FORCING POSITIVE PARTITION RELATIONS

BY

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Abstract. We show how to force two strong positive partition relations on ω₁ and use them in considering several well-known open problems.

In [32] Sierpiński proved that the well-known Ramsey Theorem [27] does not generalize to the first uncountable cardinal by constructing a partition \([ω₁]^2 = K₀ ∪ K₁\) with no uncountable homogeneous sets. Sierpiński’s partition has been analyzed in several directions. One direction was to improve this relation so as to get much stronger negative partition relations on \(ω₁\). The direction taken in this paper is to prove stronger and stronger positive relations on \(ω₁\) which do not appear to be refutable by Sierpiński’s partition. The first result of this kind is due to Dushnik and Miller [9] who proved

\[ω₁ \rightarrow (ω₁, ω)^2.\]

This was later improved by Erdős and Rado [11] to

\[ω₁ \rightarrow (ω₁, ω + 1)^2.\]

In [17] Hajnal proved the following result which shows that the Erdős-Rado theorem is, in a sense, a best possible result of this sort in ZFC:

CH implies \(ω₁ \leftrightarrow (ω₁, ω + 2)^2\).

Problem 8 of Erdős and Hajnal [12, 13] asks whether \(ω₁ \leftrightarrow (ω₁, ω + 2)^2\) can be proved without the continuum hypothesis, i.e., whether \(ω₁ \rightarrow (ω₁, ω + 2)^2\) is consistent with ZFC. The first result on this problem is due to Laver [24] who proved that

\(\text{MA}_{ℵ₁} \Rightarrow ω₁ \rightarrow (ω₁, \{ω₁\})^2.\)

This result was improved by Hajnal (see [24]) to

\(\text{MA}_{ℵ₁} \Rightarrow ω₁ \rightarrow (ω₁, \{ω₁\})^2 \text{ for all } α < ω₁.\)

Clearly these results leave open the problem whether \(ω₁ \rightarrow (ω₁, ω + 2)^2\) is consistent. In this paper we shall prove the consistency of

\[ω₁ \rightarrow (ω₁, α)^2 \text{ for all } α < ω₁.\]
Let us now consider the following relation introduced by Fred Galvin (it can be considered as a dual of the usual $\rightarrow$ relation). Let $\phi$ and $\psi$ be order types and let $r$ and $\kappa$ be cardinals. Then the symbol

\[ \phi \rightarrow^* (\psi)_{<\kappa} \]

means: If $\phi = \text{tp} A$, and if $[A]' = \bigcup_{i \in I} K_i$ is a disjoint partition such that $|K_i| < \kappa$ for all $i \in I$, then there is a $B \subseteq A$ such that $\text{tp} B = \psi$ and $|[B]' \cap K_i| \leq 1$ for all $i \in I$. Let $\phi \rightarrow^*(\psi)^\kappa$ iff $\phi \rightarrow^*(\psi)_{<\kappa^+}$.

It is easily seen that $\phi \rightarrow (\psi)^\kappa$ implies $\phi \rightarrow^* (\psi)_{<\kappa}$.

Hence, $\omega_1 \rightarrow^*(\omega_1)^2$ is a weakening of $\omega_1 \rightarrow (\omega_1)^2$, and it is not “obviously” refuted by Sierpiński’s partition. However, Galvin (unpublished) proved that

$\text{CH}$ implies $\omega_1 \rightarrow^*(\omega_1)^2$.

He asked whether $\omega_1 \rightarrow^*(\omega_1)^2$ is a theorem of ZFC or not. We answer this question by proving the consistency of $\omega_1 \rightarrow^* (\omega_1)_{<\aleph_0}$, which is, in a sense, best possible since $\omega_1 \rightarrow^* (3)_{<\aleph_0}$.

The next problem we consider is the well-known $S$-space problem from general topology [22, 28, 30]. It essentially asks for a strong partition property of $\omega_1$. To state this problem we need some definitions. A topological space $X$ is hereditarily separable iff every subspace of $X$ has a countable dense subset. $X$ is called hereditarily Lindelöf iff for every family $\mathcal{U}$ of open subsets of $X$, there is a countable $\mathcal{U}_0 \subseteq \mathcal{U}$ such that $\bigcup \mathcal{U}_0 = \bigcup \mathcal{U}$. The $S$-space problem asks whether every regular hereditarily separable topological space is hereditarily Lindelöf. A counterexample to this problem is called an $S$-space. The problem has been intensively studied since the late 1960’s, and its present formulation is due to several mathematicians [22, 28, 30]. The first example of an $S$-space was constructed by M. E. Rudin [29] using a Suslin tree. Since then a number of constructions have appeared using various assumptions such as $\Diamond$, $\text{CH}$, .... Also a number of partial nonexistence results have appeared using mainly $\text{MA} + \neg \text{CH}$ (see [22, 28, 30]). In this paper we shall prove the consistency of:

Every regular hereditarily separable topological space is hereditarily Lindelöf.

Hence the $S$-space problem is undecidable on the basis of the usual axioms of set theory. Working independently and somewhat later, J. Baumgartner proved the consistency of $\text{ZFC} + \{\text{there are no weak-HFD's}\}$. (HFD’s form an important class of subspaces of $\{0,1\}^{\aleph_1}$ used by Hajnal and Juhász and others in constructing various sorts of $S$-spaces [22, 28].)
Next we are going to consider the problem of bounds on the cardinalities of Hausdorff spaces with no uncountable discrete subspaces. (A set $D \subseteq X$ is a discrete subspace of $X$ if for every $d \in D$ there exists an open subset $U_d$ of $X$ such that $D \cap U_d = \{d\}$.) That the cardinality of such a space has a bound was first independently noticed by Isbell [20] (for completely regular spaces), Efimov [10] and de Groot [16]. The bound they found was $2^{2^{\aleph_0}}$. This bound was improved to $2^{\aleph_0}$ first by de Groot [16] for the class of all regular spaces, and then by Hajnal and Juhász [18] for the class of all Hausdorff spaces. The natural question which remained unanswered is due to de Groot, Efimov and Isbell [12, Problem 77] and asks whether there exists a Hausdorff space of cardinality $(2^{\aleph_0})^+$ with no uncountable discrete subspaces. The first result on this problem is due to Hajnal and Juhász [19] who constructed such a space using a forcing argument. A compact example has since been constructed by Fedorcuk [14] using $\Diamond$. In this note we shall prove the consistency of:

Every Hausdorff space with no uncountable discrete subspaces has cardinality $\leq 2^{\aleph_0}$.

We shall deduce this result from the consistency of the following statement, which is of independent interest:

If $X$ is a Hausdorff space with no uncountable discrete subspace, then every point of $X$ is the intersection of countably many open subsets of $X$.

The results of this paper were proved while I was visiting the Department of Mathematics at Dartmouth College during the academic year 1980–81. I would like to express my gratitude to Professor James Baumgartner for making this visit possible. I would also like to thank Professor Fred Galvin for a very stimulating correspondence concerning a class of problems about strong partition relations on $\omega_1$, a small part of which is considered in this paper. The results of this paper were announced in [34, 35, and 36].

1. In this section we construct a model of $ZFC + MA_{\aleph_1}$ in which $\omega_1$ satisfies two strong partition relations which will be used in the rest of the paper. Our forcing terminology is standard (see [5, 21, 23]). All undefined terms concerning the partition calculus can be found in [37]. If $A, B \subseteq \omega_1$, then by $A \otimes B$ we denote $\{\{\alpha, \beta\} : \alpha \in A, \beta \in B, \alpha \neq \beta\}$. If $K \subseteq [\omega_1]^2$ and $\alpha \in \omega_1$, then $K(\alpha)$ denotes the set $\{\beta < \omega_1 : \{\alpha, \beta\} \in K\}$. The symbol

$\omega_1 \rightarrow (\omega_1, (\omega_1, \fin \omega_1))^2$

means: For any partition $[\omega_1]^2 = K_0 \cup K_1$ either

1. there is an $A \in [\omega_1]^{\omega_1}$ such that $[A]^2 \subseteq K_0$, or else
2. there is an $A \in [\omega_1]^{\omega_1}$ and a family $\mathbb{B}$ of $\aleph_1$ disjoint finite subsets of $\omega_1$ such that $((\alpha) \otimes F) \cap K_1 \neq \emptyset$ for all $\alpha \in A$ and $F \in \mathbb{B}$ with $\alpha < \min F$.

The set $A$ which satisfies condition (2) is called a bad set. The purpose of this section is to prove the following theorem.
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THEOREM 1. If ZF is consistent, then so is ZFC plus the following statements simultaneously:

(i) $MA + 2^{\omega_1} = \aleph_2$,

(ii) $\omega_1 \rightarrow (\omega_1, (\omega_1; \text{fin} \omega_1))^2$,

(iii) $\omega_1 \rightarrow (\omega_1)^2_{\aleph_0}$.

If $A$ is a set, then $\mathcal{C}_A$ denotes the set of all finite partial functions from $A$ into 2 ordered by $\supseteq$. Thus $\mathcal{C}_{\omega_1}$ is the standard poset for adding $\aleph_1$ Cohen reals. If $\alpha < \beta \leq \omega_1$, then we let

$$\mathcal{C}_{\alpha, \beta} = \{ p \in \mathcal{C}_{\omega_1} : p \subseteq [\alpha, \beta) \}.$$ 

Let $\mathcal{C}_{\beta} = \mathcal{C}_{0, \beta}$.

Let $\mathcal{E}$ denote the set of all pairs $(a, A)$ where $a$ is a countable closed subset of $\omega_1$ and $A$ is a closed and unbounded subset of $\omega_1$. We order $\mathcal{E}$ by

$$\langle a, A \rangle \leq \langle b, B \rangle \quad \text{iff} \quad b = a \cap (\text{max}(b) + 1) \cup A \subseteq B \cup a \backslash b \subseteq B.$$ 

Then $\mathcal{E}$ is the Jensen closed unbounded set poset [8]. It is clear that $\mathcal{E}$ is a $\sigma$-closed poset. Moreover, every countable set $\mathcal{E}_0 \subseteq \mathcal{E}$ of pairwise compatible elements has a greatest lower bound $(a, A) \in \mathcal{E}$ defined by

$$a = \bigcup \{ b : \exists B (\langle b, B \rangle \in \mathcal{E}_0) \} \quad \text{and} \quad A = \bigcap \{ B : \exists b (\langle b, B \rangle \in \mathcal{E}_0) \}.$$ 

Let $G_\mathcal{E}$ be a generic subset of $\mathcal{E}$. Then

$$C_{G_\mathcal{E}} = \bigcup \{ a : \exists A (\langle a, A \rangle \in G_\mathcal{E}) \}$$

is a closed unbounded subset of $\omega_1$ which is almost included in every club subset of $\omega_1$ from the ground model [8].

