REYE CONGRUENCES

BY

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Abstract. This paper studies the congruences of lines which are included in two distinct quadrics of a given generic three-dimensional projective space of quadrics in \( \mathbb{P}^3 \).

Introduction. A Reye congruence is classically defined as the set \( R(W) \) of lines included in at least two distinct quadrics of a given projective space \( W \) of dimension three of quadrics in \( \mathbb{P}^3 \).

These congruences were introduced by T. Reye [R] and studied by G. Fano [F1, F2] who showed, in particular, that the generic Reye congruence is an Enriques surface. Fano's proofs are not satisfactory from a modern point of view. More important perhaps is the fact that Fano seems to have believed that the generic Enriques surface is a Reye congruence. A simple count of parameters shows that this is not the case. Indeed, the choice of \( W \) depends on \( \dim(G(3,9)) \) parameters—where \( G(3,9) \) denotes the Grassmannian of projective spaces of dimension three in \( \mathbb{P}^9 \)—so that after dividing by the action of \( PGL(3) \), it appears that the family of Reye congruences depends on at most \( \dim G(3, 9) - \dim PGL(3) = 9 \) parameters, i.e., one less than for the family of all Enriques surfaces.

The purpose of this paper is to understand the special features held by an Enriques surface which is a Reye congruence.

A careful description of the Picard group of the Reye congruence associated to a generic web \( W \) leads to the following results:

Theorem 1. The Reye congruences form a nine-dimensional family of Enriques surfaces which coincides generically with the family of Enriques surfaces of special type, i.e., which contains an elliptic pencil \( |P| \) and a smooth rational curve \( \theta \) such that \( P \cdot \theta = 2 \).

Theorem 2. The family of Reye congruences coincides generically with the family of Enriques surfaces which contains an elliptic pencil \( |P| \) and a smooth rational curve \( \theta \) such that \( P \cdot \theta = 6 \).

The K3 surfaces which arise as the étale double cover of the generic Reye congruence have a history of their own. They can be realized in \( \mathbb{P}^3 \) as quartic surfaces, an equation of which is given by the vanishing of a \( 4 \times 4 \) symmetrical...
determinant with linear entries in four variables. These quartic surfaces are classically called symmetroids. They were investigated for the first time by A. Cayley [Ca]. Our paper contains, in particular, a geometrical proof of a result due to Cayley.

**Theorem 3.** A K3 is the étale double cover of a generic Reye congruence if and only if it can be realized as the double cover of $\mathbb{P}^3$ branched over a sextic curve which splits into two cubics which intersect transversally and have a totally tangent smooth conic which does not contain any of the points of intersection of the cubics.

It appears that the symmetroids are the quartic surfaces that M. Artin and D. Mumford [A-M] considered in order to construct a counterexample to the Luröth problem in dimension three.

1. **The space of quadrics in $\mathbb{P}^2$ and $\mathbb{P}^3$**. This preliminary section gives a description of the space of quadrics in $\mathbb{P}^2$ and $\mathbb{P}^3$. Proofs can be found in [T]. We let $\mathbb{P}^9 = \text{PH}_0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$ be the space of quadrics in $\mathbb{P}^3$ and let $\mu_4 \subset \mathbb{P}^9$ be the set of quadrics of $\mathbb{P}^3$ of rank $\leq 4 - i$. This gives a stratification $\mathbb{P}^9 \supset \mu_1 \supset \mu_2 \supset \mu_3$ which we now describe.

1.1. **The degree and dimension of $\mu_i$**. A system of coordinates $(X_i)$ of $\mathbb{P}^3$ is chosen in order to identify a quadric $q \in \mathbb{P}^9$ with a symmetrical matrix $(q_{i,j})$ and its corresponding quadratic form

$$q(X) = \sum_{i,j} q_{i,j}X_iX_j \quad \text{where } X = (X_i).$$

(1.1.1) Since $\mu_1 = \{q = (q_{i,j}), \det q = 0\}$, $\mu_1$ is a quartic hypersurface $\mu_4^8$ of $\mathbb{P}^9$.

(1.1.2) A quadric $q \in \mu_2$ is an unordered couple of planes. Let $\mathbb{P}^3$ be the dual space of $\mathbb{P}^3$. There is a map

$$\mathbb{P}^{3^\vee} \times \mathbb{P}^{3^\vee} \to \text{PH}_0(\mathbb{P}^{3^\vee}, \mathcal{O}_{\mathbb{P}^{3^\vee}}(2))^\vee \simeq \text{PH}_0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$$

whose image is $\mu_2$ and branch locus $\mu_3$. This map is defined by the linear system of symmetrical divisors of bidegree $(1, 1)$ in $\mathbb{P}^{3^\vee} \times \mathbb{P}^{3^\vee}$.

The following easy lemma allows us to conclude that $\mu_2$ is a variety $\mu_6^{10}$ of dimension six and degree 10 in $\mathbb{P}^9$.

**Lemma 1.1.2.** The intersection number of $2n$ divisors of bidegree $(1, 1)$ in $\mathbb{P}^n \times \mathbb{P}^4$ is $\binom{2n}{n}$.

(1.1.3) The variety $\mu_3$ is the Veronese image of $\mathbb{P}^{3^\vee}$. It is a smooth variety $\mu_8^3$ of dimension three and degree eight in $\mathbb{P}^9$.

(1.1.4) The stratification of $\mathbb{P}^9$ by rank will henceforth be denoted by $\mathbb{P}^9 \supset \mu_4^8 \supset \mu_6^{10} \supset \mu_8^3$.

1.2. **The singularities of $\mu_i$**. As an algebraic variety, $\mu_4^8$ carries the stratification $\mu_8^3 \supset \text{sing} \mu_4^8 \supset \text{sing(sing} \mu_4^8) \supset \cdots$. It is well known that

$$\mu_6^{10} = \text{sing} \mu_4^8, \quad \mu_3^3 = \text{sing} \mu_6^{10}, \quad \mu_8^3 \text{ is smooth}.$$

The following properties give a description of the singularities of $\mu_4^8$ and $\mu_6^{10}$.
Proposition 1.2.1. (i) \( \mu_{10}^6 \) is the singular locus of \( \mu_4^8 \).
(ii) \( \mu_{10}^6 - \mu_3^3 \) is the set of double points of \( \mu_4^8 \).
(iii) \( \mu_3^3 \) is the set of triple points of \( \mu_4^8 \).
(iv) The tangent space of \( \mu_4^8 \) at \( q_0 \in \mu_4^8 - \mu_{10}^6 \) is
\[
T_{q_0}(\mu_4^8) = \{ q \in \mathbb{P}^9; \text{sing } q_0 \subseteq q \}.
\]
(v) The tangent cone of \( \mu_4^8 \) at \( q_0 \in \mu_{10}^6 \) is
\[
C_{q_0}(\mu_4^8) = \{ q \in \mathbb{P}^9; q \text{ tangent to sing } q_0 \}.
\]

