WEIGHTED LEBESGUE AND LORENTZ NORM INEQUALITIES
FOR THE HARDY OPERATOR

BY
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Abstract. Characterizations are obtained for those pairs of weight functions \( w, v \) for which the Hardy operator \( Tf(x) = \int_0^x f(t) \, dt \) is bounded from the Lorentz space \( L^{p,r}((0,\infty), v \, dx) \) to \( L^{q,s}((0,\infty), w \, dx) \), \( 0 < p, q, r, s < \infty \). The modified Hardy operators \( T_\eta f(x) = x^{-\eta} f(x) \) for \( \eta \) real are also treated.

1. Introduction. We characterize weighted Lebesgue and Lorentz norm inequalities for the Hardy operator \( Tf(x) = \int_0^x f(t) \, dt \) and the modified Hardy operators \( T_\eta f(x) = x^{-\eta} f(x) \), \( \eta \) real, \( x > 0 \) and \( f \) nonnegative. For Lebesgue norms much is already known. For example if \( 1 < p < q < \infty \) we have the result of J. S. Bradley, K. Andersen, B. Muckenhoupt, M. Artola, G. Talenti and G. Tomaselli (see [1, 3, 5, 6, 7 and 8]) which states that

\[
\left( \int_0^\infty T_f(x)^q w(x) \, dx \right)^{1/q} \leq C \left( \int_0^\infty f(x)^p v(x) \, dx \right)^{1/p} \quad \text{for all } f \geq 0
\]

if and only if the nonnegative weight functions \( w, v \) satisfy

\[
\sup_{x > 0} \left( \int_x^\infty w \left( \int_0^x v^{1-p'} \right)^{-1/p'} \right)^{1/q} = A < \infty.
\]

Moreover, \( A \leq C \leq p^{1/q} (p')^{1/p'} A \) if \( C \) is the least constant for which (1.1) holds. As usual we take \( 0 \cdot \infty = 0, 0^0 = 0 \) and interpret \( (\int_0^x v^{1-p'})^{1/p'} \) as \( \|X_{(a,b)} v^{-1}\|_{L^{p,r}(x)} \) in the case \( p' = \infty \). The theorems below contain characterizations of (1.1) for all \( p \) and \( q \) satisfying \( 0 < p, q < \infty \).

Our first theorem on weighted Lorentz norm inequalities (see below for definitions) for the Hardy operator \( T \) makes use of a condition (see (1.4)) suggested by a recent work of H.-M. Chung, R. Hunt and D. Kurtz [2].

**Theorem 1.** Suppose \( 0 < p, q \leq \infty \) (where \( q = \infty \) if \( p = \infty \)), \( 1 < r < \infty \) and \( 1 \leq s < \infty \). If the weights \( w, v \geq 0 \) satisfy

\[
\|f\|_{L^{p,q}(w)} \leq C \|f\|_{L^{r,s}(v)} \quad \text{for all } f \geq 0,
\]

(1.3)
then the pair \( w, v \) satisfies

\[
\sup_{x>0} \left( \int_x^\infty w \right)^{1/p} \left\| x(0, x)^{v^{-1}} \right\|_{L^{1/r}(v)} = A < \infty \\
\text{and } v > 0 \text{ a.e. on } (0, x) \text{ if } \int_x^\infty w > 0.
\]

Conversely, (1.4) implies (1.3) if and only if \( q \geq \max\{r, s\} \).

This theorem leaves open the cases where \( \min\{r, s\} < 1 \) or \( q < \min\{r, s\} \). The next two theorems give partial results in each case. The first shows that if the basic index \( r \) is less than one, then (1.3) holds if and only if the weight pair \( w, v \) is trivial. The second theorem characterizes (1.3) for \( q < \min\{r, s\} \) in the special case when \( r = s \).

**Theorem 2.** If \( 0 < r < 1 \) and \( 0 < p, q, s \leq \infty \), then (1.3) holds if and only if the weight pair \( w, v \) is trivial in the sense that \( v = \infty \) a.e. on any interval \( (0, x) \) such that \( \int_x^\infty w > 0 \).

