DISCONTINUOUS TRANSLATION INVARIANT FUNCTIONALS

BY

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Dedicated to Professor Shigeki Yano on his sixtieth birthday

ABSTRACT. Let $G$ be an infinite σ-compact locally compact group. We shall study the existence of many discontinuous translation invariant linear functionals on a variety of translation invariant Fréchet spaces of Radon measures on $G$. These spaces include the convolution measure algebra $M(G)$, the Lebesgue spaces $L^p(G)$, where $1 \leq p \leq \infty$, and certain combinations thereof. Among other things, it will be shown that $C(G)$ has many discontinuous translation invariant functionals, provided that $G$ is amenable. This solves a problem of G. H. Meisters.

Let $G$ be a locally compact group. In the present paper, we shall consider various translation invariant spaces of functions and measures on $G$, and prove the existence of discontinuous invariant functionals on such spaces. Among other things, it will be shown that $C(G)$ admits “many” discontinuous left-invariant functionals, provided that $G$ is an infinite, σ-compact, locally compact amenable group (our hypotheses are a little weaker than amenability). This resolves the problem of G. H. Meisters [11] as to whether there exists a discontinuous invariant functional on $C(\mathbb{R})$, where $\mathbb{R}$ denotes the circle group. As far as the compact groups are concerned, the problems for $C(G)$ and for $L^\infty(G)$ are equivalent (see Remark (IV) stated below). As some basic references on this subject, we refer to E. Granirer's papers [3, 4] on invariant means, the nice survey article [14] by Meisters, and G. S. Woodward's paper [22] on discontinuous invariant functionals on a variety of invariant spaces.

Let $G$ be an arbitrary locally compact (LC) group. All of the Lebesgue spaces $L^p(G)$, $1 \leq p \leq \infty$, are taken with respect to a fixed left Haar measure $\lambda_G$ on $G$. If $G$ is compact, we normalize $\lambda_G$ so that $\lambda_G(G) = 1$. Let $C(G)$ and $C_c(G)$ denote the Banach space of all continuous bounded (complex-valued) functions on $G$, and the subspace of all members of $C(G)$ with compact supports, respectively. We denote by $C_0(G)$ the uniform closure of $C_c(G)$ in $C(G)$. Thus the dual of $C_0(G)$ may be identified with the convolution measure algebra $M(G)$ of $G$ (cf. Hewitt and Ross [7]).

A net $(f_\alpha)$ in $C_c(G)$ converges to the zero function if and only if (i) $f_\alpha \to 0$ uniformly on $G$ and (ii) the $f_\alpha$'s are eventually supported by a fixed compact set (depending on the net). This topology for $C_c(G)$ is different from the uniform topology, and will be used throughout the paper unless otherwise mentioned. The continuous dual of $C_c(G)$ will be denoted $C'_c(G)$. An element of $C'_c(G)$ is often called
a Radon measure on \( G \) (cf. [16]). If a linear subspace \( X \) of \( C'_c(G) \) is equipped with a topological-linear-space topology which is stronger than the weak topology \( \sigma(X, C_c(G)) \), we call \( X \) a \emph{topological linear space on} \( G \). Normed, Banach, and Fréchet spaces on \( G \) are defined in a similar fashion. Notice that if \( X \) and \( Y \) are two Fréchet spaces on \( G \) such that \( X \subset Y \), then the identity map of \( X \) into \( Y \) is continuous by the closed graph theorem. We shall regard all of the spaces \( L^p(G) \subset L^1_{loc}(G) \) and \( M(G) \) as linear subspaces of \( C'_c(G) \). In particular, if \( f \in L^1_{loc}(G) \), then

\[
\langle \phi, f \rangle = \langle \phi, f \lambda_G \rangle = \int \phi(x) f(x) \, d\lambda_G(x) \quad (\phi \in C'_c(G)).
\]

Similarly, \( M(G) + L^p(G) \) denotes the algebraic sum of \( M(G) \) and \( L^p(G) \) in \( C'_c(G) \).

For each \( \phi \in C'_c(G) \) and \( a \in G \), define \( \phi \) and \( \phi_a \in C'_c(G) \) by setting \( \phi(x) = \phi(ax) \) and \( \phi_a(x) = \phi(xa) \) for all \( x \in G \). Let \( \delta_a \) denote the unit point-measure at \( a \). Thus we have

\[
\langle \phi, \delta_a \ast \mu \rangle = \langle \phi, \mu \rangle \quad \text{and} \quad \langle \phi, \mu \ast \delta_a \rangle = \langle \phi_a, \mu \rangle
\]

for \( \phi \in C'_c(G) \) and \( \mu \in C'_c(G) \). Here follow some other definitions which we shall need later.

(a) A subset \( X \) of \( C'_c(G) \) is left invariance if \( \delta_a \ast X \subset X \) for all \( a \in G \). Right- (or two-sided) invariance is defined similarly.

(b) For a subset \( X \) of \( C'_c(G) \), we define

\[
T_r(X) = \text{sp}\{ f - \delta_a \ast f : f \in X \& a \in G \}, \quad T_r(X) = \text{sp}\{ f - f \ast \delta_a : f \in X \& a \in G \},
\]

and \( T(X) = T_r(X) + T_r(X) \), where “sp” stands for linear span.

(c) If \( X \) and \( Y \) are two topological spaces such that \( X \subset Y \) set-theoretically, we shall write “\( \subset \)” for the closure in the smaller space \( X \) and “\( \sim \)” for the closure in the larger space \( Y \).

(d) For a topological linear space \( X \), \( X' \) denotes the continuous dual of \( X \). If \( X \) is a normed space and \( r > 0 \), we define \( B_r(X) = \{ f \in X : \| f \| \leq r \} \).