LEMMA 1. Let $\mathcal{C} = \mathcal{C}_{\omega_1}$ be the standard poset for adding $\aleph_1$ Cohen reals. Let $G_{\mathcal{C}}$ be a $V$-generic subset of $\mathcal{C}$. Let $\mathcal{E}$ be the Jensen club set poset in $V[G_{\mathcal{C}}]$. Let $(in V) \mathcal{D}$ be a $\sigma$-closed, c.c.c. poset and let $G_\mathcal{D}$ be a $V[G_{\mathcal{C}}]$-generic subset of $\mathcal{D}$. Let, in $V[G_{\mathcal{C}}]$, $[\omega_1]^2 = K_0 \cup K_1$ and $[\omega_1]^2 = \bigcup_{\xi < \omega_1} [\omega_1]^{\omega_1}$ be two disjoint partitions such that the first partition has no bad sets, while each color of the second partition is finite. Let $G_{\mathcal{E}}$ be a $V[G_{\mathcal{C}}][G_{\mathcal{D}}]$-generic subset of $\mathcal{E}$. In $V[G_{\mathcal{C}}][G_{\mathcal{D}}][G_{\mathcal{C}}]$, we define

$$\mathcal{S}_0 = \{ s \in [\omega_1]^{\aleph_1} : s \text{ is separated by } C_{\mathcal{E}} \cup [s]^2 \subseteq K_0 \},$$

$$\mathcal{S}_1 = \{ s \in [\omega_1]^{\aleph_1} : s \text{ is separated by } C_{\mathcal{E}} \cup \forall \xi < \omega_1 \ | \ [s]^2 \cap J_\xi | < 1 \}. $$ 

Let $\mathcal{S}_0$ and $\mathcal{S}_1$ be partially ordered by $\supseteq$. Then both $\mathcal{S}_0$ and $\mathcal{S}_1$ are c.c.c. posets in $V[G_{\mathcal{C}}][G_{\mathcal{D}}][G_{\mathcal{C}}]$.

PROOF. We first prove the lemma for the poset $\mathcal{S}_0$. So let, in $V[G_{\mathcal{C}}][G_{\mathcal{D}}][G_{\mathcal{C}}]$, $\langle s_\xi : \xi < \omega_1 \rangle$ be an $\omega_1$-sequence of elements of $\mathcal{S}_0$. By the standard $\Delta$-system argument we may assume that $s_\xi$'s are disjoint, increasing and of the same cardinality $n$, where $1 \leq n < \omega$. 

From now on we work in $V[G\subseteq G_2]$ and fix an $\mathcal{E}$-name $\langle \hat{s}_\xi, \xi < \omega_1 \rangle$ for the sequence $\langle \xi_{\hat{a}}, \xi < \omega_1 \rangle$ and a condition $\langle a_0, A_0 \rangle \in \mathcal{F}$ which forces that $n$ and $\langle \xi_{\hat{a}}, \xi < \omega_1 \rangle$ have the above properties. We shall find $\langle b, B \rangle \leq \langle a_0, A_0 \rangle$ such that

$$\langle b, B \rangle \not\in \mathcal{E}, \exists \xi < \omega_1 (\hat{s}_\xi \otimes x \subseteq K_0).$$

This will finish the proof of Lemma 1 for the poset $S_0$.

Note that $S$ is defined in $V[G\subseteq G_2]$, hence it is not necessarily $\sigma$-closed in $V[G\subseteq G_2]$. Instead of $\mathcal{E}$ we shall work with the set of all $\langle a, A \rangle \in \mathcal{E}$ such that $A \in V$. The restriction causes no loss of generality since this set is dense in $\mathcal{E}$. By an abuse of notation we denote it also by $\mathcal{E}$.

Let $\theta$ be a big enough regular cardinal, and let $N_0$ be a countable elementary submodel of $H_\theta$ such that $N_0 \cap V[G\subseteq G_2] \in V[G\subseteq G_2]$, and such that $\mathcal{E}_\theta$, $\langle a_0, A_0 \rangle$, $\hat{s}_\xi$, $\xi < \omega_1$, $K_0$, $K_1 \in N_0$. Since $V[G\subseteq G_2]$ is a c.c.c. extension of $V[G\subseteq G_2]$, such a submodel exists. Let $\delta_0 = N_0 \cap \omega_1$, and let $F \subseteq \omega_1 \setminus \delta_0$ be a fixed set of size $n$. Let

$$\mathcal{W}_F = \{\langle a, A \rangle \in \mathcal{E} \cap N_0 : \langle \langle a, A \rangle \not\in (a_0, A_0) \rangle \} \cup \{\langle a, A \rangle \leq \langle a_0, A_0 \rangle \& \exists s \in \delta_0 \exists \xi < \delta_0 (s \otimes F \subseteq K_0 \& \langle a, A \rangle \not\in \hat{s}_\xi = s)\}.$$ 

CLAIM 1. $\mathcal{W}_F$ is a dense open subset of $\mathcal{E} \cap N_0$.

PROOF. Let $\langle b, B \rangle \leq \langle a_0, A_0 \rangle$ be a given element of $\mathcal{E} \cap N_0$. By induction on $0 \leq i \leq n$, for each strictly increasing sequence $\langle x_1, \ldots, x_{n-i} \rangle$ of ordinals $< \omega_1$, we define the statements $\Phi_i(x_1, \ldots, x_n)$ as follows:

$$\Phi_n(x_1, \ldots, x_n) \text{ if } \exists \xi < \omega_1 \exists \alpha \exists a \exists \xi \in \langle a, A \rangle \leq \langle b, B \rangle \langle \langle a, A \rangle \not\in \hat{s}_\xi = \{x_1, \ldots, x_n\} \rangle;$$

$$\Phi_{n-i}(x_1, \ldots, x_{n-i}) \text{ if } \{y < \omega_1 : \Phi_{n-i+1}(x_1, \ldots, x_{n-i}, y)\} = \mathbb{N}, \text{ for } 0 \leq i \leq n.$$ 

We shall prove that $\Phi_0$ holds.

Starting from $N_0$ we build a strictly increasing, continuous sequence $\langle N_\alpha : \alpha < \omega_1 \rangle$ of countable elementary submodels of $H_\theta$. Let $\delta_\alpha = N_\alpha \cap \omega_1$ for $\alpha < \omega_1$, and let $D = \{\delta_\alpha : \alpha < \omega_1\}$. Then $D$ is a closed unbounded subset of $\omega_1$. Since $\mathcal{E} \times \mathbb{R}$ is a c.c.c. poset, we can find $\langle a', A' \rangle \leq \langle b, B \rangle$ and $\gamma < \omega_1$ such that $A' \setminus \gamma \subseteq D$. Choose $\langle a, A \rangle \leq \langle a', A' \rangle$, $\xi < \omega_1$, and $\{z_1, \ldots, z_n\} \subseteq \omega_1 \setminus \gamma$ such that

$$\langle a, A \rangle \not\in \hat{s}_\xi = \{z_1, \ldots, z_n\}.$$ 

This shows that $\Phi_{n-i}(z_1, \ldots, z_n)$ holds. By induction on $i$ we shall now show that $\Phi_{n-i}(z_1, \ldots, z_n)$ holds for all $0 \leq i < n$. Note that $\{z_1, \ldots, z_n\}$ is separated by $D$, so we can find $\alpha_1 < \cdots < \alpha_n < \omega_1$ such that $\delta_{\alpha_i} \leq z_i < \delta_{\alpha_{i+1}}$ for all $i = 1, \ldots, n$. So let us assume that $\Phi_{n-i+1}(z_1, \ldots, z_{n-i+1})$ holds for some $0 < i < n$. Let $Z_{n-i+1} = \{z < \omega_1 : \Phi_{n-i+1}(z_1, \ldots, z_{n-i+1}, z)\}$. Then $z_{n-i+1} \in Z_{n-i+1}$ by the assumption. Note that the parameters in the definition of $Z_{n-i+1}$ are all members of $N_{\alpha_{n-i+1}}$, which implies $Z_{n-i+1} \subseteq N_{\alpha_{n-i+1}}$. Hence we must have $N_{\alpha_{n-i+1}} \not\in Z_{n-i+1}$ is uncountable. Hence $Z_{n-i+1}$ is really uncountable. This shows that $\Phi_{n-i}(z_1, \ldots, z_n)$ holds and finishes the induction step.

Thus, in particular, $\Phi_i(z_1)$ holds, and by repeating the above argument we conclude that $\{z < \omega_1 : \Phi_i(z)\}$ is uncountable. Hence $\Phi_0$ holds.
We have already noted that the parameters of each statement $\Phi_i (0 \leq i \leq n)$ are members of $\mathbb{N}_0$. Hence, if we let $Y_i = \{ y < \omega_1 : \Phi_i(y) \}$ then $Y_i \in \mathbb{N}_0$, and by the fact that $\Phi_0$ holds, $Y_1$ is uncountable. We claim that, for some $y \in Y_1 \cap \delta_0$, \((y) \otimes F \subseteq K_0\). Otherwise the following holds:

$$N_0 \models \forall \delta < \omega_1 \exists F_\delta \in [\omega_1 \setminus \delta]'' \forall y \in Y_1 \cap \delta (\{ y \} \otimes F_\delta) \cap K_1 \neq \emptyset.$$  

Since $N_0 < H_\theta$, this sentence also holds in $H_\theta$. However, this easily gives a bad set with respect to $[\omega_1]^2 = K_0 \cup K_1$, contradicting the assumption that $V[G_\mathfrak{c}][G_2]$ is a property $K$ extension of $V[G_2]$ which contains no bad sets. So pick $y_1 \in Y_1 \cap \delta_0$ such that $\{ y_1 \} \otimes F \subseteq K_0$. Let $Y_2 = \{ y < \omega_1 : \Phi_2(y_1, y) \}$. Then by the assumption that $\Phi_2(y_1)$ holds, $Y_2$ is uncountable. Clearly $Y_2 \in \mathbb{N}_0$. By repeating the above argument, we can find a $y_2 \in Y_2 \cap \delta_0$ such that $\{ y_2 \} \otimes F \subseteq K_0$. Proceeding in this way we construct $y_1 < y_2 < \cdots < y_n$ such that \[
\{ y_1, \ldots, y_n \} \otimes F \subseteq K_0, \quad \text{and} \quad \Phi_n(y_1, \ldots, y_n) \text{ holds.}
\]

Hence $N_0 \models \Phi(y_1, \ldots, y_n)$. This means that we can find $\xi < \delta_0$ and $\langle a, A \rangle \leq \langle b, B \rangle$ such that $\langle a, A \rangle \in \mathcal{S} \cap \mathbb{N}_0$ and

$$\langle a, A \rangle \Vdash_{\mathcal{S}} \hat{s}_\xi = \{ y_1, \ldots, y_n \}.$$  

This shows that $\langle a, A \rangle \in \mathcal{G}_\mathfrak{c}$ and completes the proof of Claim 1.

