STRONG MARTINGALE CONVERGENCE
OF GENERALIZED CONDITIONAL EXPECTATIONS
ON VON NEUMANN ALGEBRAS

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Abstract. Accardi and Cecchini generalized the concept of conditional expectations on von Neumann algebras. In this paper we give some conditions for strong convergence of increasing or decreasing martingales of Accardi and Cecchini’s conditional expectations.

Introduction. The study of conditional expectations and martingale convergence theorems in the operator algebraic framework was initiated by Umegaki’s earlier works [14, 15]. Since then, the martingale convergence theorems on von Neumann algebras have been developed by several authors. For example, the almost sure type martingale convergence was obtained by Lance [9] and Dang-Ngoc [4]. The strong martingale convergence was completed in [13] by one of the authors.

The conditional expectation of a von Neumann algebra $M$ onto a von Neumann subalgebra $N$ does not generally exist given a faithful normal state (or semifinite weight) $\varphi$ on $M$. Indeed its existence is equivalent to the invariance of $N$ under the modular automorphism group associated with $\varphi$ (see [12]). To recover this drawback of noncommutative conditional expectations, Accardi and Cecchini [1] generalized the concept of conditional expectations on von Neumann algebras by using the Tomita-Takesaki theory [11]. Accardi and Cecchini’s conditional expectation $\varepsilon: M \to N$ with respect to $\varphi$ always exists but is not necessarily a projection onto $N$ and lacks a useful property that $\varepsilon(axb) = a\varepsilon(x)b$ for $a, b \in N$ and $x \in M$.

Araki [2] established the convergence of modular operators and modular conjugations given an increasing net of von Neumann subalgebras and a faithful normal state. By extending the arguments in [2], we have obtained in [8] the martingale type convergence of modular automorphism groups under an increasing or decreasing net of von Neumann subalgebras and a faithful normal semifinite weight. On the same lines, we investigate in this paper the strong convergence of martingales of Accardi and Cecchini’s conditional expectations (called here generalized conditional expectations).

In §1 of this paper we state the definition of the generalized conditional expectation with respect to a faithful normal semifinite weight and give easy technical
lemmas. In §2 we establish necessary and sufficient conditions for the strong convergence of an increasing martingale of generalized conditional expectations. In §3 we show the strong convergence of a decreasing martingale of generalized conditional expectations with suitable assumptions. Finally we have an application to some convergence properties for nonabelian $K$-flows.

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1. Preliminaries. Let $M$ be a von Neumann algebra and $\varphi$ be a faithful normal semifinite weight on $M$. We use the usual notations $n_\varphi = \{x \in M: \varphi(x^*x) < \infty\}$ and $m_\varphi = n_\varphi^* n_\varphi$. Let $(\mathcal{H}_\varphi, \pi_\varphi)$ be the GNS representation of $M$ induced by $\varphi$ where the canonical injection of $n_\varphi$ into $\mathcal{H}_\varphi$ is denoted by $x \mapsto x_\varphi$. Then $\mathcal{H}_\varphi = \{x_\varphi: x \in n_\varphi \cap n_\varphi^*\}$ is an achieved left Hilbert algebra and $\pi_\varphi(M)$ is its left von Neumann algebra. Let $\Delta_\varphi, J_\varphi$ and $\sigma_\varphi^t$ be the modular operator, the modular conjugation and the modular automorphism group, respectively, associated with $\varphi$.