In the sequel, \( G \) will always denote an infinite \( \sigma \)-compact LC group, unless otherwise mentioned. The following two lemmas are central to all of our results. Their natural “two-sided” versions are obvious and will be left to the reader.

**Lemma 1.** Let \( X \) and \( Y \) be a left-invariant Fréchet space and a left-invariant Banach space on \( G \), respectively. Suppose that

(i) \( X \subset Y \) and \( T_r(X) \subset \text{sp}(F) + T_r(Y) \)

for some countable subset \( F \) of \( Y \). Then there exist a natural number \( N \) and a compact subset \( K \) of \( G \) such that \( W_{N,K} \) contains a relative neighborhood of \( 0 \) in \( T_r(X) \) (with respect to \( X \)), where

(ii) \( W_{N,K} = \{ \sum_{k=1}^N (h_k - \delta_{a_k} \ast h_k) : h_k \in B_N(Y) \& a_k \in K \forall k \} \).

**Proof.** Let \( f_1, f_2, \ldots \) be an enumeration of the elements of \( F \). Since \( G \) is \( \sigma \)-compact, we can write \( G = \bigcup_{n=1}^\infty K_n \), where the \( K_n \)'s are compact subsets of \( G \) with \( K_n \subset K_{n+1} \) for all \( n \geq 1 \). Given a natural number \( n \), define

\[
V_n = \left\{ \sum_{j=1}^n c_j f_j : c_j \in \mathbb{C} \& |c_j| \leq n \forall j \right\}.
\]
Likewise, define \( W_n \) by (ii) with \( N \) and \( K \) replaced by \( n \) and \( K_n \), respectively. Notice that \( V_n \) is compact. Since the algebraic sum of a compact set and a closed set is closed in any topological linear space, it follows that \( V_n + W_n^- \) is closed in \( Y \).

By (i), the Fréchet space \( T_j(X)^- \) is covered by the sets \( V_n + W_n^- \). Since \( T_j(X)^- \) is contained in the Banach space \( Y \) on \( G \), it follows from the closed graph theorem that each \( (V_n + W_n^-) \cap T_j(X)^- \) is closed in \( T_j(X)^- \). We, therefore, infer from the Baire category theorem that at least one of the sets \( V_n + W_n^- \) contains a nonempty (relative) open subset of \( T_j(X)^- \). Accordingly, there exists a natural number \( r \) such that \( V_r + W_r^- \) contains a neighborhood \( U \) of \( 0 \in T_j(X)^- \).

Now we claim that \( U \subset W_n^- \) for some \( N \gg r \). To confirm this, first notice that the union of all \( W_n^- \) forms a linear subspace of \( Y \). Call this union \( S \) and put \( F_p = \{ f_1, \ldots, f_p \} \) for \( p = 1, 2, \ldots, r \). If \( F_p \subset S \), then \( F_p \subset W_m^- \) for some \( m \); hence \( V_r \subset W_n^- \) for some \( n > m \) by (1) and (ii); hence \( V_r + W_r^- \subset W_r^- \subset W_n^- \), where \( N = n + r \). Thus \( F_p \subset S \) implies \( U \subset W_n^- \) for some \( N \). So assume that \( F_p \) is not contained in \( S \). Then we may suppose that, for some \( p \leq r \), \( F_p \) is a maximal subset of \( F_r \) which is linearly independent modulo \( S \). After repeating a similar argument as above, we then obtain

\[
U \subset V_r + W_r^- \subset \text{sp}(F_p) + W_n^-
\]

for some \( N \gg r \). Notice that \( T_j(X) \subset T_j(Y) \subset S \); hence

\[
T_j(X) \cap \left[ \text{sp}(F_p) + W_n^- \right] = T_j(X) \cap W_n^-
\]

by the linear independence of \( F_p \) modulo \( S \). We infer from (2) and (3) that \( U \cap T_j(X) \subset W_n^- \). Since \( [T_j(X)^-] \cap W_n^- \) is closed in \( T_j(X)^- \), it follows that \( U \subset W_n^- \), as desired.

**Lemma 2.** Let \( X \) be a left-invariant Fréchet space on \( G \), and let \( Y \) be a left-invariant Banach space on \( G \) such that

\[
(C_0) \quad X \subset Y \text{ and } T_j(X)^- \subset \text{sp}(F) + T_j(Y)
\]

for some countable subset \( F \) of \( Y \). Suppose

\[
(C_1) \text{ the unit closed ball } B_1(Y) \text{ is } \sigma(Y, C_c)-\text{compact;}
\]

\[
(C_2) \text{ the orbit } \{ \delta_x \ast f : x \in G \} \text{ of each } f \in X \text{ is bounded in } X;
\]

and also (if \( G \) is noncompact)

\[
(C_3) \text{ for which } f \in X, \lim_{x \to \infty} \delta_x \ast f = 0 \text{ in } \sigma(X, C_c).
\]

Let \( W_{N,K} \subset Y \) be as in the conclusion of Lemma 1. Then:

(i) \( W_{N,K} \) is \( \sigma(Y, C_c)-\text{compact}; \)

(ii) if \( G \) is noncompact, \( W_{2N,K} \) contains a neighborhood of \( 0 \in X \);

(iii) if \( G \) is compact, there exists a neighborhood \( V \) of \( 0 \in X \) such that \( h - \lambda_G \ast h \in W_{2N,K} \) for all \( h \in V \).

