BRAUER’S HEIGHT CONJECTURE
FOR p-SOLVABLE GROUPS

BY

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ABSTRACT. We complete the proof of the height conjecture for p-solvable groups, using the classification of finite simple groups.

Introduction. The height conjecture is the statement that a p-block of a finite group has an abelian defect group if and only if all ordinary irreducible characters in the block have height zero.

While a proof of this conjecture for general finite groups seems remote, considerable progress has been made toward proving it for p-solvable groups. Fong [5] proved that all characters in a block with abelian defect group have height zero in a p-solvable group, and he proved the converse direction for the principal block [5] and for solvable groups in the case that p is the largest prime divisor of the group order [6].

Recently [24, 8], the converse direction has been established for all solvable groups. In this paper we prove the converse direction for all p-solvable groups, assuming the classification of finite simple groups.

In its general outline this paper resembles [8], where we proved the height conjecture for solvable groups. The reader is assumed to have some familiarity with [8].

Now we state our main results, the analogs of the main results of [8].

THEOREM A. Suppose that N ≤ G, that G/N is p-solvable, that ϕ ∈ Irr(N), and that \( p \mid (\chi(1)/\varphi(1)) \) for all \( \chi \in \text{Irr}(G \mid \varphi) \). Then the p-Sylow subgroups of G/N are abelian.

THEOREM B. Let B be a p-block of a p-solvable group with defect group D. If every ordinary irreducible character in B has height zero, then D is abelian.

THEOREM C. Suppose that N ≤ G, that G/N is p-solvable, and that ϕ ∈ Irr(N). Suppose that \( e \) is an integer such that \( p^{e+1} \) does not divide \( \chi(1)/\varphi(1) \) for all \( \chi \in \text{Irr}(G \mid \varphi) \). Then the derived length of a p-Sylow subgroup of G/N is at most \( 2e + 1 \).

THEOREM D. Let B and D be as in Theorem B. If every ordinary irreducible character in B has height at most \( e \), then the derived length of D is at most \( 2e + 1 \).
Theorems B, C and D follow from Theorem A as in [8], so the rest of this paper is devoted to the proof of Theorem A.

The next proposition, essentially proved by Fong [6], describes the minimal counterexample to Theorem A. Note that $N$ and $\varphi$ in the statement of Theorem A correspond to $Z$ and $\lambda$ in the statement of Proposition 0, and that $N$ in the statement of Proposition 0 does not correspond to any subgroup in the statement of Theorem A.

**Proposition 0.** Let $G$ be a minimal counterexample to Theorem A. Then $G$ has normal subgroups $Z \leq N \leq K$, and $Z$ has a faithful linear character $\lambda$, such that the following conditions are satisfied:

1. $Z = O_p(G)$ is cyclic and central in $G$.
2. $N/Z$ is a self-centralizing p-chief factor of $G$.
4. $G = O^p(G)$.
5. If $V = \operatorname{Irr}(N/Z)$, the irreducible $GF(p)[G/N]$-module dual to $N/Z$, then every element of $V$ is centralized by some $p$-Sylow subgroup of $G/N$.
6. $p | \chi(1)$ for all $\chi \in \operatorname{Irr}(G|\lambda)$.

**Proof.** This follows as in Steps 1–4 of the proof of [8, Theorem 4.4]. The assumption in that theorem, that $p = 3$, is irrelevant in Steps 1–4, as is the assumption that $G/Z$ is solvable rather than merely $p$-solvable.

The notation of Proposition 0 is used in the following summary of the contents of this paper.

After some preliminary lemmas on simple groups in §1, we consider in §2 the case that $V$ is an imprimitive $GF(p)[G/N]$-module. We use a variety of facts about permutation groups and character degrees of groups of Lie type to show that $G$ must be solvable.

In §3 we consider the case that $V$ is a primitive $GF(p)[G/N]$-module and $F(G/N) = F^*(G/N)$, where $F$ and $F^*$ denote the Fitting and generalized Fitting subgroups. We use a variant of the estimation technique in [8, §2] to show that $G$ must be solvable.

In §4 we examine the remaining case that $V$ is primitive and $F(G/N) \neq F^*(G/N)$. We use standard facts about orders, automorphisms, and multipliers of groups of Lie type and a result on permutation groups from §2 to show that $\operatorname{Irr}(G|\lambda)$ contains a character of degree divisible by $p$. This contradicts condition (6) in Proposition 0 and so completes the proof of Theorem A.

1. This section contains some general lemmas which are useful in working with nonsolvable $p$-solvable groups.

**Lemma 1.1.** Let $p$ be a prime number and let $n$ be a positive integer. Suppose that neither of the following situations occurs:

i. $n = 6$ and $p = 2$.
ii. $n = 2$ and $p$ is a Mersenne prime.
Then there is a prime number \( r \) such that \( r \mid p^n - 1 \) and \( r \mid p^m - 1 \) for \( 0 < m < n \). Such a prime number \( r \) is called a primitive divisor of \( p^n - 1 \).

**Proof.** See [8, Lemma 3.3].

**Lemma 1.2.** Let \( S \) be a simple adjoint group of Lie type. Let \( d = |Z(G)| \), where \( G \) is the universal group of the same type as \( S \). Then:

(i) \( d \mid |S| \).

(ii) If \( p \) is a prime number and \( p \mid |S| \), then \( p > d \).

(iii) There exists a prime number \( r > 3 \) such that \( r \mid |S| \), \( r \mid d \), and \( r \) is greater than the order \( l \) of the group of field automorphisms of \( S \).

**Proof.** To prove (i) and (ii) we may assume that \( G = A_n(q) \) or \( G = A_n(2, q) \) (see [9, p. 491]). Then \( d = (n + 1, q \pm 1) \), and we may assume that \( n > 3 \). Since \( q^j - 1 \) divides \( |G| \) whenever \( j \) is even and \( j \leq n + 1 \), it follows that \( (q^4 - 1)(q^2 - 1) \) divides \( |G| \) and \( n + 1 < p - 1 \). Then \( d^2 \| |G| \), \( d \| |G/Z(G)| \), and \( p > n + 1 \). This proves (i) and (ii).

To prove (iii), write \( q = q_0^l \) for a prime number \( q_0 \) and a positive integer \( l \). If \( G \) is not \( A_n(q) \) or \( 2A_n(q) \), there is an integer \( m > 2 \) such that \( (q^m + 1) \| |G| \). If \( G = 2A_n(q) \), there is an integer \( m > 3 \) such that \( m > n \) and \( (q^m + 1) \| |G| \). In either case, let \( r \) be a primitive divisor of \( q^{2m} - 1 = q_0^{2ml} - 1 \), allowing \( r = 7 \) if \( q^2 = 26 \). Then \( r \| |G| \), \( 2ml < r - 1 \), and \( r > 5 \). Then \( r \| |S| \) by (i), and \( r > l \), the order of the group of field automorphisms of \( S \). Also \( r > 5 > d \) if \( G = 2A_n(q) \) and \( r > 2m > n + 1 > d \) if \( G = 2A_n(q) \).

Thus, we may assume that \( G = A_n(q) \), so that \( q^{n+1} - 1 \) divides \( |G| \). If \( l(n + 1) > 3 \), let \( r \) be a primitive divisor of \( q^{n+1} - 1 = q_0^{(n+1)} - 1 \). Then \( l(n + 1) < r - 1 \), \( r > 5 \), \( r > l \), and \( r > n + 1 > d \). If \( l(n + 1) < 3 \), then \( d < 3 \) and \( l < 3 \), so we can let \( r \) be any prime greater than 3 which divides \( |S| \).

**Lemma 1.3.** Let \( S \) be a nonabelian simple group which admits a coprime automorphism of prime order \( p \). Then \( S \) is an adjoint group of Lie type, \( S \) admits a field automorphism of order \( p \), and \( \text{Out}(S) \) has a cyclic and central \( p \)-Sylow subgroup.