Let $N$ be a unital von Neumann subalgebra of $M$. The conditional expectation $\epsilon: M \to N$ with respect to $\varphi$ is a unique norm one normal projection $\epsilon$ of $M$ onto $N$ such that $\varphi(x) = \varphi(\epsilon(x))$ for all $x \in M_+$. According to Takesaki [12], there exists the conditional expectation $\epsilon: M \to N$ with respect to $\varphi$ if and only if $\varphi \uparrow N$ is semifinite and $\sigma_\varphi^t(N) = N$ for every $t \in \mathbb{R}$. Recently the concept of conditional expectations has been extended by Accardi and Cecchini [1]. Now suppose only that $\varphi = \varphi \uparrow N$ is semifinite, and let $(\mathcal{H}_\varphi, \pi_\varphi)$ and $J_\varphi$ be the GNS representation of $N$ and the modular conjugation associated with $\varphi$. Since the canonical injection of $n_\varphi = n_\varphi \cap N$ into $\mathcal{H}_\varphi$ is taken as the restriction of $x \mapsto x_\varphi$ to $n_\varphi$, $\mathcal{H}_\varphi$ is identified with a subspace of $\mathcal{H}_\varphi$. Let $P$ be the orthogonal projection of $\mathcal{H}_\varphi$ onto $\mathcal{H}_\varphi$, then $P \in \pi_\varphi(N)'$ and $\pi_\varphi(x) = \pi_\varphi(x)P$ for all $x \in N$. The conditional expectation $\epsilon: M \to N$ with respect to $\varphi$ is introduced in [1] by

$$\pi_\varphi(\epsilon(x)) = J_\varphi PJ_\varphi \pi_\varphi(x)J_\varphi, \quad x \in M,$$

which is a faithful normal completely positive map of $M$ into $N$ such that $\epsilon(1) = 1$ and $\varphi(x) = \varphi(\epsilon(x))$ for all $x \in M_+$ (cf. [1, Theorem 7.5]). We here call $\epsilon$ the generalized conditional expectation with respect to $\varphi$. This coincides with the conditional expectation (as a norm one projection onto $N$) with respect to $\varphi$ whenever the latter exists.

In the following lemmas, we assume that $\psi = \varphi \uparrow N$ is semifinite and $\epsilon: M \to N$ is the generalized conditional expectation with respect to $\varphi$.

**Lemma 1.** If $x \in n_\varphi$, then $\epsilon(x) \in n_\varphi$ and

$$(\epsilon(x))_\varphi = J_\varphi PJ_\varphi x_\varphi.$$

**Proof.** Let $x \in n_\varphi$. Then $\epsilon(x) \in n_\varphi$ follows from

$$\psi(\epsilon(x)^*\epsilon(x)) \leq \psi(\epsilon(x^*x)) = \varphi(x^*x) < \infty.$$
Choosing a net \( \{a_j\} \) in \( m_\psi \cap N_+ \) with \( a_j \not\to 1 \), we have
\[
(e(x))_\psi = \lim_j J_\psi \pi_\psi (a_j) J_\psi (e(x))_\psi = \lim_j \pi_\psi (e(x)) J_\psi (a_j)_\psi
= \lim_j J_\psi P J_\psi \pi_\psi (x) J_\psi (a_j)_\psi = \lim_j J_\psi P \pi_\psi (a_j) J_\psi x_\psi
= J_\psi P J_\psi x_\psi. \quad \text{Q.E.D.}
\]

**Lemma 2.** If \( \gamma \) is a \(*\)-automorphism of \( M \) and \( \varepsilon : M \to \gamma^{-1}(N) \) is the generalized conditional expectation with respect to \( \varphi \circ \gamma \), then \( \varepsilon = \gamma^{-1} \circ \varepsilon \circ \gamma \).

**Proof.** Let \( \tilde{\varphi} = \varphi \circ \gamma, \tilde{\psi} = \tilde{\varphi} \downarrow \gamma^{-1}(N) \) and \( \tilde{P} \) be the orthogonal projection of \( \mathcal{K}_{\tilde{\varphi}} \) onto \( \mathcal{K}_{\tilde{\psi}} \). There is a unitary operator \( U \) of \( \mathcal{K}_{\tilde{\varphi}} \) onto \( \mathcal{K}_{\tilde{\psi}} \) such that \( UX_\varphi = (\gamma(x))_\varphi \) for all \( x \in \mathfrak{n}_\tilde{\varphi} \). Then it is readily seen that \( P = U \tilde{P} U^*, J_\varphi = U \tilde{J_\psi} U^* \) and \( \pi_\varphi (\gamma(x)) = U \pi_\varphi (x) U^* \) for all \( x \in M \). Hence
\[
\pi_\psi (\gamma(\varepsilon(x))) = U \pi_\varphi (\varepsilon(x)) U^* = UJ_\varphi \tilde{P} J_\psi \pi_\varphi (x) J_\psi \tilde{J_\psi} U^* = J_\psi P J_\psi \pi_\psi (\gamma(x)) J_\psi \pi_\psi (x) J_\psi = \pi_\psi (\varepsilon(\gamma(x))), \quad x \in M,
\]
so that we have \( \gamma \circ \varepsilon = \varepsilon \circ \gamma \). \quad \text{Q.E.D.}

