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FINITELY GENERATED EXTENSIONS OF PARTIAL DIFFERENCE FIELDS

BY

PETER EVANOVICH[1]

ABSTRACT. A proof of the following theorem is given: If $\mathfrak{M}$ is a finitely generated extension of a partial difference field $\mathfrak{K}$ then every subextension of $\mathfrak{M}/\mathfrak{K}$ is finitely generated. An integral measure of partial difference field extensions having properties similar to the dimension of field extensions and the limit degree of ordinary difference field extensions and a new method of computing transformal transcendence degree are developed.

1. Introduction. The objective here is to show that subextensions of finitely generated difference field extensions are finitely generated (Theorem 1).

Throughout we adopt the definitions and notation developed by Cohn [**3**] and Bentsen [**1, 2**]. In particular, a difference field $\mathfrak{K} = (K; \sigma_1, \ldots, \sigma_n)$ is a field K and a finite number, $\sigma_1, \ldots, \sigma_n$, $n \geqslant 1$, of pairwise commuting isomorphisms of K into itself. If $\mathfrak{M} = (M; \sigma_1, \ldots, \sigma_n)$ is an extension of $\mathfrak{K}$, $\Delta\mathfrak{M}/\mathfrak{K}$ denotes the degree of transformal transcendence of $\mathfrak{M}/\mathfrak{K}$ [**2**, pp. 18–30; **3**, pp. 118–126], and if S is a subset of M, $\mathfrak{K}\langle S \rangle$ denotes the subextension of $\mathfrak{M}/\mathfrak{K}$ generated by S [**1**, §§2 and 3].

The proof of Theorem 1, when $\mathfrak{M}$ has one transforming operator, is given by Cohn [**3**, pp. 145–146]. The notion of limit degree is of prime importance in Cohn's proof and is inductively extended here (§3) for partial difference field extensions.

If $\mathfrak{K}$ is inversive (each σ_i is an automorphism of K) [**1**, §3] and $\mathfrak{M}$ is the inversive closure of a finitely generated extension of $\mathfrak{K}$, then Theorem 1 implies the following corollary: Every inversive subextension of $\mathfrak{M}/\mathfrak{K}$ is the inversive closure of a finitely generated extension of $\mathfrak{K}$. For suppose $\mathfrak{M}$ is the inversive closure of $\mathfrak{K}\langle S \rangle$, S a finite subset of M, and $\mathfrak{L}$ is an inversive subextension of $\mathfrak{M}/\mathfrak{K}$. From Theorem 1 $\mathfrak{L} \cap \mathfrak{K}\langle S \rangle$ is a finitely generated extension of K and the inversive closure of $\mathfrak{L} \cap \mathfrak{K}\langle S \rangle$ in $\mathfrak{M}$ is $\mathfrak{L}$, for if $x \in L$ then there exist positive integers $k_1, \ldots, k_n$ such that $y = \sigma_1^{k_1} \cdots \sigma_n^{k_n}(x) \in K\langle S \rangle$ and therefore $y \in L \cap K\langle S \rangle$. This corollary, when the underlying extension of $\mathfrak{M}/\mathfrak{K}$ is separable algebraic, follows from Theorem 2.1 of [**6**]. In the case of separable algebraic extensions, Theorem 2.1 of [**6**] is stronger than Theorem 1 of this paper in that the transforming operators are not required to commute on M.

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The following notation will be adopted throughout the paper. if $\mathcal{K} = (K; \sigma_1, \dots, \sigma_n)$ is a difference field, $S \subseteq K$, and $\sigma_{i_1}, \dots, \sigma_{i_k}$ is a nonempty subcollection of $\sigma_1, \dots, \sigma_n$ then

$$S^{(\sigma_{i_1}, \dots, \sigma_{i_k})} = \left\{ \sigma_{i_1}^{j_1}, \dots, \sigma_{i_k}^{j_k}(s) | s \in S \text{ and } j_1, \dots, j_k \text{ are nonnegative integers} \right\}.$$

For $k = 0, 1, 2, \dots,$ define $S_k = \bigcup_{i=0}^{k} \sigma_n^i (S^{(\sigma_1, \dots, \sigma_{n-1})})$. (If $n = 1$, $S^{(\sigma_1, \dots, \sigma_{n-1})} = S$.) Also, if $\mathcal{K}$ is a subfield of $\mathcal{L} = (L; \sigma_1, \dots, \sigma_n)$ and T is a subset of the field L then the underlying field of $\mathcal{K}\langle T \rangle$ will be denoted by $K\langle T \rangle$.

Finally, if $\mathbf{i} = (i_1, \dots, i_n)$, $\mathbf{j} = (j_1, \dots, j_n)$ are vectors with integral coordinates, $\mathbf{i} \geqslant \mathbf{j}$ will mean that $i_k \geqslant j_k$, $k = 1, \dots, n$.

2. Limit transcendence degree. Let $\mathcal{L} = (L; \sigma_1, \dots, \sigma_n)$ be a finitely generated extension of $\mathcal{K} = (K; \sigma_1, \dots, \sigma_n)$ and S a finite set of generators of $\mathcal{L}/\mathcal{K}$. For $k = 1, 2, \dots,$ let

$$\delta_k = \Delta\big(K(S_k); \sigma_1, \dots, \sigma_{n-1}\big) / \big(K(S_{k-1}); \sigma_1, \dots, \sigma_{n-1}\big).$$

(We adopt the conventions that, when $n = 1$, $(K(S_k); \sigma_1, \dots, \sigma_{n-1}) = K(S_k)$, δ_k is the transcendence degree of $K(S_k)/K(S_{k-1})$ and bases of transformal transcendence are transcendence bases.) Since σ_n is an isomorphism and, for $k = 1, 2, \dots,$ $K(S_{k+1})$ and $K(S_k)$ can be obtained by adjoining the same set of elements to $\sigma_n(K(S_k))$ and $\sigma_n(K(S_{k-1}))$, respectively, it follows that $\delta_{k+1} \leqslant \delta_k$, $k = 1, 2, \dots.$ Define the *limit (transformal) transcendence degree* of $\mathcal{L}/\mathcal{K}$, $\Delta_l \mathcal{L}/\mathcal{K}$, by

$$\Delta_l \mathcal{L}/\mathcal{K} = \min\{\delta_k | k = 1, 2, \dots\}.$$

Since $(K(S_k); \sigma_1, \dots, \sigma_{n-1})/(K(S_{k-1}); \sigma_1, \dots, \sigma_{n-1})$ is finitely generated, $k = 1, 2, \dots,$ $\Delta_l \mathcal{L}/\mathcal{K}$ is a nonnegative integer.

LEMMA 1. $\Delta_l \mathcal{L}/\mathcal{K} = \Delta \mathcal{L}/\mathcal{K}$.

PROOF. Choose k_0 such that if $k \geqslant k_0$ then $\delta_k = \Delta_l \mathcal{L}/\mathcal{K}$, and select $T \subseteq S$ such that $\sigma_n^{k_0}(T)$ is a basis of transformal transcendence of

$$\big(K(S_{k_0}); \sigma_1, \dots, \sigma_{n-1}\big) / \big(K(S_{k_0 - 1}); \sigma_1, \dots, \sigma_{n-1}\big).$$

Let $p \in \{0, 1, 2, \dots\}$, then $\sigma_n^{k_0 + p}(T)$ is a basis of transformal transcendence of

$$\big(\sigma_n^p\big(K(S_{k_0})\big); \sigma_1, \dots, \sigma_{n-1}\big) / \big(\sigma_n^p\big(K(S_{k_0 - 1})\big); \sigma_1, \dots, \sigma_{n-1}\big)$$

and so must contain a basis of transformal transcendence of

$$\big(K(S_{k_0 + p}); \sigma_1, \dots, \sigma_{n-1}\big) / \big(K(S_{k_0 + p - 1}); \sigma_1, \dots, \sigma_{n-1}\big).$$

Since $\Delta_l \mathcal{L}/\mathcal{K} = \delta_{k_0} = \delta_{k_0 + p}$, it follows that $\sigma_n^{k_0 + p}(T)$ is a basis of transformal transcendence of $(K(S_{k_0 + p}); \sigma_1, \dots, \sigma_{n-1})/(K(S_{k_0 + p - 1}); \sigma_1, \dots, \sigma_{n-1})$. It also follows that $\bigcup_{p=0}^{\infty} \sigma_n^{k_0 + p}(T^{\sigma_1, \dots, \sigma_{n-1}})$ is algebraically independent over K and, thus, $\sigma_n^{k_0}(T)$ is transformally algebraically independent over K. Since $\sigma_n^{k_0}(T)$ can be extended to a basis of transformal transcendence of $\mathcal{L}/\mathcal{K}$, it follows that

$$\Delta_l \mathcal{L}/\mathcal{K} = \big|\sigma_n^{k_0}(T)\big| \leqslant \Delta \mathcal{L}/\mathcal{K}.$$

(As we shall see, $\sigma_n^{k_0}(T)$ is a basis of transformal transcendence of $\mathcal{L}/\mathcal{K}$.)

To show that $\Delta\,\mathcal{L}/\mathcal{K} \leqslant \Delta_l \mathcal{L}/\mathcal{K}$, again choose k_0 such that $\delta_k = \Delta_l \mathcal{L}/\mathcal{K}$ if $k \geqslant k_0$. Let $\Delta = \Delta\mathcal{L}/\mathcal{K}$ and $c = \Delta(K(S_{k_0-1}); \sigma_1,\ldots,\sigma_{n-1})/(K; \sigma_1,\ldots,\sigma_{n-1})$. Then, for $p = 0, 1, 2, \ldots,$

$$\Delta\Big(K\big(S_{k_0+p}\big); \sigma_1,\ldots,\sigma_{n-1}\Big)/\,(K; \sigma_1,\ldots,\sigma_{n-1}) = (p+1)\delta_{k_0} + c.$$

If Z is a basis of transformal transcendence of $\mathcal{L}/\mathcal{K}$ contained in S then, for $p = 0, 1, 2, \ldots,$ $\bigcup_{i=0}^{k_0+p}\sigma_n^i(Z)$ is a transformally independent set over $(K; \sigma_1,\ldots,\sigma_{n-1})$. Since $|\bigcup_{i=0}^{k_0+p}\sigma_n^i(Z)| = (k_0 + p + 1)\Delta$,

$$(p+1)\delta_{k_0} + c \geqslant (k_0 + p + 1)\Delta.$$

Letting $p \to +\infty$ we see $\Delta_l \mathcal{L}/\mathcal{K} = \delta_{k_0} \geqslant \Delta$.

