

HAEFLIGER STRUCTURES AND LINEAR HOMOTOPY

BY

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ABSTRACT. The notion of linear-homotopy into a classifying space is introduced and used to give a precise classification of Haefliger structures. Appendix on the product theorem for simplicial spaces and realizations of bisimplicial spaces.

Introduction. A concrete model for a classifying space is usually defined as the realization of a simplicial space. It is obtained by glueing continuous families of standard geometric simplexes, and thus, we have a nice coherent notion of linear simplex lying in it. This extra “linear structure” helps in trying to understand a classifying space geometrically and not only up to homotopy type. Thinking of points moving (‘pl-ishly’) along these simplexes, we get the notion of linear-homotopy, which, applied to Segal’s model for the classifying space of a groupoid ([S] or (1.3)), gives a finer classification than Haefliger’s classical one.

THEOREM B. *For any topological groupoid G and a locally compact space X , there is a one-to-one natural correspondence between linear-homotopy classes of maps from X to BG , and G -structures on X .*

This is the main result of the paper (observe (2.1)), so let us make it precise:

Given a simplicial space A , its geometric realization $|A|$ is obtained from $\prod A_n \times \Delta^n$ identifying with all the simplicial relations ([S] or (1.1)). Intuitively, think of $|A|$ as having two directions: a “continuous” one given by the topologies of the A_n ’s; and a “linear” one induced by the second factors. More precisely, consider the underlying simplicial set of A : A^δ (read A -discrete) obtained by forgetting the topologies of the A_n ’s. The set-theoretical identity gives us a map $|A^\delta| \rightarrow |A|$. $|A^\delta|$ represents the “linear direction” of $|A|$.

DEFINITION. A homotopy $H: X \times I \rightarrow |A|$ is a *linear-homotopy* if, for every $x \in X$, the path $H_x: I \rightarrow |A|$ factors through $|A^\delta| \rightarrow |A|$.

In terms of foliations: a linear-homotopy is one in which every point moves along its leaf (see §1(ii) for discussion and examples).

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G -structures were defined by Haefliger in [H] (there, he called them Γ -structures but here G will denote the groupoid). Haefliger's work opened broad perspectives to the theory of foliations through

THEOREM A (HAEFLIGER'S CLASSIFICATION). *There is a one-to-one natural correspondence between homotopy classes of maps from X to BG , and homotopy classes of G -structures on X .*

We venture a proof of Theorem A when X is locally compact. It follows formally from Theorem B:

The correspondence is given by pulling back a universal G -structure on BG , thus, homotopic maps induce homotopic G -structures. On the other hand, suppose f_0 and $f_1: X \rightarrow BG$ induce G -structures ξ_0 and ξ_1 homotopic through a G -structure ξ on $X \times I$. Use Theorem B to classify ξ with a map $H: X \times I \rightarrow BG$, and then use it again to join the endmaps of H to f_0 and to f_1 (through linear-homotopies—which are good enough). \square

In principle, Theorem B should enable us to work with Haefliger structures in terms more precise than “up to homotopy” (e.g., foliations up to isotopy). Unfortunately, we still do not know the invariants of linear-homotopy necessary to undertake such projects.

In the case where G is a topological group, there is a simple application of Theorem B. Using the fact that (for a paracompact space X) homotopy of principal G -bundles (i.e., of G -structures) implies isomorphism, Theorems A and B yield that homotopy and linear-homotopy induce the same relation among maps from X to BG , that is to say, every homotopy into BG can be transformed into a linear-homotopy.

The paper is self-contained. §2 can be considered as a precise proof of Theorem A. There, we develop the techniques necessary to prove the extra part of Theorem B which is completed in §3.

The Product Theorem (Theorem 2, Appendix) is essentially used twice within the text. Its particular application to the proof of Theorem B (part (ii) of Proposition 1, which is the “support” of the local compactness hypothesis) could have easily been left to the reader. But we preferred a general treatment in an Appendix to fill the void of a suitable reference in the literature.

Without the persistent encouragement and advice of Solomon Jekel, this work would have never achieved its present form. I thank him deeply. Also, I must acknowledge Daniel Kan's impressive introduction to the subject.

1. Definitions and examples. The definitions in (i) are well known [S]; we rush through them only to precise notation. In (ii) we discuss a little further the notion of linear-homotopy and detail the relevant examples. Haefliger structures are treated in (iii).