Proposition 1.2.2. (i) \( \mu_3^3 \) is the singular locus of \( \mu_{10}^6 \).
(ii) The tangent space of \( \mu_{10}^6 \) at \( q_0 \in \mu_{10}^6 - \mu_3^3 \) is
\[
T_{q_0}(\mu_{10}^6) = \{ q \in \mathbb{P}^9; \text{sing } q_0 \subseteq q \}.
\]
(iii) The tangent cone of \( \mu_{10}^6 \) at \( q_0 \in \mu_3^3 \) is
\[
C_{q_0}(\mu_{10}^6) = \{ q \in \mathbb{P}^9; q \text{ tangent to } q_0 \text{ along a line} \}.
\]
This is a quartic cone over a Veronese surface.
(iv) The tangent space of \( \mu_3^3 \) at \( q_0 \in \mu_3^3 \) is
\[
T_{q_0}(\mu_3^3) = \{ q \in \mu_{10}^6; q \text{ contains the plane defining } q_0 \}.
\]

1.3. The spaces of conics in \( \mathbb{P}^2 \). The space of conics of \( \mathbb{P}^2 \) has a natural stratification
\[
\mathbb{P}^5 = \mathbb{P}H^0(\mathbb{P}^2; \mathcal{O}_{\mathbb{P}^2}(2)) \supseteq \mu_4^4 \supseteq \mu_3^2 = \text{sing } \mu_4^4,
\]
where \( \mu_4^4 \) is the cubic fourfold of singular conics and \( \mu_3^2 \), the Veronese image of \( \mathbb{P}^2 \), is the quartic surface of double lines.

With obvious notations:
\[
T_{q_0}(\mu_3^2) = \{ q \in \mathbb{P}^5; \text{sing } q_0 \subseteq q \} \quad \text{for any } q_0 \in \mu_3^2 - \mu_3^3.
\]
\[
T_{q_0}(\mu_3^3) = \{ q \in \mathbb{P}^5; \text{sing } q_0 \subseteq q \} \quad \text{for any } q_0 \in \mu_3^2.
\]
\[
C_{q_0}(\mu_3^3) = \{ q \in \mathbb{P}^5; q \text{ tangent to } q_0 \} \quad \text{for any } q_0 \in \mu_3^2.
\]

2. Hessian and Steinerian surfaces of a web. This section describes the generic section of \( \mu_4^8 \) by a three-dimensional projective space.

2.1. The Hessian surface.

Definition 2.1.1. The Hessian surface \( H = H(W) \) of a web \( W \) (i.e. a three-dimensional projective space) of quadrics in \( \mathbb{P}^3 \) is the set of singular quadrics of \( W \).
\[
H(W) = W \cap \mu_4^8.
\]
If \( W \subseteq \mu_4^8 \), either there exists a point \( x \in \mathbb{P}^3 \) which is a singular point of all the quadrics of \( W \) or the base locus of \( W \) is of dimension at least one. From now on, we assume that \( W \) is not included in \( \mu_4^8 \), so that \( H(W) \) is a quartic surface in \( W \).

The surface \( H(W) \) is also classically called a symmetroid for the following reason: choose a basis \( (q_0, \ldots, q_3) \) of \( W \) and let \( q(\lambda) = \Sigma_i \lambda_i q_i \) for \( \lambda = (\lambda_i) \) a system of
coordinates of $W$. Then an equation of $H$ is $\det(\sum \lambda_i q_i) = 0$, i.e., $H$ is given by the vanishing of a $4 \times 4$ symmetrical determinant whose entries are linear forms in the four variables $\lambda_i$.

The main properties of these quartic surfaces were described by Cayley [Ca]. We hope that our approach will show more clearly the geometric properties of these symmetroids.

(2.1.2) The singularities of $H(W)$. As a corollary of the description of $\text{sing } \mu_4^W$ given in §1, we have:

(2.1.2.1) $W$ intersects transversally $\mu_4^W$ at $q \in W \cap (\mu_4^W - \mu_5^W)$ if and only if $\text{sing } q$ is not a basepoint of $W$. Conversely, if $x$ is a basepoint of $W$, there exists at least one quadric $q \in W$ which is singular at $x$ and any such quadric is a singular point of $H(W)$.

(2.1.2.2) Any quadric $q \in W \cap (\mu_5^W - \mu_3^W)$ is a double point of $H(W)$ unless $W \subseteq C_{\mu_5^W}$, i.e., $W$ has a basepoint $x \in \text{sing } q$ at which all the quadrics of $W$ are tangent to $q$, in which case $q$ is a point of $H(W)$ of multiplicity $\geq 3$. Moreover, $W$ intersects transversally $\mu_5^W$ at $q$ if and only if there is no quadric $q' \neq q, q' \in W$, such that $\text{sing } q \subseteq q'$.

(2.1.2.3) A quadric $q \in W \cap \mu_3^W$ is a triple point of $H(W)$ unless $W \subseteq C_{\mu_3^W}$, i.e., all quadrics of $W$ are tangent to $q$ at a fixed point $x \in q$.

In particular, we have

**Proposition 2.1.2.** Let $W$ be a web of quadrics in $\mathbb{P}^3$. Assume that

(i) $W$ is basepoint free.

(ii) $\cap W$ is a double line of a quadric $q \in W \cap \mu_5^W$ such that $\cap W$ is a non other quadric $q' \in W$, $q' \neq q$, such that $\cap q' \subseteq q'$.

Then $H(W)$ is a quartic surface with exactly ten ordinary double points.

**Definition 2.1.2.** A regular web is a web satisfying (i) and (ii) of Proposition 2.1.2.

(2.1.3) This subsection describes a smooth model of $H(W)$ when $W$ is regular. We first recall a desingularization of $\mu_4^W$ defined by A. Tjurin [T]. We let

$$\bar{\mu}_4 = \{(x, q); x \in \text{sing } q \} \subseteq \mathbb{P}^3 \times \mu_4^W.$$ 

Clearly $\bar{\mu}_4$ is the complete intersection of four divisors of bidegree $(1,1)$ in $\mathbb{P}^3 \times \mathbb{P}^9$.

The smoothness of $\bar{\mu}_4$ follows from Lemma 2.1.3, which is an easy consequence of the Jacobian criterion.

**Lemma 2.1.3.** Let $X = \{(x, y); \varphi_i(x, y) = 0, i = 1, \ldots, p\}$ be the intersection of $p$ divisors of bidegree $(1,1)$ in $\mathbb{P}^n \times \mathbb{P}^m$. Then $(x, y)$ is a smooth point of $X$ if and only if there is no $\lambda = (\lambda_1)$ such that $\varphi(\lambda)(x, \cdot) = \varphi(\lambda)(y, \cdot) = 0$ where $\varphi(\lambda) = \sum \lambda_i \varphi_i$.