Define $R \subseteq (\mathcal{S} \cap \mathbb{N}_0) \times \delta_0 \times [\delta_0]^n$ by

$$R(\langle a, A \rangle, \xi, s) \iff \langle a, A \rangle \Vdash_{\mathcal{S}} \hat{s}_\xi = s.$$  

Since $\mathcal{S} \cap \mathbb{N}_0$ and $R$ can be coded using only a countable amount of information, we can find some $\alpha < \omega_1$ such that

$$\mathcal{S} \cap \mathbb{N}_0 \in V[G_{\mathfrak{c}_\alpha}] \quad \text{and} \quad R \in V[G_{\mathfrak{c}_\alpha}][G_2].$$

In $V[G_{\mathfrak{c}_\alpha}]$ we choose a mapping $\pi$ which maps $\mathcal{G}_{\alpha, \alpha + \omega}$ isomorphically into a dense subset of $\langle \langle a, A \rangle \in \mathcal{S} \cap \mathbb{N}_0 : \langle a, A \rangle \leq \langle a_0, A_0 \rangle \rangle$. Let

$$G = G_{\mathfrak{c}_{\alpha, \alpha + \omega}} = G_{\mathfrak{c}_\alpha} \cap \mathcal{G}_{\alpha, \alpha + \omega}.$$  

Then $G$ is a $V[G_{\mathfrak{c}_\alpha}]$-generic subset of $\mathcal{G}_{\alpha, \alpha + \omega}$. So $\pi''G$ is a countable pairwise compatible subset of $\mathcal{S}$. Hence $\pi''G$ has a lower bound $\langle \bar{a}, \bar{A} \rangle$ in $\mathcal{S}$.

Note that, for each $F \in [\omega_1 \setminus \delta_0]^n$, $\mathcal{G}_F$ is definable from $F$, $K_0$, $K_1$, $\mathcal{S} \cap \mathbb{N}_0$, $\langle a_0, A_0 \rangle$, $\delta_0$, and $R$. Hence, for each $F \in [\omega_1 \setminus \delta_0]^n$, $\mathcal{G}_F \in V[G_{\mathfrak{c}_\alpha}][G_2]$. Since $G$ is also a $V[G_{\mathfrak{c}_\alpha}][G_2]$-generic subset of $\mathcal{G}_{\alpha, \alpha + \omega}$, it follows that $\pi''G$ intersects each $\mathcal{G}_F$ for $F \in [\omega_1 \setminus \delta_0]^n$. Hence, for each $F \in [\omega_1 \setminus \delta_0]^n$, there is an $\langle a, A \rangle \in \mathcal{G}_F$ such that

$$\langle a, A \rangle \leq \langle \bar{a}, \bar{A} \rangle.$$  

Pick $\langle b, B \rangle \leq \langle \bar{a}, \bar{A} \rangle$, $\eta < \omega_1$, and $F \in [\omega_1 \setminus \delta_0]^n$ such that $\langle b, B \rangle \Vdash_{\mathcal{S}} \hat{s}_\eta = F$. Let $\langle a, A \rangle \in \mathcal{G}_F$ be such that $\langle b, B \rangle \leq \langle a, A \rangle$. Thus, for some $s \in [\delta_0]^n$ and $\xi < \delta_0$, we have

$$s \otimes F \subseteq K_0 \quad \text{and} \quad \langle a, A \rangle \Vdash_{\mathcal{S}} \hat{s}_\xi = s.$$  

Hence

$$\langle b, B \rangle \Vdash_{\mathcal{S}} \hat{s}_\xi \otimes \hat{s}_\eta \subseteq K_0.$$  

This completes the proof of Lemma 1 for the case of poset $\mathcal{S}_0$.  

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The proof that $\mathcal{S}$ is a c.c.c. poset in $V[G_0][G_2][G_5]$ is similar, so we mention only the main differences. Again we start with a given sequence $\langle s^\xi: \xi < \omega_1 \rangle$ of elements of $\mathcal{S}_1$. We may assume the $s^\xi$'s form a $\Delta$-system with root $s$, and if we let $t^\xi = s^\xi \setminus s$ for $\xi < \omega_1$, then the $t^\xi$'s are strictly increasing and of the same cardinality $n$, where $1 \leq n < \omega$.

Working in $V[G_0][G_2][G_5]$, we fix $\delta$-names $\langle \dot{s}^\xi: \xi < \omega_1 \rangle$ and $\langle \dot{t}^\xi: \xi < \omega_1 \rangle$ for the sequences $\langle s^\xi: \xi < \omega_1 \rangle$ and $\langle t^\xi: \xi < \omega_1 \rangle$, and a condition $\langle a_0, A_0 \rangle \in \dot{\mathcal{D}}$ which forces that $n, s, \langle \dot{s}^\xi: \xi < \omega_1 \rangle$ and $\langle \dot{t}^\xi: \xi < \omega_1 \rangle$ have the above properties. We need to find $\langle b, B \rangle \leq \langle a_0, A_0 \rangle$ such that

$$\langle b, B \rangle \Vdash \exists \xi < \eta < \omega_1 \forall \xi < \omega_1 \left[ [\dot{s}^\xi \cup \dot{s}^\eta]^2 \cap J^\xi \right] \leq 1.$$

We fix a cardinal $\theta$ and a countable elementary symbol $N_0$ of $H_\theta$ as before. Let

$$D = \{ \delta < \omega_1: \lim(\delta) \& \forall \alpha < \delta \forall \xi < \omega_1 \left( J^\xi \cap [\delta]^2 \neq \emptyset \Rightarrow J_s(\alpha) \subseteq \delta \right) \}.$$

Then $D$ is a closed unbounded subset of $\omega_1$ such that $D \in N_0$. Now we fix an $F \in [\omega_1 \setminus \delta]^{\omega_1}$ which is separated by $D$ such that

$$\forall \xi < \omega_1 \left[ [s \cup F]^2 \cap J^\xi \right] \leq 1,$$

and define

$$\mathcal{D}_F = \left\{ \langle a, A \rangle \in \dot{\mathcal{D}} \cap N: \langle \langle a, A \rangle \perp \langle a_0, A_0 \rangle \rangle \right\} \cup \left\{ \langle a, A \rangle \leq \langle a_0, A_0 \rangle \& \exists t \subseteq \delta_0 \exists \xi < \delta_0 \left( \forall \xi < \omega_1 \left[ [s \cup t \cup F]^2 \cap J^\xi \right] \leq 1 \& \langle a, A \rangle \Vdash \dot{t}^\xi = t \right) \right\}.$$

As before we claim that $\mathcal{D}_F$ is a dense open subset of $\dot{\mathcal{D}} \cap N_0$. So let $\langle b, B \rangle \leq \langle a_0, A_0 \rangle$ be a given element of $\dot{\mathcal{D}} \cap N_0$. The statements $\Phi_n(x_1, \ldots, x_{n-1})$ ($0 \leq i \leq n$) are defined as before. The proof that $\Phi_0$ holds is also the same.

Let $Y_1 = \{ y < \omega_1: \Phi_1(y) \}$. Then $Y_1$ is uncountable and $Y_1 \in N_0$. Let $\{\xi_1, \ldots, \xi_{k_1}\}$ be a list of all $\xi < \omega_1$ such that $[s \cup F]^2 \cap J^\xi \neq \emptyset$. Since each $J^\xi$ is finite, and since $Y_1 \cap \delta_0$ is infinite, we can find $y_1 \in Y_1 \cap \delta_0$ such that

$$\left( \{y_1\} \otimes (s \cup F) \right) \cap J^\xi = \emptyset \text{ for all } \xi \in \{\xi_1, \ldots, \xi_{k_1}\}.$$

We claim that

$$\left[ [s \cup \{y_1\} \cup F]^2 \cap J^\xi \right] \leq 1 \text{ for all } \xi < \omega_1.$$

Otherwise, let $\xi < \omega_1$ be such that $[s \cup \{y_1\} \cup F]^2 \cap J^\xi$ contains two different edges $l_0$ and $l_1$. It is clear that $l_0$ and $l_1$ cannot both be subsets of $s \cup \{y_1\}$ since some condition from $\dot{\mathcal{D}}$ forces this set to be a subset of a member of $\mathcal{S}_1$. From the definition of $D$ it easily follows that $\max l_0 = \max l_1$. Hence, $\max l_0 = \max l_1 \in F$. Hence, for some $i < 2$, $l_i \in s \otimes F$, which means that $\xi \in \{\xi_1, \ldots, \xi_{k_1}\}$. It follows that

$$\left( \{y_1\} \otimes (s \cup F) \right) \cap J^\xi = \emptyset.$$

Consequently, $\min l_0 \neq y_1$ and $\min l_1 \neq y_1$, which yields the contradiction $l_1, l_2 \in [s \cup F]^2$. 

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Now let $Y_2 = \{ y < \omega_1 : \Phi_2(y_1, y) \}$. Then $Y_2$ is uncountable and $Y_2 \in N_0$. Working as above, we can find $y_2 \in Y_2 \cap \delta$ such that

$$\left[ s \cup \{ y_1, y_2 \} \cup F \right]^2 \cap J_\xi \leq 1 \quad \text{for all} \ \xi < \omega_1.$$ 

Proceeding in this way we construct $y_1 < y_2 < \cdots < y_n < \delta_0$ such that

$$\forall \xi < \omega_1 \left[ \left[ s \cup \{ y_1, \ldots, y_n \} \cup F \right]^2 \cap J_\xi \leq 1, \right.$$ 

and

$$\Phi_n(y_1, \ldots, y_n) \text{ holds.}$$

Hence $N_0 \models \Phi_n(y_1, \ldots, y_n)$. So we can find $\xi < \delta_0$ and $\langle a, A \rangle \leq \langle a_0, A_0 \rangle$ such that $\langle a, A \rangle \in \mathcal{D} \cap N_0$ and

$$\langle a, A \rangle \Vdash \{ y_1, \ldots, y_n \}$$

This shows that $\langle a, A \rangle \in \mathcal{U}_F$. Hence, $\mathcal{U}_F$ is a dense open subset of $\mathcal{D} \cap N_0$.