COROLLARY 1. *If $\mathcal{L}/\mathcal{K}$ is a finitely generated extension then $\Delta_l \mathcal{L}/\mathcal{K}$ is independent of the finite set of generators and the transforming operator chosen as σ_n which are used to define $\Delta_l L/K$.*

COROLLARY 2. *If $\mathfrak{M}$ is a finitely generated extension of $\mathcal{K}$, and $\mathcal{L}$ a subextension of $\mathfrak{M}/\mathcal{K}$ then*

$$\Delta_l \mathfrak{M}/\mathcal{K} = \Delta_l \mathfrak{M}/\mathcal{L} + \Delta_l \mathcal{L}/\mathcal{K}.$$

COROLLARY 3 (TO THE PROOF OF LEMMA 1). *If $\mathcal{L} = \mathcal{K}\langle S\rangle$, $|S| < \infty$, is a finitely generated extension of the difference field $\mathcal{K}$, then there is a finite subset $Z \subseteq S$ such that Z is a basis of transformal transcendence of $\mathcal{L}/\mathcal{K}$ and if k_0 is a positive integer such that*

$$\Delta_l \mathcal{L}/\mathcal{K} = \Delta(K(S_k); \sigma_1,\ldots,\sigma_{n-1})/\,(K(S_{k-1}); \sigma_1,\ldots,\sigma_{n-1})$$

if $k \geqslant k_0$ then $\sigma_n^k(Z)$ is a basis of transformal transcendence of

$$(K(S_k); \sigma_1,\ldots,\sigma_{n-1})/\,(K(S_{k-1}); \sigma_1,\ldots,\sigma_{n-1}) \quad \textit{if } k \geqslant k_0.$$

PROOF. It follows from the proof of Lemma 1 that if

$$\Delta_l \mathcal{L}/\mathcal{K} = \Delta\Big(K\big(S_{k_0}\big); \sigma_1,\ldots,\sigma_{n-1}\Big)/\,\Big(K\big(S_{k_0-1}\big); \sigma_1,\ldots,\sigma_{n-1}\Big)$$

$Z \subseteq S$ is chosen so that $\sigma_n^k(Z)$ is a basis of transformal transcendence of $(K(S_k); \sigma_1,\ldots,\sigma_{n-1})/(K(S_{k-1}); \sigma_1,\ldots,\sigma_{n-1})$ if $k \geqslant k_0$ then $\sigma_n^{k_0}(Z)$ is a basis of transformal transcendence of $\mathcal{L}/\mathcal{K}$. Since σ_n is an isomorphism, it follows that $|Z| = |\sigma_n^{k_0}(Z)| = \Delta\mathcal{L}/\mathcal{K}$, and since $\sigma_n(K) \subseteq K$ any relation of transformal dependence of the elements of Z over $\mathcal{K}$ yields a relation of transformal dependence of the elements of $\sigma_n^{k_0}(Z)$ over $\mathcal{K}$ and it follows that Z is a basis of transformal transcendence of $\mathcal{L}/\mathcal{K}$.

EXAMPLE. Let K be the field of complex numbers $x_0, x_1, x_2, \ldots; y_0, y_1, y_2, \ldots$ algebraically independent over K and $t_1, t_2, \ldots$ elements in the algebraic closure of $K(x_0, x_1, \ldots; y_0, y_1, \ldots)$ such that $t_j^2 = x_j + y_j$, $j = 1, 2, \ldots$. If $L = K(x_0, x_1, \ldots; y_0, y_1, \ldots; t_1, t_2, \ldots)$ and σ is the isomorphism of L into itself defined by $\sigma(z) = z$ if $z \in K$, $\sigma(x_j) = x_{j+1}$, $\sigma(y_j) = y_{j+1}$ and $\sigma(t_j) = t_{j+1}$ then σ defines difference fields on L and K. Let $\mathcal{K} = (\mathcal{K}; \sigma)$ and $\mathcal{L} = (L; \sigma)$.

Note that $\mathfrak{L} = \mathfrak{K}\langle S\rangle$, $S = \{x, y, t_1\}$, and $Z = \{x, y\} \subseteq S$ is a basis of transformal transcendence of $\mathfrak{L}/\mathfrak{K}$. However, when $k \in \{1, 2, \ldots\}$, $\sigma^k(Z)$ is not a transcendence basis of $K(S_k)/K(S_{k-1})$ since $\{\sigma(x), \sigma(y)\} = \{x_1, y_1\}$ is not algebraically independent over $K(x, y, t_1)$. In fact, (x_1, y_1) satisfies $X + Y = t_1^2$.

Let $\mathfrak{L} = (L; \sigma_1, \ldots, \sigma_n)$ be a difference field extension of $\mathfrak{K} = (K; \sigma_1, \ldots, \sigma_n)$, $S \subseteq L$, $t \in \{1, 2, \ldots, n\}$ and i_t, $i_{t+1}, \ldots, i_n$ positive integers. We will use $S^*(i_n, i_{n-1}, \ldots, i_t)$ to represent

$$\bigcup_{i=0}^{i_n-1} \sigma_n^i\left(S^{(\sigma_1, \ldots, \sigma_{n-1})}\right) \cup \bigcup_{i=0}^{i_{n-1}-1} \sigma_n^{i_n}\sigma_{n-1}^i\left(S^{(\sigma_1, \ldots, \sigma_{n-2})}\right)$$
$$\cup \cdots \cup \bigcup_{i=0}^{i_t-1} \sigma_n^{i_n} \cdots \sigma_{t-1}^{i_{t-1}}\sigma_t^i\left(S^{(\sigma_1, \ldots, \sigma_{t-1})}\right)$$

and $S(i_n, i_{n-1}, \ldots, i_t) = S^*(i_n, i_{n-1}, \ldots, i_{t+1}, i_t + 1)$.

Note, if $t = n$ and i_n is a positive integer then $S(i_n) = S_{i_n}$ and $S^*(i_n) = S_{i_n - 1}$.

Let $\mathfrak{L} = (L; \sigma_1, \ldots, \sigma_n)$ be a finitely generated extension of $\mathfrak{K} = (K; \sigma_1, \ldots, \sigma_n)$, $\mathfrak{L} = \mathfrak{K}\langle S\rangle, |S| < \infty$, and k_n be chosen so that

$$\Delta = \Delta_l \mathfrak{L}/\mathfrak{K} = \Delta\big(K(S(i_n)); \sigma_1, \ldots, \sigma_{n-1}\big)/\big(K(S^*(i_n)); \sigma_1, \ldots, \sigma_{n-1}\big)$$

if $i_n \geqslant k_n$.

Suppose $n > 1$. Since

$$\begin{aligned}\Delta\big(K(S(k_n)); \sigma_1, \ldots, \sigma_{n-1}\big)&/\big(K(S^*(k_n)); \sigma_1, \ldots, \sigma_{n-1}\big)\\ &= \Delta_l\big(K(S(k_n)); \sigma_1, \ldots, \sigma_{n-1}\big)/\big(K(S^*(k_n)); \sigma_1, \ldots, \sigma_{n-1}\big)\end{aligned}$$

there exists a positive integer k_{n-1} such that if $i_{n-1} \geqslant k_{n-1}$ then

$$\Delta = \Delta\big(K(S(k_n, i_{n-1})); \sigma_1, \ldots, \sigma_{n-2}\big)/\big(K(S^*(k_n, i_{n-1})); \sigma_1, \ldots, \sigma_{n-2}\big).$$

Since σ_n is an isomorphism and both $K(S(i_n, i_{n-1}))$ and $K(S^*(i_n, i_{n-1}))$ can be obtained by adjoining the same set of elements to $\sigma_n^{i_n - k_n}(K(S(k_n, i_{n-1})))$ and $\sigma_n^{i_n - k_n}(K(S^*(k_n, i_{n-1})))$, respectively, it follows that

$$\begin{aligned}\Delta &= \Delta_l\big(K(S(i_n)); \sigma_1, \ldots, \sigma_{n-1}\big)/\big(K(S^*(i_n)); \sigma_1, \ldots, \sigma_{n-1}\big)\\ &\leqslant \Delta\big(K(S(i_n, i_{n-1})); \sigma_1, \ldots, \sigma_{n-2}\big)/\big(K(S^*(i_n, i_{n-1})); \sigma_1, \ldots, \sigma_{n-2}\big)\\ &\leqslant \Delta\big(\sigma^{i_n - k_n}K(S(k_n, i_{n-1})); \sigma_1, \ldots, \sigma_{n-2}\big)/\big(\sigma^{i_n - k_n}K(S^*(k_n, i_{n-1})); \sigma_1, \ldots, \sigma_{n-2}\big)\\ &= \Delta \quad \text{if } i_n \geqslant k_n.\end{aligned}$$

Thus, if $i_n \geqslant k_n$ and $i_{n-1} \geqslant k_{n-1}$

$$\Delta = \Delta\big(K(S(i_n, i_{n-1})); \sigma_1, \ldots, \sigma_{n-2}\big)/\big(K(S^*(i_n, i_{n-1})); \sigma_1, \ldots, \sigma_{n-2}\big).$$

Furthermore, if $Z \subseteq S$ is chosen so that $\sigma_n^{k_n}\sigma_{n-1}^{k_{n-1}}(Z)$ is a basis of transformal transcendence of

$$\big(K(S(k_n, k_{n-1})); \sigma_1, \ldots, \sigma_{n-2}\big)/\big(K(S^*(k_n, k_{n-1})); \sigma_1, \ldots, \sigma_{n-2}\big)$$

then $\sigma_n^{i_n}\sigma_{n-1}^{i_{n-1}}(Z)$ is a basis of transformal transcendence of

$$\big(K(S(i_n, i_{n-1})); \sigma_1, \ldots, \sigma_{n-2}\big)/\big(K(S^*(i_n, i_{n-1})); \sigma_1, \ldots, \sigma_{n-2}\big)$$

if $i_n \geqslant k_n$ and $i_{n-1} \geqslant k_{n-1}$. From Corollary 3 it follows that $\sigma_n^{i_n}(Z)$ is a basis of transformal transcendence of $(K(S(i_n)); \sigma_1, \ldots, \sigma_{n-1})/(K(S^*(i_n)); \sigma_1, \ldots, \sigma_{n-1})$ if $i_n \geqslant k_n$, and thus, Z is a basis of transformal transcendence of $\mathcal{L}/\mathcal{K}$.

We can inductively extend these ideas to show there exist positive integers k_1, $k_2, \ldots, k_n$ and a set $Z \subseteq S$ such that

(1) Z is a basis of transformal transcendence of $\mathcal{L}/\mathcal{K}$,

(2) if $t \in \{1, 2, \ldots, n\}$ and $i_j \geqslant k_j$, $j = t, \ldots, n$, then $\sigma_n^{i_n} \cdots \sigma_t^{i_t}(Z)$ is a basis of transformal transcendence of

$$\left(K(S(i_n, \ldots, i_t)); \sigma_1, \ldots, \sigma_{t-1}\right) / \left(K(S^*(i_n, \ldots, i_t)); \sigma_1, \ldots, \sigma_{t-1}\right)$$

and $\Delta(\mathcal{L}/\mathcal{K})$ is the transcendence degree of this extension.