We let $\pi_1; \bar{\mu}_4^W \rightarrow \mathbb{P}^3$ and $\pi_2; \bar{\mu}_4^W \rightarrow \mu_4^W$ be the natural projection maps.

**Proposition 2.1.4.** Assume $W$ is regular. Then $\tilde{H}(W)$ is a minimal desingularization of $H(W)$. It is a K3 surface.
PROOF. We apply Lemma 1.3. First, we choose a basis \((\varphi_i)\) of \(W\), and let \(\varphi(\lambda) = \sum \lambda_i \varphi_i\) so that
\[
H(W) = \left\{ (x, \lambda); \frac{\partial \varphi(\lambda)}{\partial x_i} (x) = 0 \text{ for any } i \right\}.
\]
Assume that \((x_0, \lambda_0)\) is a singular point of \(H(W)\). Then there is \(a = (a_i)\) such that
\[
\sum a_i \frac{\partial \varphi(\lambda)}{\partial x_j} (x_0) = 0 \quad \text{for all } \lambda,
\]
and
\[
\sum a_i \frac{\partial \varphi(\lambda_0)}{\partial x_j} (x) = 0 \quad \text{for all } x,
\]
or, equivalently,
\[
\sum a_i \frac{\partial \varphi_i}{\partial x_i} (x_0) = 0, \quad \frac{\partial \varphi(\lambda_0)}{\partial x_j} (a) = 0 \quad \text{for all } j.
\]
Then if \(a = x_0\), \(x_0\) is a basepoint of \(W\). If \(a \neq x_0\), \(\varphi(\lambda_0)\) is singular along the line \(l\) containing \(a\) and \(x_0\); moreover, the orthogonality of \(a\) and \(x_0\) for all the quadrics of \(W\) implies that there exists \(\lambda_1 \neq \lambda_0\) such that \(\varphi(\lambda_1)\) contains \(l\). In both cases, \(W\) would not be regular. Therefore \(H(W)\) is smooth. Since it is a complete intersection of divisors of bidegree \((1,1)\) in \(P^3 \times W\), it is a K3 surface.

2.2 The Steinerian surface.

DEFINITION 2.2.1. The Steinerian surface \(S(W)\) of a web \(W\) is defined by
\[
S(W) = \sigma \circ \pi_2^{-1}(H(W)) \subset P^3.
\]
This is the set of singular points of singular quadrics of \(W\). It is also called the Jacobian surface of \(W\) for the following reason:
\[
S(W) = \left\{ x \in P^3 \text{ such that } \exists \lambda, \frac{\partial \varphi(\lambda)}{\partial x_i} (x) = 0 \text{ for } i = 0, \ldots, 3 \right\}
\]
\[
= \left\{ x \in P^3 \text{ such that } \frac{\partial (\varphi_0, \ldots, \varphi_3)}{\partial (x_0, \ldots, x_3)} (x) = 0 \right\}.
\]
In particular, \(S(W)\) is a quartic surface in \(P^3\). The Steiner map is \(\sigma : H(W) \to S(W)\) which associates to a singular quadric of \(W\) its singular locus. It is a birational map.

PROPOSITION 2.2.2. The Steinerian surface of a regular web \(W\) is a smooth quartic surface if and only if there is no line on \(H(W)\).

PROOF. Since we are making a local study of \(S(W)\) at one of its points \(x_0\), we can choose coordinates \((x, y, z, t)\) in \(P^3\) such that \(x_0 = (0, 0, 0, 1)\). We can also choose a basis \((\varphi_i)\) of \(W\) such that
\[
\varphi_0 = P(x, y, z), \quad \varphi_2 = \beta yt + P_2(x, y, z),
\]
\[
\varphi_1 = \alpha xt + P_1(x, y, z), \quad \varphi_3 = \gamma t^2 + tQ_3(x, y, z) + P_3(x, y, z),
\]
where \( \alpha, \beta, \gamma \) are constants and \( P_0, P_1, P_2, P_3 \) (resp. \( Q_3 \)) are homogeneous polynomial in \( x, y, t \) of degree two (resp. one). Then it is easily checked that
\[
\frac{\partial (\varphi_0, \ldots, \varphi_3)}{\partial (x_0, \ldots, x_3)} = -2t^3 \alpha \beta \gamma \partial_2 P_0 + R,
\]
where \( R \) is a homogeneous polynomial in \( (x, y, z, t) \) of partial degree in \( t \) less than two. Therefore \( x_0 \in \text{sing}(S(W)) \) if and only if
(i) \( \gamma = 0 \), or (ii) \( \alpha \beta = 0 \), or (iii) \( \partial_2 P_0 = 0 \).

Case (i). \( \gamma_0 = 0 \) if and only if \( x_0 \) is a basepoint of \( W \).

Case (ii). Say \( \alpha = 0 \). Then \( \varphi_0, \varphi_1 \) generate a pencil of quadrics of \( W \) which are singular at \( x_0 \). This defines a line \( l \) on \( H(W) \). Consider the restriction of this pencil to a plane which does not contain \( x_0 \). It is a pencil of conics which, by the fact that \( W \) is regular, contains exactly three singular conics so that \( l \) is a line going through three of the double points of \( H(W) \). Conversely, it is easily checked that for any regular web \( W \), the only possible lines on \( H(W) \) are lines through three of its double points, and to each such line is associated a singular point of \( S(W) \).

Case (iii). \( \partial_2 P_0 = 0, \gamma \neq 0 \): \( P_0 \) is a quadric of rank less than two which is singular along the line \( l = \{x = y = 0\} \). The intersection of the polar hyperplanes of \( x_0 \) with respect to the quadrics of \( W \) is given by
\[
\alpha x = \beta y = 0, \quad 2\gamma t + Q_3(x, y, z) = 0,
\]
hence there exists \( x_1 \neq x_0, x_1 \in l \) such that \( (x_0, x_1) \) are orthogonal for all the quadrics of \( W \). This line \( l \) contradicts the regularity of \( W \).

(2.3) We will now introduce another surface which will help in understanding the properties held by \( S(W) \). We further identify a quadric \( q = (q_{ij}) \) with its associated bilinear form
\[
q(X, Y) = \sum_{i,j} q_{ij} X_i Y_j, \quad X = (X_j), \quad Y = (Y_j).
\]
We define
\[
\tilde{S} = \tilde{S}(W) = \{(x, y) \in \mathbb{P}^3 \times \mathbb{P}^3, q(x, y) = 0 \text{ for all } q \in W\}.
\]
A direct consequence of Lemma 2.1.3 is

**Proposition 2.3.1.** \( \tilde{S}(W) \) is a K3 surface if and only if \( W \) is regular.

**Corollary 2.3.2.** For any regular web \( W \), \( \tilde{S}(W) \) is the unramified double cover of an Enriques surface.