**Proof.** By [10, p. 169] the sporadic and alternating groups have no coprime automorphisms. By [12] the simple group \( 2F_4(2)' \) has no coprime automorphism. Thus \( S \) is an adjoint group of Lie type. If \( S \) is a Suzuki or Ree group then \( \text{Aut}(S) \) is generated by the inner and field automorphisms of \( S \) (see [23, 18, 19]). Thus, we may assume that \( S \) is not a Suzuki or Ree group. In particular, \( p > 3 \).

By [20, p. 608], we have \( D < F < \text{Out}(S) \), where \( D \) is the image in \( \text{Out}(S) \) of the group of diagonal automorphisms of \( S \), and \( F \) is the image in \( \text{Out}(S) \) of the group generated by the diagonal and field automorphisms of \( S \). Moreover \( |D| = d \), where \( d \) is as in Lemma 1.2, and \( \text{Out}(S)/F \) is isomorphic to the group of graph automorphisms of \( S \), a \( \{2, 3\} \)-group.

Since \( p > 3 \) and \( p > d \) by Lemma 1.2(ii), it follows that \( S \) admits a field automorphism of order \( p \). Since graph and field automorphisms commute [10, p. 169] and since \( D < F \) and \( p > d \), the rest of Lemma 1.3 follows.
Corollary 1.4. Let $S$ be a nonabelian simple group with Schur multiplier $M$. Then there is a prime number $r$ such that $r \mid |S|$, $r \nmid |M|$, and $r \nmid |\text{Out}(S)|$.

Proof. This is clear if $S$ is sporadic, alternating, or $2F_4(2)'$, since then both $M$ and $\text{Out}(S)$ are $\{2, 3\}$-groups.

Otherwise, $S$ is an adjoint group of Lie type. By [11, p. 280], any prime divisor of $|M|$ is 2, 3, or a divisor of $d$. Thus the result follows from Lemma 1.2 and the description of $\text{Out}(S)$ in the proof of Lemma 1.3.

Lemma 1.5. Let $G$ be a finite group. Let $F(G)$ and $F^*(G)$ denote the Fitting and generalized Fitting subgroups of $G$. If $L/W$ is a chief factor of $G$ such that $L = L'$ and $W = Z(L)$ then $L \leq F^*(G)$. Conversely, if $F^*(G) \neq F(G)$, then $F^*(G)$ contains a perfect subgroup $L$ such that $L/Z(L)$ is a chief factor of $G$.

Proof. See [3, p. 128].

2. In this section we show that the $GF(p)[G/N]$-module $V$ of Proposition 0 must be primitive. We first record several lemmas which will be needed in the proof of Theorem 2.5, the main result of this section.

Lemma 2.1. Let $G$ be a nonsolvable group which acts faithfully on a finite vector space $V$. Suppose $G$ acts transitively on $V \setminus \{0\}$. Then the (unique) nonsolvable composition factor of $G$ is not a Suzuki group.

Proof. See the discussion preceding [13, Proposition 5.1].

Lemma 2.2. Let $G$ be a transitive permutation group on a set $\Omega$ of $n$ points, and let $P \in \text{Syl}_p(G)$ for some prime $p$ dividing $|G|$. If $P$ has $f$ fixed points on $\Omega$, then $f = \left(\frac{n - 1}{2}\right)$.

Proof. This follows from [14, Corollary 2].

Lemma 2.3. Let $G$ be a primitive permutation group on $\Omega$, with degree $n$ and socle $N$. Then one of the following occurs:

(i) $N$ is elementary abelian of order $p^d$ and regular; $n = p^d$ where $p$ is prime.

(ii) $N = T_1 \times \cdots \times T_m$, where $T_1, \ldots, T_m$ are isomorphic to a fixed simple group $T$. Moreover, either

(a) $T$ is the socle of a primitive group $G_0$ of degree $n_0$ and $G \leq G_0 \wr S_m$ (with the product action), where $n = n_0^m$, or

(b) $m = kl$ and $n = |T^{k-1}|l$. The permutation group induced by $G$ on $\{T_1, \ldots, T_m\}$ has $\{T_1, \ldots, T_k\}$ as a block of imprimitivity. The group induced on the set of blocks is transitive.

Proof. See Theorem 4.1 and Remark 2 following Theorem 4.1 in [4]. In (ii)(a) the statement that $G_0 \wr S_m$ acts with the product action means that $G_0 \wr S_m$ acts on $\Omega = \Omega_n^m$, where $|\Omega_0| = n_0$. The base group of the wreath product acts componentwise on $\Omega_n^m$, while $S_m$ acts by permuting coordinates. See [4, p. 5] for a formal definition of “product action”.

The following impressive result does not depend on the classification of simple groups.
LEMMA 2.4. Let $G$ be a uniprimitive permutation group of degree $n$. Then

$$|G| < \exp\left(4\sqrt{n} \log^2 n\right).$$

**Proof.** This is [2, Corollary 3.3].

Another important ingredient in the proof of Theorem 2.5 will be the lower bounds found by Landazuri and Seitz for the smallest degree of a nontrivial projective representation of a simple group of Lie type. Their results are tabulated in [17, p. 419]. We will not reproduce their table here, except to note a misprint; the bound for $PSO(2n + 1, q')$, $q > 5$, should read $q^{2(n-1)} - 1$, as in [17, Lemma 3.3].

**Definition.** In this paper $J$ denotes the affine semilinear group over $GF(8)$. Thus $J$ is a solvable group of order 168, which acts 2-transitively on 8 points.

**Theorem 2.5.** Let $G$ be a transitive permutation group on a finite set $\Omega$. Suppose $|G : O_p(G)| = p$ and $G = O_p^r(G)$ for an odd prime $p$. Suppose each subset of $\Omega$ is stabilized by an element of order $p$ in $G$. Then $p = 3$, $|\Omega| = 8$, and $G \cong J$.

**Proof.** Let $G$ be a counterexample to the theorem. The proof will be carried out in a series of steps.

**Step 1.** $G$ is primitive on $\Omega$.

**Proof.** We write $\Omega$ as a disjoint union of blocks so that $G$ acts as a primitive group on the set of blocks. We may assume that each block contains more than one point.

By induction on $|\Omega|$, the conclusion of the theorem is valid for the action of $G$ on the set of blocks. Thus $p = 3$, we may write $\Omega = B_1 \cup \cdots \cup B_8$, and $G$ acts as $J$ on the set of blocks. Choose $\Delta \leqslant \Omega$ to consist of 2 points from $B_1$, one point from $B_2$, and one point from $B_3$. Any element of order 3 in $G$ which stabilizes $\Delta$ must stabilize $B_1$, $B_2$, and $B_3$. This contradicts the fact that elements of order 3 in $J$ have only two fixed points in the action of $J$ on 8 points.

**Step 2.** Let $|\Omega| = n$. Then:

(i) $2^{n/3} < |\text{Syl}_p(G)|$.

(ii) $2^{4n} < |\text{Syl}_p(G)|$ if $p > 3$.

(iii) If $G$ is not 2-transitive on $\Omega$, then $n \leqslant 10^8$.

**Proof.** By Lemma 2.2, an element of order $p$ in $G$ fixes less than $n/2$ points of $\Omega$. Thus, an element of order $p$ in $G$ has at most $2n/3$ cycles on $\Omega$ if $p = 3$ and at most $.6n$ cycles on $\Omega$ if $p > 3$. It follows that the number of ordered pairs $(\langle g \rangle, \Delta)$, such that $\langle g \rangle \in \text{Syl}_p(G)$, $\Delta \leqslant \Omega$, and $g$ fixes $\Delta$, is at most $2^{2n/3} |\text{Syl}_p(G)|$ if $p = 3$ and at most $2^{6n} |\text{Syl}_p(G)|$ if $p > 3$. Since the number of such ordered pairs must exceed the number of subsets of $\Omega$, parts (i) and (ii) follow.