**Theorem 3.** Suppose \( 0 < p, q \leq \infty \) (where \( q = \infty \) if \( p = \infty \)) and \( 1 \leq r < \infty \). Then

\[
\| T^q \|_{L^{r,q}(w)} \leq C \| f \|_{L^r(v)} \quad \text{for all } f \geq 0
\]
in the case \( q \geq r \) and if and only if

\[
\sup_{x>0} \left( \int_x^\infty w \right)^{1/p} \left( \int_0^{x^{1-r}} v^{1-r} \right)^{1/r} = A < \infty
\]

where \( 1/p = 1/q - 1/r \) in the case \( 0 < q < r \). The sup in (1.7) is taken over all positive increasing sequences \( \{x_k\} \) and

\[
\tilde{w}(x) = -\frac{d}{dx} \left( \int_x^\infty w \right)^{a/p} = \frac{a}{p} \left( \int_x^\infty w \right)^{(a/p)-1} w(x).
\]

We now turn to the modified Hardy operators \( T^q f(x) = x^{-q} T^q f(x) \). Andersen and Muckenhoupt have characterized weighted weak type inequalities for these operators [5]. For \( \eta \leq 0 \), \( T^q f \) satisfies the following monotonicity condition—\( T^q f \) is nonnegative for nonnegative \( f \)—and this allows us to replace weighted Lorentz norms of \( T^q f \) by weighted Lebesgue norms of \( T^q f \) and Theorems 1–3 can now be used. Indeed, for \( f \geq 0 \), (2.6) below shows that the \( L^{p,q}(w) \) norm of \( T^q f \) coincides with the \( L^q(w) \) norm of \( T^q f \) where \( w^\eta(x) = x^{-\eta} \tilde{w}(x) \) and \( \tilde{w} \) is as in Theorem 3.

For \( \eta > 0 \), \( T^q f \) does not satisfy the above monotonicity condition and, consequently, weighted Lorentz norms of \( T^q f \) are harder to deal with (Lebesgue norms of \( T^q f \) are of course equal to weighted Lebesgue norms of \( T^q f \)). There is, however, one case in which a weighted Lorentz norm inequality for \( T^q f, \eta > 0 \), can be reduced by duality to an inequality for an operator satisfying the monotonicity condition. The case \( q = \infty \) of the following theorem is contained in [5].
Theorem 4. Suppose \( \eta > 0 \) and \( 1 \leq r \leq \min\{p, q\} \). Then
\[
\|T_f\|_{L^{r,q}(\nu)} \leq C\|f\|_{L^{(p,q)}(\mu)} \quad \text{for all } f \geq 0
\]
if and only if
\[
\|s^{-\eta}X_{(x,\infty)}(s)\|_{L^{r,q}(\nu)} \left( \int_0^x v^{1-r}\right)^{1/r} \leq C \quad \text{for all } x > 0,
\]
where the second factor on the left side of (1.9) is interpreted as \( \|X_{(0,x)}v^{-1}\|_{L^{(p,q)}(\nu)} \) if \( r = 1 \).

Note that in the case \( q = \infty \), (1.9) becomes
\[
\sup_{0 < x < s < \infty} s^{-\eta} \left( \int_x^s v^{1-r}\right)^{1/r} \leq C,
\]
which is easily seen to coincide with the condition of Andersen and Muckenhoupt in [5, Theorem 2]. It may be useful to point out that for \( q < \infty \),
\[
\|s^{-\eta}X_{(x,\infty)}\|_{L^{r,q}(\nu)} = \etaq \int_x^\infty \left( s^{-\eta} \left( \int_x^s w^{1-r}\right)^{1/r} \right)^q ds.
\]