**Proof.** For each open subset \( U \) of \( G \) with compact closure, let \( C_U \) denote the subspace of all \( \phi \in C_c(G) \) such that \( \text{supp} \phi \subset U^- \). Then \( C_U \) forms a Banach space with respect to the uniform norm and the imbedding of \( C_U \) into \( C_c(G) \) is continuous. Since \( Y \) is a Banach space contained in \( C_c(G) \), the closed graph theorem assures that
there exists a finite constant \( D = D_U \) such that
\[
|\langle \phi, h \rangle| \leq D \|\phi\|_{\infty} \cdot \|h\|_Y \quad (\phi \in C_U, h \in Y).
\]

Now we claim that the mapping
\[
(x, h) \to \delta_x * h: G \times B_1(Y) \to C'_c(G)
\]
is continuous when \( B_1(Y) \) is equipped with the relative weak topology \( \sigma(Y, C'_c) \big| B_1(Y) \). Suppose that \((x_\alpha, h_\alpha)\) is a convergent net in \( G \times B(Y) \) with limit \((x, h)\). We need to prove that \( \lim \delta_{x_\alpha} * h_\alpha = \delta_x * h \) in \( C'_c(C) \). Given \( \phi \in C'_c(G) \), choose an open subset \( U \) of \( G \) such that \( \overline{\operatorname{supp}(\phi)} \subseteq U \). Since \( x_\alpha \to x \) in \( G \), it is obvious that \( x_\alpha \in U \) for eventually all \( \alpha \)’s. It follows from (4) that
\[
|\langle \phi, \delta_{x_\alpha} * h_\alpha \rangle - \langle \phi, \delta_x * h \rangle| = |\langle \phi, \delta_{x_\alpha} - \delta_x \rangle * h_\alpha \rangle| = |\langle \phi, \delta_{x_\alpha} - \delta_x \rangle| * h_\alpha \rangle| \leq \|\phi\|_{\infty} \cdot \|\delta_{x_\alpha} - \delta_x\| \cdot \|h_\alpha\|_Y \leq D \|\phi\|_{\infty} \|\delta_{x_\alpha} - \delta_x\|_\infty \cdot \|h_\alpha \|_Y
\]
for eventually all \( \alpha \)’s. Since \( \delta_{x_\alpha} \to \delta_x \) uniformly and \( h_\alpha \to h \) in \( \sigma(Y, C'_c) \), the last inequalities confirm our claim.

Put \( E_K = \{ h - \delta_x * h : h \in B_1(Y) \} \). From condition \((C_1)\) and the claim just established, we infer that \( E_K \) is \( \sigma(Y, C'_c) \)-compact whenever \( K \) is a compact subset of \( G \). The set \( W_{N,K} \) in the conclusion of Lemma 1 is the \( N \)-fold algebraic sum of \( NE_K \), and is therefore \( \sigma(Y, C'_c) \)-compact. This confirms (i).

Now we shall prove (ii) and (iii) assuming that \( (C_1) \ast \)
\[
W_{N,K} = \left[ Y + \operatorname{sp}(\lambda_G) \right] \cap W_{N,K}^* \ast
\]
where \( W_{N,K}^* \ast \) denotes the \( \sigma(C'_c, C_c) \)-closure of \( W_{N,K} \) in \( C'_c(G) \). Of course, \((C_1) \ast \) is weaker than \((C_1)\) but is strong enough to guarantee the norm-closedness of \( W_{N,K} \) in \( Y \). Now choose and fix a convex neighborhood \( V \) of \( 0 \in X \) such that \( \overline{V} \cap T_f(X)^{-1} \subseteq W_{N,K} \). Each \( f \in X \) has bounded orbit in \( X \) by \((C_2)\). Accordingly there exists a natural number \( n = n_f \) such that \( n^{-1}(\delta_x * f) \in V \) for all \( x \in G \). Hence
\[
n^{-1}(f - \delta_x * f) \in (V - V) \cap T_f(X) \subseteq W_{N,K} \quad (x \in G).
\]
Suppose \( G \) is noncompact. Then \((C_3) \), combined with (6) and \((C_1) \ast \), yields \( n^{-1}f \in W_{N,K} \). Since \( f \in X \) is arbitrary, it follows that \( X \subseteq \bigcup_{n=1}^{\infty} \bigcup_{n=1}^{\infty} W_{N,K} \). But \( W_{N,K} \) is norm-closed in \( Y \) by \((C_1) \ast \), and \( X \) is a Fréchet space contained in the Banach space \( Y \); hence \( X \cap W_{N,K} \) is closed in \( X \) (cf. the proof of Lemma 1). It follows from the Baire category theorem that \( X \cap W_{N,K} \) has nonempty interior in \( X \). Since \( W_{N,K} - W_{N,K} \subseteq W_{2N,K} \), we conclude that \( W_{2N,K} \) contains a neighborhood of \( 0 \) in \( X \), which established (ii).

Finally assume \( G \) is compact. Then \( C'_c(G) = C(G) \), \( C'_c(M(G)) = M(G) \), and \((C'_c, C_c) \ast \) is nothing but the weak-* topology of \( M(G) \). Choose a net \((\tau_\alpha)\) of probability measures on \( G \), each with finite support, such that \( \tau_\alpha \to \lambda_G \) weak-* . Then \( n^{-1}(f - \tau_\alpha * f) \in W_{N,K} \) for each \( \alpha \) by (6) and the convexity of \( V \). Passing to the weak-* limit and making use of \((C_1) \ast \), we obtain \( n^{-1}(f - \lambda_G * f) \in W_{N,K} \). Notice that \( \lambda_G * X \subseteq \lambda_G * M(G) = \operatorname{sp}(\lambda_G) \). Since \( f \in X \) is arbitrary, it follows that at once that
\[
X \subseteq \bigcup_{n=1}^{\infty} \bigcup_{n=1}^{\infty} \left\{ c \lambda_G : |c| \leq 1 \right\} + W_{N,K}.
\]
Using \((\text{C}_1)^*\) and (7), we repeat a similar argument as in the last paragraph to conclude that \(\text{sp}(\lambda_G) + W_{2N,K} \) contains a neighborhood \(V'\) of \(0 \in X\). Thus each \(h \in V'\) has a representation of the form \(h = c\lambda_G + \mu\) for some \(c \in \mathbb{C}\) and some \(\mu \in W_{2N,K}\). But it is obvious that \(\langle 1, \mu \rangle = 0\) for all \(\mu \in W_{2N,K} \subset T_f(M(G))\). Therefore \(\langle 1, h \rangle = c\langle 1, \lambda_G \rangle + \langle 1, \mu \rangle = c\), and

\[h - \lambda_G \ast h = h - \langle 1, h \rangle \lambda_G = \mu \in W_{2N,K},\]

as desired.

