$ has positive measure for all discs Δ centered at $\lambda\}$. For the set E_0 it is clear that $\sigma(S) = \sigma(N) = E_0$ and, as a consequence, $\pi_0(S^*)$ has no interior. It follows easily that S^* has the single valued extension property. Now let δ be any closed disc whose interior intersects E_0. If P denotes the orthogonal projection of $L^2(E_0)$ onto $L_a^2(E_0)$, then from (3.3)

$$PG(\bar{\delta})L^2(E_0) \subset M(S^* : \bar{\delta}). \tag{3.5}$$

However, $f \in L_a^2(E_0)$ is orthogonal to $PG(\bar{\delta})L^2(E_0)$ if and only if f vanishes on $\delta \cap E_0$. Since $L_a^2(E_0)$ has the uniqueness property mentioned above, it is clear that $PG(\bar{\delta})L^2(E_0)$ is dense in $L_a^2(E_0)$. It is now plain that $M(S^* : \bar{\delta})$ is not closed, since otherwise (3.5) implies $M(S^* : \bar{\delta}) = L_a^2(E_0)$ and this would lead to the contradiction $\sigma(S^* : \bar{\delta}) = \bar{\delta} \cap \bar{E}_0 = \bar{E}_0$.

The preceding example also shows that there exist seminormal operators T with the s.v.e.p. having cyclic vectors x such that $\sigma(T : x)$ is a proper subset of $\sigma(T)$. Indeed, suppose $T = S^*$ where S is as in the above example. Let δ be a closed disc, whose interior intersects E_0 with $\delta \cap E_0$ a proper subset of E_0. Let $y \in G(\bar{\delta})L^2(E_0)$ be a cyclic vector for the operator N^* restricted to $G(\bar{\delta})L^2(E_0)$. Then $x = PG(\bar{\delta})y$ is cyclic for S^* and $\sigma(S^* : x) \subset \bar{\delta} \cap \bar{E}_0 \subsetneq \sigma(S^*) = \bar{E}_0$.

EXAMPLE 2. A subnormal operator with the most curious local spectral theory is the dual Bergman operator. This operator is defined as follows. let Δ_0 be the closed unit disc $\Delta_0 = \{\lambda: |\lambda| \leq 1\}$. Set $B^2(\Delta_0) = L^2(\Delta_0) \ominus L_a^2(\Delta_0)$. The operator $Sf(z) = zf(z)$ acting on $L_a^2(\Delta_0)$ is called the Bergman operator. The dual of S is the subnormal operator $Bf(z) = \bar{z}f(z)$ acting on $B^2(\Delta_0)$. For a discussion of some of the remarkable properties of B we refer to [**2** and **9**].

Here we mention only that B is a generalized scalar operator [**2**] which means that B and B^* have C^∞-functional calculi (see [**8**, Chapters 3 and 4]). In particular, the local spectral spaces associated with a closed subset δ of the plane for both B and B^* are closed subspaces of $B^2(\Delta_0)$. Further, if, for example, δ is a closed disc with $\varnothing \neq \delta \cap \Delta_0 \neq \Delta_0$, then $M(B:\delta)$ and $M(B^*:\bar{\delta})$ are nontrivial. Thus B and B^* have a wealth of nontrivial invariant subspaces arising out of their local spectral theory.

4. Local spectral subspaces obtained from the Cartesian decomposition. Let $T = X + iY$ be the Cartesian decomposition of a pure seminormal operator. It is a well-known result of Putnam [**16**] (see also [**7**, p. 57]) that the operator X (and Y) is an absolutely continuous selfadjoint operator. The latter means that if $X = \int_{\mathbf{R}} t\, dE(t)$ is the spectral resolution of X, then $E(\beta) = 0$ whenever β is a Borel set of Lebesgue measure zero. Consequently, there is a Borel set E_X in $\mathbf{R}$ with $E(E_X) = I$ and such that $\int_\beta dE(t) = I$ with $\beta \subset E_X$ implies $E_X \backslash \beta$ has Lebesgue measure zero. The set E_X (really E_X is an equivalence class of Borel sets) will be called the absolutely continuous support of X.

Our final result is the following extension of Theorem 1 of [**1**] and a result of Putnam [**17**].