If $G$ is not 2-transitive, then part (i) and Lemma 2.4 imply (iii).

**Step 3.** $G$ does not have an elementary abelian regular normal subgroup.

**Proof.** Assume first that $G$ is not solvable. Let $n = q^m$ for a prime number $q$. Since $|GL(m, q)| \leqslant q^{m^2}$, Step 2 yields $2^{2m/3} < q^{m^2 + m}$, or

$$(\log 2/3) q^m < (m^2 + m) \log q.$$
Since \( G \) is nonsolvable, \( m \geq 2 \), and it is easy to see that \( (*) \) holds only if \( q^m \) is \( 3^2, 3^3, 3^4, 5^2, 7^2 \) or \( 2^m \) for some \( m \leq 7 \). In none of these cases is \( |GL(m, q)| \) divisible by the cube of the order of a simple group or by the order of a simple group which admits a coprime automorphism. Thus \( G = O^p(G) \) can’t have a nonsolvable chief factor.

Hence \( G \) is solvable and [8, Lemma 3.1] implies that \( p = 3, n = 8 \) and \( G \cong J \).

**Step 4.** \( G \) has a simple socle.

We adopt the notation of Lemma 2.3. Assume \( G \) does not have a simple socle. By Step 3, \( G \) falls under case (ii)(a) of Lemma 2.3 for \( m > 1 \), or under case (ii)(b) of Lemma 2.3.

Suppose first that \( G \) falls under case (ii)(a) with \( m > 1 \). Let \( \Omega_0 \) be the set permuted by \( G_0 \), so that \( \Omega \) may be identified with the cartesian product \( \Omega_0^m \). Let \( \alpha \) and \( \beta \) be distinct points in \( \Omega_0 \). For \( \Delta = \{1, 2, \ldots, m\} \), define \( \omega \in \Omega \) by the condition that \( \omega_i = \alpha \) for \( i \in \Delta \) and \( \omega_i = \beta \) for \( i \notin \Delta \). Define \( \eta \in \Omega \) by the condition that \( \eta_i = \alpha \) for all \( i \leq m \). Choose \( x \in G \) such that \( x \) has order \( p \) and \( x \) stabilizes the subset \( \{\omega, \eta\} \) of \( \Omega \). Then \( x \) must stabilize \( \Delta \) in its action on \( \{1, 2, \ldots, m\} \).

Since \( G \) acts transitively on \( \{T_1, \ldots, T_m\} \), the action of \( G \) on \( \{T_1, \ldots, T_m\} \) satisfies the hypotheses of Theorem 2.5. By induction on \( n \), it follows that \( m = 8 \) and \( p = 3 \). Thus \( T \) is a Suzuki group. The classification of the maximal subgroups of the Suzuki groups [23, Theorem 9] yields that \( n_0 \geq 8^2 + 1 = 65 \). Thus \( n > 10^8 \), contradicting Step 2(iii).

Next suppose (ii)(b) of Lemma 2.3 holds. It is possible that \( T \) admits a coprime automorphism of order \( p \). In this case \( |T| > |Sz(8)| = 29,120 \) and \( |T|^2 > 10^8 \). By Step 2, \( n = |T^{(k-1)l}| < 10^8 \), so \( k = 2, l = 1 \), and \( \text{Soc}(G) = T_1 \times T_2 \). Then \( G \leq \text{Aut}(T_1 \times T_2) \leq \text{Aut} T_1 \times \text{Aut} T_2 \). Since \( O^p(G) = G \), Lemma 1.3 implies that \( |G| = p |T|^2 \) and \( |\text{Syl}_p(G)| = |T|^2 \). Since \( |T| > 29,120 \), this contradicts Step 2(ii).

Thus we assume that \( T \) does not admit a coprime automorphism of order \( p \). If \( l > 1 \), our assumption that \( O^p(G) = G \) implies that an element of order \( p \) in \( G \) permutes the \( l \) blocks \( T_1 \times \cdots \times T_k, \ldots, T_{k(l-1)+1} \times \cdots \times T_{k^2} \) nontrivially. Hence \( l \geq p \). If \( l = 1 \), an element of order \( p \) in \( G \) permutes \( \{T_1, \ldots, T_k\} \) nontrivially, since \( O^p(G) = G \) and \( T \) does not admit a coprime automorphism of order \( p \). Hence \( k \geq p \). In either case \((k-1)l \geq p-1 \).

If \( p \geq 7 \), then \( n = |T^{(k-1)l}| > 60^6 > 10^8 \). If \( p = 5 \), then \( |T| \neq 60 \) and so \( n = |T^{(k-1)l}| > |T|^4 > 10^8 \). If \( p = 3 \), then \( |T| > |Sz(8)| = 29,120 \) and \( n = |T^{(k-1)l}| > |T|^2 > 10^8 \). Hence, \( n > 10^8 \) and we are done by Step 2(iii).

**Step 5.** Conclusion.

By Lemmas 2.3 and 1.2, \( |G| = p |T| \) and \( T \) admits a field automorphism of order \( p \).

First suppose \( T = Sz(q) \) for an odd power \( q \) of 2. Let \( \alpha \in \Omega \). Then \( n = |G : G_\alpha| = |T : T_\alpha| \). By [23, Theorem 9], \( n = |T : T_\alpha| \geq q^2 + 1 \), so Step 2(i) yields a contradiction. Hence, for the rest of this step we suppose \( T \neq Sz(q) \) and, in particular, \( p > 3 \).

Let \( L(T) \) be the lower bound for the smallest degree of a nontrivial projective representation of \( T \) given in [17, p. 419]. Thus in the notation of [17], \( L(T) \leq l(T, p) \) and \( L(T) \) is the number which actually appears in the table in [17, p. 419]. Let \( T = G(q) \) be an adjoint group of type \( G \) over the field of \( q \) elements. Since \( p > 3 \),
\( q \geq 32 \) and \( q \geq 243 \) if \( T \) has type \( 2G_2 \). If \( G \) is of exceptional Lie type then \( L(T) \geq 10^4 \) and \( |T| \leq L(T)^{10} \). Thus \( 2^{4L(T)} > |T| \), which contradicts Step 2(ii).

Hence, \( T \) is a classical group. If \( T \) has type \( A_m, B_m, C_m, D_m, 2A_m \) or \( 2D_m \) for \( m \geq 2 \), then it is immediate from [17, p. 419] that \( n > L(T) \geq (q^m - 1)/2 > 500 \). As \( q \geq 32 \), this implies that \( \log(2n) \geq m \log q > 3m \).

By the order formulas, \( |T| \approx \#IB_m(q) \approx q^{4m - 2} = \approx (q^m - 1)^2 > 3n^2 \log(3n) \).

By Step 2, \(|T| > 24n\). Thus \( 3n^2 \log(3n) > 24n \), contradicting \( n > 500 \).

Thus \( T = \text{PSL}(2, q) \). If \( q \) is odd then \( q \approx 243 \) and \( 2^{4L(T)} > |T| \). If \( q \) is even then \( 2^{4L(T)} = 2^{4(q-1)} > |T| \) for \( q > 32 \). Thus \( T = \text{SL}_2(32) \). Since \( G \) is primitive on \( \Omega \), \( T \triangleleft G \) is transitive on \( \Omega \). If \( T \) were not doubly transitive on \( \Omega \), then \( n > 2(q - 1) > 60 \). Since \( 2^{4(60)} > |T| \), \( T \) must be doubly transitive on \( \Omega \). By [4, Theorem 5.3], \( n = 33 \).

Now let \( x \in G = \text{Aut}(\text{SL}_2(32)) \) have order \( p = 5 \). Since \( 5 \mid |T| \) and \( T \) is transitive on \( \Omega \), it follows from [16, Lemma 13.8] that the fixed points of \( x \in \Omega \) form a single orbit under \( C_T(x) \approx S_5 \). Since the number of fixed points of \( x \) is congruent to \( 3 \) mod 5, \( x \) has 3 fixed points in \( \Omega \). Then no set of size 4 in \( \Omega \) is stabilized by an element of order 5 in \( G \). This contradiction completes the proof of Theorem 2.5.