We now give some definitions. If \( f \) denotes a measurable function defined on a measure space \((M, \mu)\), the distribution function \( f_\ast \) and the nonincreasing rearrangement \( f^\ast \) of \( f \) with respect to \( \mu \) are given by (see e.g. [4, Chapter V])
\[
f_\ast(s) = |\{ |f| > s \}|_\mu = \int_{\{|f| > s \}} d\mu, \quad f^\ast(t) = \inf \{ s; f_\ast(s) \leq t \}.
\]
For \( 0 < p < \infty \), \( 0 < q \leq \infty \), the Lorentz space \( L^{p,q}(\mu) \) consists of all functions \( f \) satisfying \( \|f\|_{L^{p,q}(\mu)} < \infty \), where
\[
\|f\|_{L^{p,q}(\mu)} = \begin{cases} \left[ \int_0^\infty \frac{q}{p} t^{q/p-1} f^\ast(t)^q \, dt \right]^{1/q} & \text{for } 0 < q < \infty, \\ \sup_{t > 0} t^{1/p} f^\ast(t) & \text{for } q = \infty \end{cases}
\]
Note that
\[
\|f\|_{L^{p,q}(\mu)} = \|f\|_{L^p(\mu)} = \left( \int_M |f|^p \, d\mu \right)^{1/p}.
\]
We shall need the following basic relationship between \( L^{p,q} \) and \( L^{p',q'} \), where \( 1/p + 1/p' = 1 = 1/q + 1/q' \), \( 1 < p < \infty \) and \( 1 < q < \infty \). (See e.g. [2, inequality (2.3)].)
\[
C^{-1}\|f\|_{L^{p,q}(\mu)} \leq \sup_{\|g\|_{L^{p',q'}(\mu)} = 1} \left| \int fg \, d\mu \right| \leq C\|f\|_{L^{p,q}(\mu)}.
\]
2. Proofs of theorems.

Proof of Theorem 1. Applying the change of variable $t = f_*(s)$ to the right side of (1.10) and integrating by parts we obtain [2, (2.4)]

$$\|f\|_{L^{p,q}(\mu)} = \begin{cases} \left[ \int_0^\infty qs^{q-1}f_*(s)^{q/p} \, ds \right]^{1/q} & \text{for } 0 < q < \infty, \\ \sup_{s \geq 0} sf_*(s)^{1/p} & \text{for } q = \infty, \end{cases}$$

(or simply evaluate the two iterated integrals of $qs^{q-1}(q/p)t^{q/p-1}$ over the set $\{(t, s); 0 < s < f^*(t), 0 < t\}$). We now prove (1.3) $\Rightarrow$ (1.4). Let $f$ be nonnegative on $(0, \infty)$. Then $T(X_{(0,x)}f) \geq f_0^x f$ on $[x, \infty)$ and so $T(X_{(0,x)}f)_*(\xi) \geq f_0^\xi w$ for $0 \leq \xi < f_0^x f$. Inequality (1.3), together with (2.1), yields

$$\|f\|_{L^{p,q}(\nu)} \geq C^{-1}||Tf||_{L^{p,q}(\nu)} \geq C^{-1}\left[ \int_0^\lambda \left( \int_x^\infty w \right)^{q/p} q t^{q-1} \, dt \right]^{1/q}, \quad \left( \lambda = \int_0^x f \right) = C^{-1}\left( \int_x^\infty w \right)^{1/p} \left( \int_0^\lambda f_0^\lambda v^{-1} \right)$$

and (1.4) now follows easily upon using (1.11) for $0 < p, q < \infty$. The cases $p = \infty$ or $q = \infty$ are established by simple modifications of this argument.