Definition. If $\mathcal{L} = \mathcal{K}\langle S \rangle$ is a finitely generated difference field extension of K, and $Z \subseteq S$ is chosen to satisfy (1) and (2) above for a set of positive integers k_1, $k_2, \ldots, k_n$ then Z is called a *limit basis of transformal transcendence* of $\mathcal{L}/\mathcal{K}$.

Corollary 4. *Suppose* $\mathcal{L} = (L; \sigma_1, \ldots, \sigma_n)$ *is a finitely generated extension of* $\mathcal{K} = (K; \sigma_1, \ldots, \sigma_n)$, $\mathcal{L} = \mathcal{K}\langle S \rangle$, $|S| < \infty$, *and* $Z \subseteq S$ *is a limit basis of transformal transcendence chosen to satisfy* (2) *above for the positive integers* $k_1, \ldots, k_n$. *If* $i_1, \ldots, i_n$ *are positive integers with* $i_j \geqslant k_j$, $j = 1, \ldots, n$, *then* $T = Z^{(\sigma_1, \ldots, \sigma_n)} \sim Z(i_n, \ldots, i_1)$ *is an algebraically independent set over* $K(S(i_n, \ldots, i_1))$.

Proof. If not, there exist positive integers $i_1, \ldots, i_n$ such that $i_j \geqslant k_j$, $j = 1, \ldots, n$, and $T = Z^{(\sigma_1, \ldots, \sigma_n)} \sim Z(i_n, \ldots, i_1)$ is an algebraically dependent set over $K(S(i_n, \ldots, i_1))$. Choose a finite set $U \subseteq T$ such that U is an algebraically dependent set over $K(S(i_n, \ldots, i_1))$. Let t be the largest integer for which there is an element of U whose order with respect to σ_t is greater than i_t. (The definition of T guarantees the existence of t.) Then let l_t denote the maximum of the orders of the elements of U with respect to σ_t.

Note that if $\sigma_1^{j_1} \cdots \sigma_n^{j_n}(z)$, $z \in Z$, is an element of U then $j_k = i_k$ if $k > t$ and $j_t \leqslant l_t$. Letting $\sigma_n^{i_n} \cdots \sigma_{t+1}^{i_{t+1}} \sigma_t^{l_t} = \sigma_t^{l_t}$ and $(i_n, \ldots, i_{t+1}, l_t) = (l_t)$ if $t = n$, and $A^{(\sigma_1, \ldots, \sigma_0)} = A$ if $A \subseteq L$ we see that

$$U \cap \sigma_n^{i_n} \cdots \sigma_{t+1}^{i_{t+1}} \sigma_t^{l_t}\left(Z^{(\sigma_1, \ldots, \sigma_{t-1})}\right) \neq \varnothing$$

(so $U \cap \sigma_n^{i_n} \cdots \sigma_{t+1}^{i_{t+1}} \sigma_t^{l_t}(S^{(\sigma_1, \ldots, \sigma_{t-1})}) \neq \varnothing$), and

$$\begin{aligned} U &\subseteq Z(i_n, \ldots, i_{t+1}, l_t) \\ &= Z^*(i_n, \ldots, i_{t+1}, l_t) \cup \sigma_n^{i_n} \cdots \sigma_{t+1}^{i_{t+1}} \sigma_t^{l_t}\left(Z^{(\sigma_1, \ldots, \sigma_{t-1})}\right) \\ &\subseteq S^*(i_n, \ldots, i_{t+1}, l_t) \cup \sigma_n^{i_n} \cdots \sigma_{t+1}^{i_{t+1}} \sigma_t^{l_t}\left(S^{(\sigma_1, \ldots, \sigma_{t-1})}\right) \\ &= S(i_n, \ldots, i_{t+1}, l_t). \end{aligned}$$

Since U is an algebraically dependent set over $K(S(i_n, \ldots, i_1))$ and $K(S(i_n, \ldots, i_1)) \subseteq K(S^*(i_n, \ldots, i_{t+1}, l_t))$ it follows that U is an algebraically dependent set over $K(S^*(i_n, \ldots, i_{t+1}, l_t))$. Since $U \cap \sigma_n^{i_n} \cdots \sigma_{t+1}^{i_{t+1}} \sigma_t^{l_t}(Z^{(\sigma_1, \ldots, \sigma_{t-1})}) \neq \varnothing$ and

$$U \subseteq K\left(S^*(i_n, \ldots, i_{t+1}, l_t)\right) \cup \sigma_n^{i_n} \cdots \sigma_{t+1}^{i_{t+1}} \sigma_t^{l_t}\left(Z^{(\sigma_1, \ldots, \sigma_{t-1})}\right)$$

we see that $\sigma_n^{i_n} \cdots \sigma_{t+1}^{i_{t+1}} \sigma_t^{l_t}(Z^{(\sigma_1,\ldots,\sigma_{t-1})})$ is an algebraically dependent set over $K(S^*(i_n,\ldots,i_{t+1}, l_t))$ and hence, $\sigma_n^{i_n} \cdots \sigma_{t+1}^{i_{t+1}} \sigma_t^{l_t}(Z)$ is not a basis of transformal transcendence of

$$\big(K\big(S(i_n,\ldots,i_{t+1}, l_t)\big); \sigma_1,\ldots,\sigma_{t-1}\big)/\big(K\big(S^*(i_n,\ldots,i_{t+1}, l_t)\big); \sigma_1,\ldots,\sigma_{t-1}\big).$$

Since $i_j \geqslant k_j$ if $j > t$ and $l_t > i_t \geqslant k_t$, this contradicts Z being a limit basis of transformal transcendence satisfying (2) above for the positive integers $k_1, k_2,\ldots,k_n$.

REMARK. Throughout the remainder of this paper a transcendence basis of an extension $\mathcal{L}/\mathcal{K}$ will always be selected so as to be a limit transcendence basis of $\mathcal{L}/\mathcal{K}$.

3. An invariant of difference field extensions. Let $\mathcal{M} = (M; \sigma_1,\ldots,\sigma_n)$ be a difference field and $\mathcal{L} = (L; \sigma_1,\ldots,\sigma_n)$, $\mathcal{K} = (K; \sigma_1,\ldots,\sigma_n)$ be difference subfields of $\mathcal{M}$ with $L \supseteq K$. We will inductively define a measure, $\mathrm{ld}_n(\mathcal{M}/\mathcal{K})$, of difference field extensions with values in the set consisting of the positive integers and $+\infty$ and having the following properties:

(ld1). *If there is a finite set $S \subseteq M$ such that $\mathcal{M} = \mathcal{L}\langle S\rangle$ then there exists a finitely generated extension $\mathcal{K}'$ of $\mathcal{K}$ contained in $\mathcal{L}$ such that* $\mathrm{ld}_n(\mathcal{M}/\mathcal{L}) = \mathrm{ld}_n(\mathcal{K}'\langle S\rangle/\mathcal{K}')$.

(ld2). *If S is a set of elements in M then* $\mathrm{ld}_n(\mathcal{L}\langle S\rangle/\mathcal{K}\langle S\rangle) \leqslant \mathrm{ld}_n(\mathcal{L}/\mathcal{K})$ *and* $\mathrm{ld}_n(\mathcal{L}\langle S\rangle/\mathcal{L}) \leqslant \mathrm{ld}_n(\mathcal{K}\langle S\rangle/\mathcal{K})$. *Equality will hold in both if S is a transformally algebraically independent set over $\mathcal{L}$.*

(ld3). *If $\mathcal{L}'$ is a difference field with subfield $\mathcal{K}'$ and ϕ is an isomorphism of $\mathcal{L}$ onto $\mathcal{L}'$ such that $\phi(K) = K'$ then* $\mathrm{ld}_n(\mathcal{L}/\mathcal{K}) = \mathrm{ld}_n(\mathcal{L}'/\mathcal{K}')$.

(ld4). *If $\mathcal{L}/\mathcal{K}$ is finitely generated then $0 = \Delta\mathcal{L}/\mathcal{K}$, the degree of transformal transcendence of $\mathcal{L}/\mathcal{K}$, if and only if* $\mathrm{ld}_n(\mathcal{L}/\mathcal{K}) < \infty$.

(ld5). $\mathrm{ld}_n(\mathcal{M}/\mathcal{K}) = \mathrm{ld}_n(\mathcal{M}/\mathcal{L}) \cdot \mathrm{ld}_n(\mathcal{L}/\mathcal{K})$.

If $n = 1$, so $\mathcal{M}, \mathcal{L}, \mathcal{K}$ are ordinary difference fields, we define ld_1 to be ordinary limit degree [**3**, Chapter 5, §16]. The verifications of (ld1), (ld2), and (ld3) for ld_1 are identical to the verifications of these properties in the induction step to follow except that where induction hypotheses are used one should invoke the proper part of Theorem III of the introduction in [**3**]. Verification of (ld4) can be found on p. 136 of [**3**] and (ld5) is Theorem XII of Chapter 5 in [**3**].

Suppose now ld_{n-1} has been defined for difference field extensions having $n - 1$ transforming operators.

Let $\mathcal{L} = (L; \sigma_1,\ldots,\sigma_n)$ be an extension of $\mathcal{K} = (K; \sigma_1,\ldots,\sigma_n)$. Assume first that $\mathcal{L}/\mathcal{K}$ is finitely generated, say $\mathcal{L} = \mathcal{K}\langle S\rangle$ where S is some finite subset of L. For each nonnegative integer m, $\mathcal{N}_m = (K(S_m); \sigma_1,\ldots,\sigma_{n-1})$ is a finitely generated extension of $\mathcal{N}_{m-1}$. (Define $K(S_{-1}) = K$.) Using (ld3) and (ld2) we have $\mathrm{ld}_{n-1}(\mathcal{N}_m/\mathcal{N}_{m-1}) \geqslant \mathrm{ld}_{n-1}(\mathcal{N}_{m+1}/\mathcal{N}_m)$. Let a be the limit as $m \to \infty$ of $\mathrm{ld}_{n-1}(\mathcal{N}_m/\mathcal{N}_{m-1})$: $a = \infty$ or a is a positive integer.

The proof that a above is independent of the choice of generators of $\mathcal{L}/\mathcal{K}$ is exactly the proof of this fact for ld_1, found on p. 136 of [**3**], when dimension of field extension is replaced by ld_{n-1} and "(a) of Theorem III" is replaced by (ld5) of the inductive assumption. We then define $\mathrm{ld}_n(\mathcal{L}/\mathcal{K}) = a$.

If $\mathcal{L}/\mathcal{K}$ is not finitely generated we define $\mathrm{ld}_n(\mathcal{L}/\mathcal{K})$ to be the maximum of $\mathrm{ld}_n(\mathfrak{M}/\mathcal{K})$ when $\mathfrak{M}/\mathcal{K}$ is a finitely generated subextension of $\mathcal{L}/\mathcal{K}$, if the maximum exists and ∞ if it does not. As in the case of limit degrees of ordinary difference fields it follows that if $\mathcal{L}/\mathcal{K}$ is finitely generated then $\mathrm{ld}_n(\mathcal{L}/\mathcal{K})$ is also the maximum of the limit degrees of the finitely generated subextensions of $\mathcal{L}/\mathcal{K}$ (see [**3**, p. 136]).