2. **Increasing case.** In this section we discuss the martingale convergence of generalized conditional expectations for the increasing case. We fix an increasing net \( \{N_\alpha\} \) of unital von Neumann subalgebras of \( M \) and let \( N_\infty = \bigvee_\alpha N_\alpha \). Let \( \varphi \) be a faithful normal semifinite weight on \( M \). Assume that \( \psi_\alpha = \varphi \downarrow N_\alpha \) is semifinite for each \( \alpha \) and hence also \( \psi_\infty = \varphi \downarrow N_\infty \) is semifinite. For each \( \alpha \), we take \( n_\alpha = n_\varphi \cap N_\alpha, m_\alpha = n_\alpha^* n_\alpha \), the GNS representation \( (\mathcal{K}_{\alpha}, \pi_\alpha) \) of \( N_\alpha \), the left Hilbert algebra \( \mathfrak{H}_\alpha = \{x_\alpha : x \in n_\alpha \cap n_\alpha^*\} \), the modular operator \( \Delta_\alpha \), the modular conjugation \( J_\alpha \) and the modular automorphism group \( \sigma_\alpha \) associated with \( \psi_\alpha \), then \( \{\mathcal{K}_\alpha\} \) is an increasing net of subspaces of \( \mathcal{K}_\varphi \). Let \( P_\alpha \) be the orthogonal projection of \( \mathcal{K}_\varphi \) onto \( \mathcal{H}_\alpha \). We further take \( n_\infty, m_\infty, (\mathcal{K}_\infty, \pi_\infty), \mathfrak{H}_\infty, \Delta_\infty, J_\infty, \sigma_\infty \) and \( P_\infty \) analogously associated with \( \psi_\infty \). The generalized conditional expectations \( e_\alpha : M \to N_\alpha \) and \( e_\infty : M \to N_\infty \) with respect to \( \varphi \) are given as follows:
\[
\pi_\alpha (e_\alpha(x)) = J_\alpha P_\alpha J_\alpha \pi_\psi (x) J_\alpha J_\alpha, \quad \pi_\infty (e_\infty(x)) = J_\infty P_\infty J_\infty \pi_\psi (x) J_\infty J_\infty, \quad x \in M.
\]

Under the above assumptions and notations, we now state

**Theorem 3.** The following conditions are equivalent:
(i) \( \bigcup_\alpha \mathfrak{H}_\alpha (\subset \mathfrak{H}_\infty) \) is a core of \( \Delta/2 \);
(ii) \( \|e_\alpha(x) - e_\infty(x)\| \to 0 \) for every \( x \in n_\varphi \);
(iii) \( s\text{-lim}_\alpha e_\alpha(x) = e_\infty(x) \) for every \( x \in M \);
(iv) \( \|f \circ e_\alpha - f \circ e_\infty\| \to 0 \) for every \( f \in M_* \).

If \( \varphi \) is bounded (i.e., \( \varphi(1) < \infty \)), then the above conditions (i)–(iv) are fulfilled.

**Proof.** (i) \( \Rightarrow \) (ii). If \( x \in n_\varphi \), then by Lemma 1 we have
\[
(e_\alpha(x))_\varphi = J_\alpha P_\alpha J_\alpha x_\varphi, \quad (e_\infty(x))_\varphi = J_\infty P_\infty J_\infty x_\varphi.
\]
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Therefore the condition (ii) is equivalent to \( s\lim_{a} J_{a} P_{a} = J_{x} P_{x} \), and so we may assume \( N_{x} = M \) and show \( s\lim_{a} J_{a} P_{a} = J_{x} \). Let \( f \in M_{x}^{+} \) with \( f \leq \varphi \) be given. By [16, Theorem 3.2], there exists a unique \( h \in m_{x} \) with \( 0 \leq h \leq 1 \) such that

\[
f(x) = \frac{1}{2} \varphi(hx + xh), \quad x \in m_{x} \cap m_{x}^{*}.
\]

For each \( \alpha \), there further exists a unique \( h_{\alpha} \in m_{\alpha} \) with \( 0 \leq h_{\alpha} \leq 1 \) such that

\[
f(x) = \frac{1}{2} \varphi(h_{\alpha}x + xh_{\alpha}), \quad x \in m_{\alpha} \cap m_{x}^{*}.
\]

Under the assumption (i), the following results (1)–(3) have been proved in [8].