We leave the remainder of the proof of Lemma 1 to the reader since the rest of the proof for $S_1$ is like the proof for $S_0$.

Now we are going to describe a mixed iteration $\langle \mathcal{P}_a : a < \omega_2 \rangle$ of Cohen and Jensen partially ordered sets. This will be done by induction on $a$, and for this purpose let $E$ and $O$ denote the sets of all even and odd ordinals $< \omega_2$, respectively.

If $a = 0$, then $\mathcal{P}_a = \emptyset$.

If $a \in E$, then $A_{a+1}$ is the set of all functions $p$ with domain $a + 1$ such that

$$p \upharpoonright a \in \mathcal{P}_a \text{ and } \{ p(a) : \gamma < \omega_1 \} \text{ is a member of } \mathcal{C}_{(a) \times \omega_1}.$$ 

If $a \in O$, then $\mathcal{P}_{a+1}$ is the set of all functions $p$ with domain $a + 1$ such that

$$p \upharpoonright a \in \mathcal{P}_a \text{ and } \{ p(a) : \gamma \in \text{Jensen club set poset } \mathcal{C}_a \}.$$ 

If $p, q \in \mathcal{P}_{a+1}$, let $p \leq q$ iff $p \upharpoonright a \leq q \upharpoonright a$ and $p \upharpoonright a \Vdash \mathcal{P}_a p(a) \supseteq q(a)$.

If $a \in O$, then $\mathcal{P}_{a+1}$ is the set of all functions $p$ with domain $a + 1$ such that

$$p \upharpoonright a \in \mathcal{P}_a \text{ and } \{ p(a) : \gamma \in \text{Jensen ordering} \}.$$ 

If $a$ is a limit ordinal with $\text{cf} a = \omega$, then $\mathcal{P}_a$ is the set of all functions $p$ with domain $a$ such that $p \upharpoonright \beta \in \mathcal{P}_\beta$ for all $\beta < a$, and for some $\gamma < \omega_1$, $\mathcal{P}_a p(\beta) = \emptyset$ for all $\beta \in [\gamma, a) \cap E$.

If $a$ is a limit ordinal with $\text{cf} a > \omega$, then $\mathcal{P}_a$ is the set of all functions $p$ with domain $a$ such that $p \upharpoonright \beta \in \mathcal{P}_\beta$ for all $\beta < a$, and for some $\gamma < \omega_1$, $\mathcal{P}_a p(\beta) = \emptyset$ for all $\beta \in [\gamma, a) \cap E$ and $\mathcal{P}_a p(\beta) = \langle \emptyset, \omega_1 \rangle$ for $\beta \in [\gamma, a) \cap O$.

In both limit cases we put $p \leq q$ iff $p \upharpoonright \beta \leq q \upharpoonright \beta$ for all $\beta < a$.

From now on let $a \leq \omega_2$ be a fixed ordinal. For $p \in \mathcal{P}_a$ we define $\text{supp}(p) = \{ \beta < a : p(\beta) \neq \emptyset \}$ if $\beta \in E$ and $p(\beta) \neq \langle \emptyset, \omega_1 \rangle$ if $\beta \in O$. Then it is easily checked that $\text{supp}(p) \cap E$ is finite, and that $\text{supp}(p) \cap O$ is at most countable.

We say that $p \in \mathcal{P}_a$ is a determined condition if, for every $\beta \in \text{supp}(p) \cap E$, there is some $s_\beta(p) \in \mathcal{C}_{(\gamma) \times \omega_1}$ such that $p(\beta) = s_\beta(p)$ (more precisely, $p(\beta) = (s_\beta(p))$). If $p \in \mathcal{P}_a$ is a determined condition, then by $\sigma(p)$ we denote

$$\bigcup \{ s_\beta(p) : \beta \in \text{supp}(p) \cap E \}.$$
considered as a member of $C(\alpha) := C(E \cap \alpha) \times \omega$. By induction on $\alpha$ it is easily seen that the set of all determined conditions is dense in $\mathcal{P}_\alpha$. So from now on we shall always work with determined conditions, and use $\mathcal{P}_\alpha$ informally to denote also the set of all determined numbers of $\mathcal{P}_\alpha$.

Note that the function $\sigma: \mathcal{P}_\alpha \rightarrow C(\alpha)$ is order preserving, and has the property that if $\sigma(p) = r$ and $r' \leq r$, then for some $p' \leq p$, $\sigma(p') = r'$. Hence, forcing with $\mathcal{P}_\alpha$ can be considered as forcing first with $C(\alpha)$ and then with $\{ p \in \mathcal{P}_\alpha: \sigma(p) \in G_{C(\alpha)} \}$.

**Lemma 2.** For every $f \in V^{\mathcal{P}_\alpha}$ and $p_0 \in \mathcal{P}_\alpha$ with the property $p_0 \Vdash \omega \rightarrow \text{On}$, there are $g \in V^{C(\alpha)}$ and $p \leq p_0$ such that $\sigma(p) = \sigma(p_0)$ and $p \Vdash \omega \rightarrow \omega$.

**Proof.** If $p \in \mathcal{P}_\alpha$ and if $r \in C(\alpha)$ is compatible with $\sigma(p)$, then $p \land r$ denotes the following member $q$ of $\mathcal{P}_\alpha$. If $\beta \in \text{On}$, then $q(\beta) = \sigma(p)(\beta)$. If $\beta \in \text{On}$, then $q(\beta) = \sigma(p)(\beta)$ $\cup$ $(r \upharpoonright \{ \beta \} \times \omega)$. It is clear that this is a well-defined condition and that $p \land r \leq p$.

**Claim 2.** Assume $p$, $p' \in \mathcal{P}_\alpha$ are such that $p' \leq p$. Then there exists a $q \in \mathcal{P}_\alpha$, with $q < p$ such that $\sigma(q) = \sigma(p)$, and $p' = q \land \sigma(p')$.

**Proof.** We define $q \upharpoonright \beta$ by induction on $\beta < \alpha$. Assume $q \upharpoonright \beta$ is defined. If $\beta \in \text{On}$, let $q(\beta) = q(\beta)$. If $\beta \in \text{On}$, let $q(\beta)$ be a $\mathcal{P}_\beta$-name for a member of the Jensen poset $\mathcal{D}_\beta$ which is equal to $p(\beta)$ if $p \upharpoonright \beta$ is a member of $G_{\mathcal{P}_\beta}$, and equal to $p(\beta)$ otherwise. If $\beta$ is a limit ordinal, let $q(\beta) = \bigcup \{ q(\gamma): \gamma < \beta \}$. Now by induction on $\beta < \alpha$ one easily checks that

$$q \upharpoonright \beta \in \mathcal{P}_\beta, \quad q \upharpoonright \beta \leq q \upharpoonright \beta, \quad \sigma(q \upharpoonright \beta) = \sigma(q \upharpoonright \beta),$$

and

$$p' \upharpoonright \beta \leq (q \land \sigma(p')) \upharpoonright \beta \leq p' \upharpoonright \beta.$$

This completes the proof of Claim 2.

Starting from $p_0$, by induction on $n < \omega$, we define a sequence $\langle p_n: n < \omega \rangle$ of members of $\mathcal{P}_\alpha$, and for each $n < \omega$ sequences $\langle r^n_\xi: \xi < \delta_n \rangle$ and $\langle x^n_\xi: \xi < \delta_n \rangle$ of members of $C(\alpha)$ and $\text{On}$, respectively, such that

1. $p_{n+1} \leq p_n$,
2. $\sigma(p_{n+1}) = \sigma(p_n)$,
3. $p_{n+1} \land r^n_{\xi} \Vdash \omega \rightarrow \omega$,
4. $\{ r^n_\xi: \xi < \delta_n \}$ is a maximal antichain below $\sigma(p_n)$.

Let us first see how to prove the lemma using such sequences. By induction on $\beta \leq \alpha$ we construct $p \upharpoonright \beta \in \mathcal{P}_\beta$ such that $p \upharpoonright \beta \leq p_n \upharpoonright \beta$ and $\sigma(p \upharpoonright \beta) = \sigma(p_n \upharpoonright \beta)$ for all $n < \omega$. Assume $p \upharpoonright \beta$ is constructed. If $\beta \in \text{On}$, let $p(\beta) = p_0(\beta)$. If $\beta \in \text{On}$, let $p(\beta)$ be a $\mathcal{P}_\beta$-name for the greatest lower bound of $\{ p_n(\beta): n < \omega \}$ in $\mathcal{D}_\beta$. If $\beta$ is a limit ordinal, let $p \upharpoonright \beta = \bigcup \{ p \upharpoonright \gamma: \gamma < \beta \}$. Then it is easily checked that $p \upharpoonright \beta$ is a well-defined condition, and that $p = p \upharpoonright \alpha$ has the properties $p \leq p_n$ and $\sigma(p) = \sigma(p_n)$ for all $n < \omega$. Since $p$ is uniquely determined by $\langle p_n: n < \omega \rangle$, we shall denote it by $\land_{n < \omega} p_n$.

Define $\dot{g} \in V^{C(\alpha)}$ to be a function from $\omega$ into the ordinals such that

$$\| \dot{g}(n) = r^n_\xi \| = r^n_\xi \quad \text{for} \quad n < \omega \quad \text{and} \quad \xi < \delta_n.$$

Then by (1)–(4) we have

$$p \Vdash \omega \rightarrow \omega \quad \text{and} \quad p \Vdash \omega \rightarrow \omega \rightarrow (\omega \rightarrow \omega).$$

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So we are left with the construction of \( \langle p_n : n < \omega \rangle \). Assume \( p_n \) is defined. By induction on \( \xi < \delta_n \) we define sequences \( \langle q^n_\xi : \xi < \delta_n \rangle \), \( \langle r^n_\xi : \xi < \delta_n \rangle \) of members of \( \mathcal{C}(\alpha) \), \( \mathcal{C}(\alpha) \) and \( \mathcal{O}_n \), respectively, such that

(5) \( q^n_\xi \leq q^n_\zeta \leq p_n \) for \( \xi < \zeta \),

(6) \( \sigma(q^n_\xi) = \sigma(p_n) \),

(7) \( r^n_\xi \leq \sigma(p_n) \),

(8) \( r^n_\xi \perp r^n_\zeta \) for \( \xi \neq \zeta \),

(9) \( q^n_\xi \cap r^n_\xi \not\models f(n) = x^n_\xi \).