**Corollary 2.6.** Suppose \( |G : O_p'(G)| = p \) and \( G = O_p'(G) \) for an odd prime \( p \). Suppose \( G \) acts faithfully and imprimitively on a finite vector space \( V \) of characteristic \( p \) so that each \( v \in V \) is centralized by a \( p \)-Sylow subgroup of \( G \). Then \( G \) is solvable.

**Proof.** Let \( V = V_1 \oplus \cdots \oplus V_n \) be an imprimitivity decomposition for the action of \( G \). Let \( G_1 \) be the stabilizer in \( G \) of \( V_1 \) and let \( C = \text{Core}_G(G_1) \). Let \( \Omega = \{1, 2, \ldots, n\} \). Let \( \Delta \leq \Omega \). By choosing a vector whose nonzero components correspond to \( \Delta \), we see that \( (G/C) \Omega \) satisfies the hypotheses of Theorem 2.5. As in [8, Lemma 3.2] \( C \) acts transitively on \( V_1 - \{0\} \). By Lemma 2.1, \( C \) is solvable. Thus \( G \) is solvable.

3. Let \( G \) and \( N \) be as in Proposition 0. Suppose that \( F^*(G/N) = F(G/N) \). Theorem 3.1 below shows that \( G \) must be solvable. The groups \( G \) and \( K \) below correspond to \( G/N \) and \( K/N \) in Proposition 0.

**Theorem 3.1.** Let \( |G : O_p'(G)| = p \) and \( G = O_p'(G) \) for an odd prime \( p \). Suppose that \( V \) is a faithful irreducible primitive \( GF(p)[G] \)-module. Suppose \( p \mid |C_G(x)| \) for all \( x \in V \). If \( F^*(G) = F(G) \), then \( G \) is solvable.

**Proof.** Let \( K = G' \). The hypotheses imply that \( K \) is the unique maximal normal subgroup of \( G \). The proof is carried out in a series of steps.

**Step 1.** There is a unique maximal normal abelian subgroup \( Z \) of \( G \). Furthermore, \( Z \) is cyclic and \( Z = Z(K) \).

**Proof.** As in Step 2 of [8, Theorem 2.3].

**Step 2.** Let \( E/Z \) be a chief factor of \( G \), let \( B = C_G(E) \) and let \( C = C_G(E/Z) \). Then:

(i) \( E/Z \) is an elementary abelian \( q \)-group for a prime \( q \) and \( E \leq K \).

(ii) \( BE = C \leq K \) and \( B \cap E = Z \).

(iii) \(|E/Z| = q^{2n}\) for an integer \( n \).
(iv) $K/C$ is isomorphic to a subgroup of the symplectic group $\text{Sp}(2n, q)$.

Proof. If $Z = K$, the conclusion of the theorem is satisfied, so we assume $Z < K$.

Since $K$ is the unique maximal normal subgroup of $G$, $E \leq K$. Since $E/Z$ is a chief factor of $G$ and $Z \leq Z(K)$, $E$ is nilpotent or $E/Z$ is a direct sum of isomorphic nonabelian simple groups. In either case $E \leq F^*(G)$ by Lemma 1.5. The hypotheses and Step 1 yield that $E$ is nilpotent but nonabelian. The rest of the proof follows that of Step 4 in [8, Theorem 2.3].

Step 3. There exist $E_1, \ldots, E_m \leq G$ such that:

(i) $E_i/Z$ is a chief factor of $G$ for each $i$.

(ii) $[E_i, E_j] = 1$ when $i \neq j$.

(iii) $M/Z = E_1/Z \times \cdots \times E_m/Z$, where $M$ is defined to be $E_1E_2\cdots E_m$.

(iv) $C_G(M) = Z$ and $C_{G/Z}(M/Z) = M/Z$.

Proof. As in Step 6 of [24, Theorem 3.3]. We remark that $M = F(G)$.

Step 4. Let $W \neq 0$ be an irreducible $Z$-submodule of $V$ and let $e = |M : Z|^{1/2}$.

Then $\dim V = te(\dim W)$ for an integer $t$.

Proof. As in Step 6 of [8, Theorem 2.3].

Step 5. Let $W$ be as in Step 4 and let $q_i$ be the prime divisor of $E_i/Z$. Then:

(i) $|Z|/(|W| - 1)$.

(ii) $q_i/(|W| - 1)$ for each $i$.

Proof. As in Step 14 of [24, Theorem 3.3].

Step 6. $|E/Z| = 4$ if and only if $C = K$. In this situation, $p = 3$.

Proof. As in Step 7 of [24, Theorem 3.3].

Step 7. Assume that $|E/Z| \neq 4$. Let $P \in \text{Syl}_p(G)$. Then:

(i) If $s$ is a prime divisor of $|F(G/C)|$, then $s \mid q^{2n} - 1$.

(ii) If $1 \neq S \in \text{Syl}_s(F(G/C))$ and if $C_S(P) = 1$, then $\dim C_{E/Z}(P) = 2n/p$.

(iii) If $G/C$ is solvable, then $1 \neq C_{G/C}(F(G/C)) \leq F(G/C) \leq K/C$.

(iii) If $F(G/C)$ is cyclic and $G/C$ is solvable, then $F(G/C) = K/C$ and $\dim C_{E/Z}(P) = 2n/p$.

Proof. As in Step 11 of [24, Theorem 3.3].

Step 8. Let $P \in \text{Syl}_p(G)$. Then:

(i) $|\text{Syl}_p(G)| \mid C_P(P) \mid V$.

(ii) $|\text{Syl}_p(G)| \geq |V|^{1/2}$.

Proof. As in Step 7 of [8, Theorem 2.3]. Note that we may replace the $\geq$ sign in (ii) by a $>$ sign, since $|V|^{1/2}$ is not a $p'$-integer.

Step 9. Let $q = 2, p = 3, n \neq 1$. Then:

(i) $n \geq 6$.

(ii) If $n \leq 7$ and $K/C$ is nonsolvable, then $|K/C| \leq 2^{28}$.

Proof. We first assume that $K/C$ is solvable. Suppose that $n = 5$. Since $p = 3$ and $|\text{Sp}(10,2)| = 3^6 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17 \cdot 31 \cdot 2^{25}$, it follows from Steps 2(iv) and 7(i),(iv) that $|F(G/C)| \mid 11 \cdot 31$ and $3 \mid 10$. Thus, $n \neq 5$ and similarly, $n \neq 2$. If $n = 4$, then $|F(G/C)| \mid 5^2 \cdot 17$ by Steps 2(iv) and 7(i). Since $C_{G/C}(F(G/C)) \leq K/C$, a 3-Sylow subgroup of $G$ must act nontrivially on the 5-Sylow subgroup of $F(G/C)$. Then Step
7(iii) yields a contradiction. Thus \( n \neq 4 \). If \( n = 3 \), then Step 7 yields that \( F(G/C) = K/C \) is cyclic of order 7 and \( G/C \) is a Frobenius group of order 21. It is easy to see that \( G/C \) has exactly two nonisomorphic faithful irreducible representations over \( GF(2) \), both of degree 3. Thus, \( E/Z \) is not an irreducible \( G/C \)-module and not a chief factor of \( G \), a contradiction.