We now verify (ld1)–(ld5) for ld_n. Let $\mathfrak{M}$, $\mathcal{L}$, $\mathcal{K}$ be as in the first paragraph of this section.

Suppose $\mathfrak{M} = \mathcal{L}\langle S\rangle$ for some finite subset S of M. Choose k_0 sufficiently large so that if $k \geqslant k_0$, $\mathrm{ld}_n(\mathfrak{M}/\mathcal{L}) = \mathrm{ld}_{n-1}(\mathfrak{N}_k/\mathfrak{N}_{k-1})$ where $\mathfrak{N}_k = (L(S_k); \sigma_1,\ldots,\sigma_{n-1})$. Choose a finite subset Y of $L(S_{k_0})$ such that

$$\mathrm{ld}_n(\mathfrak{M}/\mathcal{L}) = \mathrm{ld}_{n-1}\big(K(Y_0, S_{k_0}); \sigma_1,\ldots,\sigma_{n-1}\big)/\big(K\big(Y_0, S_{k_0-1}\big); \sigma_1,\ldots,\sigma_{n-1}\big).$$

Writing each $y \in Y$ as a quotient of polynomials in the elements of S_{k_0} with coefficients in L let Z be the finite collection of coefficients of these polynomials. Let K' be the underlying field of $\mathcal{K}' = \mathcal{K}\langle Z\rangle$. $K' \subseteq L$ and $\mathcal{K}'/\mathcal{K}$ is finitely generated. Using our inductive assumptions we have, for $k \geqslant k_0$,

$$\begin{aligned}\mathrm{ld}_n(\mathfrak{M}/\mathcal{L}) &= \mathrm{ld}_{n-1}\big(\mathfrak{N}_{k_0}/\mathfrak{N}_{k_0-1}\big)\\ &= \mathrm{ld}_{n-1}\Big(\big(K'\big(S_{k_0}\big); \sigma_1,\ldots,\sigma_{n-1}\big)/\big(K'\big(S_{k_0-1}\big); \sigma_1,\ldots,\sigma_{n-1}\big)\Big)\\ &\geqslant \mathrm{ld}_{n-1}((K'(S_k); \sigma_1,\ldots,\sigma_{n-1})/(K'(S_{k-1}); \sigma_1,\ldots,\sigma_{n-1}))\\ &\geqslant \mathrm{ld}_{n-1}\big(\mathfrak{N}_k/\mathfrak{N}_{k-1}\big) = \mathrm{ld}_n(\mathfrak{M}/\mathcal{L}).\end{aligned}$$

Hence, $\mathrm{ld}_n(\mathfrak{M}/\mathcal{L}) = \mathrm{ld}_n(\mathcal{K}'\langle S\rangle/\mathcal{K}')$, and (ld1) is established.

Let $S \subseteq M$ and $\mathcal{K}\langle S\rangle = (F; \sigma_1,\ldots,\sigma_n)$. If V is a finite subset of L then, by the induction hypothesis,

$$\begin{aligned}&\mathrm{ld}_{n-1}\big(F(V_k); \sigma_1,\ldots,\sigma_{n-1}\big)/\big(F\big(V_{k-1}\big); \sigma_1,\ldots,\sigma_{n-1}\big)\\ &\qquad\leqslant \mathrm{ld}_{n-1}\big(K(V_k); \sigma_1,\ldots,\sigma_{n-1}\big)/\big(K\big(V_{k-1}\big); \sigma_1,\ldots,\sigma_{n-1}\big)\end{aligned}$$

for each positive integer k. Since V is arbitrary it follows that $\mathrm{ld}_n(\mathcal{L}\langle S\rangle/\mathcal{K}\langle S\rangle) \leqslant \mathrm{ld}_n(\mathcal{L}/\mathcal{K})$. If S is transformally algebraically independent over $\mathcal{L}$ then $\bigcup \sigma_n^i(S)$ $(i = 0, 1,\ldots)$ is transformally algebraically independent over $(L; \sigma_1,\ldots,\sigma_{n-1})$. Hence, if V is a finite subset of L,

$$\begin{aligned}&\mathrm{ld}_{n-1}\big(K(V_k); \sigma_1,\ldots,\sigma_{n-1}\big)/\big(K\big(V_{k-1}\big); \sigma_1,\ldots,\sigma_{n-1}\big)\\ &\quad= \mathrm{ld}_{n-1}\big(F(V_k); \sigma_1,\ldots,\sigma_{n-1}\big)/\big(F\big(V_{k-1}\big); \sigma_1,\ldots,\sigma_{n-1}\big)\end{aligned}$$

for each positive integer k. Since V is arbitrary this equality is enough to insure that $\mathrm{ld}_n(\mathcal{L}\langle S\rangle/\mathcal{K}\langle S\rangle) = \mathrm{ld}_n(\mathfrak{M}/\mathcal{L})$.

For each positive integer k and finite subset V of S

$$\begin{aligned}&\mathrm{ld}_{n-1}\big(L(V_k); \sigma_1,\ldots,\sigma_{n-1}\big)/\big(L\big(V_{k-1}\big); \sigma_1,\ldots,\sigma_{n-1}\big))\\ &\quad\leqslant \mathrm{ld}_{n-1}\big(K(V_k); \sigma_1,\ldots,\sigma_{n-1}\big)/\big(K\big(V_{k-1}\big); \sigma_1,\ldots,\sigma_{n-1}\big),\end{aligned}$$

with equality if S and hence $\cup \sigma_n^i(V)$ $(i = 1,\dots,k)$ is transformally algebraically independent over $\mathcal{L}$ and $(L; \sigma_1,\dots,\sigma_{n-1})$, respectively. Hence, $\mathrm{ld}_n(\mathcal{L}\langle S\rangle/\mathcal{L}) \leqslant \mathrm{ld}_n(\mathcal{K}\langle S\rangle/\mathcal{K})$ with equality if S is transformally independent over $\mathcal{L}$. This establishes (ld2).

(ld3) is easy.

To verify (ld4) suppose $\mathcal{L} = \mathcal{K}\langle S\rangle$ where S is a finite set and, for $k = 1,2,\dots,$ let $\mathfrak{N}_k = (K(S_k); \sigma_1,\dots,\sigma_{n-1})$ and $\delta_k = \Delta\mathfrak{N}_k/\mathfrak{N}_{k-1}$.

If $\Delta\mathcal{L}/\mathcal{K} = 0$ then there is a k_0 such that if $k \geqslant k_0$ then $\delta_k = 0$ since the transcendence degree and limit transcendence degree are equal. The induction hypothesis implies $\mathrm{ld}_{n-1}\mathfrak{N}_k/\mathfrak{N}_{k-1} < \infty$ if $k \geqslant k_0$ and hence, $\mathrm{ld}_n\mathcal{L}/\mathcal{K} < \infty$.

If $\mathrm{ld}_n\mathcal{L}/\mathcal{K} < \infty$ then there is a k_0 such that if $k \geqslant k_0$ then $\mathrm{ld}_{n-1}\mathfrak{N}_k/\mathfrak{N}_{k-1} < \infty$, and then the induction hypothesis implies $\delta_k = 0$ if $k \geqslant k_0$. Hence $\Delta\mathcal{L}/\mathcal{K} = 0$.

The verification of (ld5) is exactly that of the proof of (ld5) for ordinary difference fields given by Cohn in [**3**, Theorem XII, p. 137] provided the appropriate modifications are made, including replacing ordinary transcendence degree by partial transcendence degree [**2**, pp. 18–30], replacing the dimension of field extensions by ld_{n-1}, and invoking our induction hypothesis when Cohn invokes Theorem III of the introduction of [**3**].

4. Finitely generated extensions of difference fields. Throughout this section $\mathfrak{M} = (M; \sigma_1,\dots,\sigma_n)$ will be a finitely generated extension of $\mathcal{K} = (K; \sigma_1,\dots,\sigma_n)$ and $\mathcal{L} = (L; \sigma_1,\dots,\sigma_n)$ will be subextension of $\mathfrak{M}/\mathcal{K}$.

The following notation will be adopted. If $S \subseteq M$, k is an integer such that $n \geqslant k \geqslant 1$ and $\mathbf{i} = (i_k, i_{k-1},\dots,i_1)$ is a k-tuple of positive integers then

$$S^*(k,\mathbf{i}) = \bigcup_{i=0}^{i_k-1} \sigma_k^i(S^{(\sigma_1,\dots,\sigma_{k-1})}) \cup \bigcup_{j=1}^{k-1} \sigma_k^{i_k}\cdots\sigma_{k-j+1}^{i_{k-j+1}}\left[\bigcup_{i=0}^{i_{k-j}-1} \sigma_{k-j}^i(S^{(\sigma_1,\dots,\sigma_{k-j-1})})\right]$$

and

$$S(k,\mathbf{i}) = S^*(k,(i_k,\dots,i_2,i_1+1)).$$

In the definitions of $S^*(k,\mathbf{i})$ and $S(k,\mathbf{i})$ given above and in the remainder of this section the symbol $S^{(\sigma_1,\dots,\sigma_0)}$ is used to denote S, and if $k = 1$, $S^*(k,\mathbf{i}) = \cup_{i=0}^{i_k-1}\sigma_k^i(S)$.

Let $S \subseteq M$, m, j_1, k_1 be integers with $n \geqslant m > j_1 \geqslant k_1 \geqslant 1$, $a_{j_1}, a_{j_1-1},\dots,a_{k_1}$ nonnegative integers and $i_m = i_{k_0-1},\dots,i_{j_1+1}$ positive integers. If $\mathbf{a}_1 = (a_{j_1},\dots,a_{k_1})$ and $\mathbf{i}_1 = (i_{k_0-1},\dots,i_{j_1+1})$ then

$$S^*(m,\mathbf{i}_1,\mathbf{a}_1) = \bigcup_{i=0}^{i_m-1} \sigma_m^i(S^{(\sigma_1,\dots,\sigma_{m-1})}) \cup \bigcup_{j=1}^{m-j_1-1} \sigma_m^{i_m}\cdots\sigma_{m-j+1}^{i_{m-j+1}}\left[\bigcup_{i=0}^{i_{m-j}-1} \sigma_{m-j}^i(S^{(\sigma_1,\dots,\sigma_{m-j-1})})\right]$$

and

$$S(m,\mathbf{i}_1,\mathbf{a}_1) = S^*(m,\mathbf{i}_1,\mathbf{a}_1) \cup \sigma_m^{i_m} \cdots \sigma_{j_1+1}^{i_{j_1}+1}(T)$$

where

$$T = \bigcup \sigma_{j_1}^{\alpha_{j_1}} \cdots \sigma_{k_1}^{\alpha_{k_1}}(S^{(\sigma_1,\ldots,\sigma_{k_1-1})}) \qquad (0 \leqslant \alpha_j \leqslant a_j, j = k_1,\ldots,j_1).$$

(If $j_1 = m-1$, take $S^*(m,\mathbf{i}_1,\mathbf{a}_1) = \bigcup_{i=0}^{i_m-1}\sigma_m^i(S^{(\sigma_1,\ldots,\sigma_{m-1})})$.) If $k_1 > 1$, $i_{k_1-1},\ldots,i_1$ are positive integers and $\mathbf{i}_2$ denotes $(i_{k_1-1},\ldots,i_1)$, we define

$$S^*(m,\mathbf{i}_1,\mathbf{a}_1,\mathbf{i}_2) = S^*(m,\mathbf{i}_1,\mathbf{a}_1) \cup A$$

where

$$A = \bigcup_{\alpha_{j_1},\ldots,\alpha_{k_1}} \sigma_m^{i_m} \cdots \sigma_{j_1+1}^{i_{j_1}+1}\sigma_{j_1}^{\alpha_1} \cdots \sigma_{k_1}^{\alpha_{k_1}}(S^*(k_1-1,\mathbf{i}_2)),$$

$0 \leqslant \alpha_j \leqslant a_j, j = k_1,\ldots,j_1$ and $S(m,\mathbf{i}_1,\mathbf{a}_1,\mathbf{i}_2) = S^*(m,\mathbf{i}_1,\mathbf{a}_1,(i_{k_1-1},\ldots,i_2,i_1+1))$.