We abbreviate left- [two-sided] invariant linear functional as LILF [TILF]. It is evident that Lemma 2 as well as Lemma 1 has a natural “two-sided” analogue. All that follows are corollaries to Lemma 2.

**Theorem 1.** There exist uncountably many TILF’s on \(M(G)\) whose restrictions to one of \(M_a(G)\) or \(M_d(G)\) are linearly independent modulo the continuous functionals on \(M_a(G)\) or \(M_d(G)\).

**Proof.** The symbols \(M_a(G)\) and \(M_d(G)\) denote the absolutely continuous and discrete measures on \(G\), respectively. Notice that \(M_a(G)\) is a closed two-sided ideal while \(M_d(G)\) is a closed two-sided invariant subalgebra of \(M(G)\).

By Zorn’s lemma, \(T(M_a(G))^{-}\) contains a subset \(F_a\) which is maximal with respect to the linear independence modulo \(T(M_a(G))\). Similarly there exists a subset \(F_d\) of \(T(M_d(G))^{-}\) which is maximal with respect to the linear independence modulo \(T(M_d(G))\). If \(G\) is discrete, then \(M_a(G) = M_d(G) = M(G)\), so we shall choose \(F_a = F_d\). If \(G\) is nondiscrete, it is obvious that \(F_a \cap F_d = \emptyset\) and that \(F_a \cup F_d\) is linearly independent modulo \(T(M(G))\). Notice that

\[T(M_a(G))^{-} = \text{sp}(F_a) + T(M_a(G)) \subset \text{sp}(F_a) + T(M(G)),\]

and similarly for \(M_d(G)\) and \(F_d\).

Now suppose that \(X\) is any two-sided invariant, closed subspace of \(M(G)\) and that there exists a countable subset \(F\) of \(M(G)\) such that \(T(X)^{-} \subset \text{sp}(F) + T(M(G))\). We then claim that the unit ball \(B(X)\) of \(X\) is not weak-* dense in \(B(M(G))\). Indeed, by a natural “two-sided” version of Lemma 2, there exists a weak-* compact subset \(W\) of \(T(M(G))\) such that: if \(G\) is noncompact, then \(B(X) \subset W\); and if \(G\) is compact, then \(\mu - \lambda_G \ast \mu \in W\) for all \(\mu \in B(X)\). First assume \(G\) is noncompact. Since \(X\) is weak-* compact, we then have \(B(X)^{*} \subset W\), where \(B(X)^{*}\) denotes the weak-* closure of \(B(X)\) in \(M(G)\). Since \(W \subset T(M(G))\), we conclude that \(\langle 1, \mu \rangle = 0\) for all \(\mu \in B(X)^{*}\); hence \(B(X)\) is not weak-* dense in \(B(M(G))\). Next assume that \(G\) is compact (and infinite). Then \(\mu - \lambda_G \ast \mu \in W\) for all \(\mu \in B(X)^{*}\), again by the weak-* compactness of \(W\). Taking discrete parts, we obtain \(\mu_d \in (W_d)^{-} \subset T(M_d(G))\) for all \(\mu \in B(X)^{*}\). A similar argument as above therefore shows that \(B(X)\) is not weak-* dense in \(B(M(G))\).

By the Hahn-Banach convexity theorem, both \(B_1(M_a(G))\) and \(B_1(M_d(G))\) are weak-* dense in \(B_1(M(G))\). It follows from the above paragraph that neither \(F_a\) nor \(F_d\) is countable. In order to obtain uncountably many TILF’s on \(M(G)\) with the desired properties, it is sufficient to extend \(\{\mu + T(M(G)) : \mu \in F_a \cup F_d\}\) to a Hamel base of the quotient space \(M(G)/T(M(G))\). (Notice here that the dimension of \(T(M)^{-} / T(M)\) plays an important role.) This completes the proof.
For two normed spaces $X$ and $Y$ on $G$, we define a norm on $X \cap Y$ by setting $\|f\| = \|f\|_X + \|f\|_Y$ for $f \in X \cap Y$. Notice that if both $X$ and $Y$ are Banach spaces on $G$, then so is $X \cap Y$.

**Theorem 1.** Suppose $G$ is noncompact (and $\sigma$-compact). Let $X \subset L^1(G)$ be a left-invariant Fréchet space on $G$ whose topology is induced by countably many, left-invariant seminorms [e.g., $X = L^1 \cap C_0$]. If $X$ contains an element with nonzero Haar integral, then there exist uncountably many TILF's on $L^1(G)$ whose restrictions to $X$ are linearly independent modulo $X'$.

**Proof.** Unfortunately the closed unit ball of $L^1(G)$ is not $\sigma(C_c, C_r)$-compact, unless $G$ is discrete. So we replace $L^1(G)$ by $Y = M(G)$. Since the topology of $X$ is induced by left-invariant seminorms, each element of $X$ has bounded (left) orbit in $X$. Since $X \subset L^1(G) \subset M(G)$ and $G$ is noncompact, $X$ also satisfies condition $(C_3)$ in Lemma 2.