Thus, we may assume that \( K/C \) is nonsolvable and \( 2 \leq n \leq 7 \). There exists an integer \( d \), and a chief factor \( R/T \) of \( G/C \) such that \( R/T \) is isomorphic to the direct product of \( d \) copies of a nonabelian simple group. Since \( |K/C| \) divides \( |\text{Sp}(14, 2)| \) and \( 3 \mid |K/C| \), it follows that \( |K/C| \) divides \( 2^{49} \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 43 \cdot 127 \). Hence, \( d \leq 2 \) and since \( O^3(G/C) = G/C \) we have \( d = 1 \). Hence \( R/T \) is isomorphic to a Suzuki group which admits an automorphism of order 3. By the order formulas for the Suzuki groups and the bound on \( |K/C| \) above, it follows that \( R/T = \text{Sz}(8) \) and \( K/R \) and \( T/C \) are both solvable. Since \( |\text{Out}(\text{Sz}(8))| = 3 \), we may replace \( R \) and \( T \) by \( K \) and \( Cd \), respectively, so that \( K/T = \text{Sz}(8) \) and \( T/C \) is solvable.

**Step 10.** Let \( q = 2 \), \( p \neq 3 \) and \( n \neq 1 \). Then:

(i) \( n \geq 4 \).

(ii) If \( n = 4 \), then \( K/C \) is solvable.

(iii) If \( n = 5 \) and \( K/C \) is nonsolvable, then \( p = 5 \) and \( K/C \cong \text{Sz}(32) \).

**Proof.** First suppose that \( n = 3 \). Since \( |K/C| \) divides \( |\text{Sp}(6, 2)| = 2^9 \cdot 3^4 \cdot 5 \cdot 7 \), the order formulas [9, p. 491] and Lemma 1.2 show that \( K/C \) involves no simple group which admits a coprime automorphism. Since \( G = O^p(G) \), it follows that \( K/C \) has no nonabelian simple chief factor. As \( G = O^p(G/C) \) and \( |K/C| \) is not divisible by the fifth power of the order of a nonabelian simple group, every chief factor of \( K/C \) must be solvable, so \( K/C \) is solvable. As \( p \mid |\text{Aut}(E/Z)| \) and \( p \neq 3 \), \( p \) must be 5, 7 or 31. By Step 7(i), 7(ii), \( O_3(G/C) \) is elementary abelian of order \( 3^4 \) and \( p = 5 \). Hence, an element of order \( p \) in \( G \) has no fixed points on \( 0_3(G/C) \). By Step 7(ii), 5 | 6, a contradiction. Thus \( n \neq 3 \). Similarly \( n \neq 2 \).

If \( n = 4 \), the arguments of the preceding paragraph show that \( K/C \) is solvable. If \( n = 5 \), the arguments of the preceding paragraph show that \( K/C \) is solvable or that a composition series for \( K/C \) has a unique nonsolvable factor, which is isomorphic to \( SL_2(32) \).

Thus, we may assume that \( n = 5 \), \( p = 5 \), and \( K/C \) has a unique nonsolvable composition factor, isomorphic to \( SL_2(32) \). If \( F(G/C) = 1 \), then \( F^*(G/C) = SL_2(32) \). Since \( C_{G/C}(F^*(G/C)) = F^*(G/C) \), by [3, Theorem 13.12], it follows that \( G/C \cong \text{Aut}(SL_2(32)) \) and \( K/C \cong SL_2(32) \).

We may assume that \( F(G/C) \neq 1 \). Under this assumption we will show that \( K/C \) acts faithfully on an extraspecial group of order \( 2^{11} \).
Since $E/Z$ is elementary abelian and $Z \leq Z(E)$, each commutator of $E$ has order 2 and $|E'| = 2$. An application of Fitting’s lemma to the coprime action of $F(G/C)$ on $O_{2}(E)/E'$ yields that $E/E' = (E_0/E') \times (Z/E')$ for some $E_0 \triangleleft G$. Since $E/Z$ is chief and $E$ is nonabelian, $E' = Z(E_0) = \Phi(E_0)$. Since $|E'| = 2$, $E_0$ is extraspecial of order $2^1$ and $K/C$ acts faithfully on $E_0$.

By [15, p. 357], $K/C$ is isomorphic to a subgroup of one of the two orthogonal groups $O^+ (10, 2)$ or $O^- (10, 2)$. By [15, p. 248], neither $|O^+ (10, 2)|$ nor $|O^- (10, 2)|$ is divisible by $|SL_2(32)|$. Thus $K/C$ has no nonsolvable composition factor, completing the proof of this step.

**Step II.** If $q^n$ is 5, 7, 11, 32 or 33, then $K/C$ is solvable. Also $q^n \neq 3$.

**Proof.** Suppose that $q^n$ is 5, 7, 11, 32 or 33 and $K/C$ is nonsolvable. Our assumption that $O^p(G) = G$ implies that $K/C$ involves a simple group which admits a coprime automorphism of order $p \neq q$, or that $|K/C|$ is divisible by the $p$th power of the order of a nonabelian simple group. Since $K/C$ is subgroup of $Sp(2n, q)$, the order formulas [9, p. 491] yield a contradiction.

If $q^n = 3$, then $|Aut(E/Z)| = 48$. Since $p$ divides $|Aut(E/Z)|$, this contradicts the hypothesis that $p \neq 2$ and $p \nmid |K|$.

**Step 12.** Conclusion.

We may choose an integer $k \geq 0$ such that $|E_i/Z| = 4$ if and only if $i \leq k$. We let $C_0 = K$ and define $C_i$ to be the centralizer in $C_{i-1}$ of $E_i/Z$, for $1 \leq i \leq m$. By Step 2(iv) applied to $E_i/Z$, $C_{i-1}/C_i$ is isomorphic to a subgroup of $Sp(2n_i, q_i)$ for each $i$. By Steps 6 and 3, $C_k = K$ and $C_m = M$. Since $|Sp(2n, q)| < q^{2n^2+n}$, we have $|\text{Syl}_p(G)| \leq |K|$ and

$$(1) \quad \log(|\text{Syl}_p(G)|) \leq \log|Z| + 2k \log 2 + \sum_{i=k+1}^{m} (2n_i^2 + 3n_i) \log q_i.$$ 

By Steps 4 and 8, we have

$$(2) \quad \log(|\text{Syl}_p(G)|) > 2^{k-1} \left( \prod_{i=k+1}^{m} q_i^{n_i} \right) \log |W|.$$ 

By Step 5, $q_i \leq |Z| < |W|$ for all $i$ and thus

$$(3) \quad 2k \log 2 + \sum_{i=k+1}^{m} (2n_i^2 + 3n_i) \log q_i > \left( -1 + 2^{k-1} \prod_{i=k+1}^{m} q_i^{n_i} \right) \log |W|$$

and

$$(4) \quad 1 + 2k + \sum_{i=k+1}^{m} (2n_i^2 + 3n_i) > 2^{k-1} \prod_{i=k+1}^{m} q_i^{n_i}.$$ 

We let $l = \sum_{i=k+1}^{m} n_i$, so that (4) yields $1 + 2k + 2l^2 + 3l > 2^{k+1} - 1$ and hence $k + l \leq 8$. If $l = 0$, then $K = C_m = M$ and $G$ is solvable. We may assume that $l \geq 1$.

Suppose first that $n_{k+1} = 1$. By Step 11, $q_{k+1} \geq 5$. Then (4) yields

$$6 + 2k + 2(l - 1)^2 + 3(l - 1) > 2^{k+1} \cdot 5 \cdot 3^{l-1}.$$ 

Hence $l \leq 2$. If $l = 2$, then $q_{k+2} \geq 5$ by Step 11, and (4) gives the contradiction $11 + 2k > 2^{k-1} + 5^2$. Thus $l = 1$ and $q_{k+1} = 5, 7$ or 11 by (4). Since $C_K = K$ and

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\[ C_{k+1} = M, \] it follows from Step 11 that \( G \) is solvable. We may assume that \( n_i \geq 2 \)
for all \( i > k \).