The ordinal \( \delta_n \) is a countable ordinal determined by the fact that \( \{ r^n_\xi : \xi < \delta_n \} \) is a maximal antichain below \( \sigma(p_n) \). Assume \( q^n_\xi \)'s, \( r^n_\xi \)'s and \( x^n_\xi \)'s are defined for every \( \xi < \xi < \omega \). If \( \{ r^n_\xi : \xi < \xi \} \) is a maximal antichain below \( \sigma(p_n) \), we let \( \delta_n = \zeta \) and \( p_{n+1} = \bigwedge_{\xi < \delta_n} q^n_\xi \). Clearly (1)–(4) are satisfied. So let us assume \( \{ r^n_\xi : \xi < \delta_n \} \) is not a maximal antichain below \( \sigma(p_n) \). Pick \( r \leq \sigma(p_n) \) which is incompatible with each \( r^n_\xi \) (\( \xi < \delta \)). Let \( q = \bigwedge_{\xi < \delta_n} q^n_\xi \). Choose \( q' \leq q \wedge r \) and \( x^n_\xi \) such that \( q' \not\models f(n) = x^n_\xi \). By Claim 2 we can find \( q^n_\xi \leq q \) such that \( \sigma(q^n_\xi) = \sigma(q) = \sigma(p_n) \) and \( q' = q^n_\xi \cap \sigma(q^n_\xi) \). Let \( r^n_\xi = \sigma(q^n_\xi) \). It is clear that (5)–(9) are satisfied. This completes the proof of Lemma 2.

Note that, in particular, Lemma 2 shows that \( \mathcal{P}_\alpha \) preserves \( \aleph_1 \). If \( \alpha < \omega_2 \), let

\[
\mathcal{P}_{\alpha, \omega_2} = \{ p \upharpoonright (\omega_2 \setminus \alpha) : p \in \mathcal{P}_\alpha \}.
\]

Let \( \mathcal{P}_{\alpha, \omega_2} \) be ordered in \( V_{\ell_{\mathcal{P}}^\alpha} \) by the ordering \( \leq \) as defined as follows:

\[
q' \leq q \iff \exists p \in G_{\ell_{\mathcal{P}}^\alpha}(p \cup q' \leq p \cup q).
\]

Then \( \mathcal{P}_{\omega_2} = \mathcal{P}_\alpha \star \mathcal{P}_{\alpha, \omega_2} \) and Lemma 2 easily gives

\[
\#_{\ell_{\mathcal{P}}^\alpha} \mathcal{P}_{\alpha, \omega_2} \text{ is equivalent to } \mathcal{P}_{\omega_2}.
\]

**LEMMA 3.** Assume CH. Then \( \mathcal{P}_{\omega_2} \) satisfies the \( \aleph_2 \)-chain condition.

**Proof.** Since elements of \( \mathcal{P}_{\omega_2} \) have countable supports, a standard application of Fodor’s Lemma shows that we may restrict ourselves to proving that, for each \( \alpha < \omega_2 \), \( \mathcal{P}_\alpha \) satisfies the \( \aleph_2 \)-c.c.

So let \( \alpha < \omega_2 \) and let \( \mathcal{P}_\alpha \subseteq \mathcal{P}_\alpha \) be the set of all \( p \in \mathcal{P}_\alpha \) such that for every \( \beta \in O \), there is a \( \mathcal{C}(\beta) \)-name \( a_\beta^p \) for a countable closed subset of \( \omega_1 \) and a \( \mathcal{C}(\beta) \)-name \( A_\beta^p \) for a closed and unbounded subset of \( \omega_1 \) such that \( p \upharpoonright \beta \#_{\ell_{\mathcal{P}}^\alpha} p(\beta) = \langle a_\beta^p, A_\beta^p \rangle \).

**Claim 3.** For every \( q \in \mathcal{P}_\alpha \) there is a \( p \in \mathcal{P}_\alpha \) such that \( p \leq q \) and \( \sigma(p) = \sigma(q) \).

**Proof.** We prove the claim by induction on \( \alpha \).

Assume \( \alpha = \beta + 1 \). If \( \beta \in E \), there is nothing to be proved. So assume \( \beta \in O \). By Lemma 2 we can find \( q' \leq q \upharpoonright \beta \) and a \( \mathcal{C}(\beta) \)-name \( a_\beta^p \) such that \( \sigma(q') = \sigma(q) = \sigma(q) \) and

\[
q' \not\models A_\beta^p \text{ the first coordinate of } q(\beta) \text{ is equal to } a_\beta^p.
\]

By the induction hypothesis we can find a \( p' \in \mathcal{P}_\beta \) such that \( p' \leq q' \) and \( \sigma(p') = \sigma(q') \). Let

\[
p = p' \cup \left\{ \left( \beta, \left( a_\beta^p, A_\beta^p \right) \right) \right\},
\]
where \( A^\beta_p \) is a \( \mathfrak{P}_\beta \)-name such that \( \mathbb{P}_\alpha \downarrow A^\beta_p \) is the second coordinate of \( q(\beta) \). Then \( p \in \mathfrak{P}_\alpha \), \( p \leq q \) and \( \sigma(p) = \sigma(q) \).

First assume \( \alpha = \omega \). Let \( \langle \alpha_n : n < \omega \rangle \) be a strictly increasing sequence of ordinals cofinal with \( \alpha \) such that \( \alpha_0 = 0 \). By induction on \( n < \omega \) we construct a sequence \( \langle p_n : n < \omega \rangle \) of elements of \( \mathfrak{P}_\alpha \) such that \( p_0 = q \) and

1. \( p_{n+1} \leq p_n \) and \( \sigma(p_{n+1}) = \sigma(p_n) \),
2. \( p_{n+1} \uparrow \alpha_{n+1} \in \mathfrak{P}_\alpha \).

Assume \( p_n \) is defined. By the induction hypothesis we can find \( p'_{n+1} \in \mathfrak{P}_{\alpha_{n+1}} \) such that \( p_{n+1} \leq p'_{n+1} \alpha_{n+1} \) and \( \sigma(p'_{n+1}) = \sigma(p_n \uparrow \alpha_{n+1}) \). Let \( p_{n+1} \in \mathfrak{P}_\alpha \) be defined by \( p_{n+1} \alpha_{n+1} = p'_{n+1} \) and \( p_{n+1}(\beta) = p_n(\beta) \) for all \( \beta \in [\alpha_{n+1}, \alpha) \). Then \( p_{n+1} \leq p_n \), \( \sigma(p_{n+1}) = \sigma(p_n) \), and \( p_{n+1} \alpha_{n+1} \in \mathfrak{P}_{\alpha_{n+1}} \).

Define \( p \in \mathfrak{P}_\alpha \) as follows. If \( \beta \in E \), let \( p(\beta) = q(\beta) \). So suppose \( \beta \in O \), and let \( w < \omega \) be such that \( \beta \in [\alpha_w, \alpha_{w+1}) \). Let \( \mathfrak{P}_\alpha \) be a \( \mathfrak{C}(\beta)-\)name for the closure of \( \bigcup \{ A^\beta_m : n < i < \omega \} \), and let \( A^\beta_p \) be a \( \mathfrak{P}_\beta \)-name for \( \bigcap \{ A^\beta_m : n < i < \omega \} \). Let \( p(\beta) = \langle A^\beta_m, A^\beta_{m+1} \rangle \). Then \( p \in \mathfrak{P}_\alpha \), \( p \leq q \) and \( \sigma(p) = \sigma(q) \).

Now assume \( \alpha > \omega \). Since \( \text{supp}(q) \) is countable, there is a \( \gamma < \alpha \) such that \( \text{supp}(q) \subseteq \gamma \). Using the induction hypothesis, we can find a \( p' \in \mathfrak{P}_\gamma \) such that \( p' \leq q \gamma \) and \( \sigma(p') = \sigma(q \gamma) = \sigma(q) \). Define \( p \in \mathfrak{P}_\alpha \) by \( p \gamma = p' \) and \( p(\beta) = \emptyset \) for all \( \beta \in [\gamma, \alpha) \cap E \) and \( p(\beta) = \langle \emptyset, \omega_1 \rangle \) for all \( \beta \in [\gamma, \alpha) \cap O \). Then \( p \in \mathfrak{P}_\alpha \), \( p \leq q \) and \( \sigma(p) = \sigma(q) \). This proves the claim.

Suppose \( p, q \in \mathfrak{P}_\alpha \) are such that \( p(\beta) = q(\beta) \) for every \( \beta \in E \) and \( \mathbb{P}_\alpha \downarrow A^\beta_p = A^\beta_q \) for every \( \beta \in O \). We claim that then \( p \) and \( q \) are compatible in \( \mathfrak{P}_\alpha \). To see this let us define \( p' \in \mathfrak{P}_\alpha \) as follows. If \( \beta \in E \), let \( p'(\beta) = p(\beta) = q(\beta) \). If \( \beta \in O \), we choose \( p'(\beta) \) to satisfy \( \mathbb{P}_\alpha p'(\beta) = \langle A^\beta_p, A^\beta_{p'} \rangle \). Then clearly \( p' \in \mathfrak{P}_\alpha \) and \( p' \leq p, q \).

Since there are only \( \aleph_1 \mathfrak{C}(\alpha) \)-names of countable closed subsets of \( \omega_1 \), this finishes the proof of Lemma 3.

Now we are ready to finish the proof of Theorem 1. Assume GCH holds. Let \( \langle \mathfrak{P}_\alpha : \alpha < \omega_2 \rangle \) be the iteration defined above and let \( \mathfrak{P} = \mathfrak{P}_{\omega_2} \). Then in \( \mathbb{V}^{\mathfrak{P}} \), \( 2^{\aleph_0} = 2^{\aleph_1} = \aleph_3 \) holds. Working in \( \mathbb{V}^{\mathfrak{P}} \), we define a finite support iteration \( \langle \mathfrak{P}_\xi : \xi < \omega_2 \rangle \) of c.c.c. posets of size \( \leq \aleph_1 \), à la Solovay and Tennenbaum [33], such that if \( \mathfrak{P}_\xi = \mathfrak{P}_{\omega_2} \), then \( \mathbb{V}^{\mathfrak{P}} \) satisfies (i)-(iii) of Theorem 1.