Let $F$ be a subset of $T_f(X)$ which is maximal with respect to the linear independence modulo $T_f(X)$. Let $F_0$ be a subset of $F$ which is maximal with respect to the linear independence modulo $T(M(G))$. Then we have

$$T_f(X)^{-} = \text{sp}(F) + T_f(X) \subset \text{sp}(F_0) + T(M(G)).$$

Therefore the proofs of Lemmas 1 and 2 apply to the present situation *mutatis mutandis*. Since $X$ contains an element with nonzero Haar integral by the hypotheses, it follows from Lemma 2 that the set $F_0$ cannot be countable. This completes the proof.

**Remarks.** (I) Let $G$ be an arbitrary LC group. If there exists a closed normal subgroup $H$ of $G$ such that $G/H$ is infinite and $\sigma$-compact, then $M(G)$ has uncountably many TILF's whose restrictions to $M_a(G)$ are linearly independent modulo $[M_a(G)]'$. The proof is routine.

(II) It is possible to prove that if $G$ is a noncompact LCA group, then every translation invariant linear operator on $L^1(G)$ is continuous. (We omit the proof although it is nontrivial.) Thus the study of invariant functionals and the study of invariant operators are somewhat different (cf. [9]).

**Theorem 2.** Suppose $1 < p \leq \infty$. Then the following assertions are equivalent.

(i) Each LILF on $L^p(G)$ is either a constant multiple of the Haar integral (if $G$ is compact) or the zero functional (if $G$ is noncompact).

(ii) The dimension of the quotient space $L^p(G)/T_f(L^p(G))$ is at most countable.

(iii) There exist a natural number $N$, a finite constant $C$, and a compact subset $K$ of $G$ such that each $f \in L^p(G)$ has a representation of the form

$$f = a + \sum_{k=1}^{N} \left(f_k - \delta_{x_k} * f_k\right),$$

where $a \in C$ [$a = 0$ if $G$ is noncompact], $x_k \in K$, $f_k \in L^p(G)$, and $\|f_k\|_p \leq C\|f\|_p$ for all $k$'s.

**Proof.** The space $X = Y = L^p(G)$ is a left-invariant Banach space on $G$ and possesses the three properties $(C_1)$, $(C_2)$ and $(C_3)$ in Lemma 2. The only exceptional
case arises when \( G \) is noncompact and \( p = 0 \), because then \( L^\infty(G) \) does not satisfy (C3). However, this difficulty may be circumvented as follows.

Suppose that \( G \) is noncompact (and \( \sigma \)-compact) and that \( T_\delta(C_0(G))^c \subset \text{sp}(F) + T_\delta(L^\infty(G)) \) for some countable subset \( F \) of \( L^\infty(G) \). It is obvious that \( X = C_0(G) \) satisfies conditions (C2) and (C3) in Lemma 2. It follows from Lemma 2 with \( Y = L^\infty(G) \) that there exists a natural number \( N \) and a compact subset \( K \) of \( G \) such that \( B_1(C_0(G)) \) is contained in the weak-* compact set \( W_{N,K} \) of \( T_\delta(L^\infty(G)) \). But then \( B_1(L^\infty(G)) \subset W_{N,K} \), since \( B_1(C_0(G)) \) is weak-* dense in \( B_1(L^\infty(G)) \). We have thus proved that (ii) \( \Rightarrow \) (iii) holds even in the exceptional case under discussion. This completes the proof.

**Theorem 2**. Let \( G \) be a \( \sigma \)-compact, noncompact, amenable LC group, and let \( 1 < p \leq \infty \). Then there exist uncountably many LILFs on \( L^1(G) + L^p(G) \) whose restrictions to \( L^p \cap C_0(G) \) are linearly independent modulo the continuous linear functionals on \( L^p \cap C_0(G) \).

**Proof.** The space \( Y = M(G) + L^p(G) \) forms a Banach space on \( G \) with respect to the intermediate norm defined by

\[
\|f\| = \inf\{\|g\|_M + \|h\|_p : g \in M(G), h \in L^p(G), \text{and } g + h = f\}.
\]

It is easy to check that \( X = L^p \cap C_0(G) \) and \( Y \) satisfy conditions (C1)-(C3) in Lemma 2.

Suppose, by way of contradiction, that \( T_\delta(X)^c \subset \text{sp}(F) + T_\delta(Y) \) for some countable subset \( F \) of \( Y \). Then Lemma 2 yields a \( \sigma(Y,C_\sigma) \)-compact subset \( W_{N,K} \) of \( T_\delta(Y) \) such that \( B_1(L^p \cap C_0) \subset W_{N,K} \). By using an appropriate approximate identity, one checks that \( B_1(L^p \cap C_0) \) is \( \sigma(C'_\sigma,C_\sigma) \)-dense in \( B_1(L^p \cap L^\infty) \); hence \( B_1(L^p \cap L^\infty) \subset W_{N,K} \). The remainder of the proof may be accomplished by modifying Woodward’s method in [22, Theorem 1], as follows:

Since \( B_1(L^p \cap L^\infty) \subset W_{N,K} \), each element \( f \) of \( L^p \cap L^\infty \) can be written in the form

\[
f = \mu + \sum_{k=1}^N (h_k - \delta_{x_k} \ast h_k),
\]

where \( \mu \in M(G) \), \( h_k \in L^p(G) \) and \( x_k \in K \) for all \( k \)‘s. Since \( G \) is assumed to be amenable, it satisfies the Følner condition [5]. We can, therefore, find a sequence \( (E_n) \) of compact subsets of \( G \) such that

\[
\lambda_G(E_n \Delta x^{-1}E_n) < n^{-1}\lambda_G(E_n) \quad (x \in K, n \in \mathbb{N}),
\]

where \( \Delta \) stands for symmetric difference of sets. There is no loss of generality in assuming that \( \lambda_G(E_n) > n^2 \) for all \( n \geq 1 \). Writing \( q = p/(p-1) \), we infer from (9) and Hölder’s inequality that \( h \in L^p(G) \) and \( x \in K \) imply

\[
\left\| \int_{E_n} (h - \delta_x \ast h) \, dx \right\| \leq \int_{E_n \Delta x^{-1}E_n} \|h\|_p \lambda_G(E_n \Delta x^{-1}E_n)^{1/q}' h\|_p \leq n^{-1/q}\lambda_G(E_n)^{1/q}' h\|_p
\]
for all \( n \geq 1 \). It follows from (8) that \( f \in L^p \cap L^\infty \) implies
\[
\left| \int_{E_n} f \, dx \right| \leq \|\mu\| + n^{-1/q} \lambda_G(E_n)^{1/q} \sum_{k=1}^{N} \|h_k\|_p
\]
\[
\leq C_f \cdot n^{-1/q} \lambda_G(E_n)^{1/q} \quad (n = 1, 2, \ldots),
\]
where \( C_f \) is a finite constant depending only on \( f \). Therefore the Banach-Steinhaus theorem yields a finite constant \( C \) such that
\[
\left| \int_{E_n} f \, dx \right| \leq C n^{-1/q} \lambda_G(E_n)^{1/q} (\|f\|_p + \|f\|_\infty)
\]
for all \( f \in L^p \cap L^\infty \) and all \( n \)’s. In particular, choosing \( f \) to be the indicator function of \( E_n \), we obtain
\[
\lambda_G(E_n) \leq C n^{-1/q} \lambda_G(E_n)^{1/q} \left[ \lambda_G(E_n)^{1/p} + 1 \right]
\]
\[
\leq 2 C n^{-1/q} \lambda_G(E_n) \quad (n = 1, 2, \ldots).
\]
Since \( q \) is a finite positive number, the last inequalities give us the desired contradiction.

**Theorem 3.** Let \( X \) be a left-invariant closed subspace of \( L^\infty(G) \) such that \( B_1(X) \) is weak-* dense in \( B_1(L^\infty) \) [e.g., \( C_0(G), C_u(G), C(G) \), etc.]. Suppose either:

(i) \( G \) is compact and \( L^\infty(G) \) has two linearly independent LILF’s,

(ii) \( G \) is noncompact and \( L^\infty(G) \) has a nonzero LILF, or

(iii) \( L^\infty(G) \) has a LILF which does not annihilate \( C(G) \).

Then there exist uncountably many LILF’s on \( L^\infty(G) \) whose restrictions to \( X \) are linearly independent modulo \( X' \).

**Proof.** A moment’s glance at the proof of Theorem 2 shows that the desired conclusion is certainly true if either (i) or (ii) holds. So it will suffice to show that (iii) implies (i) if \( G \) is compact, and (ii) if \( G \) is noncompact. If \( G \) is discrete, then there is nothing to prove. So assume \( G \) is nondiscrete.

Passing to a nondiscrete metrizable quotient of \( G \) (cf. [7, p. 71]), we can find a compact subset \( E \) of \( G \) such that \( \lambda_G(E) > 0 \) and \( E \) has empty interior. Then no finitely many translates of \( E \) cover \( G \) (a.e.) by the Baire category theorem. Since the maximal ideal space of \( L^\infty(G) \) is compact, it follows that there exists a nonzero complex homomorphism \( \Phi \) of \( L^\infty(G) \) such that \( \Phi(\delta_x \cdot \xi_E) = 0 \) for all \( x \in G \). \( (\xi_E \) denotes the indicator function of \( E \).) It is easy to see that such a \( \Phi \) can be chosen so that \( \Phi(f) = f(e) \) for all \( f \in C(G) \), where \( e \) is the identity element of \( G \). Define a mapping \( \Phi' \) from \( L^\infty(G) \) into \( l^\infty(G) \) by setting
\[
(\Phi'(f))(x) = \Phi(\delta_{x^{-1}} \cdot f) \quad (f \in L^\infty(G) \text{ and } x \in G).
\]
Thus \( \Phi' \xi_E = 0 \) and \( (\Phi'f)(x) = f(x) \) for all \( f \in C(G) \). Moreover, \( f \in L^\infty(G), a \in G, \) and \( x \in G \) imply
\[
\Phi'(\delta_a \cdot f)(x) = \Phi(\delta_{x^{-1}} \cdot \delta_a \cdot f) = (\Phi'f)(a^{-1}x)
\]
by (11). It follows immediately that \( \Phi[T_i(L^\infty(G))] \subset T_i(l^\infty(G)) \).
Now suppose that either (i) or (ii) fails to hold. Then $L^\infty(G) = \text{sp}(\xi_E) + T_t(L^\infty(G))$ by Theorem 2 with $p = \infty$. Since $C(G) \subset L^\infty(G)$ and $\Phi^t\xi_E = 0$, it follows that $C(G) = \Phi'[C(G)] \subset T_t(L^\infty(G))$. This contradicts (iii) and the proof is complete.

**Corollary 4.** Let $G$ be a $\sigma$-compact, infinite, LC group. If $G$ is amenable as a discrete group, a fortiori, if $G$ is abelian, then there exist uncountably many LILF’s on $L^\infty(G)$ whose restrictions to $C_0(G)$ are linearly independent modulo $[C_0(G)]'$. 

**Proof.** Obvious from the definition of amenability.

**Remarks.** (III) Whenever $G$ is a nondiscrete $\sigma$-compact LC group, there exist uncountably many TILF’s on $M(G) + L^\infty(G)$ whose restrictions to $L^1(G)$ are linearly independent modulo $[L^1(G)]'$. This may be proved by modifying the proof of Theorem 1.