Suppose \( n_{k+1} = 2 \), so that \( q_{k+1} \geq 3 \) by Step 10. Now (4) becomes
\[ 15 + 2k + 2(l - 2)^2 + 3(l - 2) > 2^{k-1} \cdot 3^2 \cdot 2^{l-2} \]
and \( l \leq 5 \). But then Step 10 yields that \( q_i \geq 3 \) for \( i > k + 1 \) and (4) implies that
\[ 15 + 2k + 2(l - 2)^2 + 3(l - 2) > 2^{k-1} \cdot 3^l \]
and \( l \leq 3 \). By the last paragraph \( l = 2 \). Then \( q_{k+1} \) is 3 or 5 by inequality (4). If \( q_{k+1} = 5 \), then (3) and (4) yield that \( k = 0 \) and \( 5^{14} > |W|^{23/2} \), whence \( |W| < 11 \), contradicting Step 5. Thus \( q_{k+1} = 3 \). Since \( C_k = K \) and \( C_{k+1} = M \), Step 11 implies that \( K/C \) and \( G \) are solvable. We may assume that \( n_i \geq 3 \) for all \( i > k \).

Suppose that \( n_{k+1} = 3 \), so that \( q_{k+1} \geq 3 \) by Step 10. Inequality (4) yields that
\[ 28 + 2k + 2(l - 3)^2 + 3(l - 3) > 2^{k-1} \cdot 3^3 \cdot 2^{l-3} \]
and that \( l < 6 \). By the last paragraph \( l = 3 \). Then \( 28 + 2k > 2^{k-1}q_{k+1}^2 \) by (4). Hence
\( q_{k+1} = 3 \) and \( k \leq 1 \). Since \( C_k = K \) and \( C_{k+1} = M \), Step 11 implies that \( K/C \) and \( G \) are solvable. Hence \( n_i \geq 4 \) for all \( i > k \).

Suppose \( n_{k+1} = 4 \). Then
\[ 45 + 2k + 2(l - 4)^2 + 3(l - 4) > 2^{k-1}q_{k+1}^4 \cdot 2^{l-4} \]
by (4) and \( l < 8 \). By the last paragraph \( l = 4 \). Then \( q = 2 \) or \( 3 \) and \( k = 0 \) if \( q = 3 \). If \( q = 3 \), then inequality (3) becomes \( 3^{44} > |W|^{79/2} \), contradicting Step 5. Hence \( q = 2 \), and \( K/M \) and \( G \) are solvable. Hence \( n_i \geq 5 \) for all \( i > k \).

Now \( m = k + 1 \), since \( k < k + l \leq 8 \). If \( n_{k+1} = 8 \), then \( k = 0 \) and \( q_1 = 2 \) by (4). Inequality (3) becomes \( 2^{152} > |W|^{127} \), contradicting Step 5. Thus \( 5 \leq n_{k+1} \leq 7 \).

Suppose \( n_{k+1} = 6 \). By (4), \( k \leq 1 \). If \( k = 1 \), then \( q_{k+1} = 2 \) and (3) implies that \( 2^{92} > |W|^3 \), a contradiction. Thus \( k = 0 \) and (3) becomes \( 2^{90} > |W|^3 \). Since \( |W| \) is a power of \( p \), it follows that \( |W| = p \) and \( p = 3 \), \( 5 \) or \( 7 \). Since \( |Z| = |W| = p \), we have \( P \leq C_G(Z) \). By Step 2, \( |K/M| \) divides \( |\text{Sp}(12, 2)| \) and thus \( |K/Z| \) \( 2^{48} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 31 \). If \( p = 5 \) or \( 7 \), then \( P \) must fix and centralize an \( r \)-Sylow subgroup of \( K/Z \), whenever \( r = 5, 7, 13 \) or \( 17 \). Thus
\[ |\text{Syl}_p(G)| = |K/Z : C_{K/Z}(P)| \leq 2^{48} \cdot 3^8 \cdot 11 \cdot 31 \leq 2^{70} \]
Inequality (2) now has \( 2^{70} > |W|^{32} \) and so \( |W| < 5 \), a contradiction. Thus \( p = 3 = |W| \). By Step 9, \( |\text{Syl}_3(G)| \leq |K/Z| \leq 2^{40} \) and inequality (2) yields \( 2^{40} > 3^{32} \), a contradiction. Hence \( n_{k+1} \neq 6 \). Similarly \( n_{k+1} \neq 7 \).

Thus \( n_{k+1} = 5 \) and \( q_{k+1} = 2 \). By Steps 9 and 10, we may assume that \( p = 5 \) and \( K/M \cong SL_2(32) \). Since \( |K/M| < 2^{15} \), a modification of the argument used to derive inequality (3) shows that
\[ 2k(\log 2 + 25 \log 2 > (-1 + 16 \cdot 2^k) \log |W|) \]
This contradicts the fact that \( |W| \geq p = 5 \) and completes the proof of the theorem.
4. In this section consider the case that $V$ is a primitive $GF(p)[G/N]$-module
and $P(G/N) \neq F(G/N)$. The group $G$ in the statements of Propositions 4.1 and
4.2 corresponds to $G/O_p(N)$ in the setting of Proposition 0.

**Proposition 4.1.** Let $G = O^p(G)$ and $|G : O_p(G)| = p$ for an odd prime $p$. Suppose
that $L/W$ is a nonabelian simple chief factor of $G$. Suppose that $\mu \in \text{Irr}(W)$ is
invariant in $G$. Then some character in $\text{Irr}(G/\mu)$ has degree divisible by $p$.

**Proof.** By applying a character triple isomorphism [16, Theorem 11.28] we may
suppose that $\mu$ is linear and faithful, so that $W \leq Z(G)$. We must produce
$\chi \in \text{Irr}(L/\mu)$ such that $p | I_G(\chi)$. We will do this in a series of steps.

Step 1. We may assume that $L = L'$.

**Proof.** Since $(L/W)' = L/W$, we have $L = L'W$ and $L'/L' \cap W = L/W$. Sup-
pose there were a character $\chi' \in \text{Irr}(L | p_{L \cap W})$ such that $p | I_G(\chi')$. Then $L =$ $L'W \leq I_G(\chi')$, and since $L/L' \cong W/L' \cap W$ is cyclic, there exists $\chi_1 \in \text{Irr}(L)$
which extends $\chi'$. Since $\chi_1 | W$ extends $\chi'(1)\mu | L \cap W$, we may choose a linear character $\nu$ of $L/L' \cong W/L' \cap W$ so that $\chi_1 \nu \in \text{Irr}(L | \mu)$. Set $\chi = \chi_1 \nu$. Then $\chi$ extends $\chi'$,$\chi \in \text{Irr}(L | \mu)$, and $I_G(\chi) \leq I_G(\chi')$. Thus $p | I_G(\chi)$.

Step 2. $L/W$ is an adjoint group of Lie type.

**Proof.** This follows from $G = O^p(G)$ and Lemma 1.3.

Step 3. There is an automorphism $\sigma$ of $L/W$ and a prime number $r$ such that
$p | r | W$, $r | L | \sigma$, and $r | I_{L/W}(\sigma)$.

**Proof.** By Step 2, $L/W$ is an adjoint group over $GF(q^p)$ where $q$ is a prime
power; of course, $q$ is not a power of $p$. Let $G(q^p)$ be the simply connected group of
the same type, so that $G(q^p)$ is a central extension of $L/W$. Then there is a simply
connected and simple (in the sense of algebraic groups) algebraic group $G$ and an
endomorphism $\sigma$ of $G$ such that $G_{\sigma}$, the fixed point group, is $G(q)$ (see [22, pp.
82–83]). Let $\tau = \sigma^p$. Then $G_{\tau}$ is finite [21, 10.6] and $G_{\tau} = G(q^p)$ by [21, 11.16,
11.13]. Moreover, $G_{\tau}$ admits $\sigma$ and the restriction of $\sigma$ to $G_{\tau}$ has order $p$. Let
d$_{\tau} = d(q^p) = |Z(G_{\tau})|$. By Lemma 1.2, $d_{\tau} = |G_{\tau}/Z(G_{\tau})|$, so $p | |G_{\tau}|$. Consequently, $C_{G_{\tau}/Z(G_{\tau})}(\sigma) \cong G_{\sigma}/G_{\sigma} \cap Z(G_{\tau})$ and $\sigma$ induces an automorphism of order $p$ on $G_{\sigma}/Z(G_{\tau}) \cong L/W$.