1. \( h_{\varphi} \in D(\Delta), (h_{\alpha})_{\varphi} \in D(\Delta), \| (h_{\alpha})_{\varphi} - h_{\varphi} \| \to 0 \) and \( \| \Delta_{\alpha}(h_{\alpha})_{\varphi} - \Delta_{\varphi} h_{\varphi} \| \to 0 \).

2. The set \( \{ (1 + \Delta)(h_{\varphi}) \} \) is total in \( \mathcal{K}_{\varphi} \) where \( h \in m_{x} \) with \( 0 \leq h \leq 1 \) is taken as above for any \( f \in M_{x}^{+} \) with \( f \leq \varphi \).

3. Let \( \Delta_{\alpha} = \Delta_{\alpha} P_{\alpha} + (1 - P_{\alpha}) \). Then \( s\lim_{a} \Delta_{\alpha}^{\varphi} = \Delta_{\varphi}^{\varphi} \) uniformly for \( t \) in any finite interval.

We now define vector-valued continuous functions \( F_{a}(z) \) on the strip \( 0 \leq \text{Re} z \leq 1 \), analytic in the interior, by

\[
F_{a}(z) = e^{z} \left( \Delta_{\alpha}^{\varphi}(h_{\alpha})_{\varphi} - \Delta_{\alpha}^{\varphi} h_{\varphi} \right), \quad 0 \leq \text{Re} z \leq 1.
\]

Since

\[
\| F_{a}(it) \| \leq e^{-t^{2}} \| (h_{\alpha})_{\varphi} - h_{\varphi} \| + e^{-t^{2}} \| \Delta_{\alpha}^{\varphi} - \Delta_{\varphi} \| h_{\varphi} \|, \\
\| F_{a}(1 + it) \| \leq e^{1 - t^{2}} \| \Delta_{\alpha}(h_{\alpha})_{\varphi} - \Delta_{\varphi} h_{\varphi} \| + e^{1 - t^{2}} \| \Delta_{\alpha}^{\varphi} - \Delta_{\alpha} \| \Delta_{\alpha} h_{\varphi} \|, \quad t \in \mathbb{R},
\]

we have

\[
e^{1/4} \| J_{a}(h_{\varphi})_{\varphi} - J_{a} h_{\varphi} \| = e^{1/4} \| \Delta_{a}^{\varphi/2}(h_{\alpha})_{\varphi} - \Delta_{a}^{\varphi/2} h_{\varphi} \| = \| F_{a}(1/2) \| \\
\leq \sup_{t \in \mathbb{R}} \left( \| F_{a}(it) \|, \| F_{a}(1 + it) \| \right) \to 0
\]

by using (1) and (3). Hence

\[
\| J_{a} P_{a} h_{\varphi} - J_{a} h_{\varphi} \| \leq \| (h_{\alpha})_{\varphi} - h_{\varphi} \| + \| J_{a}(h_{\alpha})_{\varphi} - J_{a} h_{\varphi} \| \to 0.
\]

Moreover, the set \( \{ h_{\varphi} \} \), taken for any \( f \in M_{x}^{+} \) with \( f \leq \varphi \), is total in \( \mathcal{K}_{\varphi} \), because (2) implies

\[
\left\{ h_{\varphi} \right\} = \left( (1 + \Delta_{\varphi})^{-1} \mathcal{K}_{\varphi} \right) = \{ 0 \}.
\]

Thus \( s\lim_{a} J_{a} P_{a} = J_{x} \) is proved.