We inductively extend the definitions of $S(\cdot)$ and $S^*(\cdot)$ as follows. Let $S \subseteq M$; m, j_1, $k_1,\ldots,j_t$, k_t be integers with $n \geqslant m > j_1$, $j_l \geqslant k_l$, $l = 1,\ldots,t$, $k_l > j_{l+1}+1$, $l = 1,\ldots,t$, and $k_t \geqslant 1$; a_{j_1}, $a_{j_1-1},\ldots,a_{k_1}$; a_{j_2}, $a_{j_2-1},\ldots,a_{k_2}$; $\ldots$; a_{j_t}, $a_{j_t-1},\ldots,a_{k_t}$ nonnegative integers; and $i_m = i_{k_0-1},\ldots,i_{j_1+1}$; $i_{k_1-1},\ldots,i_{j_2+1}$; $\ldots$; $i_{k_{t-1}-1},\ldots,i_{j_t+1}$ positive integers. For $l = 1,\ldots,t$ we write $\mathbf{a}_l$ and $\mathbf{i}_l$ for $(a_{j_l}, a_{j_l-1},\ldots,a_{k_l})$ and $(i_{k_{l-l}-1},\ldots,i_{j_l+1})$, respectively. Then

$$S^*(m,\mathbf{i}_1,\mathbf{a}_1,\ldots,\mathbf{i}_t,\mathbf{a}_t) = S^*(m,\mathbf{i}_1,\mathbf{a}_i) \cup B$$

where

$$B = \bigcup_{\alpha_{j_1},\ldots,\alpha_{k_1}} \sigma_m^{i_m} \cdots \sigma_{j_1+1}^{i_{j_1}+1}\sigma_{j_1}^{\alpha_{j_1}} \cdots \sigma_{k_1}^{\alpha_{k_1}}(C),$$

$0 \leqslant \alpha_j \leqslant a_j, j = k_1,\ldots,j_1$ and $C = S^*(k_1-1,\mathbf{i}_2,\mathbf{a}_2,\ldots,\mathbf{i}_t,\mathbf{a}_t)$. Also,

$$S(m,\mathbf{i}_1,\mathbf{a}_1,\ldots,\mathbf{i}_t,\mathbf{a}_t) = S^*(m,\mathbf{i}_1,\mathbf{a}_1) \cup D$$

where

$$D = \bigcup_{\alpha_{j_1},\ldots,\alpha_{k_1}} \sigma_m^{i_m} \cdots \sigma_{j_1+1}^{i_{j_1}+1}\sigma_{j_1}^{\alpha_{j_1}} \cdots \sigma_{k_1}^{\alpha_{k_1}}(E),$$

$0 \leqslant \alpha_j \leqslant a_j, j = k_1,\ldots,j_1$ and $E = S(k_1-1,\mathbf{i}_2,\mathbf{a}_2,\ldots,\mathbf{i}_t,\mathbf{a}_t)$.

Now, let $k_t > 1$, i_{k_t-1}, $i_{k_t-2},\ldots,i_1$ be positive integers and let $\mathbf{i}_{t+1}$ denote $(i_{k_t-1},\ldots,i_1)$. Then,

$$S^*(m,\mathbf{i}_1,\mathbf{a}_1,\ldots,\mathbf{i}_t,\mathbf{a}_t,\mathbf{i}_{t+1}) = S^*(m,\mathbf{i}_1,\mathbf{a}_1) \cup F$$

where

$$F = \bigcup_{\alpha_{j_1},\ldots,\alpha_{k_1}} \sigma_m^{i_m} \cdots \sigma_{j_1+1}^{i_{j_1}+1}\sigma_{j_1}^{\alpha_{j_1}} \cdots \sigma_{k_1}^{\alpha_{k_1}}(G),$$

$0 \leqslant \alpha_j \leqslant a_j, j = k_1,\ldots,j_1$ and $G = S^*(k_1-1,\mathbf{i}_2,\mathbf{a}_2,\ldots,\mathbf{i}_t,\mathbf{a}_t,\mathbf{i}_{t+1})$. Also,

$$S(m,\mathbf{i}_1,\mathbf{a}_1,\ldots,\mathbf{i}_t,\mathbf{a}_t,\mathbf{i}_{t+1}) = S^*\big(m,\mathbf{i}_1,\mathbf{a}_1,\ldots,\mathbf{i}_t,\mathbf{a}_t,\big(i_{k_t-1},\ldots,i_2,i_1+1\big)\big).$$

Finally, if $\mathbf{a} = (a_m, \ldots, a_j)$ with $m \geq j$ we define

$$S(m, \mathbf{a}) = \bigcup_{0 \leq \alpha_l \leq a_l} \sigma_m^{\alpha_m} \cdots \sigma_j^{\alpha_j}\left(S^{(\sigma_1, \ldots, \sigma_{j-1})}\right).$$

If $m = n$ we suppress the use of n in the above notation and write, for example, $S^*(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t)$ for $S^*(n, \mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t)$.

REMARK. The notation developed above conforms in appearance to its usage in the remainder of this section. In particular the $\mathbf{a}_j$-vectors will be regarded as fixed and the $\mathbf{i}_j$-vectors will be allowed to vary. Further, the definitions of C and G above may not seem to conform to the above notations since there are no $\mathbf{i}_1$ and $\mathbf{a}_1$ vectors present. C and G can be viewed, respectively, as $S^*(k_1 - 1, \mathbf{i}'_1, \mathbf{a}'_1, \ldots, \mathbf{i}'_{t-1}, \mathbf{a}'_{t-1})$ and $S^*(k_1 - 1, \mathbf{i}'_1, \mathbf{a}'_1, \ldots, \mathbf{i}'_{t-1}, \mathbf{a}'_{t-1}, \mathbf{i}'_t)$ where $\mathbf{i}'_j = \mathbf{i}_{j+1}$, $j = 1, \ldots, t$ and $\mathbf{a}'_j = \mathbf{a}_{j+1}$, $j = 1, \ldots, t-1$.

EXAMPLE. To illustrate the above definitions let $n = 5$ and a_2, a_4 be fixed nonnegative integers. For positive integers i_5, i_3, i_2 and i_1,

$$S(3, (i_3, i_2, i_1)) = S^*(3, (i_3, i_2, i_1 + 1))$$
$$= \bigcup_{i=0}^{i_3-1} \sigma_3^i\left(S^{(\sigma_1, \sigma_2)}\right) \cup \sigma_3^{i_3}\left[\bigcup_{i=0}^{i_2-1} \left(S^{(\sigma_1)}\right)\right] \cup \sigma_3^{i_3}\sigma_2^{i_2}\left[\bigcup_{i=0}^{i_1} \sigma_1^i(S)\right],$$

$$S^*(5, (i_5), (a_4), (i_3), (a_2)) = \bigcup_{i=0}^{i_5-1} \sigma_5^i\left(S^{(\sigma_1, \ldots, \sigma_4)}\right) \cup \bigcup_{j=0}^{a_4} \sigma_5^{i_5}\sigma_4^j\left(\bigcup_{k=0}^{i_3-1} \sigma_3^k\left(S^{(\sigma_1, \sigma_2)}\right)\right),$$

$$S(5, (i_5), (a_4), (i_3), (a_2))$$
$$= \bigcup_{i=0}^{i_5-1} \sigma_5^i\left(S^{(\sigma_1, \ldots, \sigma_4)}\right) \cup \bigcup_{j=0}^{a_4} \sigma_5^{i_5}\sigma_4^j\left[\bigcup_{k=0}^{i_3-1} \sigma_3^k\left(S^{(\sigma_1, \sigma_2)}\right) \cup \bigcup_{l=0}^{a_2} \sigma_3^{i_3}\sigma_2^l\left(S^{(\sigma_1)}\right)\right]$$

and

$$S(5, (i_5), (a_4), (i_3), (a_2), (i_1))$$
$$= \bigcup_{i=0}^{i_5-1} \sigma_5^i\left(S^{(\sigma_1, \ldots, \sigma_4)}\right) \cup \bigcup_{j=0}^{a_4} \sigma_5^{i_5}\sigma_4^j\left[\bigcup_{k=0}^{i_3-1} \sigma_3^k\left(S^{(\sigma_1, \sigma_2)}\right) \cup \bigcup_{l=0}^{a_2} \sigma_3^{i_3}\sigma_2^l\left(\bigcup_{p=0}^{i_1} \sigma_1^p(S)\right)\right].$$

LEMMA 2. *Let S be a finite set of generators of $\mathfrak{M}/\mathfrak{K}$; $j_1, k_1, \ldots, j_t, k_t$ integers with $n > j_1$, $j_l \geq k_l$, $l = 1, \ldots, t$, $k_l > j_{l+1} + 1$, $l = 1, \ldots, t$, and $k_t \geq 1$; $a_{j_1}, a_{j_1-1}, \ldots, a_{k_1}; \ldots; a_{j_t}, a_{j_t-1}, \ldots, a_{k_t}$ nonnegative integers; and U a finite subset of L. There exist positive integers $b_n, b_{n-1}, \ldots, b_{j_1+1}; b_{k_1-1}, \ldots, b_{j_2+1}; \ldots; b_{k_{t-1}-1}, \ldots, b_{j_t+2}$ ($b_n, b_{n-1}, \ldots, b_{j_1+1}; b_{k_1-1}, \ldots, b_{j_2+1}; \ldots; b_{k_{t-2}-1}, \ldots, b_{j_{t-1}+1}$ if $j_t + 2 = k_{t-1}$), a nonnegative integer a_{j_t+1} and a finite subset V of L containing U such that if $i_n, i_{n-1}, \ldots, i_{j_1+1}; i_{k_1-1}, \ldots, i_{j_2+1}; \ldots; i_{k_{t-1}-1}, \ldots, i_{j_t+2}$ ($i_n, i_{n-1}, \ldots, i_{j_1+1}; i_{k_1-1}, \ldots, i_{j_2+1}; \ldots; i_{k_{t-2}-1}, \ldots, i_{j_{t-1}+1}$ if $j_t + 2 = k_{t-1}$) are positive integers with each $i_j \geq b_j$ and W is a finite subset of L containing V then*

$$L \cap K\langle W\rangle(S(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t))$$
$$\subseteq K\langle W\rangle\left(S\left(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_{t-1}, \mathbf{a}_{t-1}, \left(i_{k_{t-1}-1}, \ldots, i_{j_t+2}\right), \left(a_{j_t+1}\right)\right)\right)$$

where $\mathbf{i}_m = (i_{k_{m-1}-1},\ldots,i_{j_m+1})$ *and* $\mathbf{a}_m = (a_{j_m},\ldots,a_{k_m})$, $m = 1,\ldots,t$ *and* $k_0 - 1 = n$. $(L \cap K\langle W\rangle(S(\mathbf{i}_1,\mathbf{a}_1,\ldots,\mathbf{i}_t,\mathbf{a}_t))$

$$\subseteq K\langle W\rangle\big(S\big(\mathbf{i}_1,\mathbf{a}_1,\ldots,\mathbf{i}_{t-1},\big(a_{j_{t-1}},\ldots,a_{k_{t-1}},a_{j_t+1}\big)\big)\big)$$

if $j_t + 2 = k_{t-1}$.)