(IV) $L^\infty(G)$ has a discontinuous LILF if and only if so does $C_0(G)$, the Banach space of all left uniformly continuous bounded functions on $G$. The “only if” part is a special case of Theorem 3. The proof of the “if” part requires Cohen’s factorization theorem [2]; see also [6].

(V) Let $\text{LIM}(L^\infty(G))$ denote the set of all left-invariant means on $L^\infty(G)$. As is well known, if $G$ is an infinite LC group which is amenable as a discrete group, then the dimension of [the linear space spanned by] $\text{LIM}(L^\infty(G))$ is quite “huge”; see [1,3–5,18–21]. This can also happen for some nonamenable groups.

Suppose that $G$ and $H$ are two infinite compact groups, and that $G$ is amenable as a discrete group. For each $M \in \text{LIM}(L^\infty(G))$, define

$$\langle f, M \times \lambda_H \rangle = \left( \int f(x,y) \, d\lambda_H(y), M_x \right) \quad (f \in L^\infty(G \times H)).$$

It is obvious that this is well defined and that the correspondence $M \to M \times \lambda_H$ is an isomorphism of $\text{LIM}(L^\infty(G))$ into $\text{LIM}(L^\infty(G \times H))$. We do not know whether condition (iii) in Theorem 3 characterizes the amenable groups.

(VI) Let $G$ be a free group with two generators (cf. [7]). It is a well-known fact that $l^\infty(G)$ has no left-invariant mean. In fact, the zero functional is the only one LILF on $l^\infty(G)$. Therefore the conclusion of Theorem 3 fails to hold for some groups.

Let $a$ and $b$ be the free generators of $G$, and let $f \in l^\infty(G)$ be given. We claim that $f$ has a representation of the form

$$f(x) = g(ax) - g(x) + h(bx) - h(x) \quad (x \in G)$$

for some $g$ and $h \in l^\infty(G)$. To confirm this, notice that $G$ is the union of the sets $\{e\}$, $A$ and $B$, where $A[B]$ is the set of all reduced words starting with a nonzero power of $a[b]$. Therefore we can write $f = f_1 + f_2$, where $f_1, f_2 \in l^\infty(G)$, $\text{supp} f_1 \subset A \cup \{e\}$, and $\text{supp} f_2 \subset B \cup \{e\}$. Using the fact that $G$ is the union of the disjoint sets $b^n(A \cup \{e\})$ for $n = 0, \pm 1, \ldots$, we define

$$h(b^n x) = \begin{cases} f_1(x) & \text{if } x \in A \cup \{e\} \text{ and } n > 0, \\ 0 & \text{if } x \in A \cup \{e\} \text{ and } n \leq 0. \end{cases}$$
Similarly, define
\[ g(a^n x) = \begin{cases} f_2(x) & \text{if } x \in B \cup \{ e \} \text{ and } n > 0, \\ 0 & \text{if } x \in B \cup \{ e \} \text{ and } n \leq 0. \end{cases} \]

One checks that \( f_1(x) = h(bx) - h(x) \) and \( f_2(x) = g(ax) - g(x) \) for all \( x \in G \). Hence \( f = f_1 + f_2 \) admits a decomposition of the desired form. It follows immediately that the only LILF on \( l^n(G) \) is the trivial functional.

(VII) Let \( G \) be an arbitrary LC group. Suppose \( G \) contains a closed normal subgroup \( H \) such that \( G/H \) is compact and \( C(G/H) \) admits a discontinuous LILF. Then there exist uncountably many LILF's on \( C_c(G) \) which are linearly independent modulo \( C_c(G) \). This may be proved by using Theorem (15.21) of [7] and applying Lemma 2 (cf. [17]).

**Examples.** (a) Let \( A_1(\mathbb{R}) \) denote the space of all \( f \in L^1 \cap C_0(\mathbb{R}) \) such that \( \text{supp} \hat{f} \subset [-1, 1] \) and \( f^{(n)} \in L^p(\mathbb{R}) \) for all \( n \in \mathbb{N} \). Here \( \hat{f} \) denotes the Fourier transform of \( f \). It is evident that \( A_1(\mathbb{R}) \) forms an invariant Fréchet space on \( \mathbb{R} \) with respect to the seminorms
\[ p_n(f) = \|f\|_1 + \|f'\|_1 + \cdots + \|f^{(n)}\|_1 \quad (n = 1, 2, \ldots ). \]

Let \( Y \) denote the linear span in \( C_c(\mathbb{R}) \) of all \( M(\mathbb{R}) + L^p(G) \) with \( 1 \leq p < \infty \). Then \( Y \) has uncountably many TILF's whose restrictions to \( A_1(\mathbb{R}) \) are linearly independent modulo \( [A_1(\mathbb{R})]' \).

To prove this, put \( Y_m = M(\mathbb{R}) + L^m(\mathbb{R}) \) for \( m \geq 1 \). Then \( Y_m \subset Y_{m+1} \) and \( Y \) is the union of all \( Y_m \) with \( m \in \mathbb{N} \). Accordingly, \( Y \) may be regarded as the strict inductive limit of the invariant Banach spaces \( Y_m (m \in \mathbb{N}) \). Suppose by way of contradiction that the above-stated result is false. Then we can modify the proofs of Lemmas 1 and 2 to obtain three natural numbers \( N, q, n \) with the following property: each \( f \in A_1(\mathbb{R}) \) can be written in the form
\[ f = \sum_{j=1}^N (\mu_j - \delta_{x_j} * \mu_j) + \sum_{j=1}^N (h_j - \delta_{x_j} * h_j), \]

where \( \mu_j \in M(\mathbb{R}) \), \( h_j \in L^q(\mathbb{R}) \), \( x_j \in [-N, N] \), \( \|\mu_j\|_M \leq Np_n(f) \), and \( \|h_j\|_q \leq Np_n(f) \) for all \( j \)'s.