Conversely, fix $f \geq 0$ in $L^{r,s}(\nu)$ and suppose $q \geq \max\{r, s\}$. If $w \neq 0$, then $\int_x^\infty w > 0$ for some $x > 0$ and from (1.4) and (1.11) we have

$$\int_0^x f \leq \int_0^x f_0^\lambda v^{-1} \leq \|f\|_{L^{r,s}(\nu)} \|X_{(0,x)}v^{-1}\|_{L^{r,s}(\nu)} < \infty.$$ 

Thus we can choose $x_k$ such that $Tf(x_k) = \int_0^{x_k} f = 2^k$ for all $k$ in $Z$ satisfying $2^k < \int_0^\infty f$. We suppose $0 < q < \infty$, the case $q = \infty$ being an easy modification of the following argument. From (2.1) we have

$$\|Tf\|_{L^{p,q}(\nu)} = q \int_0^\infty s^{q-1}(Tf)_*(s)^{q/p} \, ds \leq C \sum_k 2^k q \left( \int_{\{T > 2^k\}} w \right)^{q/p} \leq C \sum_k \left( \int_{x_{k-1}}^{x_k} f v^{-1} \right)^q \left( \int_{x_k}^\infty w \right)^{q/p} \leq C \sum_k \|f_k\|_{L^{r,s}(\nu)} \|X_{(0,x_k)}v^{-1}\|_{L^{r,s}(\nu)} \left( \int_{x_k}^\infty w \right)^{q/p}$$

by (1.11), where

$$f_k = X_{(x_{k-1}, x_k)} f \leq CA \sum_k \|f_k\|_{L^{r,s}(\nu)} \leq CA \|f\|_{L^{r,s}(\nu)}$$

by the following lemma, which is a slight extension of Lemma 2.5 in [2].
**Lemma 1 [2].** Let \((M, \mu)\) be a measure space. Suppose \(q \geq \max\{r, s\}\) and \(\{E_k\}\) is a sequence of disjoint measurable subsets of \(M\). Then

\[
\sum_k \|X_{E_k}f\|_{L^{r, \mu}(\mu)} \leq \|f\|_{L^{r, \mu}(\mu)}.
\]

**Proof.**

\[
\sum_k \|X_{E_k}f\|_{L^{r, \mu}(\mu)} = \sum_k \left( \int_0^\infty \left( \int_{E_k} f(t)^{\gamma/r} \, dt \right)^{\nu/s} \, ds \right)^{q/s}
\]

by Minkowski, since \(q \geq s\)

\[
\leq \left( \int_0^\infty \left( \sum_k \left( \int_{E_k} f(t)^{\gamma/r} \, dt \right)^{q/r} \right)^{s/r} \, ds \right)^{q/s}
\]

It remains to show that if \(q < \max\{r, s\}\), then (1.4) does not imply (1.3). We consider two cases: \(q < s\) and \(s \leq q < r\). If \(q < s\), set \(v \equiv 1\) on \((0, \infty)\) and let \(w\) be such that the product on the left side of (1.4) is identically 1, i.e. \(w(x) = (p/r')x^{-p/r'-1}\). Let

\[
f_{\alpha, \beta}(x) = x^{-\alpha}(1 + |\log x|)^{-\beta},
\]

where \(\alpha = 1/r\) and \(q < 1/\beta < s\). Now if \(x_i = t^{-1/\eta}(1 + |\log t|)^{-\beta/\alpha}\), then \(f_{\alpha, \beta}(x_i) \approx t\), so using (2.1),

\[
\|f_{\alpha, \beta}\|_{L^{r, \mu}(v)} = s \int_0^\infty \left\{ f_{\alpha, \beta} > t \right\}^{\gamma/r} t^{s-1} \, dt
\]

\[
\approx \int_0^\infty x_i^{\gamma/r} t^{s-1} \, dt = \int_0^\infty \frac{dt}{t(1 + |\log t|)^{\beta s}} < \infty.
\]

On the other hand,

\[
Tf_{\alpha, \beta}(x) \approx x^{1/r}(1 + |\log x|)^{-\beta}
\]

and for \(y_i = t^r(1 + |\log t|)^{\beta r}\), we have \(Tf_{\alpha, \beta}(y_i) \approx t\). Thus

\[
\|Tf_{\alpha, \beta}\|_{L^{q, \mu}(w)} = q \int_0^\infty \left\{ Tf_{\alpha, \beta} > t \right\}^{q/p} t^q \, dt
\]

\[
\approx \int_0^\infty \left[ \int_{y_i}^{x_i} \frac{p}{r} x^{-p/r'-1} \, dx \right]^{q/p} t^{q-1} \, dt
\]

\[
= \int_0^\infty \frac{dt}{t(1 + |\log t|)^{\beta q}} = \infty,
\]

so the weight pair \((w, v)\) satisfies (1.4) but not (1.3).