Proof. Throughout this proof W will be used to denote finite subsets of L, t.d. and Δ will be used to denote transcendence degree and transformal transcendence degree, respectively, the notation $\mathbf{i}_m$ will be reserved for positive integral vectors of the form $(i_{k_{m-1}-1},\ldots,i_{j_m+1})$, $m = 1,\ldots,t$, $k_0 - 1 = n$ and $\mathbf{a}_m = (a_{j_m},\ldots,a_{k_m})$, $m = 1,\ldots,t$.

The proof is by induction on $k = k_t - 1$. The proof for $k = 0$ and the induction step are carried out simultaneously.

For positive integral vectors $\mathbf{i}_1,\ldots,\mathbf{i}_t$ define

$$\delta(\mathbf{i}_1,\ldots,\mathbf{i}_t) = \Delta\big(L\big(S(\mathbf{i}_1,\mathbf{a}_1,\ldots,\mathbf{i}_t,\mathbf{a}_t)\big);\sigma_1,\ldots,\sigma_k\big)/\big(L\big(S^*(\mathbf{i}_1,\mathbf{a}_1,\ldots,\mathbf{i}_t,\mathbf{a}_t)\big):\sigma_1,\ldots,\sigma_k\big)\big).$$

(If $k = 0$,

$$\delta(\mathbf{i}_1,\ldots,\mathbf{i}_t) = \text{t.d. } L\big(S(\mathbf{i}_1,\mathbf{a}_1,\ldots,\mathbf{i}_t,\mathbf{a}_t)\big)/L\big(S^*(\mathbf{i}_1,\mathbf{a}_1,\ldots,\mathbf{i}_t,\mathbf{a}_t)\big).)$$

Let $\mathbf{i}_m$ and $\mathbf{i}'_m$ for $m = 1,\ldots,t$, be positive integral vectors such that $\mathbf{i}'_m \geqslant \mathbf{i}_m$, $m = 1,\ldots,t$, and τ be the isomorphism $\prod \sigma_p^{i'_p - i_p}$ $(k_{m-1} - 1 \geqslant p \geqslant j_m + 1,\ m = 1,\ldots,t)$. Since

$$\big(L\big(S(\mathbf{i}'_1,\mathbf{a}_1,\ldots,\mathbf{i}'_t,\mathbf{a}_t)\big);\sigma_1,\ldots,\sigma_k\big)$$

and

$$\big(L\big(S^*(\mathbf{i}'_1,\mathbf{a}_1,\ldots,\mathbf{i}'_t,\mathbf{a}_t)\big);\sigma_1,\ldots,\sigma_k\big)$$

can be obtained by adjoining the same set of elements to

$$\big(\tau\big(L\big(S(\mathbf{i}_1,\mathbf{a}_1,\ldots,\mathbf{i}_t,\mathbf{a}_t)\big)\big);\sigma_1,\ldots,\sigma_k\big)$$

and

$$\big(\tau\big(L\big(S^*(\mathbf{i}_1,\mathbf{a}_1,\ldots,\mathbf{i}_t,\mathbf{a}_t)\big)\big);\sigma_1,\ldots,\sigma_k\big),$$

respectively, it follows that $\delta(\mathbf{i}'_1,\ldots,\mathbf{i}'_t) \leqslant \delta(\mathbf{i}_1,\ldots,\mathbf{i}_t)$, and if A is a basis of transformal transcendence of

$$\big(L\big(S(\mathbf{i}_1,\mathbf{a}_1,\ldots,\mathbf{i}_t,\mathbf{a}_t)\big);\sigma_1,\ldots,\sigma_k\big)/\big(L\big(S^*(\mathbf{i}_1,\mathbf{a}_1,\ldots,\mathbf{i}_t,\mathbf{a}_t)\big);\sigma_1,\ldots,\sigma_k\big)$$

then $\tau(A)$ contains a basis of transformal transcendence of

$$\big(L\big(S(\mathbf{i}'_1,\mathbf{a}_1,\ldots,\mathbf{i}'_t,\mathbf{a}_t)\big);\sigma_1,\ldots,\sigma_k\big)/\big(L\big(S^*(\mathbf{i}'_1,\mathbf{a}_1,\ldots,\mathbf{i}'_t,\mathbf{a}_t)\big);\sigma_1,\ldots,\sigma_k\big).$$

Hence, we can choose positive integral vectors $\mathbf{c}_m = (c_{k_m-1},\ldots,c_{j_m+1})$, $m = 1,\ldots,t$, and a finite subset Z of

$$\bigcup_{0 \leqslant \alpha_i \leqslant a_i} \sigma_{j_1}^{\alpha_{j_1}}\cdots\sigma_{k_1}^{\alpha_{k_1}}\cdots\sigma_{j_t}^{\alpha_{j_t}}\cdots\sigma_{k_t}^{\alpha_{k_t}}(S)$$

such that if $\mathbf{i}_m \geqslant \mathbf{c}_m$, $m = 1,\dots,t$, then $\delta(\mathbf{i}_1,\dots,\mathbf{i}_t) = \delta(\mathbf{c}_1,\dots,\mathbf{c}_t)$ and $\rho(Z)$ where

$$\rho = \prod_{m=1}^{t} \sigma_{k_m - 1}^{i_{k_m} - 1} \cdots \sigma_{j_m + 1}^{i_{j_m} + 1}$$

is a basis of transformal transcendence of

$$\big(L(S(\mathbf{i}_1,\mathbf{a}_1,\dots,\mathbf{i}_t,\mathbf{a}_t)); \sigma_1,\dots,\sigma_k\big) / \big(L(S^*(\mathbf{i}_1,\mathbf{a}_1,\dots,\mathbf{i}_t,\mathbf{a}_t)); \sigma_1,\dots,\sigma_k\big).$$

(If $k = 0$ the argument given above, with $L(S(\mathbf{i}_1,\mathbf{a}_1,\dots,\mathbf{i}_t,\mathbf{a}_t))$ replacing

$$\big(L(S(\mathbf{i}_1,\mathbf{a}_1,\dots,\mathbf{i}_t,\mathbf{a}_t)); \sigma_1,\dots,\sigma_k\big),$$

$L(S^*(\mathbf{i}_1,\mathbf{a}_1,\dots,\mathbf{i}_t,\mathbf{a}_t))$ replacing

$$\big(L(S^*(\mathbf{i}_1,\mathbf{a}_1,\dots,\mathbf{i}_t,\mathbf{a}_t)); \sigma_1,\dots,\sigma_k\big)$$

and the word transcendence replacing the words transformal transcendence, guarantees positive integral vectors $\mathbf{c}_1,\dots,\mathbf{c}_t$ and a finite set

$$Z \subseteq \bigcup_{0 \leqslant \alpha_i \leqslant a_i} \sigma_{j_1}^{\alpha_{j_1}} \cdots \sigma_{k_1}^{\alpha_{k_1}} \cdots \sigma_{j_t}^{\alpha_{j_t}} \cdots \sigma_{k_t}^{\alpha_{k_t}}(S)$$

such that if $\mathbf{i}_m \geqslant \mathbf{c}_m$, $m = 1,\dots,t$, then $\delta(\mathbf{i}_1,\dots,\mathbf{i}_t) = \delta(\mathbf{c}_1,\dots,\mathbf{c}_t)$ and $\rho(Z)$ is a transcendence base for $L(S(\mathbf{i}_1,\mathbf{a}_1,\dots,\mathbf{i}_t,\mathbf{a}_t))/L(S^*(\mathbf{i}_1,\mathbf{a}_1,\dots,\mathbf{i}_t,\mathbf{a}_t))$.)

For positive integral vectors $\mathbf{i}_1,\dots,\mathbf{i}_t$ let $Y = Y(\mathbf{i}_1,\dots,\mathbf{i}_t) = \rho(Z^{(\sigma_1,\dots,\sigma_k)})$. (Recall that if $k = 0$ $Z^{(\sigma_1,\dots,\sigma_k)} = Z$.) Henceforth, when the symbol Y is used in conjunction with an $S(\cdot)$ or $S^*(\cdot)$ the vectors $\mathbf{i}_j$, $j = 1,\dots,t$, used to define Y will be the vector $\mathbf{i}_j$, $j = 1,\dots,t$, used to define $S(\cdot)$ or $S^*(\cdot)$. For example, when we write $K(Y, S(\mathbf{i}_1,\mathbf{a}_1,\dots,\mathbf{i}_t,\mathbf{a}_t))$, Y means $Y(\mathbf{i}_1,\dots,\mathbf{i}_t)$, and when we write $K(Y, S^*(\mathbf{i}'_1,\mathbf{a}_1,\dots,\mathbf{i}'_t,\mathbf{a}_t,\mathbf{i}'_{t+1}))$, Y means $Y(\mathbf{i}'_1,\dots,\mathbf{i}'_t)$. If $\mathbf{i}_j \geqslant \mathbf{c}_j$, $j = 1,\dots,t$, then the degree of transformal transcendence of

$$\big(L(Y, S(\mathbf{i}_1,\mathbf{a}_1,\dots,\mathbf{i}_t,\mathbf{a}_t)); \sigma_1,\dots,\sigma_k\big) / \big(L(Y, S^*(\mathbf{i}_1,\mathbf{a}_1,\dots,\mathbf{i}_t,\mathbf{a}_t)); \sigma_1,\dots,\sigma_k\big)$$

is zero and so this extension has finite ld_k. (If $k = 0$, the transcendence degree of

$$L(Y, S(\mathbf{i}_1,\mathbf{a}_1,\dots,\mathbf{i}_t,\mathbf{a}_t))/L(Y, S(\mathbf{i}_1,\mathbf{a}_1,\dots,\mathbf{i}_t,\mathbf{a}_t))$$

is zero and this extension has finite degree.)