(i) *Simplicial spaces, topological categories and groupoids.* Let \mathbf{n} denote the category naturally associated to the ordered set $\{0, 1, \dots, n\}$. Analogously, \mathbf{N} has object set $\{0, 1, \dots\}$.

(1.1) A *simplicial space* A is a collection $\{A_n\}$ of spaces together with *face maps* f^* : $A_m \rightarrow A_n$ for every functor $f: \mathbf{n} \rightarrow \mathbf{m}$. Its *realization* is the space $|A| = \prod_n A_n \times \Delta^n / \sim$, where $(a, f_*(t)) \sim (f^*(a), t)$ for every $f: \mathbf{n} \rightarrow \mathbf{m}$, $a \in A_m$ and $t \in \Delta^n$.

Given a point $x \in |A|$, there is a unique nondegenerate simplex of A , $\text{car}(x)$ (read *carrier* of x), such that $(\text{car}(x), t_x)$ represents x , t_x is an interior point of Δ^n , and $n = \dim(\text{car}(x))$.

(1.2) Given a *topological category* (*top-cat*) C , $\text{Ob}(C)$ ($\text{Mor}(C)$) denotes its object (morphism) space; $S & T: \text{Mor}(C) \rightarrow \text{Ob}(C)$ denote the source and target maps. If every morphism $c \in \text{Mor}(C)$ has an inverse c^{-1} , and if the involution $c \mapsto c^{-1}$ is continuous, C is called a *groupoid*. G will denote a groupoid throughout this work.

X will denote a space, or the groupoid naturally associated to it: $\text{Ob}(X)$ and $\text{Mor}(X)$ are copies of X and all structural maps are the identity map of X (X as a groupoid is the initial object among top-cats having object space X). The context will always clarify the meaning of X (although that, in a categorical sense, the difference is superfluous).

The *nerve* of a top-cat C is the simplicial space NC having $(NC)_n = \text{Functors}\{\mathbf{n} \rightarrow C\}$ with the subspace topology from $\text{Mor}(C)^{\text{Mor}(\mathbf{n})}$. And the *realization* of C is $|C| = |NC|$.

Observe that if $x \in |C|$, then $|\text{car}(x)|(t_x) = x$.

The explicit model for the classifying space of a groupoid that we use is Segal's generalization of Milnor's join construction [S]:

(1.3) Given a topological category C , let $C_{\mathbf{N}}$ denote the subtop-cat of $\mathbf{N} \times C$ obtained by deleting all nonidentity morphisms whose first component is an identity. Then define $BC = |C_{\mathbf{N}}|$.

(ii) *Linear homotopy*. X^δ (read *X-discrete*) denotes the space with the same pointset as X but with the discrete topology. Thus, for a simplicial space A , we have $A^\delta = \{A_n^\delta\}$ and there is an obvious simplicial map $A^\delta \rightarrow A$. The analogous notation applies to top-cats.

Let A be a simplicial space (or a top-cat). Consider a homotopy $H: X \times I \rightarrow |A|$. We always have a set theoretical commutative diagram:

$$(1.4) \quad \begin{array}{ccc} X^\delta \times I & \xrightarrow{H'} & |A^\delta| \\ \downarrow & & \downarrow \\ X \times I & \xrightarrow{H} & |A| \end{array}$$

H is a *linear-homotopy* if and only if H' is continuous.

Linear-homotopy is a natural equivalence relation to consider among maps into "classifying spaces". Here are some examples.

For a discrete simplicial space, linear-homotopy is homotopy in the usual sense.

On the other extreme, a linear-homotopy into a space $Y = |Y|$ is constant (as a homotopy). Thus, linear-homotopy is equality among spaces.

(1.5) Natural transformations induce linear-homotopies. This observation is due to Segal [S] (where we borrowed the term from). Let $T: C \times \mathbf{1} \rightarrow D$ be a (continuous)

natural transformation— C and D are topological categories. The Product Theorem (Theorem 2, Appendix) gives us a homotopy $|T|: |C| \times I \rightarrow |D|$. But we also have a map $|T^\delta|: |C^\delta| \times I \rightarrow |D^\delta|$ through which $|T|'$ clearly factors (see (1.4)). Whence, $|T|$ is, in fact, a linear-homotopy.