**Proof.** If \( W \) is a basepoint free, \( i: (x, y) \to (y, x) \) is a fixed-point free involution of \( \tilde{S} \). The corollary follows from the fact that the quotient of a K3 surface by a fixed-point free involution is an Enriques surface.

(2.3.3) If \( (\varphi_i) \) is a basis of \( W \), and if \( \pi_0: \tilde{S}(W) \to \mathbb{R}^3 \) is obtained by projection into one of its factors, the image of \( \tilde{S}(W) \) is
\[
\left\{x \in \mathbb{P}^3 \text{ such that } \exists y = (y_i) \text{ with } \sum_i y_i \frac{\partial \varphi_i}{\partial x_i}(x) = 0 \text{ for all } j\right\},
\]
which is precisely \( S(W) \).
The following diagram summarizes the different surfaces which have been introduced:

$$\begin{align*}
\tilde{S}(W) & \quad \tilde{H}(W) \\
\downarrow \pi_0 & \quad \pi_1 \leftarrow \pi_2 \downarrow \\
S(W) & \quad \sigma \quad H(W)
\end{align*}$$

where $\pi_0$, $\pi_1$, $\pi_2$ and $\sigma$ are birational maps and, when $W$ is regular, $\tilde{S}(W)$ and $\tilde{H}(W)$ are isomorphic $K3$ surfaces.

(2.4) On the Picard group of $S(W)$.

(2.4.1) This paragraph will assume that $W$ is a regular web for which $S(W)$ is smooth. Such webs will be called excellent. The surface $S(W)$ is then isomorphic to $\tilde{S}(W)$ and contains:

1. the rational curves $\theta_i$, singular loci of the ten quadrics of $W \cap \mu_1^6$;
2. the system of hyperplane sections $|t|\sigma$ of $S$;
3. the image $T_i \theta_i$ of the system of hyperplane sections of $H(W)$;
4. ten pencils of elliptic curves $|E_i|$ obtained as residual intersection of $S$ with the planes containing $\theta_i$.

The following relations should be clear:

(i) $\eta_{\tilde{H}}^2 = \eta_S^2 = 4$;
(ii) $|\eta_H| = |\theta_i + E_i|$;
(iii) $\eta_H \cdot \theta_i = 0$, $\theta_i \cdot \theta_j = 0$ for $i \neq j$;
(iv) $\eta_S \cdot \theta_i = 1$, $\eta_S \cdot E_i = 3$ for all $i$;
(v) $E_i \cdot E_j = 2$ if $i \neq j$.

**Proposition 2.4.1.** $2\eta_S = 3\eta_H - \sum_i \theta_i$ in Pic($S$).

**Proof** (cf. [T]). Let $P$ be the linear system of polar cubics of $H$. For any $q_0 \in W$ the polar cubic $P(q_0)$ of $q_0$ with respect to $H$ satisfies the following characteristic property:

$$P(q_0) \cap H = \{ q \in H; q \neq q_0, qq_0 \text{ is tangent to } H(W) \text{ at } q \}.$$

From the description of $T_q H$ given in §1, we see that

$$P(q_0) \cap H = \sigma^{-1}(q_0 \cap S(W)),$$

from which the proposition follows.

(2.4.2) The proof of (2.4.1) shows more precisely that $P = W[\mu]$. Therefore, since a generic web is not invariant under any projective automorphism of $P^3$, we have

**Proposition 2.4.2.** The smooth Steinerian surfaces form a nine-dimensional locally closed set of the Hilbert scheme of smooth polarized $K3$ surfaces of degree four in $P^3$.

(2.4.3) We define the enveloping cone $\Gamma_q$ of a point $q \in H(W)$ to be the closure in $P^1$ of the sets of lines tangent to $H(W)$ at a point $q' \neq q$ and containing $q$. Let $q_0 \in \text{sing}(H(W))$. After an appropriate choice of coordinates in $P^3$, we can assume that $q = (0, 0, 0, 1)$ and an equation of $H(W)$ is

$$x_0^2 A_2(x_1, x_2, x_3) + 2x_0 B_1(x_1, x_2, x_3) + C_4(x_1, x_2, x_3) = 0,$$
where $A_2, B_3, C_4$ are homogeneous polynomials of degrees 2, 3, 4 in $(x_1, x_2, x_3)$. Then $\Gamma_q$ is the cone of equation $B_3^2 - A_2C_4 = 0$. It is a sextic cone totally tangent to the tangent cone $A_2 = 0$.

**Proposition 2.4.3 (Cayley’s Property).** Let $W$ be a regular web. Then the enveloping cone of the double point $q_i \in \text{sing } H(W)$ splits into two cubic cones totally tangent to the (irreducible) tangent cone of $H(W)$ at $q_i$.

**Proof.** By Proposition 1.2.1, $\Gamma_q$ is tangent to $H(W)$ along a curve $\gamma_i$ which is the inverse image of $q_i \cap S(W)$ under the Steiner map. Since $q_i$ splits into two planes, $q_i \cap S = E_i + E_i' + 2\theta_i$, where $E_i, E_i'$ are the residual cubics cut by the planes of $q_i$. Therefore, $\gamma_i$ splits into the images of $E_i, E_i'$ on $H$. The irreducibility of the tangent cone is a consequence of the regularity of $W$.

**Corollary 2.4.4.** The Hessian surface of an excellent web is a double cover of $\mathbb{P}^2$ branched along a sextic curve which is totally tangent to a smooth conic and which splits into two cubic curves which intersect transversally at points not lying on the conic.

**Theorem 2.4.5.** Let $H$ be a quartic surface in $\mathbb{P}^3$ with ten rational double points $p_i$. Let $f: \tilde{H} \to H$ be a minimal desingularization of $H$ and assume that $\tilde{H}$ is a K3 surface. Let $|\eta_H| = |\ell \cdot \ell_H(1)|$ and let $\theta_i$ be the fundamental cycle of $f^{-1}(p_i)$. Assume, moreover, that there exists an effective divisor $\eta_S$ such that

\[ 2\eta_S = 3\eta_H - \sum \theta_i \quad \text{in } \text{Pic}(\tilde{H}). \]

Then $H$ is the Hessian surface of a web.