If $k > 0$ (ld2) and the definition of ld_k guarantee the existence of nonnegative integral vectors $\mathbf{d}(1,1),\dots,\mathbf{d}(1,t)$ with $\mathbf{d}(1,j) \geqslant \mathbf{c}_j$, $j = 1,\dots,t$, nonnegative integers $a(1,1),\dots,a(1,k)$ and a positive integer α such that if $\mathbf{i}_j \geqslant \mathbf{d}(1,j)$, $j = 1,\dots,t$, and $i_j \geqslant a(1,j)$, $j = 1,\dots,k$, then

(1)

$$\begin{aligned} \alpha &= \mathrm{ld}_k\big(L(Y, S(\mathbf{i}_1,\mathbf{a}_1,\dots,\mathbf{i}_t,\mathbf{a}_t)); \sigma_1,\dots,\sigma_k\big)/ \\ &\qquad \big(L(Y, S^*(\mathbf{i}_1,\mathbf{a}_1,\dots,\mathbf{i}_t,\mathbf{a}_t)); \sigma_1,\dots,\sigma_k\big) \\ &= L\big(Y, S(\mathbf{i}_1,\mathbf{a}_1,\dots,\mathbf{i}_t,\mathbf{a}_t,(i_k,\dots,i_1))\big) : L\big(Y, S^*(\mathbf{i}_1,\mathbf{a}_1,\dots,\mathbf{i}_t,\mathbf{a}_t,(i_k,\dots,i_1))\big) \end{aligned}$$

Select a finite subset $V^{(1)}$ of L containing U such that if $W \supseteq V^{(1)}$,

$$\begin{aligned} \alpha = \mathrm{ld}_k\big(K\langle W\rangle(Y, S(\mathbf{d}(1,1),\mathbf{a}_1,\dots,\mathbf{d}(1,t),\mathbf{a}_t)); \sigma_1,\dots,\sigma_k\big)/ \\ K\langle W\rangle\big(Y, S^*(\mathbf{d}(1,1),\mathbf{a}_1,\dots,\mathbf{d}(1,t),\mathbf{a}_t); \sigma_1,\dots,\sigma_k\big) \end{aligned} \tag{2}$$

and

$$\begin{aligned}\alpha = {} & K\langle W\rangle(Y, S(\mathbf{d}(1,1), \mathbf{a}_1, \ldots, \mathbf{d}(1,t), \mathbf{a}_t, (a(1,k), \ldots, a(1,1)))): \\ & K\langle W\rangle(Y, S^*(\mathbf{d}(1,1), \mathbf{a}_1, \ldots, \mathbf{d}(1,t), \mathbf{a}_t, (a(1,k), \ldots, a(1,1)))).\end{aligned} \tag{3}$$

If $W \supseteq V^{(1)}$, $\mathbf{i}_j \geqslant \mathbf{d}(1, j)$, $j = 1, \ldots, t$, and $i_j \geqslant a(1, j)$, $j = 1, \ldots, k$, then

$$\begin{aligned}\alpha = {} & \mathrm{ld}_k(K\langle W\rangle(Y, S(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t)); \sigma_1, \ldots, \sigma_k)/ \\ & (K\langle W\rangle(Y, S^*(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t)); \sigma_1, \ldots, \sigma_k)\end{aligned} \tag{2'}$$

and

$$\begin{aligned}\alpha = {} & K\langle W\rangle(Y, S(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t, (i_k, \ldots, i_1))): \\ & K\langle W\rangle(Y, S^*(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t, (i_k, \ldots, i_1))).\end{aligned} \tag{3'}$$

In fact, if $i_j \geqslant d(1, j)$, $j = 1, \ldots, t$, then by (1), (2) and (ld2) we have

$$\begin{aligned}\alpha &= \mathrm{ld}_k(L(Y, S(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t)); \sigma_1, \ldots, \sigma_k)/ \\ &\qquad (L(Y, S^*(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t)); \sigma_1, \ldots, \sigma_k) \\ &\leqslant \mathrm{ld}_k(K\langle W\rangle(Y, S(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t)); \sigma_1, \ldots, \sigma_k)/ \\ &\qquad (K\langle W\rangle(Y, S^*(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t); \sigma_1, \ldots, \sigma_k)) \\ &\leqslant \mathrm{ld}_k(K\langle W\rangle(Y, S(\mathbf{d}(1,1), \mathbf{a}_1, \ldots, \mathbf{d}(1,t), \mathbf{a}_t); \sigma_1, \ldots, \sigma_k)/ \\ &\qquad (K\langle W\rangle(Y, S^*(\mathbf{d}(1,1), \mathbf{a}_1, \ldots, \mathbf{d}(1,t), \mathbf{a}_t); \sigma_1, \ldots, \sigma_k)) = \alpha\end{aligned}$$

which verifies (2′) for the appropriate $\mathbf{i}_t$. An analogous argument can be used to establish (3′) for sufficiently large $\mathbf{i}_l$ and i_j.

Now, let $\mathbf{i}_j \geqslant \mathbf{d}(1, j)$, $j = 1, \ldots, t$, and $x \in L \cap K\langle W\rangle(S(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t))$. Then there exist $i_1, \ldots, i_k$ with $i_j \geqslant a(1, j)$, $j = 1, \ldots, k$, such that

$$x \in L \cap K\langle W\rangle(S(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t, (i_k, \ldots, i_1))).$$

It will be shown that

$$\begin{aligned}x &\in K\langle W\rangle(Y, S(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t, (i_k, \ldots, i_2), (a(1,1)))) \\ &= K\langle W\rangle(Y, S(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t, (i_k, \ldots, i_2, a(1,1)))).\end{aligned}$$

(If $k = 1$ replace $K\langle W\rangle(Y, S(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t, (i_k, \ldots, i_2), (a(1,1))))$ above and below with $K\langle W\rangle(Y, S(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, (a_{j_t}, \ldots, a_2, a(1,1))))$. Note that when $k = 1$, $k_t = 2$.) If not, there is a $j_1 > a(1, 1)$ such that

$$x \in K\langle W\rangle(Y, S(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t, (i_k, \ldots, i_2, j_1)))$$

but

$$x \notin K\langle W\rangle(Y, S^*(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t, (i_k, \ldots, i_2, j_1))).$$

Using Theorem III of the introduction in [3] we have

$$\begin{aligned}\alpha = K\langle W\rangle(Y, S(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t, (i_k, \ldots, i_2, j_1))):\\ K\langle W\rangle(Y, S^*(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t, (i_k, \ldots, i_2, j_1)))\\ > K\langle W\rangle(Y, S(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t, (i_k, \ldots, i_2, j_1))):\\ K\langle W\rangle(Y, S^*(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t, (i_k, \ldots, i_2, j_1)), x)\\ \geqslant L(Y, S(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t, (i_k, \ldots, i_2, j_1))):\\ L(Y, S^*(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t, (i_k, \ldots, i_2, j_1))) = \alpha,\end{aligned}$$

a contradiction.

Further, if $\mathbf{i}_j \geqslant \mathbf{d}(1, j), j = 1, \ldots, t, i_j \geqslant a(1, j), j = 1, \ldots, k$, and

$$x \in L \cap K\langle W\rangle(Y, S(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t, (i_k, \ldots, i_2), (a(1, 1))))$$

(see the last parenthetical remark for the correct notation when $k = 1$), then x can be expressed as a quotient, P/Q, of polynomials in the elements of

$$D = Y \sim \rho Z(k, i_k, \ldots, i_2, a(1, 1)) \qquad \left(\rho = \prod_{m=1}^{t} \sigma_{k_m-1}^{i_{k_m}-1} \cdots \sigma_{j_m+1}^{i_{j_m}+1}\right)$$

with coefficients in $K\langle W\rangle(S(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t, (i_k, \ldots, i_2), (a(1, 1))))$. Setting $xQ = P$, noting that D is algebraically independent over

$$L(S(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t, (i_k, \ldots, i_2), (a(1, 1))))$$

(Corollary 4, §2) and equating corresponding nonzero coefficients (assuming $x \neq 0$), we see that

$$x \in K\langle W\rangle(S(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t, (i_k, \ldots, i_2), (a(1, 1)))).$$

If $k = 0$, replace (1) above with (1′) below.

$$\alpha = L(Y, S(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t)): L(Y, S^*(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t)) \tag{1′}$$

if $\mathbf{i}_j \geqslant \mathbf{d}(1, j), j = 1, \ldots, t$. Arguments analogous to those used above show there exists a finite set $V \subseteq L$ such that if $W \supseteq V$, (1′) remains valid when L is replaced by $K\langle W\rangle$. As above, one can show that if $\mathbf{i}_j \geqslant \mathbf{d}(1, j), j = 1, \ldots, t$, then

$$L \cap K\langle W\rangle(S(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t)) \subseteq K\langle W\rangle(S^*(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t)).$$

Since $\sigma_1, \ldots, \sigma_{t_j-1}$ map $S^*(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t)$ into itself, it follows that if $\mathbf{c}_t \geqslant \mathbf{a}_t$ then

$$L \cap K\langle W\rangle(S(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_{t-1}, \mathbf{a}_{t-1}, \mathbf{i}_t, \mathbf{c}_t)) \subseteq K\langle W\rangle(S^*(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t)).$$

Hence,

$$\begin{aligned}L \cap K\langle W\rangle\left(S^*\left(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_{t-1}, \mathbf{a}_{t-1}, \left(i_{k_t-1}, \ldots, i_{j_t+2}, i_{j_t+1} + 1\right), \mathbf{a}_t\right)\right)\\ \subseteq K\langle W\rangle(S^*(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t)).\end{aligned}$$

Inductively it follows that if $p \geqslant 0$ then

$$\begin{aligned}L \cap K\langle W\rangle\left(S^*\left(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_{t-1}, \mathbf{a}_{t-1}, \left(i_{k_t-1}, \ldots, i_{j_t+2}, i_{j_t+1} + p\right), \mathbf{a}_t\right)\right)\\ \subseteq K\langle W\rangle(S^*(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \mathbf{a}_t)).\end{aligned}$$

Hence, if $\mathbf{i}_j \geqslant \mathbf{d}(1, j), j = 1,\ldots,t$, and a_{j_t+1} is the last component of $\mathbf{d}(1, t)$ then

$$\begin{aligned} &L \cap K\langle W\rangle(S(\mathbf{i}_1, \mathbf{a}_1,\ldots,\mathbf{i}_t, \mathbf{a}_t)) \\ &\quad\subseteq L \cap K\langle W\rangle\big(S^*\big(\mathbf{i}_1, \mathbf{a}_1,\ldots,\mathbf{i}_{t-1}, \mathbf{a}_{t-1}, \big(i_{k_t-1},\ldots,i_{j_t+2}, i_{j_t+1}+1\big), \mathbf{a}_t\big)\big) \\ &\quad\subseteq K\langle W\rangle\big(S^*\big(\mathbf{i}_1, \mathbf{a}_1,\ldots,\mathbf{i}_{t-1}, \mathbf{a}_{t-1}, \big(i_{k_t-1},\ldots,i_{j_t+2}, a_{j_t+1}\big), \mathbf{a}_t\big)\big) \\ &\quad\subseteq K\langle W\rangle\big(S\big(\mathbf{i}_1, \mathbf{a}_1,\ldots,\mathbf{i}_{t-1}, \mathbf{a}_{t-1}, \big(i_{k_t-1},\ldots,i_{j_t+2}\big), \big(a_{j_t+1}\big)\big)\big) \end{aligned}$$

which establishes the lemma for $k = 0$. (The appropriate modification in notation is necessary here when $j_t + 2 = k_{t-1}$. (See the statement of the lemma.)