Finally, let us consider Haefliger's groupoid Γ [H]. $B\Gamma$ is the “classifying space for foliations”, and it is well known that $B\Gamma^\delta$ is the leaf of the universal foliation on $B\Gamma$. Thus, linear-homotopy is precisely “moving along the leafs”. In this context, Theorem B is intuitively clear: the foliation induced by a map $f: X \rightarrow B\Gamma$ is not changed at all if we let X move along the universal foliation.

(iii) *Haefliger structures.* The following definition is easily seen to correspond exactly to the original one of Haefliger [H].

First, we associate a groupoid to an open cover:

(1.6) Given $U = \{U_\alpha\}_{\alpha \in \Sigma}$, an open cover of X , denote by $\underline{\Sigma}$ the discrete groupoid with $\text{Ob}(\underline{\Sigma}) = \Sigma$ and $\text{Mor}(\underline{\Sigma}) = \Sigma \times \Sigma$ with the obvious structural maps. Then define X_U to be the full subgroupoid of $X \times \underline{\Sigma}$ spanned by the subspace of objects: $\{(x, \alpha) | x \in U_\alpha\}$.

(1.7) **DEFINITION.** A *G-structure* on X is an equivalence class of *G-cocycles* on X ; where a *G-cocycle* on X is a pair (U, F) with U an open cover of X and $F: X_U \rightarrow G$ a functor; and where two *G-cocycles* (U_0, F_0) and (U_1, F_1) on X are equivalent if there exists a functor $F: X_{U_0 \amalg U_1} \rightarrow G$ restricting (in the obvious sense) to F_0 and F_1 . A *G-structure* on X is *numerable* if it is represented by a *G-cocycle* (U, F) with U admitting a subordinate partition of unity.

Examples of *G-structures* form a wide spectrum: maps (when G is a space); principal bundles (when G is a group); foliations (for Haefliger's Γ); manifold or geometric structures (0-dimensional foliations for obvious choices of Γ), and vector bundle morphisms are some of them. For more discussion and examples see [H or B].

2. Proof of Theorem A. Partitions of unity play an important role in the classic proof of Theorem A. For expository reasons, we did not mention them in the Introduction. But it is due time to correct this omission:

(2.1) **REMARK.** In Theorems A and B, “*G-structures*” should read “*numerable G-structures*”. Furthermore, in Theorem A, homotopies between numerable *G-structures* are understood to be numerable.

Here, instead of working with partitions of unity in the usual way, we work with (the closely related concept of) maps into the infinite simplex Δ^∞ ($= |\mathbb{N}|$, as definition). So we have

(2.2) **DEFINITION.** Given a map $p: X \rightarrow \Delta^\infty$, let X_p be the full subtop-cat of $X \times \mathbb{N}$ spanned by the subspace of objects $\{(x, i) | i \in \text{Image}(\text{car}(p(x)))\}$. A *numerated G-cocycle* on X is a pair (p, F) where $p: X \rightarrow \Delta^\infty$ is a map and $F: X_p \rightarrow G$ is a functor. Denote by $\text{NumCo}(X, G)$ the set of such.

(2.3) **REMARK.** Numerable *G-structures* on X can be considered in a natural way as equivalence classes of numerated *G-cocycles* on X under the obvious equivalence relation (see, e.g., (2.5)).

This is so because a numerated G -cocycle on X is essentially a G -cocycle together with a fixed subordinate partition of unity. (Compare (2.2) with (1.6) and (1.7), keeping in mind the natural open cover of Δ^∞ and that G is a groupoid.) (Details can be found in [B] or in the author's Thesis.)

Matching the precision gained by (2.3), we have

PROPOSITION 1. *For a locally compact space X , and any groupoid G , there is a natural bijection*

$$\text{Maps}(X, BG) \leftrightarrow \text{NumCo}(X, G).$$

PROOF. (i) Let $f: X \rightarrow BG$ be a given map. Recall that $BG = |G_N|$ and that $G_N \subset N \times G$, (1.3). Let π_1 be the projection functor from G_N to N . Then put $p = |\pi_1| \cdot f$.

To define $F: X_p \rightarrow G$, let X_f be the subtop-cat of $X \times G_N$ consisting of objects (and morphisms) (x, g) such that $g \in \text{Image}(\text{car}(f(x)))$. X_f is indeed a category for, by the definition of G_N , the image of any simplex $(n \rightarrow G_N)$ is already a subcategory of G_N .

(2.4) *Claim.* The restriction to X_f of the functor $\text{id}_X \times \pi_1: X \times G_N \rightarrow X \times N$ is an isomorphism onto X_p .