**Proof.** We want to apply Lemma 6.22, p. 374 of [B] to the map

\[ S^2H^0(\tilde{H}, \mathcal{O}_{\tilde{H}}(\eta_S)) \to H^0\left(\tilde{H}, \mathcal{O}_{\tilde{H}}\left(3\eta_H - \sum \theta_i\right)\right). \]

Let $|\eta_S| = |M| + F$ be the decomposition of $|\eta_S|$ into its moving $|M|$ and its fixed part $F$. Then $\dim |M| \geq 3$ since $\eta_S^2 = 4$. If $|M|$ is reducible, there exists an elliptic pencil $|E|$ and an integer $k \geq 3$ such that $|M| = |kE|$ (cf. [S-D]). But $\eta_S \cdot \eta_H = 6 \geq kE \cdot \eta_H$ implies $E \cdot \eta_H = 2$. Hence, $|\eta_H|$ is hyperelliptic, which is absurd. Therefore $|M|$ is irreducible and, in particular, $M^2 \geq 4$.

Let us show that $M^2 = 4$. We first remark that $|M - \eta_H|$ is empty, otherwise let $G \in |M - \eta_H|$. Then

\[ 2\eta_S = 2M = 2F = 2\eta_H + 2G + 2F = 3\eta_H - \sum \theta_i, \]

hence $\eta_H = \Sigma \theta_i + 2F + 2G$, which is absurd since no hyperplane section of $H$ can be a double curve. This implies $\eta_H \cdot (M - \eta_H) > 0$, otherwise $\eta_H \cdot M \leq 4$ and, by the Hodge index theorem, $M - \eta_H = 0$. But then by Riemann-Roch, $(M - \eta_H)^2 \leq -4$, hence $M^2 = 4$ and $H \cdot F = 0$ as desired.

To conclude that $F = 0$, notice that $\eta_S^2 = (M + F)^2 = 4$ gives $2M \cdot F + F^2 = 0$, and $2F \cdot \eta_S = 2F \cdot (M + F) = F \cdot (3\eta_H - \Sigma \theta_i)$ gives $F^2 + F \cdot \Sigma \theta_i = 0$, which is impossible since $\eta_H \cdot F = 0$ implies $F^2 < 0$, $\theta_i \cdot F \leq 0$ unless $F = 0$.
We claim now that \(|\eta_S|\) is nonhyperelliptic. Otherwise, let \(|E|\) be an elliptic pencil such that \(E \cdot \eta_S = 2\). Then it is easily checked that the nonhyperellipticity of \(|\eta_H|\) implies \(E \cdot \eta_H = 3\), hence \(E \cdot (\Sigma \theta_i) = 5\). Using the fact that \(|E|\) is obtained by moving a plane about a line of \(H\), there would be five nodes of \(H\) on a line, which is absurd. Therefore \(|\eta_S|\) is a nonhyperelliptic system of genus three.

It is now easy to complete the proof of Theorem 2.4.5 by using Beauville’s lemma. This part of the proof is omitted.

**Corollary 2.4.6.** Let \(H\) be the double cover of \(\mathbb{P}^2\) branched along two cubic curves which intersect transversally and have a totally tangent conic which does not contain any of the intersection points of the cubic curves. Then \(H\) is a Hessian surface.

**Proof.** Let \(H \to \mathbb{P}^2\) be the covering map. Let \((\theta_i), i = 1, \ldots, 9\), be the inverse image of the intersection points of the two cubic curves, let \(\theta_0\) be the inverse of the totally tangent conic, and let \(|E|\) be the proper transform of the obvious pencil of elliptic curves. Finally let \(|M| = |\sigma^*c_{p^*}(1)|\). Then

\[2\eta_S = 3\eta_H - \sum_i \theta_i\]

where \(|\eta_S| = |E + \theta_0|\), \(|\eta_H| = |M + \theta_0|\)

and apply Theorem 2.4.5.

**Remark.** (2.4.6) shows that the quartic surfaces considered by M. Artin and D. Mumford in [A-M] are precisely Hessian surfaces.

**Proposition 2.4.5.** Let \(H\) be the Hessian surface of a regular web and let \(h\) be a smooth hyperplane section of \(H\). Then there exists a double cover \(f: \Sigma \to M\) branched along \(h \cup \text{sing} H\). Moreover, \(\Sigma\) is a surface of general type such that \(p_g = 1, q = 0, K_{\Sigma}^2 = 2\).

**Proof.** Proposition 2.4.1 clearly holds for regular webs. Therefore

\[h + \sum_i \theta_i = 2(2\eta_H - \eta_S)\]

in \(\text{Pic} \tilde{\Sigma}\).

Therefore there exists a double cover \(\Sigma\) of \(\tilde{\Sigma}\) branched along \(h + \Sigma_i \theta_i\). By the projection formula,

\[p_g(\Sigma) = h^0(\tilde{\Sigma}, \mathcal{O}_{\tilde{\Sigma}}(2\eta_H - \eta_S)) + h^0(\tilde{\Sigma}, \mathcal{O}_{\tilde{\Sigma}}) = 1 + h^0(\tilde{\Sigma}, \mathcal{O}_{\tilde{\Sigma}}(2\eta_H - \eta_S)),\]

\[q(\Sigma) = h^1(\tilde{\Sigma}, \mathcal{O}_{\tilde{\Sigma}}(2\eta_H - \eta_S)) + h^1(\tilde{\Sigma}, \mathcal{O}_{\tilde{\Sigma}}) = h^1(\tilde{\Sigma}, \mathcal{O}_{\tilde{\Sigma}}(2\eta_H - \eta_S)).\]

Assume there exists \(F \in |2\eta_H - \eta_S|\). Then \(\eta_S + \Sigma_i \theta_i = \eta_H + F\), and, multiplying by \(\theta_j, -1 = F \cdot \theta_j\) for \(j = 1, \ldots, 10\). Therefore \(F = \Sigma \theta_i + F, F_i > 0, \text{and } \eta_S = \eta_H + F_1\), which is clearly absurd. This shows that \(p_g(\Sigma) = 1\), and \(q(\Sigma) = 0\) follows by Riemann-Roch. Moreover, one has

\[K_{\Sigma} = 2(2\eta_H - \eta_S)^2 = -8.\]

The surface \(\Sigma\) is obtained from \(\Sigma\) by blowing-down the ten exceptional curves of the first kind defined by the \(\theta_i\)'s. This concludes the proof.

**3. Reye congruences.** This section describes the generic section of \(\mu^6_{10} \subset \mathbb{P}^9\) by a projective space of dimension five.

**3.1. Definition and the Picard group of a Reye congruence.**
Proposition 3.1.2. Let $W$ be a web without basepoints. Then $R(W)$ is the set of lines of $\mathbb{P}^3$ which are included in two distinct quadrics of $W$.

Proof. Let $l$ be the line defined by a point $(x, y) \in \mathcal{S}$. A quadric of $W$ of equation $\varphi = 0$ contains $l$ if and only if $\varphi(x) = \varphi(y) = 0$, so there exist at least two distinct quadrics of $W$ containing $l$.