If $k = 1$ it follows from our work above that

$$L \cap K\langle W\rangle(S(\mathbf{i}_1, \mathbf{a}_1,\ldots,\mathbf{i}_t, \mathbf{a}_t)) \subseteq K\langle W\rangle\big(S\big(\mathbf{i}_1, \mathbf{a}_1,\ldots,\mathbf{i}_t, \big(a_{j_t},\ldots,a_2, a(1,1)\big)\big)\big)$$

if $W \supseteq V^{(1)}$ and $\mathbf{i}_j \geqslant \mathbf{d}(1, j), j = 1,\ldots,t$. Apply the induction hypothesis with $k = 0$ to find positive integers $b_n,\ldots,b_{j_1+1};\ldots;b_{k_{t-1}-1},\ldots,b_{j_t+2}$, a nonnegative a_{j_t+1} and a finite subset V of L containing $V^{(1)}$ such that if $i_n \geqslant b_n,\ldots,i_{j_1+1} \geqslant b_{j_1+1};\ldots;i_{k_{t-1}-1} \geqslant b_{k_{t-1}-1},\ldots,i_{j_t+2} \geqslant b_{j_t+2}$ and $W \supseteq V$ then

$$\begin{aligned} &L \cap K\langle W\rangle(S(\mathbf{i}_1, \mathbf{a}_1,\ldots,\mathbf{i}_t, \mathbf{a}_t)) \\ &\qquad\qquad\subseteq K\langle W\rangle\big(S\big(\mathbf{i}_1, \mathbf{a}_1,\ldots,\mathbf{i}_{t-1}, \mathbf{a}_{t-1}, \big(i_{k_t-1},\ldots,i_{j_t+2}\big), \big(a_{j_t+1}\big)\big)\big). \end{aligned}$$

(The appropriate modification in notation is necessary here if $k_{t-1} = j_t + 2$. See the statement of the lemma.) This establishes the lemma when $k = 1$.

If $k > 1$, apply the induction hypothesis to find nonnegative integral vectors $\mathbf{d}(\beta, j)$, $\beta = 2,\ldots,k$, $j = 1,\ldots,t$, nonnegative integers $a(\beta, j)$, $\beta = 2,\ldots,k$, $j = \beta,\ldots,k$ and finite subsets $V^{(\beta)}$ of L, $\beta = 2,\ldots,k$ with $\mathbf{d}(\beta, j) \geqslant \mathbf{d}(\beta - 1, j)$, $j = 1,\ldots,t$, $\beta = 2,\ldots,k$, $a(\beta, j) \geqslant a(\beta - 1, j)$, $\beta = 2,\ldots,k, j = \beta,\ldots,k$ and $V^{(\beta)} \supseteq V^{(\beta-1)}$, $\beta = 2,\ldots,k$, such that for $\beta = 1,\ldots,k-2$ if $W \supseteq V^{(\beta)}$, $\mathbf{i}_j \geqslant \mathbf{d}(\beta, j)$, $j = 1,\ldots,t$, and $i_j \geqslant a(\beta, j), j = \beta,\ldots,t$, then

$$\begin{aligned} &L \cap K\langle W\rangle\big(S\big(\mathbf{i}_1, \mathbf{a}_1,\ldots,\mathbf{i}_t, \mathbf{a}_t, \big(i_k,\ldots,i_{\beta+1}\big), (a(\beta, \beta))\big)\big) \\ &\qquad\qquad\subseteq K\langle W\rangle\big(S\big(\mathbf{i}_1, \mathbf{a}_1,\ldots,\mathbf{i}_t, \mathbf{a}_t, \big(i_k,\ldots,i_{\beta+2}\big), (a(\beta+1, \beta+1))\big)\big) \end{aligned}$$

and

$$\begin{aligned} &L \cap K\langle W\rangle(S(\mathbf{i}_1, \mathbf{a}_1,\ldots,\mathbf{i}_t, \mathbf{a}_t, (i_k), (a(k-1, k-1)))) \\ &\qquad\qquad\subseteq L \cap K\langle W\rangle\big(S\big(\mathbf{i}_1, \mathbf{a}_1,\ldots,\mathbf{i}_t, \big(a_{j_t},\ldots,a_{k_t}, a(k, k)\big)\big)\big). \end{aligned}$$

Apply the induction hypothesis again to find integral vectors $\mathbf{d}(k+1, j), j = 1,\ldots,t$ integers $a(k+1, j)$, $j = 1,\ldots,k$, and a finite subset V of L with $\mathbf{d}(k+1, j) \geqslant d(k, j), j = 1,\ldots,t$, $a(k+1, j) \geqslant a(k, j)$, $j = 1,\ldots,k$, and $V \supseteq V^{(k)}$ such that if $\mathbf{i}_j \geqslant \mathbf{d}(k+1, j), j = 1,\ldots,t$, then

$$\begin{aligned} &L \cap K\langle W\rangle\big(S\big(\mathbf{i}_1, \mathbf{a}_1,\ldots,\mathbf{i}_t, \big(a_{j_t},\ldots,a_{k_t}, a(k, k)\big)\big)\big) \\ &\qquad\qquad\subseteq K\langle W\rangle\big(S\big(\mathbf{i}_1, \mathbf{a}_1,\ldots,\mathbf{a}_{t-1}, \big(i_{k_t-1},\ldots,i_{j_t+2}\big), \big(a_{j_t+1}\big)\big)\big) \end{aligned}$$

where $a_{j_{t+1}}$ is the last coordinate of $\mathbf{d}(k+1, t)$. [If $k_{t-1} = j_t + 2$ this last containment should read

$$L \cap K\langle W\rangle\big(S\big(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_t, \big(a_{j_t}, \ldots, a_{k_t}, a(k,k)\big)\big)\big)$$
$$\subseteq K\langle W\rangle\big(S\big(\mathbf{i}_1, \mathbf{a}_1, \ldots, \mathbf{i}_{t-2}, \big(a_{j_t-1}, \ldots, a_{k_{t-1}}, a_{j_{t+1}}\big)\big)\big).\big]$$

THEOREM 1. *If $\mathfrak{M}/\mathfrak{K}$ is finitely generated and $\mathfrak{L}/\mathfrak{K}$ is a subextension of $\mathfrak{M}/\mathfrak{K}$ then $\mathfrak{L}/\mathfrak{K}$ is finitely generated.*

PROOF. The proof is by induction on n (the number of transforming operators). When $n = 1$, Theorem 1 is Theorem XVIII of Chapter 5 in [3].

Let $n > 1$, and S be a finite set of generators of $\mathfrak{M}/\mathfrak{K}$. Using a proof similar to that of the above lemma when $k = 0$ one can find a finite subset $V^{(1)}$ of L and integers $a(1,1), \ldots, a(1,n)$ such that if $\mathbf{i}_1 \geqslant (a(1,n), \ldots, a(1,1))$ then

$$L \cap K\langle W\rangle(S(\mathbf{i}_1)) \subseteq K\langle W\rangle(S((i_n, \ldots, i_2), (a(1,1))))$$

for any finite subset W of L containing $V^{(1)}$. Proceeding inductively and using Lemma 2 we can find subsets $V^{(2)}, \ldots, V^{(n)}$ and sets of nonnegative integers $a(k,k), \ldots, a(k,n)$, $k = 2, \ldots, n$, with $V^{(k)} \supseteq V^{(k-1)}$ and $a(k,k) \geqslant a(k-1,k), \ldots, a(k,n) \geqslant a(k-1,n)$ for $k = 2, \ldots, n$ such that for $k = 2, \ldots, n-1$ if W is a finite subset of L containing $V^{(k)}$ then

$$L \cap K\langle W\rangle(S((i_n, \ldots, i_k), (a(k-1, k-1))))$$
$$\subseteq K\langle W\rangle(S((i_n, \ldots, i_{k+1}), (a(k,k))))$$

and

$$L \cap K\langle W\rangle(S((i_n), (a(n-1, n-1)))) \subseteq K\langle W\rangle(S((a(n,n))))$$

if $(i_n, \ldots, i_k) \geqslant (a(k,n), \ldots, a(k,k))$ and $i_n \geqslant a(n,n)$, respectively.

Now, let $x \in L$. There exist nonnegative integers $i_1, \ldots, i_n$ with $i_k \geqslant a(k,k)$ for $k = 1, \ldots, n$ such that $x \in K(S(i_n, \ldots, i_1))$. Hence,

$$\begin{aligned} x &\in L \cap K\langle V^{(n)}\rangle(S((i_n, \ldots, i_1))) \\ &\subseteq L \cap K\langle V^{(n)}\rangle(S((i_n, \ldots, i_2), (a(1,1)))) \\ &\subseteq L \cap K\langle V^{(n)}\rangle(S((i_n, \ldots, i_3), (a(2,2)))) \\ &\subseteq \cdots \subseteq L \cap K\langle V^{(n)}\rangle(S((i_n), (a(n-1, n-1)))) \\ &\subseteq L \cap K\langle V^{(n)}\rangle(S((a(n,n)))). \end{aligned}$$

Hence, $L \subseteq K\langle V^{(n)}\rangle(S((a(n,n))))$.

Now,

$$\big(K\langle V^{(n)}\rangle(S((a(n,n))); \sigma_1, \ldots, \sigma_{n-1})\big) / \big(K\langle V^{(n)}\rangle; \sigma_1, \ldots, \sigma_{n-1}\big)$$

is finitely generated. ($\bigcup_{j=0}^{a(n,n)} \sigma_n^j(S)$ is a finite set of generators.) Hence, by the induction hypothesis $(L: \sigma_1, \ldots, \sigma_{n-1}) = (K\langle V^{(n)}\rangle; \sigma_1, \ldots, \sigma_{n-1})\langle A\rangle$ for some finite subset A of L. Since $\sigma_n(L) \subseteq L$, $L = K\langle V^{(n)}, A\rangle$.

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CENTER FOR NAVAL ANALYSES, 2000 N. BEAUREGARD STREET, ALEXANDRIA, VIRGINIA 22311