(ii) \( \Rightarrow \) (i). Suppose that \( s\lim_{a} J_{a} P_{a} = J_{x} P_{x} \) and hence \( s\lim_{a} J_{a} P_{a} J_{x} P_{x} = P_{x} \). Take the generalized conditional expectation \( \epsilon_{\infty, \alpha} : N_{x} \to N_{x} \) with respect to \( \psi_{\infty} \). Given \( x \in m_{\alpha} \cap m_{x}^{*} \), by Lemma 1 we obtain, for each \( \alpha \), \( \epsilon_{\infty, \alpha}(x) \in m_{\alpha} \cap m_{x}^{*} \) and

\[
\| \left( \epsilon_{\infty, \alpha}(x) \right)_{\varphi} - x_{\varphi} \| = \| J_{a} P_{a} J_{x} \varphi - x_{\varphi} \| \to 0, \\
\| \Delta_{\infty}^{\varphi/2}(\epsilon_{\infty, \alpha}(x))_{\varphi} - \Delta_{\infty}^{\varphi/2} x_{\varphi} \| = \| (\epsilon_{\infty, \alpha}(x^{*}))_{\varphi} - (x^{*})_{\varphi} \| = \| J_{a} P_{a} J_{x} (x^{*})_{\varphi} - (x^{*})_{\varphi} \| \to 0.
\]

Thus (i) is satisfied.
(i) ⇒ (iii). Given \( x \in M \) and \( \xi \in \mathcal{N}_\infty \), we have
\[
\| \pi_\infty (\varepsilon_x(x)) \xi - \pi_\infty (\varepsilon_\infty(x)) \xi \|
\leq \| \pi_\infty (\varepsilon_x(x)) (P_\infty - \xi) \| + \| \pi_\infty (\varepsilon_x(x)) P_\infty \xi - \pi_\infty (\varepsilon_\infty(x)) \xi \|
\leq \| x \| \| P_\infty \xi - \xi \| + \| J_\infty P_\infty \| \| P_\infty \xi - \xi \|
- \| J_\infty P_\infty \xi - \xi \|
\rightarrow 0,
\]
because \( s-lim_a J_\infty P_\infty = J_\infty P_\infty \) and hence \( s-lim_a P_\infty = P_\infty \). Thus \( s-lim_a \varepsilon_x(x) = \varepsilon_\infty(x) \).

(i) ⇒ (iv). Since \( \pi_\infty (N_\infty) \) on \( \mathcal{N}_\infty \) is a standard form of \( N_\infty \), given \( f \in M_\infty^* \) there is a \( \xi \in \mathcal{N}_\infty \) such that \( f(x) = \langle \pi_\infty(x) \xi, \xi \rangle \) for all \( x \in N_\infty \) (cf. \([7, \text{Lemma 2.10}]\)). For every \( x \in M \), we have
\[
| f(\varepsilon_x(x)) - f(\varepsilon_\infty(x)) | \leq \langle \pi_\infty (\varepsilon_x(x)) (\xi - P_\infty \xi), \xi \rangle
+ \langle \pi_\infty (\varepsilon_x(x)) P_\infty \xi, \xi \rangle - \langle \pi_\infty (\varepsilon_\infty(x)) \xi, \xi \rangle
\leq \| x \| \| P_\infty \xi - \xi \| \| \xi \|
+ \langle \pi_\infty (x) J_\infty P_\infty \xi, \xi \rangle
- \langle \pi_\infty (x) J_\infty P_\infty \xi, \xi \rangle
\leq \| x \| \| \xi \| (\| P_\infty \xi - \xi \| + 2 \| J_\infty P_\infty \xi - J_\infty P_\infty \xi \|).
\]
This shows that (i) implies (iv).