Finally, if \(s \leq q < r\), set \(v(x) = f_{\alpha, \beta}(x)\), where \(\alpha = 1\) and \(1 < \beta < r/q\), and let \(w(x) = f_{r, \delta}(x)\), where \(\gamma = p + 1\) and \(\delta = \beta p/r\). Then

\[
\left( \int_x^\infty w \right)^{1/p} \approx x^{-1}(1 + |\log x|)^{-\beta/r},
\]
and for \( s \leq r \) we have

\[
\|x_{(0,x)}^w\|_{L^r(v)} \leq \|x_{(0,x)}^w\|_{L^r(v)} = \left( \int_0^x v^{-r}\right)^{1/r} \leq Cx(1 + |\log x|)^{\beta/r},
\]

so condition (1.4) holds for the weights \( w, v \). With \( f \equiv 1 \) on \((0, \infty)\) we have

\[
\|f\|_{L^r(v)}^s = \left( \int_0^\infty v^{s/r} \right) < \infty
\]

since \( \beta > 1 \), while

\[
\|Tf\|_{L^r(w)} = \int_0^\infty \frac{dt}{t(1 + |\log t|)^{\beta/r}} = \infty \quad \text{since} \quad \frac{\beta q}{r} < 1,
\]

so (1.4), but not (1.3), holds for the weight pair \((w, v)\). This completes the proof of Theorem 1.

**Proof of Theorem 2.** Clearly (1.3) holds if the weight pair \((w, v)\) is trivial in the sense indicated in Theorem 2. Conversely suppose (1.3) holds for some \( r < 1 \). Let \( F(x) = \min(1, v(x)^{-1/r}) \) and set \( f_{a,b} = \chi_{(a,b)} F \) for \( 0 < a < b < \infty \). Suppose for the moment that \( r < s \) and \( p < \infty \). With \( f = f_{a,b} \) in (1.3) we obtain, as in (2.2), that

\[
(2.4) \quad \left( \int_a^b F \right) \left( \int_b^\infty w \right)^{1/p} \leq C\|f_{a,b}\|_{L^r(v)} \leq C\|f_{a,b}\|_{L^r(v)} = C\left( \int_a^b F^{-1} v \right)^{1/r} \quad \text{since} \quad r < s
\]

\[
\leq C(b - a)^{1/r} \quad \text{since} \quad F(x)^{-1} v(x) \leq 1 \quad \text{for all} \quad x.
\]

Now divide both sides of (2.4) by \((b - a)\) and let \((a, b)\) shrink to a Lebesgue point \( x \) of \( F \) to obtain \( F(x)(\int_x^\infty w)^{1/p} \leq 0 \), which yields \( v(x) = \infty \) if \( \int_x^\infty w > 0 \). If \( 0 < s < r \) we can modify the above argument as follows. Let \( d = \inf\{x; v(x) = 0\} \) and there is a set \( E \subset (0, d) \) of positive Lebesgue measure satisfying \( f_E v < \infty \). Suppose, in order to derive a contradiction, that such a set \( E \) exists. Let \( F(x) \) be as above but set \( f_{a,b} = \chi_{E \cap (a,b)} F \) for \( 0 < a < b < d \). Now choose \( \rho \) such that \( d < \rho < 1 \) and in (2.4) replace \( \|f_{a,b}\|_{L^r(v)} \leq C\|f_{a,b}\|_{L^r(v)} \) (which may fail for \( s < r \)) with \( \|f_{a,b}\|_{L^r(v)} \leq C\|f_{a,b}\|_{L^r(v)} \), where \( C' \) depends on \( f_E v \) and \( \rho \) as well as on \( r \) and \( s \). Arguing as before we obtain that \( \chi_E(x) F(x)(\int_x^\infty w)^{1/p} \leq 0 \) whenever \( x \) is a Lebesgue point of \( \chi_E F \). Since \( \int_x^\infty w > 0 \) for \( x < d \) we conclude that \( v = \infty \) a.e. on \( E \). Thus \( f_E v = \infty \), the desired contradiction. The case \( p = \infty \) is an easy adaptation of these arguments and this completes the proof of Theorem 2.