Since π_1 takes $\text{car}(f(x))$ isomorphically onto $\text{car}(p(x))$ for every $x \in X$, then it is clear that $(\text{id}_X \times \pi_1)|_{X_f}$ is a bijection onto X_p . That it is, moreover, an isomorphism of top-cats follows from the fact that, since N is discrete, $\text{id}_X \times \pi_1$ is open.

Observe that X_f comes equipped with a functor $\tilde{f}: X_f \rightarrow G_N$, so that we define F as the composite

$$X_p \xrightarrow{\tilde{f}} X_f \xrightarrow{\pi_2} G_N \xrightarrow{\pi_2} G.$$

This gives the correspondence $\text{Maps}(X, BG) \rightarrow \text{NumCo}(X, G)$.

(ii) Take $(p, F) \in \text{NumCo}(X, G)$. Let $\tilde{p}: X_p \rightarrow N$ be the projection functor. Observe that $(\tilde{p}, F): X_p \rightarrow N \times G$ factors through the inclusion $G_N \hookrightarrow N \times G$; so that we get a functor $\bar{F}: X_p \rightarrow G_N$.

If X is locally compact then $|X \times N| = X \times \Delta^\infty$ (Theorem 2, Appendix), and then the graph of p defines a section $s_p: X \rightarrow |X_p|$. This gives a map

$$|\bar{F}| \cdot s_p: X \rightarrow BG.$$

The rest of the proof is straightforward. \square

PROPOSITION 2. *Let X be locally compact, and G a groupoid. Let $f_0, f_1: X \rightarrow BG$ and $(p_0, F_0), (p_1, F_1) \in \text{NumCo}(X, G)$ correspond, respectively, to each other by means of Proposition 1, then*

$$(p_0, F_0) \sim (p_1, F_1) \Rightarrow f_0 \text{ is linear-homotopic to } f_1.$$

PROOF. Let $h_0, h_1: N \rightarrow N$ be the functors $h_0(n) = 2n$, $h_1(n) = 2n + 1$. Let $p: X \rightarrow \Delta^\infty$ be the map defined by

$$p = \frac{1}{2}(|h_0| \cdot p_0) + \frac{1}{2}(|h_1| \cdot p_1)$$

and let $i_k: X_{p_k} \rightarrow X_p$, $k = 0, 1$, be the obvious inclusions (restrict $\text{id}_X \times h_k$). We clearly have

(2.5) $(p_0, F_0) \sim (p_1, F_1)$ iff there exists $F: X_p \rightarrow G$ such that $F_0 = F \cdot i_0$ and $F_1 = F \cdot i_1$.

Recalling part (ii) of Proposition 1, consider the diagram

$$\begin{array}{ccccc}
 & & |X_{p_0}| & & \\
 & s_{p_0} \swarrow & \downarrow |i_0| & \searrow |\bar{F}_0| & \\
 X & \xrightarrow{s_p} & |X_p| & \xrightarrow{|\bar{F}|} & BG \\
 & s_{p_1} \searrow & \uparrow |i_1| & \nearrow |\bar{F}_1| & \\
 & & |X_{p_1}| & &
 \end{array}$$

where f_0 and f_1 appear as the top and bottom row compositions respectively. We see that the diagram commutes up to linear-homotopy:

Clearly, the left-hand side triangles are linear-homotopy commutative (all the maps except the F 's are linear-homotopy equivalences). On the other hand, the natural transformations from \bar{F}_k to $\bar{F} \circ i_k$ defined by

$$(x, n) \mapsto (n \rightarrow h_k(n), \text{id}(F_k(x, n)))$$

for $(x, n) \in \text{Ob}(X_{p_k})$ and $k = 0, 1$, produce via (1.5) the required linear-homotopies for the right-hand side triangles. \square

At this point, the proof of Theorem A given in the Introduction works with two comments on it: the first assertion is clearly valid (reconsider part (i) of Proposition 1 with id_{BG}); as for the applications of Theorem B, observe that Propositions 1 and 2 are (respectively) sufficient.