Now put \( I_m = [-m, m] \) for \( m = 1, 2, \ldots \). Then notice that the symmetric difference \( \{[-N, N] + I_m \} \Delta I_m \) has Lebesgue measure \( 2N \) and is disjoint from \( I_m \). If \( f \in A_1(\mathbb{R}) \) is as above, we infer from Hölder's inequality that
\[
\left| \int_{-m}^m f \, dx \right| \leq \sum_{j=1}^N \left| \mu \left( [I_m - x_j] \Delta I_m \right) \right| + \sum_{j=1}^N \int_{[I_m - x_j] \Delta I_m} |h_j| \, dx \\
\leq o(1) + \sum_{j=1}^N \left( \int_{[I_m - x_j] \Delta I_m} |h_j|^q \, dx \right)^{1/q} (2N)^{(q-1)/q} \\
= o(1) + o(1) = o(1) \quad \text{as } m \to \infty.
\]

Therefore, \( \int_{\mathbb{R}} f \, dx = 0 \) for all \( f \in A_1(\mathbb{R}) \), which is of course absurd.
(b) As was proved by Meisters [10], each TILF on \( C^\infty(\Pi) \) is continuous and is therefore a constant multiple of the Lebesgue integral. However, this is not true for any \( C^{(m)}(\Pi) = \{ f \in C(\Pi) : f^{(m)} \in C(\Pi) \} \), where \( m \in \mathbb{N} \).

To prove this, fix \( m \in \mathbb{N} \) and let \( Y_m \) denote the subspace of all \( f \in C^{(m-1)}(\Pi) \) such that \( f^{(m-1)} \) is absolutely continuous and \( f^{(m)} \in L^\infty(\Pi) \). Then \( Y_m \) forms an invariant Banach space on \( \Pi \) with respect to the norm

\[
\|f\|_{(m)} = \|f\|_\infty + \|f'\|_\infty + \cdots + \|f^{(m)}\|_\infty \quad (f \in Y_m).
\]

One checks that the imbedding of \( C^{(m)}(\Pi) \) into \( Y_m \) is isometric, that \( B_1(Y_m) \) is \( \sigma(Y_m, C(\Pi)) \)-compact, and that the unit ball of \( C^{(m)}(\Pi) \) is \( \sigma(Y_m, C(\Pi)) \)-dense in \( B_1(Y_m) \). Suppose that \( X = C^{(m)}(\Pi) \) has no discontinuous TILF. Then Lemmas 1 and 2 assure that each \( f \in Y_m \) can be written in the form

\[
f = \langle 1, f \rangle + \sum_{j=1}^N \left( h_j - \delta_{x_j} \ast h_j \right),
\]

where \( h_j \in Y_m \) and \( x_j \in \Pi \) for all \( j \)'s. Taking the \( m \)th derivatives of both sides of the last equation, we obtain

\[
f^{(m)} = \sum_{j=1}^N \left[ h_j^{(m)} - \delta_{x_j} \ast h_j^{(m)} \right].
\]

However, it is easy to see that \( (d/dt)^m Y_m \) consists of all \( g \in L^\infty(\Pi) \) with \( \langle 1, g \rangle = 0 \). It follows from Theorem 3 that the last expression is impossible for some \( f \in Y_m \). This \textit{reductio ad absurdum} establishes the desired result. Notice also that each TILF on \( C^{(m)}(\Pi) \) which extends to a TILF on \( L^2(\Pi) \) is necessarily continuous by [15].

(c) The following example is included here because of its contrast with the last example. Let \( G \) be a compact abelian group with dual \( \Gamma \), and let \( \{ \psi_n \} \) be a sequence of elements of \( L^\infty(\Gamma) \) such that \( 1 \leq \psi_1 \leq \psi_2 \leq \cdots \). Let \( X \) consist of all \( f \in C(G) \) such that

\[
p_n(f) = \sum_{\gamma \in \Gamma} \left| \hat{f}(\gamma) \right| \psi_n(\gamma) < \infty \quad (n = 1, 2, \ldots).
\]

It is obvious that \( X \) forms an invariant Fréchet space on \( G \) with respect to the seminorms \( p_n(\cdot) \). Suppose that there exists a compact subgroup \( H \) of \( G \) such that \( G/H \) is an infinite torsion group. Then it is easy to construct an element \( f_0 \in X \) with the following property: whenever \( F \) is a finite subset of \( G \), then \( \lambda_H(F) \ast f_0 \) is nonconstant, where \( H(F) \) is the compact subgroup of \( G \) generated by \( H \cup F \) and \( \lambda_H(F) \) is the normalized Haar measure of \( H(F) \).

Suppose by way of contradiction that there is no TILF on \( L^1(G) \) whose restriction to \( X \) is discontinuous. Then, by Lemmas 1, 2 and the Lebesgue-Radon-Nikodým theorem, we can write \( f_0 \) in the form

\[
f_0 = C + \sum_{j=1}^N \left( g_j - \delta_{x_j} \ast g_j \right),
\]

where \( C = \langle 1, f_0 \rangle \), \( g_j \in L^1(G) \) and \( x_j \in G \) for all \( j \)'s. Letting \( F = \{ x_1, \ldots, x_N \} \), we then have \( \lambda_H(F) \ast f_0 = C \) a.e. on \( G \). Since \( f_0 \) is continuous, it follows that \( \lambda_H(F) \ast f_0 = C \) everywhere on \( G \), which contradicts our choice of \( f_0 \).
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