(iii) ⇒ (ii) and (iv) ⇒ (ii). If either (iii) or (iv) holds, then \( w-lim_a \varepsilon_x(x) = \varepsilon_\infty(x) \) for all \( x \in M \). For every \( x \in \eta_\infty \), noting that \( \{(\varepsilon_x(x))_\infty\} \) is bounded in \( \mathcal{N}_\infty \), let \( \xi \) be any weak accumulation point of \( \{(\varepsilon_x(x))_\infty\} \) and choose a subnet \( \{(\varepsilon_x(x))_\infty\} \) of \( \{(\varepsilon_x(x))_\infty\} \) such that \( \varepsilon_x(x) \) is weak. Since \( w-lim_a \varepsilon_x(x) = \varepsilon_\infty(x) \), we then have \( \xi = (\varepsilon_\infty(x))_\infty \) (cf. \([10, \text{p. 28}]\)). Hence it follows that \( (\varepsilon_x(x))_\infty \rightarrow (\varepsilon_\infty(x))_\infty \) weakly. On the other hand, \( \| (\varepsilon_x(x))_\infty \| \leq \| (\varepsilon_\infty(x))_\infty \| \) holds for each \( \alpha \) by Lemma 1. Thus \( \| (\varepsilon_x(x))_\infty \| - (\varepsilon_\infty(x))_\infty \| \rightarrow 0 \).

Finally let \( \varphi \) be bounded. For every \( x \in N_\infty \), there is a net \( \{ x_j \} \) in \( \bigcup \alpha \mathcal{N}_\infty \) such that \( s-lim_j x_j = x \) and \( s-lim_j x_j^* = x^* \). We then have \( \| (x_j) - (x^*) \|_{\varphi} \rightarrow 0 \) and \( \| (x_j^*) - (x^*)^* \|_{\varphi} \rightarrow 0 \). Hence the condition (i) is satisfied. Q.E.D.

Remarks. (1) When each \( \varepsilon_x(x) \) is the conditional expectation (as a norm one projection) with respect to \( \varphi \), the martingale convergence properties (iii) and (iv) are satisfied (see \([13]\)).

(2) Under the condition (i), it can be proved (see \([8]\)) that \( s-lim_a \gamma^x(x) = \gamma^\infty(x) \) uniformly for \( t \) in any finite interval for every \( x \in \bigcup \alpha \mathcal{N}_\infty \). In this case, it further holds that \( s-lim_a \gamma^x(\varepsilon_x(x)) = \gamma^\infty(\varepsilon_\infty(x)) \) uniformly for \( t \) in any finite interval for every \( x \in M \).

3. Decreasing case. In this section we deal with the convergence of generalized conditional expectations for the decreasing case. Let \( \{ N_\alpha \} \) be a decreasing net of unital von Neumann subalgebras of \( M \) with \( N_\infty = \bigcap \alpha N_\alpha \). Let \( \varphi \) be a faithful normal semifinite weight on \( M \) such that \( \psi_\infty = \varphi \upharpoonright N_\infty \) is semifinite and hence each \( \psi_\alpha = \varphi \upharpoonright N_\alpha \) is semifinite. We use the notations \( \eta_\alpha, \eta_\infty, (\mathcal{N}_\alpha, \pi_\alpha, \Delta_\alpha, J_\alpha, P_\alpha) \) and \( \eta_\infty, \eta_\infty, (\mathcal{N}_\infty, \pi_\infty, \Delta_\infty, J_\infty, P_\infty) \) as in \( \S 2 \). We take the generalized conditional expectations \( \varepsilon_\alpha : M \rightarrow N_\alpha \) and \( \varepsilon_\infty : M \rightarrow N_\infty \) with respect to \( \varphi \).
Theorem 4. Under the above assumptions, consider the following conditions:
(i) $s\text{-lim}_a P_a = P_\infty$ (i.e., $\cap_a \mathcal{K}_a = \emptyset$);
(ii) $\|\phi(a(x)) - \phi_\infty(x)\| \to 0$ for every $x \in n_\varphi$;
(iii) $s\text{-lim}_a \phi_a(x) = \phi_\infty(x)$ for every $x \in M$;
(iv) $\|f \circ \phi_a - f \circ \phi_\infty\| \to 0$ for every $f \in M^*$.

Then the conditions (i) and (ii) are equivalent and the condition (i) implies (iii) and (iv).

If $\varphi$ is bounded, then the conditions (i)–(iii) are equivalent.