**Proof of Theorem 3.** In the case \( q > r > 1 \), the equivalence of (1.5) and (1.6) is a special case of Theorem 1. If \( q \geq r = 1 \) the equivalence of (1.5) and (1.6) can be established by the argument of Theorem 1 (see (2.2) and (2.3)) since the analogue of

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(1.11) holds in this case, i.e.

\[
\|f\|_{L^r(w)} = \sup_{\|g\|_{L^r(w)} \leq 1} \left| \int fgw \right| \quad \text{for } 1 \leq r \leq \infty.
\]

To handle the case \(0 < q < r\), we first observe that if \(h\) is nonnegative and nondecreasing on \((0, \infty)\) then

\[
\|h\|_{L^{r/q}(w)}^q = \frac{q}{p} \int_0^\infty h(x)^q g(x)^{q/p-1} w(x) \, dx,
\]

where \(g(x) = \int_0^\infty w\). This equality is established by evaluating the two iterated integrals of \(q^{q-1}(q/p)g(x)^{q/p-1}w(x)\) over the set \(\{(x, s); 0 < s < h(x), 0 < x\}\). Performing the \(s\) integration first yields the right side of (2.6), and performing the \(x\) integration first yields the right side of (2.1) since for fixed \(s\), if

\[
x(s) = \sup \{x; h(x) \leq s\}
\]

then

\[
\int_0^\infty \frac{q}{p} g(x)^{q/p-1} w(x) \, dx = g(x(s))^{q/p} = \left( \int_0^\infty w \right)^{q/p} = h_*(s)^{q/p}.
\]

Now suppose (1.5) holds and \(0 < q < r\). First assume \(r > 1\). Fix a positive increasing sequence \(\cdots x_k < x_{k+1} \cdots\) and, given a sequence of positive numbers \(a_k\), set \(f = \sum a_k x_k\sigma\), where \(x_k = x_{(x_{k-1}, x_k)}\) and \(\sigma = v^{1-r'}\). Then with \(\tilde{w}(x) = (q/p)g(x)^{q/p-1}w(x)\) we have

\[
\left( \sum_k a_k x_k \sigma \right)^{q/r} = \left\| \sum_k a_k x_k \sigma \right\|_{L^q(w)}^q \geq C\|Tf\|_{L^{r/q}(w)}^q
\]

\[
= \frac{q}{p} \int_0^\infty Tf(x)^q g(x)^{q/p-1} w(x) \, dx \quad \text{by (2.6)}
\]

\[
\geq C \sum_k Tf(x_k)^q \int_{x_k}^{x_{k+1}} \frac{q}{p} g(x)^{q/p-1} w(x) \, dx
\]

\[
\geq C \sum_k \left( \int_{x_k}^{x_{k+1}} f \right)^q \left( \int_{x_k}^{x_{k+1}} \tilde{w} \right)^{q/r}
\]

\[
= C \sum_k \left[ a_k^q \left( \int x_k \sigma \right)^{q/r} \left( \int_{x_k}^{x_{k+1}} \tilde{w} \right) \left( \int_{x_k}^{x_{k+1}} v^{1-r'} \right)^{q/r} \right]
\]

for all sequences \(\{a_k\}\) of nonnegative numbers. Since the dual of the sequence space \(l^{r/q}\) is \(l^{(r/q)'}\), (2.7) shows that the sequence, whose terms consist of the second factors \([\cdots]\) in the final sum above, is in \(l^{(r/q)'}\). Since \(q(r/q)' = \rho\), this proves (1.7). Note that (1.7) persists even if \(\int x_k \sigma = \infty\) for some \(k\) since then (1.5) easily implies \(\int_x^\infty w = 0\). Finally for \(r = 1\) we modify this argument as follows. Let \(f = \sum a_k f_k\), where \(f_k\) is supported in \([x_{k-1}, x_k]\). If now \(f_k\) is allowed to vary within the unit ball of \(L^1(v)\), (2.7) and (2.5) yield (1.7) as above.