3. Proof of Theorem B. There only remains to prove the following

PROPOSITION 3. *With notation as in Proposition 2, we have*

$$f_0 \text{ is linear-homotopic to } f_1 \Rightarrow (p_0, F_0) \sim (p_1, F_1).$$

PROOF. Let $p: X \rightarrow \Delta^\infty$ be as in Proposition 2. From (2.5) it clearly suffices to fill the following diagram:

$$\begin{array}{ccccc}
 & & X_{p_0} & & \\
 & & \downarrow j_0 & & \\
 & & C & \dashrightarrow & G \\
 & j \dashleftarrow & \uparrow j_1 & \dashrightarrow & F_1 \\
 X_p & \xleftarrow{\quad} & & \xrightarrow{\quad} & \\
 & & X_{p_1} & &
 \end{array}
 \tag{3.1}$$

Let $H: X \times I \rightarrow BG$ be a linear-homotopy from f_0 to f_1 .

(i) *Construction of C.* (a) Let A be the subsimplicial-space of $N(X \times I) \times NG_N$ generated by the simplices $\{((x, t), \text{car}(H(x, t)))\}$.

(b) Let $B = A/\sim$, where \sim is the equivalence relation:

(3.2) Given $((x, t), \sigma) \& ((x, s), \sigma) \in A_n$, for any $n \geq 0$, $((x, t), \sigma) \sim ((x, s), \sigma)$ iff $((x, t'), \sigma) \in A_n$ for every $t' \in ts$ (ts is the interval between t and s).

(c) Let C be the top-cat: $\text{Ob}(C) = B_0$, $\text{Mor}(C) = B_0 \times_X B_0$ with the obvious groupoid structure.

The fibre product $B_0 \times_X B_0$ is performed (on both sides) over the 0-component of the simplicial map $B \rightarrow NX$, obtained from the diagram:

$$\begin{array}{ccccc} A & \longrightarrow & N(X \times I) \times NG_N & \longrightarrow & N(X \times I) \longrightarrow NX \\ & \searrow & & \nearrow & \dashrightarrow \\ & & B & & \end{array}$$

Note from the definition that there is a unique simplicial map $\beta: B \rightarrow NC$ for which $\beta_0 = \text{id}_{B_0}$.

(ii) *Construction of the j's.*

(3.3) Observe that A is the nerve of $(X \times I)_H$ as constructed in (i) of Proposition 1.

Then, (2.4) together with the obvious functoriality of the construction gives us inclusions $NX_{p_k} \rightarrow A$ (where, as from now on, $k = 0, 1$). Now, the composite simplicial maps $NX_{p_k} \rightarrow A \rightarrow B \xrightarrow{\beta} NC$ are the nerves of functors $j_k: X_{p_k} \rightarrow C$.

Observing that $\text{Ob}(X_p) = \text{Ob}(X_{p_0}) \amalg \text{Ob}(X_{p_1})$, the object map of $j: X_p \rightarrow C$ is forced upon us. And on morphisms simply put $\text{Mor}(j) = (\text{Ob}(j) \circ S, \text{Ob}(j) \circ T)$.

(iii) *Construction of F.* Here, we have to look at the construction of C more closely. From the inclusion $A \subset N(X \times I) \times NG_N$ we obtain projection simplicial maps:

$$\begin{array}{ccccc} & & A & & \\ & & \downarrow \Pr_1 & & \\ & \swarrow & N(X \times I) & \searrow & NG_N \\ NX & & & & NI \end{array}$$

(3.4) \Pr_1 restricted to any component of A_n (for any $n \geq 0$) is a homeomorphism onto an open set of $X \times I = N(X \times I)_n$.

This follows from (3.3) and (2.4). (Moreover, A_0 can be considered as an open cover of $X \times I$, and A_n consists of the n -fold intersections of the “components” of A_0 .)

(3.5) Given a and b in A_n , then $a \sim b$ (in the sense of (3.2)) iff for some $x \in X$, a and b lie in the same component of $(A_x)_n$, where A_x denotes the simplicial fibre in A over x .

One implication is obvious. For the other, suppose a and b lie in the same component of $(A_x)_n$. Note from (3.4) that every component of $(A_x)_n$ is homeomorphic to an open interval in I .

Since H is a linear-homotopy, $H_x: I \rightarrow BG$ factors through $BG^\delta \rightarrow BG$. Then, the projection $\text{Pr}_2|_{A_x}: A_x \rightarrow NG_N$ factors through $NG_N^\delta \rightarrow NG_N$; thus, Pr_2 is constant on every component of $(A_x)_n$. From here, (3.5) easily follows.

(3.6) The projection $B \rightarrow NX$ is locally a homeomorphism (i.e. every point of B_n has an open neighbourhood homeomorphic to an open set of X).