Conversely, let $l$ be a line of $\mathbb{P}^3$ included in two distinct quadrics of $W$. Let $N \subset W$ be a pencil generating $W$ together with the previous quadrics. Then the restriction of $N$ to $l$ is a pencil of quadrics of $l$. By Lemma 1.1.2, there exists a unique couple of points $(x, y)$ of $l$ orthogonal for all the quadrics of $N|l$ such that $(x, y) \in \mathcal{S}$.

As a corollary of (3.1.2) and (2.3.1), we have

Proposition 3.1.3. Let $W$ be a web of quadrics in $\mathbb{P}^3$. Then $R(W)$ is an Enriques surface if and only if $W$ is regular.

(3.2) We will now proceed to describe the Picard group of $R(W)$ when $W$ is excellent. Let $p: \mathcal{S} \to R$ be the quotient map and define, after 2.4,

$$F_i = p(E_i), \quad F'_i = p(E'_i) \quad \text{and} \quad D_i = p(\theta_i).$$

The elliptic pencils $|E_i|$ are $i$-invariant, the $i$-invariant fibres being precisely $E_i$ and $E'_i$. The following lemma is a consequence of (2.4.1):

Lemma 3.2.1. There exist on $R(W)$ ten smooth rational curves $D_i$ and ten elliptic pencils $|2F_i|=|2F'_i|$ such that:

(i) $D_i \cdot F_j = 1$ for $i \neq j$;
(ii) $D_i \cdot F'_j = 3$ for $i = 1, \ldots, 10$;
(iii) $F_i \cdot F'_j = 1$ for $i \neq j$;
(iv) $D_i \cdot D_j = 2$ for $i \neq j$;
(v) Pic($R$) $\otimes \mathbb{Q}$ is generated by the $F_i$'s.

(3.2.2) The first intersection relation says that $R(W)$ is an Enriques surface of special type, i.e., there exist a smooth irreducible rational curve $\theta$ and an elliptic pencil $|P|$ such that $P \cdot \theta = 2$. Actually, using results of [Co], we have

Theorem 3.2.2. The generic Enriques surface of special type is the Reye congruence of an excellent web.

(3.2.3) The second intersection relation of (3.2.1) shows that there exist a rational curve $\theta$ and an elliptic pencil $|P|$ such that $P \cdot \theta = 6$. Conversely, we have

Theorem 3.2.3. Let $R$ be an Enriques surface which contains an elliptic pencil $|P|$ and a smooth rational curve $\theta$ such that $P \cdot \theta = 6$. Then $R$ is the minimal desingularization of a Reye congruence.

Proof. Let $p: S \to R$ be the unramified double cover of $R$. If $|P| = |2F|$, then $|p^{-1}(F)| = |E|$ is an elliptic pencil on $S$. Let $\theta_0, \theta_1$ be the components of $p^{-1}(\theta)$ and define

$$|\eta_S| = |E + \theta_0|, \quad |i(\eta_S)| = |E + \theta_1|.$$
where \(i\) is the canonical involution of \(S\). Then consider the map \(\varphi = \varphi_{\mathbb{P}^n} \times \varphi_{\mathbb{P}(\eta_5)}: \)
\[
\varphi: S \rightarrow \text{PH}^0(S, \mathcal{O}_S(\eta_5)) \times \text{PH}^0(S, \mathcal{O}_S(i(\eta_5)))
\]

The idea of the proof is to show that \(\varphi(S)\) is the complete intersection of four symmetrical divisors of bidegree \((1,1)\) so that \(\varphi_{\mathbb{P}^n}(S)\) is the surface of the corresponding web of quadrics, from which the theorem is clear.

The morphisms \(\varphi_{\mathbb{P}^n}\) and \(\varphi_{\mathbb{P}(\eta_5)}\) are easily checked to be of degree one. Let \(a, b\) be the canonical generators of the Chow ring of \(\mathbb{P}^3 \times \mathbb{P}^3\). Then the class of \(\varphi(S)\) is equal to \(4a^3b + 6a^2b^2 + 4ab^3\), as follows from \(\eta_S \cdot i(\eta_S) = 6, \eta_S^2 = 4, i(\eta_S)^2 = 4\). Therefore if we let \(L = \varphi_* \mathcal{O}_{\mathbb{P} \times \mathbb{P}}(1,1), L^2 = 20\) and \(h^4(S, L) = 12\) by Riemann-Roch. Since \(h^0(\mathbb{P}^3 \times \mathbb{P}^3, \mathcal{O}_{\mathbb{P} \times \mathbb{P}}(1,1)) = 16\), there exist at least four independent divisors of bidegree \((1,1)\) containing \(\varphi(S)\). Since the class of \(\varphi(S)\) is precisely \((a + b)^4\), it follows that \(\varphi(S)\) is equal to the intersection of these divisors.

Let \(q_i\) be the bilinear forms associated to these divisors. Let \(q_i = q_i^s + q_i^a\) be their decomposition into symmetric and antisymmetric parts. We want to prove that \(q_i^a = 0\). Since \(\varphi(S)\) lies symmetrically in \(\mathbb{P}^3 \times \mathbb{P}^3\), \(q_i^s(x, y) = q_i^a(x, y) = 0\) for any \((x, y) \in \varphi(S)\). Let us assume, for example, that \(q_i^a, i = 1, 2, 3\), are linearly independent and \(q_0^a \neq 0\). Then \(\varphi_{\mathbb{P}^n}(S)\) is a quartic surface in \(\mathbb{P}^3\) included in the quartic surface, an equation of which is
\[
\partial(q_0, \ldots, q_3)/\partial(x_0, \ldots, x_3) = 0.
\]

These quartic surfaces must coincide, which is absurd since a basepoint of \(q_1, q_2, q_3\) would give a fixed-point of the involution induced by \(i\) on \(\varphi_{\mathbb{P}^n}(S)\), hence a fixed-point on \(S\).

(3.2.4) Using the third intersection relation of (3.2.1), we have

**Proposition 3.2.4.** Let \((F_1, F_2, F_3)\) be any three of the “half-elliptic pencils” defined in (3.2.1). Then \(|F_1 + F_2 + F_3|\) or \(|F_1 + F_2 + F_3 + K_R|\) defines a morphism of degree one onto a sextic surface in \(\mathbb{P}^3\) which is double along the edges of a tetrahedron.

**Proof.** This follows directly from [Co].

**Remark.** After tedious computations, one checks that the elliptic pencils \(|2F_i|\) have no reducible fibres when no four nodes of \(H(W)\) are coplanar. Under this additional assumption, \(|F_1 + F_2 + F_3|\) and \(|F_1 + F_2 + F_3 + K_R|\) are both of degree one (cf. [Co]).

(3.2.5) The fourth intersection of (3.2.1) can be used to show

**Proposition 3.2.5.** The unramified double cover of the Reye congruence of an excellent web can be realized as the intersection of three quadrics in \(\mathbb{P}^5\) on which the involution is projective.