Now suppose (1.7) holds and \(0 < q < r\). Fix \(f \geq 0\) in \(L^r(w)\). As in the proof of Theorem 1 we can choose \(x_k\) such that \(Tf(x_k) = 2^k\) for all integers \(k\) satisfying
2^k < \int_0^\infty f. Then (2.6) yields
\[ \|Tf\|_{L^{p,q}(w)}^p = \frac{q}{p} \int_0^\infty Tf(x)^p g(x)^{q/p} w(x) \, dx = \sum_{k} \int_{x_k}^{x_{k+1}} \]
\[ \leq 2^q \sum_{k} 2^{kq} \left( \int_{x_k}^{x_{k+1}} \frac{1}{w} \right) \leq 4^q \sum_{k} \left( \int_{x_k}^{x_{k+1}} f \right)^{q/p} \left( \int_{x_k}^{x_{k+1}} \frac{1}{w} \right)^{q/p} \]
\[ \leq 4^q \sum_{k} \left( \int_{x_k}^{x_{k+1}} f \right)^{q/p} \left[ \left( \int_{x_k}^{x_{k+1}} v^{-r} \right)^{1/r} \left( \int_{x_k}^{x_{k+1}} \frac{1}{w} \right)^{1/q} \right] \]
\[ \leq 4^q B^q \left( \sum_{k} \int_{x_k}^{x_{k+1}} f \right)^{q/p} = (4B \|f\|_{L'(v)})^q \]

by an application of Holder’s inequality with exponents \( r/p \) and \( (r/q)' \). Thus (1.5) holds and this completes the proof of Theorem 3.

**Proof of Theorem 4.** Let \( f \) be nonnegative on \((0, \infty)\) and fix \( x > 0 \). Since \( T^\xi f(s) \geq s^{-\gamma} \int_0^s f(t) \, dt \) for \( s \geq x \), we have, from (1.8),
\[ \left( \int_0^\infty f(t) \, dt \right)^{r-1} \|s^{-\gamma} \chi(x, \infty)\|_{L^{p,q}(w)} \leq \|T^\xi f\|_{L^{p,q}(w)} \leq C \|f\|_{L'(v)} \]
By duality we obtain
\[ \|\chi(0, x)^{r-1} \|_{L'(v)} \|s^{-\gamma} \chi(x, \infty)\|_{L^{p,q}(w)} \leq C, \]
which is (1.9).

Conversely, we begin by noting that (1.8) is equivalent to the dual inequality
\[ (2.8) \quad \|v^{-1} T^\xi (g w)\|_{L'(v)} \leq C \|g\|_{L^{r,q}(w)} \quad \text{for all } g \geq 0, \]
where \( T^\xi \|f(x) = \int_0^\infty s^{-\gamma} g(s) \, ds \). Suppose \( g \geq 0 \) and \( \|g\|_{L^{r,q}(w)} < \infty \). Consider first the case \( r > 1 \). Provided \( v \not\equiv \infty \) on \((0, \infty)\) we can, as in the proof of Theorem 1, choose \( x_k \) such that \( \int_{x_k}^{x_{k+1}} s^{-\gamma} g(s) \, ds = 2^k \) for all integers \( k \) satisfying \( 2^k < \int_0^\infty s^{-\gamma} g(s) \, ds \).