(3.4) and (3.5) reduce this to an easy exercise in point set topology. Namely, U/\sim has the required property with U an open subset of $X \times I$ and $(x, t) \sim (x, s)$ iff $(x, t') \in U$ for all $t' \in \overline{ts}$.

Let B_x be the simplicial fibre in B over $x \in X$. The final step is to prove

(3.7) B_x is a contractible simplicial set (i.e., B_x is discrete as subsimplicial space of B , and $|B_x|$ is contractible).

(3.5) implies that $(B_x)_n$ has a point for every connected component of $(A_x)_n$ (one could write $B_x = \Pi_0(A_x)$), and (3.6) implies that every point of $(B_x)_n$ is open. Thus, B_x is discrete.

Before giving an explicit description of B_x , let us see how its contractibility gives a natural definition for F :

There is an obvious simplicial map $F': B \rightarrow NG$ (recall part (i)). Thus, F is already defined on objects ($\text{Ob}(C) = B_0$). Now, given $(a, b) \in \text{Mor}(C)$, we have that $a \& b \in (B_x)_0$ for some $x \in X$. Then, since B_x is connected, we can find a simplicial path in B_x :

$$a \cdot \underline{\gamma_0} \cdot \underline{\gamma_1} \cdot \dots \cdot \underline{\gamma_n} \cdot b,$$

where “—” can be “ \leftarrow ” or “ \rightarrow ”. Then define

$$(3.8) \quad F(a, b) = F'_1(\gamma_n)^{\epsilon_n} \circ \dots \circ F'_1(\gamma_0)^{\epsilon_0}$$

where ϵ_i is 1 or -1 according to the orientation of γ_i . Since B_x is simply-connected, any two paths as above are simplicially homotopic. And, using F'_2 , one easily sees that simplicial moves do not affect the outcome of (3.8). Thus, F is well defined. Finally, the continuity of F follows from (3.6) (any simplicial path as above can be extended over a small neighborhood of x).

To describe B_x explicitly, consider the map $\tau = |\pi_1| \circ H_x: I \rightarrow \Delta^\infty$. Then A_x is the nerve of I_τ (recall (2.2), (2.4) and (3.3)).

Let $\{U_\alpha\}_{\alpha \in \Sigma}$ be the open cover of I consisting of the connected components of $(A_x)_0 = \text{Ob}(I_\tau)$. We have a function $\Psi: \Sigma \rightarrow \mathbb{N}$, for which $\{U_\alpha | \alpha \in \Psi^{-1}(i)\}$ is the set of components of the open subset of I : $\{t \in I | i \in \text{Image}(\text{car}(\tau(t)))\}$.

Now B_x can be described as the simplicial set with a vertex u_α for each $\alpha \in \Sigma$ and an n -simplex $\langle u_{\alpha_0}, \dots, u_{\alpha_n} \rangle$ whenever $\cap_{i=0}^n U_{\alpha_i} \neq \emptyset$ and $\Psi(\alpha_0) \leq \Psi(\alpha_1) \leq \dots \leq \Psi(\alpha_n)$. To give an explicit contraction of B_x , note that we can find a finite sequence $\alpha^0, \alpha^1, \dots, \alpha^k$, such that for each $\alpha \in \Sigma$, we either have (a) $U_\alpha \subset U_{\alpha^i}$ for a unique i ; or (b) $U_\alpha \subset U_{\alpha^{i-1}} \cap U_{\alpha^i}$ and $U_\alpha \cap U_{\alpha^j} = \emptyset$ for $j \neq i-1, i$. (Put $\text{lub}(U_\alpha) = \max\{\text{lub}(U_\alpha) | 0 \in U_\alpha\}$, and proceed by induction using that each $t \in I$ is in exactly $\dim(\text{car}(\tau(t)))$ of the U_α 's.) Then, retract B_x to the path $\langle u_{\alpha^0}, u_{\alpha^1}, \dots, \langle u_{\alpha^{k-1}}, u_{\alpha^k} \rangle \rangle$ by sending u_α to u_{α^k} if (a) holds, or to the barycenter of $\langle u_{\alpha^{i-1}}, u_{\alpha^i} \rangle$ if (b) holds, and extending linearly. The path is contractible. Hence, we have completed the proof of the theorem.