Assume \( \xi < \omega_2 \) and that in \( \mathbb{V}^{\mathfrak{P}} \) we have a partition \( [\omega_1]^2 = \dot{K}_0 \cup \dot{K}_1 \) with no bad sets. Pick an even ordinal \( \alpha < \omega_2 \) such that \( \dot{K}_0, \dot{K}_1 \in \mathbb{V}^{\mathfrak{P}} \). We have already remarked that \( \mathfrak{P}_{\alpha, \omega_2} \) is, in \( \mathbb{V}^{\mathfrak{P}} \) (equivalent to) a mixed iteration of length \( \omega_2 \) of Cohen and Jensen posets. It begins by first introducing \( \aleph_1 \) Cohen reals and then adding a Jensen club. So by Lemma 1, the poset \( \dot{S}_\xi \) of all finite 0-homogeneous subsets of \( \omega_1 \), which are separated by \( C_{\xi+1} \) in \( \mathbb{V}^{\mathfrak{P}_{\omega_2+\dot{S}_\xi}} \), is a c.c.c. poset in \( \mathbb{V}^{\mathfrak{P}_{\omega_2+\dot{S}_\xi}} \). Hence, one condition \( s_0 \in \dot{S}_\xi \) forces that the generic object is uncountable. At the next step of the iteration we force with \( \langle s \in \dot{S}_\xi : s \supseteq s_0 \rangle \), which we again denote by \( \dot{S}_\xi \). We know that \( \dot{S}_\xi \) is a c.c.c. poset in \( \mathbb{V}^{\mathfrak{P}_{\omega_2+\dot{S}_\xi}} \), but we have to show that it remains c.c.c after forcing with \( \mathfrak{P}_{\alpha+2, \omega_2} \). Let us prove the following more general fact. Let \( \dot{P} \) be an arbitrary c.c.c. poset. Then \( \dot{P} \) remains c.c.c. after forcing with \( \mathfrak{P} = \mathfrak{P}_{\omega_2} \). Otherwise, pick a \( \mathfrak{P} \)-name
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As in the proof of Lemma 2, by induction on \( \gamma \) we construct a decreasing sequence \( \langle p_\gamma : \gamma < \omega_1 \rangle \) of members of \( \mathcal{P} \), a sequence \( \langle r_\gamma : \gamma < \omega_1 \rangle \) of members of \( \mathcal{G}_{E \times \omega_1} \), and a sequence \( \langle q_\gamma : \gamma < \omega_1 \rangle \) of members of \( \mathcal{D} \) such that

1. \( \sigma(p_\gamma) = \sigma(p_\delta) \) for \( \gamma < \delta < \omega_1 \),
2. \( p_\gamma \wedge r_\gamma \models \tau = q_\gamma \).

Pick an \( A \in [\omega_1]^{\aleph_1} \) such that \( r_\gamma \) and \( r_\delta \) are compatible, whenever \( \gamma, \delta \in A \). Then it is easily checked that we have reached a contradiction since \( \{ q_\gamma : \gamma \in A \} \) is an uncountable antichain of \( \mathcal{D} \).

Similarly, one defines posets for getting \( \omega_1 \rightarrow (\omega_1)^2 \) as well as the posets for getting \( \mathbf{MA}_{\aleph_1} \). This completes the proof of Theorem 1.

The following result (in ZFC) is an easy consequence of Lemma 1.

**Theorem 2.** \( \omega_1 \rightarrow (\omega_1)^2 \) for all \( \alpha < \omega_1 \).

**Proof.** Let \( [\omega_1]^2 = \bigcup_{\xi < \omega_1} J_\xi \) be a given disjoint partition such that \( |J_\xi| < \aleph_0 \) for all \( \xi < \omega_1 \). Let \( \omega \leq \alpha < \omega_1 \) be fixed. Let \( \mathcal{S} \subseteq V^{\mathcal{C}_{\omega_1}} \) be the set of all finite subsets of \( \omega_1 \) separated by \( C_\xi \) such that \( |s|^2 \cap J_\xi| \leq 1 \) for all \( \xi < \omega_1 \). By Lemma 1, \( \mathcal{S} \) is a c.c.c. poset in \( V^{\mathcal{C}_{\omega_1}} \). So we can find an \( s_0 \in \mathcal{S} \) such that \( s_0 \upharpoonright s \cup \bigcup s \) is stationary in \( \omega_1 \).

Thus, in particular, we can find a \( p_0 \in \mathcal{C}_{\omega_1} \) and a \( (\mathcal{C}_{\omega_1} \ast \mathcal{S} \ast \mathcal{S}) \)-name \( \dot{A} \) such that

\[ p_0 \upharpoonright \dot{A} \text{ is a closed subset of } \omega_1 \text{ of type } \alpha + 1 \text{ & } \forall \xi < \omega_1 \left[ |\dot{A}|^2 \cap J_\xi| \leq 1 \right]. \]

Pick a \( (\mathcal{C}_{\omega_1} \ast \mathcal{S} \ast \mathcal{S}) \)-name \( \dot{f} \) such that \( p_0 \upharpoonright \dot{f} : \alpha + 1 \rightarrow \dot{A} \) is the unique isomorphism. Let \( \langle \alpha_n : n < \omega \rangle \) be an enumeration of \( \alpha + 1 \). Now by induction on \( n < \omega \) we define a decreasing sequence \( \langle p_n : n < \omega \rangle \) of elements of \( \mathcal{C}_{\omega_1} \ast \mathcal{S} \ast \mathcal{S} \) and a sequence \( \langle \beta_n : n < \omega \rangle \) of ordinals \( < \omega_1 \) such that \( p_{n+1} \upharpoonright \dot{f}(\alpha_n) = \beta_n \), making sure that \( B = \{ \dot{B}_n : n < \omega \} \) is a closed subset of \( \omega_1 \). Then \( tp B = \alpha + 1 \) and \( \forall \xi < \omega_1 \left[ |B|^2 \cap J_\xi| \leq 1 \right] \).

This completes the proof.

**Remarks.** (1) The closed unbounded set poset was defined and first used in building c.c.c. posets in the extension by Jensen [8]. The fact that an elementary chain of submodels is useful in proving the c.c.c. property of posets with separated conditions was first realized by Shelah [1, 3]. The use of the Cohen generic reals in building conditions in \( \sigma \)-closed posets was first made explicit by Avraham [2]. The first mixed iteration of Cohen posets and \( \sigma \)-closed posets was defined by Mitchell [26]. It is clear that if we want only to preserve \( \aleph_1 \), then in the above mixed iteration the Jensen posets can be replaced by any \( \sigma \)-closed poset. If we want the iteration to have the \( \aleph_2 \)-c.c., the \( \sigma \)-closed posets must satisfy one of the standard strong \( \aleph_2 \)-chain conditions.

(2) The posets involved in Lemma 1 can also be iterated in a countable support iteration \( \langle \mathcal{S}_n : \alpha < \omega_2 \rangle \). A Laver type argument shows that \( \mathcal{S}_{\omega_2} \) preserves \( \aleph_1 \) [5, 25]. Using GCH, one then shows that \( \mathcal{S}_{\omega_2} \) satisfies the \( \aleph_2 \)-c.c.

(3) If we are not interested in the exact equiconsistency result, we could use the Proper Forcing Axiom (PFA; [6, 7, 31]) in showing that (i)–(iii) of Theorem 1 are
consistent. Namely, in this case, in Lemma 1, we can disregard $\mathcal{G}$ and $\mathcal{C}_{\omega_1}$, and directly show by the same proof that $\mathcal{S}_0$ and $\mathcal{S}_1$ are c.c.c. posets in $\mathcal{V}^\mathcal{G}$. To build a condition which will meet all the $\mathcal{B}_F$'s, we need only use $\text{MA}_{\kappa^+}$, a consequence of PFA.

(4) It is clear that the proof of Lemma 1 also shows that each (finite) power of the poset $\mathcal{S}_1$ satisfies the c.c.c. Hence the model of Theorem 1 can also satisfy the following partition property of $\omega_1$ stronger than $\omega_1 \rightarrow (\omega_1)^2_{<\aleph_0}$:

**If $[\omega_1]^2 = \bigcup_{i \in I} K_i$ is a disjoint partition where each $K_i$ is finite, then there is a decomposition $\omega_1 = \bigcup_{n < \omega} A_n$ such that $\forall n < \omega \forall i \in I \left[ [A_n]^2 \cap K_i \right] \leq 1$.**

2. This section begins with a discussion of the partition relation $\omega_1 \rightarrow (\omega_1, (\omega_1; \text{fin} \omega_1))^2$ and ends with the applications mentioned in the introduction.

For $G \subseteq [\omega_1]^2$, $\text{Chr}(G)$ denotes the chromatic number of $G$ and equals the minimal cardinal $\kappa$ for which there is a partition $\omega_1 = \bigcup_{\xi < \kappa} A_\xi$ such that $[A_\xi]^2 \cap G = \emptyset$ for all $\xi < \kappa$.

**Theorem 3.** Assume $\text{MA}_{\kappa^+}$ and $\omega_1 \rightarrow (\omega_1, (\omega_1; \text{fin} \omega_1))^2$. Then for every $G \subseteq [\omega_1]^2$ either $\text{Chr}(G) \leq \aleph_0$, or else there is an $A \in [\omega_1]^{\aleph_1}$ and a family $\mathcal{B}$ of $\aleph_1$ disjoint finite subsets of $\omega_1$ such that $\left( \{\alpha \} \otimes F \right) \cap G \neq \emptyset$ for all $\alpha \in A$ and $F \in \mathcal{B}$ with $\alpha < \min F$.

**Proof.** Given $G \subseteq [\omega_1]^2$, let $\mathcal{P}$ be the set of all finite $p \subseteq \omega_1$ such that $[p]^2 \cap G = \emptyset$. The ordering on $\mathcal{P}$ is $\supseteq$.

If $\mathcal{P}$ is a c.c.c. poset, then by $\text{MA}_{\kappa^+}$, $\mathcal{P}$ is $\sigma$-centered, so $\text{Chr}(G) \leq \aleph_0$.