Proof. (i) $\Rightarrow$ (ii). By Lemma 1, the condition (ii) is equivalent to $s\text{-lim}_a J_a P_a = J_\infty P_\infty$. For each $f \in M_\varphi^+$ with $f \leq \varphi$, there exist, by [16, Theorem 3.2], unique $h_a \in m_\alpha$ and $h_\infty \in m_\infty$ with $0 \leq h_a \leq 1$, $0 \leq h_\infty \leq 1$ such that

$$f(x) = \frac{1}{2} \varphi(h_a x + x h_a), \quad x \in n_a \cap n_\alpha^*;$$

$$f(x) = \frac{1}{2} \varphi(h_\infty x + x h_\infty), \quad x \in n_\infty \cap n_\varphi^*.$$

Under the assumption (i), the following results (1)–(3) hold (see [8]).

1. $(h_a)_\varphi \in D(\Delta_a)$, $(h_\infty)_\varphi \in D(\Delta_\infty)$, $\|(h_a)_\varphi - (h_\infty)_\varphi\| \to 0$ and $\|\Delta_\alpha(h_a)_\varphi - \Delta_\infty(h_\infty)_\varphi\| \to 0$.

2. The set $\{(1 + \Delta_\alpha)(h_\infty)_\varphi\}$ is total in $\mathcal{H}_\infty$ where $h_\infty \in m_\infty$ with $0 \leq h_\infty \leq 1$ is taken as above for any $f \in M_\varphi^+$ with $f \leq \varphi$.

3. Let $\tilde{\Delta}_a = \Delta_a P_a + (1 - P_a)$ and $\tilde{\Delta}_\infty = \Delta_\infty P_\infty + (1 - P_\infty)$. Then $s\text{-lim}_a \tilde{\Delta}_a = \tilde{\Delta}_\infty$ uniformly for $t$ in any finite interval.

Defining

$$F_a(z) = e^{-z} e^{-2(\Delta_a)_\varphi - \Delta_\infty(h_\infty)_\varphi}, \quad 0 \leq \Re z \leq 1,$$

and using (1) and (3), we have

$$\|F_a(\frac{1}{2})\| = e^{1/4} \|J_a(h_a)_\varphi - J_\infty(h_\infty)_\varphi\| \to 0$$

as in the proof of Theorem 3. Hence

$$\|J_a(h_\infty)_\varphi - J_\infty(h_\infty)_\varphi\| \leq \|\phi_a - \phi_\infty\| + \|J_a(h_a)_\varphi - J_\infty(h_\infty)_\varphi\| \to 0.$$
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(i) ⇒ (iv). Since \( \pi_\varphi(M)^* \mathcal{H} \) is dense in \( \mathcal{H} \), it suffices to prove (iv) for \( f \in M_\varphi^* \) such that

\[
f(x) = \langle \pi_\varphi(x) a' \xi, a' \xi \rangle, \quad x \in M,
\]

where \( a' \in \pi_\varphi(M)' \) and \( \xi \in \mathcal{H} \). For this \( f \), we have

\[
|f(\epsilon_\alpha(x)) - f(\epsilon_\infty(x))| = |\langle \pi_\varphi(\epsilon_\alpha(x)) \xi, a^* a' \xi \rangle - \langle \pi_\varphi(\epsilon_\infty(x)) \xi, a^* a' \xi \rangle|
\]

\[
= |\langle \pi_\varphi(x) J_\varphi J_\infty \xi, J_\infty J_\varphi J_\infty \xi \rangle - \langle \pi_\varphi(x) J_\varphi J_\infty \xi, J_\varphi J_\infty \xi \rangle| 
\]

\[
\leq \|x\| \left( \|J_\infty \xi - J_\infty \xi\| + \|\xi\| \right). 
\]

If (i) is satisfied, then it follows from the proof of (i) ⇒ (ii) that \( \|J_\alpha \xi - J_\infty \xi\| \to 0 \) for all \( \xi \in \mathcal{H} \). Therefore \( \|f \circ \epsilon_\alpha - f \circ \epsilon_\infty\| \to 0 \).

Finally if \( \varphi \) is bounded, then it is clear that (iii) implies (ii). Q.E.D.