Appendix. The Product Theorem. We give an elementary proof of Milnor's Product Theorem for simplicial spaces. Different versions of it are well known and widely used, but the following one (whose proof follows essentially Milnor's original one [M]) does not appear in the literature. The right setting to work in is that of bisimplicial spaces.

Given a bisimplicial space E (i.e., a functor $E: \Delta^{\text{op}} \times \Delta^{\text{op}} \rightarrow \text{Top}$, where Δ is the category of the ordered sets $\mathbf{n} = \{0, 1, \dots, n\}$), one has three simplicial spaces naturally associated to it: the diagonal E^Δ ; the vertical realization E^v , obtained by realizing the columns of E ; and the horizontal realization E^h .

THEOREM 1. *For any bisimplicial space E : $|E^\Delta| = |E^v| = |E^h|$.*

(Here, equality means canonical homeomorphism, realization is the usual one ([S] or (1.3)), and everything takes place in the classic category of topological spaces.)

PROOF. Consider the diagram

$$(A.1) \quad \begin{array}{ccc} & \coprod_{n,m,k} E_{n,m} \times N(\mathbf{n} \times \mathbf{m})_k \times \Delta^k & \\ \alpha \searrow & & \swarrow \beta \\ \coprod_k E_{k,k} \times \Delta^k & & \coprod_{n,m} E_{n,m} \times \Delta^n \times \Delta^m \end{array}$$

To define α and β , recall that a k -simplex of the nerve of $\mathbf{n} \times \mathbf{m}$ is a functor $\sigma: \mathbf{k} \rightarrow \mathbf{n} \times \mathbf{m}$; denote its projections by $\sigma_1: \mathbf{k} \rightarrow \mathbf{n}$ and $\sigma_2: \mathbf{k} \rightarrow \mathbf{m}$. Then put

$$\alpha(e, \sigma, t) = ((\sigma_1, \sigma_2)^*(e), t), \quad \beta(e, \sigma, t) = (e, |\sigma_1|(t), |\sigma_2|(t)).$$

(A.2) Let $\|E\|$ be the quotient of $\coprod_{n,m} E_{n,m} \times \Delta^n \times \Delta^m$ under the relation $(e, f_* t_1, g_* t_2) \sim ((f, g)^* e, t_1, t_2)$ (for all obvious choices).

Clearly, the fibres of β composed with the quotient map to $\|E\|$ correspond to the fibres of α followed by the quotient to $|E^\Delta|$. Thus, to conclude that $|E^\Delta| = \|E\|$ we show that α and β are quotient maps:

α is a quotient map for it has a section $\delta(e, t) = (e, (\text{id}_k, \text{id}_k), t)$ (which includes the obvious diagonal mappings).

With β , one runs into the classic nuisance: “the product of quotients is not necessarily a quotient”. However, we get by with the following elementary lemma, whose proof follows literally that of Hu’s book [HU, Lemma (IV, 6.1), p. 143].

DEFINITION. A quotient map $q: X \rightarrow Y$ is *locally compact* if for every pointed neighborhood (y, V) in Y there exists an open set U of X with compact closure contained in $q^{-1}(V)$, and such that $q(U)$ is an open neighborhood of y .

LEMMA. *If one of the quotient maps p, q is locally compact, then $p \times q$ is a quotient map.*

The Lemma applies to the components of β : “fix n, m ”; the hypothesis being fulfilled by the second factors. Hence, β is a quotient map and $|E^\Delta| = \|E\|$.

The theorem now follows by proving that $\|E\| = |E^h|$ (and symmetrically $\|E\| = |E^v|$). But this is obvious from the fact that the composition

$$\coprod_{n,m} E_{n,m} \times \Delta^n \times \Delta^m \rightarrow \coprod_m |\{n \mapsto E_{n,m}\}| \times \Delta^m \rightarrow |E^h|$$

is a quotient map (apply the Lemma— Δ^m is compact). \square

As a corollary to the proof we obtain a sharp generalization of Milnor’s Product Theorem [M].

THEOREM 2. *Let A and B be simplicial spaces. If the quotient map $\coprod_n B_n \times \Delta^n \rightarrow |B|$ is locally compact, then $|A \times B| = |A| \times |B|$.*

PROOF. Consider the bisimplicial space $\{E_{n,m} = A_n \times B_m\}$. The Lemma implies that $|A| \times |B| = \|E\|$ (recall (A.2)). And, on the other hand, $E^\Delta = A \times B$. \square

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