Hence, we may assume $\mathcal{P}$ is not a c.c.c. poset. Let $\{p_\alpha; \alpha < \omega_1\}$ be an uncountable antichain of $\mathcal{P}$. A standard $\Delta$-system argument shows we may assume the $p_\alpha$'s are disjoint, strictly increasing and of the same cardinality $n$ ($1 \leq n < \omega$). Let $\langle p_\alpha(i); i < n \rangle$ be the strictly increasing enumeration of $p_\alpha$, $\alpha < \omega_1$. For each $\alpha < \beta < \omega_1$, there exist $i, j < n$ such that $\langle p_\alpha(i), p_\beta(j) \rangle \in G$. This gives a coloring of $[\omega_1]^2$ into $n^2$ colors. Now an easy application of $\omega_1 \rightarrow (\omega_1, (\omega_1; \text{fin} \omega_1))^2$ completes the proof of Theorem 3.

A consequence of Theorem 3 is that, in the model of §1,

$$\omega_1 \rightarrow (\text{stationary}, (\omega_1; \text{fin} \omega_1))^2$$

holds. However, an examination of the proof of Theorem 1 shows that, in fact, in this model, the stronger relation

$$\omega_1 \rightarrow (\text{stationary}, (\text{stationary}; \text{fin} \omega_1))^2$$

holds. Let us also mention the following strengthening $(\ast)$ of $\omega_1 \rightarrow (\omega_1, (\omega_1; \text{fin} \omega_1))^2$ in a dual direction. The consistency of this strengthening will appear in a later paper.

$(\ast)$ For every partition $[\omega_1]^2 = K_0 \cup K_1$ either there is an $A \in [\omega_1]^{\aleph_1}$ such that $[A]^2 \subseteq K_0$, or else there exist $\langle A_n; n < \omega \rangle$ and $\langle \mathcal{B}_n; n < \omega \rangle$ such that:

(i) $\omega_1 \setminus \bigcup_{n < \omega} A_n$ is countable;

(ii) $\mathcal{B}_n$ is a family of $\aleph_1$ disjoint finite subsets of $\omega_1$;

(iii) $\langle \{\alpha \} \otimes F \rangle \cap K_1 \neq \emptyset$ for all $\alpha \in A_n$ and $F \in \mathcal{B}_n$ with $\alpha < \min F$.

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Let us note that it is not possible to strengthen $\omega_1 \to (\omega_1, (\omega_1; \text{fin} \omega_1))^2$ at the same time in both the direction of Theorem 3 and that of (*), i.e., there is a partition $[\omega_1]^2 = K_0 \cup K_1$ with no stationary 0-homogeneous sets, but $\omega_1$ is not a countable union of bad sets. A proof of this simple fact will also appear elsewhere.

**Theorem 4.** Assume $MA_{\aleph_1}$ and $\omega_1 \to (\omega_1, (\omega_1; \text{fin} \omega_1))^2$. Then for every partition $[\omega_1]^2 = K_0 \cup K_1$ either there is an $A \in [\omega_1]^\aleph_1$ such that $[A]^2 \subseteq K_0$, or else for every $\alpha < \omega_1$ there are $B, C \subseteq \omega_1$ such that $tp B = \alpha$, $|C| = \aleph_1$, and $[B]^2 \cup (B \otimes C) \subseteq K_1$.

In particular we have the following consequence mentioned in the introduction.

**Theorem 5.** Assume $MA_{\aleph_1}$. Then $\omega_1 \to (\omega_1, (\omega_1; \text{fin} \omega_1))^2$ implies $\omega_1 \to (\omega_1, \alpha)^2$ for all $\alpha < \omega_1$.

**Proof of Theorem 4:** Let $[\omega_1]^2 = K_0 \cup K_1$ be a given partition, and assume $[A]^2 \not\subseteq K_0$ for all $A \in [\omega_1]^\aleph_1$. For each $\alpha < \omega_1$ we shall construct $B, C \subseteq \omega_1$ such that $tp B = \omega^\alpha$, $|C| = \aleph_1$, and $[B]^2 \cup (B \otimes C) \subseteq K_1$. First we need some technical definitions and facts.

For each $1 < \alpha < \omega_1$ we fix a nondecreasing sequence $(\alpha(n); n < \omega)$ of smaller ordinals such that $\omega^\alpha = \sum_{n < \omega} \omega^{\alpha(n)}$, and if $\alpha > 1$, then $\alpha(0) \geq 1$. Also for every set $B \subseteq \omega_1$ of type $\omega^\alpha$ we fix a decomposition $B = \bigcup_n B(n)$ such that

$$B(0) < \cdots < B(n) < \cdots$$

and $tp B(n) = \omega^{\alpha(n)}$.

Let $\mathcal{V}$ be a fixed nonprincipal ultrafilter on $\omega_1$. By induction on $1 < \alpha < \omega_1$ we define a nonprincipal ultrafilter $\mathcal{U}_\alpha(B)$ on every set $B \subseteq \omega_1$ of type $\omega^\alpha$. If $\alpha = 1$, then the isomorphism of $\omega$ and $B$ induces $\mathcal{U}_\alpha(B)$. So now assume $1 < \alpha < \omega_1$ and define

$$D \in \mathcal{U}_\alpha(B) \iff \{n < \omega; D \cap B(n) \in \mathcal{U}_{\alpha(n)}(B(n))\} \in \mathcal{V}.$$ 

By induction on $\alpha$ it easily follows that $tp D = \omega^\alpha$ for every $D \in \mathcal{U}_\alpha(B)$. The following lemma is due to Hajnal [24, p. 1031]. For the sake of completeness we sketch the proof.

**Claim 4.** Let $1 < \alpha < \omega_1$ and let $B \subseteq \omega_1$ have type $\omega^\alpha$. Let $\langle D_\xi; \xi < \omega_1 \rangle$ be a sequence of elements of $\mathcal{U}_\alpha(B)$. Then there exists a $D \subseteq B$, with $tp D = \omega^\alpha$ such that $D \setminus D_\xi$ is a bounded subset of $D$ for every $\xi < \omega_1$.

**Proof.** The proof is by induction on $\alpha$. The case $\alpha = 1$ is a well-known consequence of $MA_{\aleph_1}$. So let $1 < \alpha < \omega_1$. By the induction hypothesis, for each $n < \omega$, there is an $E_n \subseteq B(n)$ of type $\omega^{\alpha(n)}$ such that $E_n \setminus D_\xi$ is bounded in $E_n$ for all $\xi < \omega_1$ with the property

$$n \in N_\xi = \{m < \omega; D_\xi \cap B(m) \in \mathcal{U}_{\alpha(m)}(B(m))\}.$$ 

Now for each $\xi < \omega_1$, we fix $f_\xi \in \omega^\omega$ with the property that for every $n \in N_\xi$, the $f_\xi(n)$-end-section of $E_n$ is a subset of $D_\xi$. Let $N \subseteq \omega$ be an infinite set almost included in each $N_\xi$, and let $f \in \omega^\omega$ eventually dominate each $f_\xi$. For $n \in N$, let $D_n$ be the $f(n)$-end-section of $E_n$. Let $D = \bigcup_{n \in N} D_n$. Then $D$ is as required.

Now we are ready for the proof of Theorem 4. By induction on $\alpha < \omega_1$, for each $A \in [\omega_1]^\aleph_1$ we shall construct $B, C \subseteq A$ such that $tp B = \omega^\alpha$, $|C| = \aleph_1$, and $[B]^2 \cup (B \otimes C) \subseteq K_1$. Since the case $\alpha = 0$ is trivial we assume $1 < \alpha < \omega_1$. Let $A \in [\omega_1]^\aleph_1$. License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
be given. By \( \omega_1 \rightarrow (\omega_1, (\omega_1; \text{fin } \omega_1))^2 \) there is an \( A_0 \in [A]^{\aleph_1} \) and a family \( \mathcal{B} \) of \( \aleph_1 \) disjoint finite subsets of \( \omega_1 \) such that \( \{\beta \} \cap F \neq \emptyset \) for all \( \alpha \in A_0 \) and \( F \in \mathcal{B} \) with \( \alpha < \min F \).

Using the induction hypothesis we recursively construct sets \( B_n, C_n \subseteq A_0 (n < \omega) \) such that:

1. \( \text{tp } B_n = \omega^{\alpha(n)} \) and \( |C_n| = \aleph_1 \);
2. \( B_n \subseteq B_{n+1} \) and \( C_n \supseteq C_{n+1} \);
3. \( B_{n+1} \subseteq C_n \);
4. \( [B_n]^2 \cup (B_n \otimes C_n) \subseteq K_1 \).

Let \( B = \bigcup_{n < \omega} B_n \). Then tp \( B = \omega^\omega \) and \( [B]^2 \subseteq K_1 \). Pick \( F \in \mathcal{B} \) such that sup \( B \leq \min F \). By the assumptions on \( A_0 \) and \( \mathcal{B} \), we have that \( B \subseteq \bigcup_{\gamma \in \gamma} K_1(\gamma) \). Hence for some \( \gamma = \gamma(F) \in F \), we have \( K_1(\gamma) \cap B \in \mathcal{B}_\alpha(B) \). By Claim 4 there is a \( D \subseteq B \) with \( \text{tp } D = \omega^\omega \) such that \( D \setminus K_1(\gamma(F)) \) is bounded in \( D \) for every \( F \in \mathcal{B} \), with sup \( B \leq \min F \). Thus, for some uncountable \( \mathcal{B}_0 \subseteq \mathcal{B} \) and \( \delta \in D \) we have \( D \setminus \delta \subseteq K_1(\gamma(F)) \) and sup \( B \leq \min F \) for all \( F \in \mathcal{B}_0 \). Let \( B^* = D \setminus \delta \) and \( C^* = \{\gamma(F); F \in \mathcal{B}_0\} \). Then \( B^*, C^* \subseteq A \), \( \text{tp } B^* = \omega^\omega \), \( |C^*| = \aleph_1 \), and \( [B^*]^2 \cup (B^* \otimes C^*) \subseteq K_1 \).

This completes the proof.

Let us now consider the following combinatorial principle introduced by Fred Galvin:

\( (**) \) There are ideals \( \mathcal{I}, \mathcal{J} \subseteq \mathcal{P}(\omega_1) \) such that:

1. \( \mathcal{I} \cap \mathcal{J} = [\omega_1]^{<\aleph_0} \);
2. \( \forall \emptyset = \{A \cup B; A \in \mathcal{I} \land B \in \mathcal{J}\} = [\omega_1]^{<\aleph_0} \);
3. \( \forall A [\omega_1]^{\aleph_0} \cap \mathcal{J} \neq \emptyset \land [\alpha]^{\aleph_0} \cap \mathcal{I} \neq \emptyset \).