**Proof.** The inverse image of \(D_i + D_j, i \neq j,\) on \(S(W)\) splits into two disjoint components \(\theta_i + i(\theta_j), \theta_j + i(\theta_i)\) so that \(|D_i + D_j|\) is an elliptic pencil (cf. [Co]). Let \(|D_i + D_j| = |2F_{ij}|\) and let \(E_{ij}\) be the inverse image of \(F_{ij}\) on \(S(W)\) so that \(|E_{ij}|\) is an elliptic pencil such that \(E_j \cdot E_{ij} = E_i \cdot E_{ij} = 4\). Then the linear system \(|E_i + E_{ij}|\) gives a morphism
\[
\varphi_{i,j}: S(W) \rightarrow \text{PH}^0(S(W), \mathcal{O}_{S(W)}(E_i + E_{ij}))^\vee \subset \mathbb{P}^5.
\]
This morphism is of degree one since \(|E_i + E_{ij}|\) is obviously not hyperelliptic. Moreover, \(\varphi_{i,j}(S(W))\) is the intersection of three quadrics, as follows from the fact that \(|E_i + E_{ij}|\) is not trigonal (cf. [S-D]). The involution induced by \(i\) is projective because \(|E_i + E_{ij}|\) is \(i\)-invariant.

**Proposition 3.2.6.** The map associated to \(|D_1 + D_2 + D_3|\) gives a morphism of degree two onto a Cayley cubic surface \(\mathcal{C}\) which is branched along \((\text{sing } \mathcal{C}) \cup \gamma\), where \(\gamma\) is generically a smooth canonical curve of genus four such that \(\gamma \cap (\text{sing } \mathcal{C}) = \emptyset\).

**Corollary 3.2.7.** The generic minimal desingularization of the double cover of \(\mathcal{C}\) branched along \((\text{sing } \mathcal{C}) \cup \gamma\), where \(\gamma\) is a canonical curve of genus four such that \(\gamma \cap (\text{sing } \mathcal{C}) = \emptyset, \gamma \subset \mathcal{C}\), coincides with the generic Reye congruence.

This follows from [Co].

3.3. A first projective model of \(R(W)\) in \(P^5\).

(3.3.1) Let \(W\) be a web of quadrics in \(P^3\). Let \(W^\perp\) be the orthogonal \(W\) in \(\text{PH}^0(P^3, \mathcal{O}_{P^3}(2))\). Then \(W^\perp\) can be identified with a five-dimensional projective space of quadrics in \(P^3\).

Consider the filtration of \(\text{PH}^0(P^3, \mathcal{O}_{P^3}(2))\) by the rank

\[
\text{PH}^0(P^3, \mathcal{O}_{P^3}(2)) = \nu_5 \supset \nu_4 \supset \nu_3.
\]

Then \(W^\perp \cap \nu_3\) is equal to

\[
\{(x, y) \in \mathcal{P}; \varphi(x, y) = 0 \text{ for all } \varphi \in W\} \subset \text{PH}^0(P^3, \mathcal{O}_{P^3}(2)).
\]

This means that \(R(W)\) lies naturally in \(P^5\):

\[
R(W) = W^\perp \cap \nu_3 \subset W = \text{PH}^0(P^3, \mathcal{O}_{P^3}(2))/W \subset \text{PH}^0(P^3, \mathcal{O}_{P^3}(2)).
\]

This allows to give another proof of

(3.3.2) \(R(W)\) is smooth if and only if \(W\) is regular.

**Proof.** \(R(W)\) is smooth if and only if \(W^\perp\) does not intersect \(\nu_3\) and intersects transversally \(\nu_3 - \nu_2\). Clearly, \(W^\perp \cap \nu_3 = \emptyset\) if and only if \(W\) has a basepoint \(x\). Moreover, for a generic basepoint \(x\) of \(W\), \((x, x) \in W^\perp \cap \nu_3\) is a quadruple point of \(R(W)\), as expected from the fact that \(x\) is an ordinary double point of \(S(W)\). Then one checks, using the description of the tangent space to \(\nu_3 - \nu_2\), that \(W\) intersects transversally \(\nu_3 - \nu_2\) at \((x, y)\) if and only if there exists no pencil of quadrics of \(W\) containing the line \(l\) joining \(x\) and \(y\) and containing a quadric singular along \(l\).

**Proposition 3.3.3.** Let \(W\) be a regular web. The linear system \(|\delta|\) of hyperplane sections of \(R(W) \subset W^\perp \cong R^5\) is numerically equivalent to \(\Sigma_i F_i/3\).

**Proof.** The images of the curves \(F_i\) are plane cubics on \(R(W) \subset P^5\). Indeed let \(P_i\) be the plane of the quadric of rank two \(q_i\) containing \(E_i\). Let \(N\) be a set of quadrics generating \(W\) with \(q_i\), and let \(N^\perp\) be the orthogonal of \(N\) in \(\text{PH}^0(P, \mathcal{O}_{P}(2))\). Consider the stratification of \(\text{PH}^0(P, \mathcal{O}_{P}(2))\) by the rank

\[
\text{PH}^0(P, \mathcal{O}_{P}(2)) = \nu_4 \supset \nu_3 \supset \nu_2.
\]

Then our claim follows from

\[
N^\perp \cap \nu_3 = \{(x, y) \in P_i \times P_i; \varphi(x, y) = 0 \text{ for all } \varphi \in N\}.
\]
Since $\Delta^2 = 10$, $\Delta \cdot F_i = 3$ for all $i$, it is easy to check that $\Delta$ is numerically equivalent to $\Sigma F_i / 3$.

**Corollary 3.3.4.** The Picard group of $R(W)$ is generated by $(\Delta, F_i)$ for an appropriate choice of $F_i$'s.

**Proof.** Changing one $F_i$ into $F_i + K_R$, if necessary, we can assume $3\Delta = \Sigma F_i + K_R$. Therefore, Pic $R$ is generated (over $\mathbb{Z}$) by $\Delta$ and the $F_i$'s: One checks that for any divisor $D$, $D$ and $(D\Delta)\Delta - \Sigma_i (DF_i)F_i$ are numerically equivalent.

**Corollary 3.3.5.** Let $W$ be an excellent web. Then $|\Delta| = |2F_i + D_i|$ for all $i$.

**Proof.** The numerical equivalence of $|\Delta|$ and $|2F_i + D_i|$ results from intersection relations. To conclude one needs to notice that $h^0(R, \mathcal{O}(\Delta - D_i)) \geq 2$ because the image of $D_i$ is included in the two hyperplanes defined by the quadrics of rank one corresponding to the two planes of $q_i$.

3.4. *A second projection model of $R(W)$ is $\mathbb{P}^5$.*

If $R(W)$ is the Reye convergence of a basepoint free web $W$, $R(W)$ lies naturally in the Grassmannian $G(1, 3)$ of lines in $\mathbb{P}^3$ by (3.1.2).