By the order formulas [2, 11.16], $|G(q^p)|$ has the form $(q^p)^N \prod_{j \geq 1} (q^{m_j} - \epsilon_j)$
where each $\epsilon_j$ is a root of 1 of order 1, 2 or 3. Choose notation so that $m_1 \geq m_2 \geq \cdots$.
If $G$ is untwisted let $r$ be a primitive divisor of $q^{m_1} - 1$. If $G$ is of type $2B_2$, $2D_n$,
$3D_4$, $2G_2$, $2F_4$, $2E_6$, let $r$ be a primitive divisor of $q^{4p} - 1$, $q^{2np} - 1$, $q^{12p} - 1$
$q^{2p} - 1$, $q^{12p} - 1$, $q^{15p} - 1$, respectively. If $G$ is of type $2A_n$, let $r$ be a primitive
divisor of $q^{2p(n+1)} - 1$ if $n$ is even and $q^{2np} - 1$ if $n$ is odd. Since $p \geq 3$, the
exceptional cases $2^6 - 1$ and $p^2 - 1$ in Lemma 1.1 do not arise. Also $r > 3$.

By the order formulas, $r | |G(q^p)|$ and $r | d(q^p)$. Hence $r | |L/W|$. Let $M$ be the
Schur multiplier of $G(q^p)$. By [11, p. 280], any prime greater than 3 which divides
$|M|$ must divide $d(q^p)$. Since $r | d(q^p)$ and $L = L'$ it follows that $r | |W|$
.

Finally, we show that $r | |C_{L/W}(\sigma)|$. Note that $|C_{L/W}(\sigma)||G(q)|$ and $|G(q)|$ has
the form $(q^p)^N \prod_{j \geq 1} (q^{m_j} - \epsilon_j)$. If $G$ is untwisted, the definition of $r$ makes it clear that
$r | |G(q)||$. If $G$ has type $2D_4$, then any prime divisor of $|G(q)|$ divides $q^{12} - 1$, so
Step 4. Let $r$ be as in Step 3 and let $\alpha \in \text{Aut}(L/W)$ we have order $p$. Then $r \mid |C_{L/W}(\alpha)|$.  

Proof. Let $\sigma$ be as in Step 3. By Lemma 1.3 and Sylow’s theorem, $\langle \sigma \rangle$ and $\langle \alpha \rangle$ are conjugate in $\text{Aut}(L/W)$. Thus $r \mid |C_{L/W}(\alpha)|$.  

Step 5. Any two elements of order $p$ in $G$ fix the same irreducible characters of $L$.  

Proof. Let $g, h \in G$ have order $p$. We may assume that $gh \in \text{OP}(G)$. By Lemma 1.3, $gh$ induces an inner automorphism of $L/W$. Hence we may choose $x \in L$ so that $gh^{-1}x$ centralizes $L/W$. Since $W \leq Z(G)$, $gh^{-1}x$ also centralizes $W$. Therefore $\langle gh^{-1}x, L, L \rangle = [L, gh^{-1}x, L] = 1$. The three subgroup lemma yields $1 = [L, L, gh^{-1}x] = [L, gh^{-1}x]$, so $gh^{-1}x$ centralizes $L$. Hence $gh^{-1}$ induces an inner automorphism of $L$, and the result follows.  

Step 6. Let $g \in G$ be a fixed element of order $p$. Let $x \in L$ be a fixed element of order $r$. Suppose that $\langle x^g \rangle$ and $\langle x \rangle$ are conjugate in $L$. Then the conclusion of Proposition 4.1 holds.  

Proof. Since $p \mid |L|$, $g$ must normalize an $L$-conjugate $\langle y \rangle$ of $\langle x \rangle$. By Step 4, $g$ does not centralize $\langle y \rangle$. By Step 3, $\langle y, W \rangle = \langle y \rangle \times W$. Let $\nu$ be a faithful linear character of $\langle y \rangle$. Let $\theta = (\mu \times \nu)^\mu$, let $c = |C_L(y)|/|\langle y \rangle \times W|$, and let $e = \nu(y)$. By the definition of induced characters, $\theta(y) = c\sum_{y \in S} e^y$, where $S$ is a $p'$-subgroup of $\text{Gal}(\mathbb{Q}(e)/\mathbb{Q})$. Also  

$$
\theta^g(y) = \sum_{y \in S} e^{\beta y},
$$

where $\beta \in \text{Gal}(\mathbb{Q}(e)/\mathbb{Q})$ has order $p$. Since the primitive $r$th roots of 1 are linearly independent over $\mathbb{Q}$, it follows that $\theta^g(y) \neq \theta(y)$, so $\theta^g \neq \theta$.

Let $\chi$ be an irreducible constituent of $\theta$ such that $\chi^g \neq \chi$. Then $\chi \in \text{Irr}(L|\mu)$. By Step 5, $\chi$ is fixed by no element of order $p$ in $G$, so $p \mid |I_G(\chi)|$.  

Step 7. Let $g$ and $x$ be as in Step 6. Suppose that $\langle x^g \rangle$ and $\langle x \rangle$ are not conjugate in $L$. Then the conclusion of Proposition 4.1 holds.  

Proof. As in Step 6, $\langle x, W \rangle = \langle x \rangle \times W$. Let $\theta = (1_{\langle x \rangle} \times \mu)^L$. Then $\theta(x) = |\text{N}_L(\langle x \rangle) : \langle x, W \rangle| \neq 0$, while $\theta(x^g) = 0$. Hence $\theta \neq \theta^g$. The conclusion of Proposition 4.1 follows as in Step 6.  

Proposition 4.2. Let $G = \text{OP}(G)$ and $\mu \in \text{Irr}(W)$ be a character of order $p$ for an odd prime $p$. Suppose that $L/W$ is a nonabelian nonsimple chief factor of $G$. Suppose that $\mu \in \text{Irr}(W)$ is invariant in $G$. Then some character in $\text{Irr}(G|\mu)$ has degree divisible by $p$.  

Proof. As in the proof of Proposition 4.1, we may assume that $\mu$ is linear and faithful and that $L = L'$. We have $L/W = \prod_{i=1}^n S_i/W$, where the $S_i/W$ are isomorphic simple groups. The $S_i$ are transitively permuted by the action of $G$.  

Step 1. $L$ is the central product of the $S_i$.  

Proof. For $i \neq j$, $x \in S_i$, $y \in S_j$, the map $y \rightarrow [x, y]$ defines a homomorphism from $S_i$ to $W$ whose kernel contains $W$. Since $S_i/W$ is simple, this homomorphism must be trivial. Thus $[S_i, S_j] = 1$. Since $\cap S_i = W$, the result follows.

Step 2. Each $S_i$ is perfect.
PROOF Since \( L \) is perfect, \( L \) is the product of the \( S_i' \). Since \( G \) permutes the \( S_i' \) transitively, \( S_i' \cap W \) is the same group \( W_0 \) for all \( i \). Then \( L/W_0 \) is the direct product of the \( S_i'/W_0 \). Thus \( |L|=|W_0|\prod|S_i'/W_i'| \), so \( W_0=W \) and so \( S_i'=S_i \) for all \( i \).

To make the remaining steps of the proof clearer we introduce an “abstract” group \( S \), isomorphic to each \( S_i' \). Thus \( S \) is perfect and \( Z(S) \cong W \).

**Step 3.** Let \( \mu_0 \) be a faithful linear character of \( Z(S) \). Let \( A \) be the centralizer in \( \text{Aut}(S) \) of \( Z(S) \). Then \( A \) has more than one orbit on \( \text{Irr}(S \mid \mu_0) \).