Galvin proved that \( \diamond \) implies \( (**) \) and that \( (**) \) has some topological applications [15, Theorem 4]. He also asked for the consistency of \( \neg (**) \). The next result shows that \( \neg (**) \) is consistent.

**Theorem 6.** \( \omega_1 \rightarrow (\omega_1; \text{fin } \omega_1)^2 \) implies \( \neg (**) \).

**Proof.** Let \( \mathcal{I} \) and \( \mathcal{J} \) be ideals satisfying (i) and (ii) of \( (**) \). For each \( \alpha < \omega_1 \) we can find disjoint \( A_\alpha \subseteq \mathcal{I} \) and \( B_\alpha \subseteq \mathcal{J} \) such that \( A_\alpha \cup B_\alpha = \alpha \). Define \( [\omega_1]^2 = K_0 \cup K_1 \) by

\[
\{\beta, \alpha\} \subseteq K_0 \text{ iff } \beta \in A_\alpha.
\]

Since \( \omega_1 \rightarrow (\omega_1; \text{fin } \omega_1)^2 \) holds, we consider the following two cases:

**Case I.** There is an \( A \in [\omega_1]^{\aleph_1} \) and a family \( \mathcal{B} \) of \( \aleph_1 \) disjoint finite subsets of \( \omega_1 \) such that \( \{\{\beta\} \cap F \} \cap K_0 \neq \emptyset \) for all \( \alpha \in A \) and \( F \in \mathcal{B} \) with \( \alpha < \min F \).

For \( F \in \mathcal{B} \) we define \( A(F) = \bigcup \{A_\alpha; \alpha \in F\} \). Then \( A(F) \in \mathcal{I} \) and \( A \cap \min F \subseteq A(F) \) for each \( F \in \mathcal{B} \). Hence \( [A]^{\aleph_0} \subseteq \mathcal{J} \), contradicting the conjunction of (i) and (iii). This shows that \( (**) \) fails in this case.

**Case II.** There is an \( A \in [\omega_1]^{\aleph_1} \) and a family \( \mathcal{B} \) of \( \aleph_1 \) disjoint finite subsets of \( \omega_1 \) such that \( \{\{\alpha\} \cap F \} \cap K_1 \neq \emptyset \) for all \( \alpha \in A \) and \( F \in \mathcal{B} \) with \( \alpha < \min F \).

Proceeding as in Case I we show that here \( [A]^{\aleph_0} \subseteq \mathcal{I} \), which again contradicts \( (**) \). This completes the proof.

The remainder of this section is devoted to the topological applications mentioned in the introduction.
Theorem 7. Assume $\omega_1 \rightarrow (\omega_1, (\omega_1; \text{fin} \omega_1))^2$. Let $X$ be a topological space with no uncountable discrete subspaces. Let $\mathcal{U}$ be a family of open subsets of $X$ such that $\bigcup \mathcal{U} = X$. Then there is a countable $\mathcal{U}_0 \subseteq \mathcal{U}$ such that $X = \bigcup \{\overline{U} : U \in \mathcal{U}_0\}$.

Proof. Assume by way of contradiction that, for every countable $\mathcal{U}_0 \subseteq \mathcal{U}$, $X \neq \bigcup \{\overline{U} : U \in \mathcal{U}_0\}$. Then by induction on $\alpha < \omega_1$, we can easily construct sequences $(U_\alpha : \alpha < \omega_1)$ and $(x_\alpha : \alpha < \omega_1)$ of members of $\mathcal{U}$ and $X$, respectively, such that

1. $x_\alpha \in U_\alpha$,
2. $x_\alpha \notin \bigcup \{\overline{U}_\beta : \beta < \alpha\}$.

Define $[\omega_1]^2 = K_0 \cup K_1$ by

$$\{\beta, \alpha \} \in K_0 \iff x_\beta \in U_\alpha.$$

Since $\omega_1 \rightarrow (\omega_1, (\omega_1; \text{fin} \omega_1))^2$ holds, we consider the following two cases:

Case I. There is an $A \in [\omega_1]^\aleph_1$ such that $[A]^2 \subseteq K_0$. Then for every $\alpha \in A$, $U_\alpha \cap \{x_\beta : \beta \in A\} = \{x_\alpha\}$. Hence $\{x_\alpha : \alpha \in A\}$ is an uncountable discrete subspace of $X$, a contradiction.

Case II. There is an $A \subseteq [\omega_1]^\aleph_1$ and a family $\mathcal{B}$ of $\aleph_1$ disjoint subsets of $\omega_1$ such that $(\{\alpha\} \otimes F) \cap K_1 
eq \emptyset$ for all $\alpha \in A$ and $F \in \mathcal{B}$ with $\alpha < \text{min} F$. For $F \in \mathcal{B}$ we define

$$U(F) = \bigcup_{\gamma \in F} U_\gamma.$$

Then for each $F \in \mathcal{B}$,

$$\{x_\alpha : \alpha \in A \cap \text{min} F\} \subseteq U(F).$$

Choose inductively an $A_0 \subseteq [A]^\aleph_1$ and, for each $\alpha \in A_0$, an $F_\alpha \in \mathcal{B}$ such that if $\beta < \alpha$ are in $A_0$, then

$$\max F_\beta < \beta < \text{min} F_\alpha \leq \max F_\alpha < \alpha.$$

Then by (1) and (2), for each $\alpha \in A_0$,

$$\left(U_\alpha \setminus U(F_\alpha) \right) \cap \{x_\beta : \beta \in A_0\} = \{x_\alpha\}.$$

Hence $\{x_\alpha : \alpha \in A_0\}$ is an uncountable discrete subspace of $X$, a contradiction. This completes the proof.

Theorem 8. Assume $\omega_1 \rightarrow (\omega_1, (\omega_1; \text{fin} \omega_1))^2$. Then every regular topological space with no uncountable discrete subspace is hereditarily Lindelöf.

Proof. Clearly it suffices to show that $X$ is Lindelöf. So let $\mathcal{U}$ be a family of open sets such that $\bigcup \mathcal{U} = X$. Since $X$ is a regular space, there is a family $\mathcal{V}$ of open subsets of $X$ such that $\bigcup \mathcal{V} = X$, and such that for every $W \in \mathcal{V}$ there is a $U(W) \in \mathcal{U}$ such that $U(W) \supseteq \overline{W}$. By Theorem 7 there is a countable $\mathcal{V}_0 \subseteq \mathcal{V}$ such that

$$X = \bigcup \{\overline{W} : W \in \mathcal{V}_0\}.$$

Hence $\mathcal{U}_0 = \{U(W) : W \in \mathcal{V}_0\}$ is a countable subfamily of $\mathcal{U}$ such that $\bigcup \mathcal{U}_0 = X$. This completes the proof.
Corollary 9. Assume $\omega_1 \to (\omega_1, (\omega_1; \text{fin } \omega_1))^2$. Then every regular hereditarily separable topological space is hereditarily Lindelöf.

Theorem 10. Assume $\omega_1 \to (\omega_1, (\omega_1; \text{fin } \omega_1))^2$. Let $X$ be a Hausdorff space with no uncountable discrete subspaces. Then every point of $X$ is the intersection of countably many open subsets of $X$.

Proof. Fix $x \in X$. Since $X$ is a Hausdorff space, for every $y \in X \setminus \{x\}$ there is an open set $U_y$ such that $y \in U_y$ and $x \not\in U_y$. By Theorem 7, applied to the space $X \setminus \{x\}$, there is a countable $Y \subseteq X \setminus \{x\}$ such that $X \setminus \{x\} = \bigcup \{U_y; y \in Y\}$. This shows that $\{x\}$ is a $G_\delta$ subset of $X$.

The following theorem is a simple consequence of Theorem 10 using a result of [18]. However, since the result we need is a relatively simple application of $(2^{\aleph_0})^+ \to (\mathcal{N}_1)^\aleph_0$, we shall give some details.

Theorem 11. Assume $\omega_1 \to (\omega_1, (\omega_1; \text{fin } \omega_1))^2$. Then every Hausdorff space with no uncountable discrete subspaces has cardinality $\leq 2^{\aleph_0}$.

Proof. Assume by way of contradiction that $X$ is a Hausdorff space of cardinality $> 2^{\aleph_0}$ with no uncountable discrete subspaces.

Let $<$ be a well-ordering of $X$. By Theorem 10, for each $x \in X$ we can fix a family $\{U^n_x; n < \omega\}$ of open subsets of $X$ such that $\{x\} = \bigcap_{n < \omega} U^n_x$. For $m, n < \omega$ and $\{x, y\} \subseteq \{x\}^2$ we let

$$\{x, y\} \in K_{mn} \iff x \not\in U^n_x \& y \not\in U^m_y.$$

Clearly, $\{x\}^2 = \bigcup_{m, n < \omega} K_{mn}$. By $(2^{\aleph_0})^+ \to (\mathcal{N}_1)^\aleph_0$, there are $m', n' < \omega$ and $D \in [X]^\aleph_1$ such that $D^2 \subseteq K_{m'n'}$. For $x \in D$, let $W_x = U^{m'}_x \cap U^{n'}_x$. Then for each $x \in D$, $W_x \cap D = \{x\}$. Hence, $D$ is a discrete subspace of $X$, a contradiction. This completes the proof.

We conclude the paper with a remark on the following partition relation (it is dual to $\omega_1 \to (\omega_1, (\omega_1; \text{fin } \omega_1))^2$), denoted by

$$\omega_1 \to (\omega_1, (\text{fin } \omega_1; \omega_1))^2.$$

This relation means: For every partition $\{\omega_1\}^2 = K_0 \cup K_1$ either

1. there is an $A \in [\omega_1]^\aleph_1$ such that $[A]^2 \subseteq K_0$, or
2. there is a family $\mathcal{E}$ of $\aleph_1$ disjoint finite subsets of $\omega_1$ and a set $B \in [\omega_1]^\aleph_1$ such that $(F \otimes \{B\}) \cap K_1 \neq \emptyset$ for all $F \in \mathcal{E}$ and $\beta \in B$ with $\max F < \beta$.

The consistency of $\omega_1 \to (\omega_1, (\text{fin } \omega_1; \omega_1))^2$ is an open problem. It is easily seen that $\omega_1 \to (\omega_1, (\text{fin } \omega_1; \omega_1))^2$ implies the dual statement of Theorem 8, i.e., that every regular space with no uncountable discrete subspaces is hereditarily separable.

The reader interested in the role of $\text{MA}_{\aleph_1}$ in the problems we have considered here can find some information in [4].

References


(Russian)