Then with \( \chi_k = \chi(x_k, x_{k+1}) \) we have
\[ \|v^{-1} T^\xi (g w)\|_{L'(v)} \leq C \sum_{k} \left( \int_{x_k}^{x_{k+1}} s^{-\gamma} g(s) \, ds \right)^{r-1} \left( \int_{x_k}^{x_{k+1}} v^{1-1/r} \right) \]
\[ \leq C \sum_{k} \|\chi_k g\|_{L^{r,q}(w)} \left[ \|s^{-\gamma} \chi(x_k, \infty)\|_{L^{r,q}(w)} \int_{x_k}^{x_{k+1}} v^{1-1/r} \right] \text{ by (1.11)} \]
\[ \leq C \sum_{k} \|\chi_k g\|_{L^{r,q}(w)} \quad \text{by (1.9)} \]
\[ \leq C \|g\|_{L^{r,q}(w)} \]
by Lemma 1 since \( r' \geq \max\{p', q'\} \). Finally for \( r = 1 \) we have
\[
\|v^{-1} T^*_\eta(gw)\|_{L^{r}(\mathbb{C})} \leq \sup_{x > 0} \|X_{(0,x)} v^{-1} T^*_\eta(gw)(x)\|
\]
\[
= \sup_{x > 0} \|X_{(0,x)} v^{-1} \|_{L^{r}(\mathbb{C})} \int_x^\infty s^{-\eta} g(s) w(s) \, ds
\]
\[
\leq \sup_{x > 0} \|X_{(0,x)} v^{-1} \|_{L^{r}(\mathbb{C})} \|s^{-\eta} X_{(x,\infty)}(s)\|_{L^{r,\eta}(\nu(w))} \|g\|_{L^{r,\eta}(\nu(w))} \text{ by (1.11)}
\]
\[
\leq C \|g\|_{L^{r,\eta}(\nu(w))} \text{ by (1.9)},
\]
and this completes the proof of Theorem 4.

**Remarks.** (I) Theorem 3 has a simple analogue in the case \( r = \infty \). If \( 0 < p, q \leq \infty \) (where \( a = \infty \) if \( p = \infty \)) then
\[
\|Tv\|_{L^{p,\eta}(\nu(w))} \leq C \|f\|_{L^{\infty}(\mathbb{C})} \text{ for all } f \geq 0
\]
if and only if \( \int_{-\infty}^\infty w = 0 \) whenever \( |\{t \in [0, x]; v(t) = 0\}| \) is positive and \( \|h\|_{L^{p,\eta}(\nu(w))} \leq C \), where \( h(x) = x \) for \( x > 0 \).

(II) In the case \( q = r \) of Theorem 3, one has \( A \leq C \leq q^{1/q} (q')^{1/q} A \) provided \( C \) is the least constant for which (1.5) holds. These inequalities are sharp. That \( A \leq C \) is obvious and to obtain the other inequality replace \( h \) by \( Tf \) in (2.6) and apply Theorem 1 of [3] (i.e. the equivalence of (1.1) and (1.2) for \( p = q \)) to obtain (1.5) with
\[
C \leq q^{1/q} (q')^{1/q} \sup_{x > 0} \left( \int_x^\infty \frac{q}{p} g(t)^{q/p-1} w(t) \, dt \right)^{1/q} \left( \int_0^x s^{-\eta} \right)^{1/q} = q^{1/q} (q')^{1/q} A.
\]
To show that this latter inequality is best possible let \( w(x) = x^{-p'/q'-1} \) and \( v(x) = 1 \) so that \( A \) in (1.6) equals \( (p/q')^{1/p} \). For \( \delta > -1/q \) define \( f_\delta(x) = \chi_{(0,x)}(x)x^\delta \), where \( t \) is chosen so that \( \|f_\delta\|_{L^{q}} = 1 \), i.e. \( t = (1 + q\delta)^{1/(1 + q\delta)} \). A computation using (2.6) shows that
\[
\lim_{\delta \to -1/q} \|Tf_\delta\|_{L^{p,\eta}(\nu(w))} = (q'/p)^{1/p} q^{1/q} (q')^{1/q'} = A q^{1/q} (q')^{1/q'}.
\]

**References**
6. M. Artola, untitled and unpublished manuscript.

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