THEOREM 4.1. *Let $T = X + iY$ be the Cartesian form of a pure seminormal operator T and E_X denote the absolutely continuous support of X. If there is a real number μ with* essinf $E_X < \mu <$ esssup E_X *and*

$$\int_{E_X} \frac{dt}{|t-\mu|} < \infty, \tag{4.1}$$

then the operator T has a nontrivial invariant subspace.

PROOF. Without loss of generality it can be assumed that $\pi_0(T) = \varnothing$ and that T^* is hyponormal. By the result of Berger and Shaw [**4**] it can be assumed that $TT^* - T^*T$ is trace class and by the result of Berger [**3**] (see also [**7**, p. 110ff]) it can be assumed that the principal function $g = g_{T^*}$ associated with T^* satisfies $0 \leq g_{T^*} \leq 1$.

The principal function g is supported on $E_X \times \mathbf{R}$ and (4.1) implies for some constant $M < +\infty$

$$\nu_g(\lambda) = \iint \frac{g(x,y)}{|x+iy-\lambda|^2}\, dx\, dy < M$$

on $\operatorname{Re}\lambda = \mu$.

As a consequence of (1.12) we learn that the operator function $C(\lambda)$ defined by (1.1) satisfies

$$\|C(\lambda)\| \leqslant \alpha < 1, \tag{4.2}$$

where α is independent of λ on $\operatorname{Re}\lambda = \mu$.

Using the notation of the proof of Proposition 1.1, it follows from (4.2) that the operator $L(\lambda)$ satisfies $\|L(\lambda)\| \leqslant \beta < 1$, where β is a constant independent of λ on $\operatorname{Re}\lambda = \mu$.

Choose $f \in \mathcal{H}$ with $d = \sqrt{D}f \neq 0$. Then

$$T_\lambda^*[L(\lambda)]^*C(\lambda)f = T_\lambda C(\lambda)f = \sqrt{D}f = d.$$

That is $(T^* - \lambda)d(\lambda) = d$ where $d(\lambda) = [L(\bar{\lambda})]^*C(\bar{\lambda})f$ is bounded and weakly continuous on $\operatorname{Re}\lambda = \mu$.

From this point on the proof is completed as in the proof of Theorem 1 of [**1**]. One uses the bounded solution $d(\lambda)$ to construct a nonzero x with $\sigma(T^* : x)$ a proper subset of $\sigma(T)$. We omit the remaining details.

EXAMPLE 1. Let E be a bounded Lebesgue measurable subset of the real line, and $a, \phi_1, \phi_2, \ldots, \phi_N$ elements in $L^\infty(E)$ with $a = \bar{a}$ and

$$\phi(t) = \sum_{j=1}^{N} |\phi_j(t)|^2 \neq 0 \quad \text{a.e. } t \in E. \tag{4.3}$$

The operator T defined on $L^2(E)$ by

$$Tf(x) = xf(x) - i\left[a(x)f(x) + \sum_{j=1}^{N} \frac{\phi_j(x)}{\pi i} \int_E \frac{\overline{\phi_j(t)}\, f(t)}{x - t}\, dt\right]$$

is a cohyponormal operator which under the assumption (4.3) is pure. Moreover, $TT^* - T^*T = (2/\pi)\sum_{j=1}^N \phi_j \otimes \phi_j$ is at most of rank N. The principal function $g = g_{T^*}$ of the operator T^* is the characteristic function of the set

$$G(a, \phi) = \{(x, y) : x \in E, a(x) - \phi(x) \leqslant y \leqslant a(x) + \phi(x)\}$$

(see, e.g., [**7**, p. 110]).

The result in Theorem 4.1 shows that the condition

$$\int_E \frac{dt}{|t - \mu|} < \infty \tag{4.4}$$

with $\operatorname{essinf} E < \mu < \operatorname{esssup} E$ implies the operator T has a nontrivial invariant subspace.

It is easy to choose examples of sets E satisfying (4.4) with $\operatorname{essinf} E < \mu < \operatorname{esssup} E$ and such that $\operatorname{Re}\lambda = \mu$ intersects $\sigma(T)$. In particular, the cohyponormal operator T satisfies (a) and (b) of Proposition 1.1 at points λ in $\sigma(T)$.

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