We will compute the homology class of $R(W)$ in $G(1, 3)$ after a brief reminder of Schubert calculus in $G(1, 3)$. After embedding $G(1, 3)$ in $\mathbb{P}^5$ by Plücker coordinates, we will compute the corresponding system of hyperplane sections of $R(W)$.

(3.4.1) *Schubert calculus in $G(1, 3)$.*

(3.4.1.1) Let $l \cap l'$ (resp. $p$, $P$) be a line (resp. a point, a plane) of $\mathbb{P}^3$. We have the following usual notations for Schubert cycles:

\[ \sigma_l = \{l'; l \cap l' \neq \emptyset\}, \quad \sigma_p = \{l'; p \in l'\}, \]
\[ \sigma_P = \{l'; l' \subset P\}, \quad \sigma_{p,P} = \{l'; p \in l' \subset P\}. \]

The cohomology ring of $G(1, 3)$ is generated by these cycles, $\sigma_{p,P} \in H_2(G(1, 3), \mathbb{Z})$, $\sigma_p, \sigma_P \in H_4(G(1, 3), \mathbb{Z})$ and $\sigma_l \in H_6(G(1, 3), \mathbb{Z})$.

(3.4.1.2) $G(1, 3)$ is embedded by Plücker coordinates as a smooth quadric hypersurface $\Omega \subset \mathbb{P}^5$. The two rulings of $\Omega$ are the two families of 2-planes $\sigma_p, \sigma_P$. Any line of $\Omega$ is of the form $\sigma_{p,P}$. The tangent space $T_l \Omega$ of $\Omega$ at $l$ is such that $(T_l \Omega) \cap \Omega = \sigma_l$.

(3.4.1.3) For any cycle $[S] \in H_4(G(1, 3), \mathbb{Z})$, we let $a, b$ be the two integers such that $[S] = a\sigma_p + b\sigma_P$. Classically, $a$ is called the order of $[S]$ and $b$ its class. We will say that $(a, b)$ is the type of $[S]$ and $a + b$ is its degree.

(3.4.2) We have the well-known result.

**Proposition 3.4.2.** The 4-cycle associated to a Reye congruence in $G(1, 3)$ is of type $(7, 3)$.

**Proof.** (Somewhat simpler than in [G-H].) The class of $R(W)$ is the number of lines of $R(W)$ in a generic plane $P$, i.e., half the number of points of

\[ \{(x, y), \varphi(x, y) = 0, \text{ for all } \varphi \in W\} \subset P \times P. \]

By Lemma 1.1.2, the class of $R(W)$ is equal to three.
The order of $R(W)$ is the number of lines $R(W)$ through a generic point $p$. Let $N(p)$ be the set of quadrics of $W$ through $p$. Let $p, x_1, \ldots, x_7$ be the basepoints of $N(p)$. Then obviously the seven lines $px_i$ belong to $R(W)$. Conversely, let $l$ be a line of $R(W)$ through $p$ and let $\pi \subset N(p)$ be a pencil of quadrics of $W$ containing $l$. Consider the residual twisted cubic $\gamma$ of the base locus of $\pi$. Then $\gamma$ does contain all the $x_i$’s if $l \neq px_i$ for all $i$. Hence, $\gamma$ is in the base locus of $N(p)$, which would imply that $W$ has some basepoint, which is not true for a generic $W$.

**Corollary 3.4.3.** The system of hyperplane sections of $R(W) \subset G(1,3) \subset \mathbb{P}^5$ is $|\Delta + K_R|$. 

**Proof.** Let us first prove that $R(W)$ is not included in any hyperplane. From the description of a hyperplane tangent to $\Omega$, it is clear that $R(W)$ can be included in a hyperplane only if this hyperplane is intersecting $\Omega$ transversally. Then $R(W)$ would be included in a smooth quadric in $\mathbb{P}^4$, hence a generic hyperplane section of $R(W)$ would be of type $(7,3)$ on a smooth quadric in $\mathbb{P}^3$. The genus of such a curve is $3 \cdot 7 - 3 - 7 + 1 = 12$ and, using the genus formula on $R(W)$, this genus should be $10/2 + 1 = 6$ since $R(W)$ is of degree 10 for the embedding in $G(1,3) = \Omega \subset \mathbb{P}^5$.

Therefore, $R(W)$ is embedded in $\mathbb{P}^5$ by a complete linear system. It is now easy to check that the images of the plane cubics $F_i$ are plane cubics of $R(W)$. Hence the system of hyperplane sections $|\Delta'|$ is (as in (3.3.3)) numerically equivalent to $|\Delta|$. Assuming that $S(W)$ is smooth, we have proved that $|\Delta| = |2F_i + D_i|$, therefore $|\Delta'| = |2F_i + D_i + K_R|$ because there is a hyperplane of $\mathbb{P}^5$ containing simultaneously $F_i$ and $F_i + K_R$, namely $T_{sing(\Omega)}(\Omega)$.

**Remark.** Let $R(W) = \Omega \subset \mathbb{P}^5$ be the Reye congruence of a good web $W$ embedded in $\mathbb{P}^5$ as a congruence of lines via Plücker coordinates. Then the generic hyperplane section $h$ of $R(W)$ is a Prym-canonical curve of genus six (cf. [D-S]) where the 2-torsion sheaf is $\mathcal{O}_h \otimes \mathcal{O}_h(K_R)$. This Prym-canonical curve lies in the ramification locus of the Prym map $P_5: M_6^{(2)} \to \mathcal{O}_5$ by a theorem of Beauville [B].

This suggests the following question: Does the image of the Reye-congruences dominate the ramification locus $B$ of $P_5$?

Notice that $\dim B = 14 = (\text{dimension of the family of Reye congruences}) + \dim \mathbb{P}^{5 \vee}$. Notice also that (dim of the family of Reye congruences) $+ \dim \Omega^{\vee} = 13$, where $\Omega^{\vee}$ is the dual variety of $\Omega$. By the fact that $R(W)$ is of type $(7,3)$, one can show that a section of $R(W)$ by a hyperplane tangent to $\Omega$ is generically a smooth trigonal curve and 13 is precisely the dimension of the trigonal locus of $M_6$.

More generally, one should look at the Hilbert scheme $\mathfrak{H}$ of polarized smooth Enriques surfaces of degree ten in $\mathbb{P}^5$ and consider the obvious map, defined in an open set $\mathfrak{V}$ of $\mathfrak{H}$, $\mathfrak{V} \to M_6^{(2)}$. This map is known to be dominant [M]. A natural problem is the study of the induced map $\mathfrak{V}/\text{PGL}(5) \to M_6^{(2)}$ especially in its relation to the Prym map $P_5: M_6^{(2)} \to \mathcal{O}_5$.

**References**


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