**PROOF.** Suppose not. Then every character in \( \text{Irr}(S \mid \mu_0) \) has the same degree \( d \). Let \( m=|\text{Irr}(S \mid \mu_0)| \). By [16, p. 84], \(|S : Z(S)| = md^2 \).

By the argument in Step 5 of Proposition 4.1, any element of \( A \) which induces an inner automorphism of \( S/Z(S) \) lies in \( \text{Inn}(S) \), so that \( A/\text{Inn}(S) \) is isomorphic to a subgroup of \( \text{Out}(S/Z(S)) \). Therefore, \( m \) divides \( |\text{Out}(S/Z(S))| \).

Let \( r \) be as in Corollary 1.4, applied to \( S/Z(S) \). Since \( r \mid |S/Z(S)| \) and \( r \mid |\text{Out}(S/Z(S))| \), it follows that \( r \mid m \) and \( r \mid d \). Let \( R \in \text{Syl}_r(S) \). Since \( r \mid |Z(S)| \), \( R \times Z(S) \) is a subgroup of \( S \). Let \( \theta = (1_R \times \mu_0)^S \). Then \( r \mid \theta(1) \), which contradicts the fact that every irreducible constituent of \( \theta \) lies in \( \text{Irr}(S \mid \mu_0) \).

**Step 4.** Let \( U \) be the permutation group on \( \{S_1, \ldots, S_n\} \) induced by the action of \( G \). Then the conclusion of Proposition 4.2 holds if \( p > 3 \) or if \( U \cong J \).

**PROOF.** Since \( \text{Op}'(G) = G \) we have \( p \mid |U| \). By Theorem 2.5 we can choose \( \Delta \leq \{S_1, \ldots, S_n\} \) so that no element of order \( p \) in \( G \) fixes \( \Delta \). Fix isomorphisms \( f_i: S \to S_i \) so that the restrictions \( f_i: Z(S) \to W \) are the same function for all \( i \). Then \( \mu_0 = f_i^{-1}(\mu) \) is a well-defined linear character of \( Z(S) \). By Step 3, we may choose \( \chi, \psi \in \text{Irr}(S \mid \mu_0) \) to lie in different \( A \)-orbits. Define \( \eta \in \text{Irr}(L \mid \mu) \) by requiring that \( \eta\big|_{S_i} = (\eta(1)/\chi(1))f_i(\chi) \) for \( S_i \in \Delta \) and \( \eta\big|_{S_i} = (\eta(1)/\psi(1))f_i(\psi) \) for \( S_i \not\in \Delta \).

Suppose \( g \in G \) fixes \( \eta \). Then there exist indices \( i, j \) such that \( S_i \in \Delta, S_j \not\in \Delta \) and \( S_g = S_j \). Let \( c(g): S_i \to S_j \) be the isomorphism given by conjugating by \( g \). Then \( f_i c(g) f_j^{-1}: S \to S, f_i c(g) f_j^{-1} \in A \), and \( f_i c(g) f_j^{-1} \) takes \( \chi \) to \( \psi \), a contradiction.

**Step 5.** Conclusion.

Let \( S, A \) and \( U \) be as above. We may assume by Step 4 that \( U \cong J, p = 3, n = 8 \) and \( S/Z(S) \cong S_8(q) \) for some odd power \( q \) of 2. If \( q > 8 \) then \( S_8(q) \) has a trivial Schur multiplier, so \( L \) is the direct product of 8 copies of \( S_8(q) \) and \( \mu = 1 \). We can write \( \{S_1, \ldots, S_8\} \) as the disjoint union of 3 sets \( \Delta_1, \Delta_2, \Delta_3 \) so that no element of order 3 in \( G \) stabilizes all 3 sets. Now choose irreducible characters \( \chi_1, \chi_2, \chi_3 \) of \( S \cong S_8(q) \) whose degrees are all different. Define \( \chi \in \text{Irr}(L \mid \mu) \) to be the direct product whose \( j \)th component is \( \chi_j \) if \( S_j \in \Delta_j \). Then \( \chi \) is not fixed by an element of order 3 in \( G \).

Thus we may assume that \( S/Z(S) \cong S_8(8) \). By the argument in the preceding paragraph, we may assume that \( Z(S) \neq 1 \). Since \( S \) is perfect, \( Z(S) \) is cyclic, and the multiplier of \( S_8(8) \) is \( \mathbf{Z}_2 \times \mathbf{Z}_2 \) by [1, Theorem 2], we have \(|Z(S)| = 2 \). Since \( |\text{Out}(S_8(8))| = 3 \) and \( \text{Aut}(S_8(8)) \) has a trivial multiplier [1, Theorem 2], it follows that every automorphism of \( S \) is inner. Let \( \Delta_1, \Delta_2, \Delta_3 \) be as in the preceding paragraph. Since \(|S/Z(S)| = 29,120 \) is not the sum of two squares, we can choose distinct characters \( \chi_1, \chi_2, \chi_3 \in \text{Irr}(S \mid \mu) \). Fix isomorphisms \( f_i: S \to S_i \) for \( 1 \leq i \leq 8 \) and define \( \chi \in \text{Irr}(L \mid \mu) \) by the condition that \( \chi\big|_{S_i} = (\chi(1)/\chi_i(1))f_i(\chi_i) \) for \( S_i \in \Delta_i \).
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Since \( A = \text{Inn}(S) \), it follows that \( \chi \) is fixed by no element of order 3 in \( G \). This completes the proof of Proposition 4.2.

**PROOF OF THEOREM A.** Let \( G \) be a minimal counterexample to Theorem A. Then \( G \) is nonsolvable by [8, Theorem A] and satisfies conditions (1)–(6) of Proposition 0. Let \( V \) be as in Proposition 0.

We may apply Corollary 2.6 and Theorem 3.1 to the action of \( G/N \) on \( V \) to deduce that \( V \) is a primitive \( GF(p)[G/N] \)-module and \( F^*(G/N) \neq F(G/N) \). By Lemma 1.5, there is a perfect subgroup \( \bar{L} \) of \( G/N \) such that \( \bar{L}/Z(\bar{L}) \) is a nonsolvable chief factor of \( G/N \). Any prime divisor of \( |Z(\bar{L})| \) divides \( |M(S)| \), the order of the Schur multiplier of a nonabelian simple composition factor of \( \bar{L} \). By Lemma 1.2 and the table in [11, p. 280], we conclude that \( p \) exceeds every prime divisor of \( |Z(\bar{L})| \). Since \( V \) is a primitive \( GF(p)[G/N] \)-module, \( Z(\bar{L}) \) is cyclic, and thus every element of order \( p \) in \( G \) centralizes \( Z(\bar{L}) \).

Let \( L \) and \( W \) be the inverse images in \( G/O_p(N) \) of \( \bar{L} \) and \( Z(\bar{L}) \). We identify the central cyclic subgroup \( Z \) of \( G \) with its image in \( G/O_p(N) \). Thus \( W \) is a normal abelian subgroup of \( G/O_p(N) \), and \( W/Z \cong Z(\bar{L}) \).

Any element of order \( p \) in \( G/O_p(N) \) centralizes both \( Z \) and \( W/Z \cong Z(\bar{L}) \). As \( p \mid |W| \) and \( G = O^p(G) \), it follows that \( W \preceq Z(G/O_p(N)) \). Thus, any linear character \( \mu \) of \( W \) which extends \( \lambda \) is invariant in \( G/O_p(N) \). We may apply Proposition 4.1 or 4.2 to \( G/O_p(N) \), \( L \), \( W \) and \( \mu \) to obtain \( \chi \in \text{Irr}(G/O_p(N) | \mu) \) such that \( p \mid \chi(1) \). Since \( \chi \) may be viewed as a character in \( \text{Irr}(G | \lambda) \), this contradicts (6) in Proposition 0 and completes the proof of Theorem A.

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