REMARKS. (1) If each \( \epsilon_\alpha \) is the conditional expectation (as a norm one projection) with respect to \( \varphi \), then the martingale convergence properties (iii) and (iv) hold (see [1,3]). In this case, \( (\epsilon_\alpha(x))_\varphi = P_\alpha x_\varphi \) and \( (\epsilon_\infty(x))_\varphi = P_\infty x_\varphi \) for all \( x \in M \). Hence the conditions (i) and (ii) are also satisfied (see the proof of (iii) ⇒ (ii) in Theorem 3).

(2) We can find a von Neumann algebra \( M \) with a cyclic and separating vector \( \xi \) and a decreasing net \( \{\mathcal{H}_\alpha\} \) of von Neumann subalgebras of \( M \) such that \( \xi \) is cyclic for each \( \mathcal{H}_\alpha \) and \( \mathcal{H}_\infty = \bigcap_\alpha \mathcal{H}_\alpha = C \xi \) (cf. [3. Example 1]). Let \( \varphi(x) = \langle x \xi, \xi \rangle \) for \( x \in M \), then \( \mathcal{H}_\alpha = \mathcal{H}_\varphi \) for each \( \alpha \) and \( \mathcal{H}_\infty = C \xi \). Thus the conditions (i)–(iii) in Theorem 4 are not satisfied for this case.

(3) Under the condition (i), it follows (see [8]) that \( s\text{-lim}_\alpha \sigma^\alpha_t(x) = \sigma^\infty_t(x) \) uniformly for \( t \) in any finite interval for every \( x \in M_\infty \), and further \( s\text{-lim}_\alpha \varphi_t(\epsilon_\alpha(x)) = \varphi_t(\epsilon_\infty(x)) \) uniformly for \( t \) in any finite interval for every \( x \in M \).

As an example we lastly consider nonabelian \( K \)-flows studied by Emch [5,6]. A nonabelian \( K \)-flow is described by \( (M, \varphi, \gamma, N_0) \) where \( M \) is a von Neumann algebra, \( \varphi \) is a faithful normal state on \( M \), \( \gamma \) is a one-parameter automorphism group of \( M \), and \( N_0 \) is a von Neumann subalgebra of \( M \) such that \( \varphi \) is \( \gamma_t \)-invariant and

\[
N_0 \subseteq \gamma_t(N_0), \quad t \geq 0,
\]

\[
\bigcap_{t \in \mathbb{R}} \gamma_t(N_0) \xi = C \xi, \quad \bigvee_{t \in \mathbb{R}} \gamma_t(N_0) = M.
\]

Here it is not necessarily assumed that \( \sigma^\varphi_t(N_0) = N_0 \) for all \( t \in \mathbb{R} \). Let \( N_t = \gamma_t(N_0) \) and \( \epsilon_t : M \to N_t \) be the generalized conditional expectation with respect to \( \varphi \). Then \( \{N_t : t \in \mathbb{R}\} \) is an increasing net of unital von Neumann subalgebras of \( M \) with \( \bigvee_{t \in \mathbb{R}} N_t = M \) and \( \bigcap_{t \in \mathbb{R}} N_t = C \xi \), and the condition (i) in Theorem 3 (resp. Theorem 4) is satisfied for \( t \to \infty \) (resp. \( t \to -\infty \)). Moreover, it follows from Lemma 2 that \( \epsilon_t = \gamma_t \circ \epsilon_0 \circ \gamma_{-t} \) for all \( t \in \mathbb{R} \). Therefore we conclude by Theorems 3 and 4 that for every \( x \in M \) and \( f \in M_\varphi^* \),

\[
ts\text{-lim}_{t \to \infty} \epsilon_t(x) = x, \quad s\text{-lim}_{t \to -\infty} \epsilon_t(x) = \epsilon_\infty(x) = \varphi(x)1,
\]

\[
\|f \circ \gamma_t \circ \epsilon_0 - f \circ \epsilon_0\| = \|f \circ \epsilon_t - f\| \to 0 \quad \text{(as } t \to \infty\text{)},
\]

\[
\|f \circ \gamma_t \circ \epsilon_0 - f(1)\varphi\| = \|f \circ \epsilon_t - f \circ \varphi\| \to 0 \quad \text{(as } t \to -\infty\text{)}.
\]
References


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