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THE STRUCTURE OF GALOIS GROUPS OF *CM*-FIELDS

BY

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ABSTRACT. A *CM*-field K defines a triple (G, H, ρ), where G is the Galois group of the Galois closure of K, H is the subgroup of G fixing K, and $\rho \in G$ is induced by complex conjugation. A "ρ-structure" identifies *CM*-fields when their triples are identified under the action of the group of automorphisms of G. A classification of the ρ-structures is given, and a general formula for the degree of the reflex field is obtained. Complete lists of ρ-structues and reflex fields are provided for $[K:\mathbb{Q}] = 2n$, with $n = 3, 4, 5$ and 7. In addition, simple degenerate Abelian varieties of *CM*-type are constructed in every composite dimension. The collection of reflex fields is also determined for the dihedral group $G = D_{2n}$, with n odd and H of order 2, and a relative class number formula is found.

Let K be a *CM*-field with $[K:\mathbb{Q}] = 2n$, and let K^c be the Galois closure of K over $\mathbb{Q}$, where $\mathbb{Q}$ is the field of rational numbers. In the theory of complex multiplication, the reflex field K' of a *CM*-type is a fundamental notion. Recall that a *CM*-type (K, Φ) specifies a set $\Phi = \{\phi_1, \ldots, \phi_n\}$ of n embeddings of K into $\mathbb{C}$ so that every embedding is among $\{\phi_1, \bar{\phi}_1, \ldots, \phi_n, \bar{\phi}_n\}$. It is known that $[K':\mathbb{Q}]$ depends upon $[K:\mathbb{Q}]$ and Φ in a rather complex way. When $K/\mathbb{Q}$ is Abelian, K' is a subfield of K. Shimura has shown that there exist *CM*-fields such that the maximal degree $[K':\mathbb{Q}] = 2^n$ occurs, and that there are cases with $K \subsetneq K'$. A general formula for the degree of the reflex field given in §1, is one of our main results.

An Abelian variety A with complex multiplication of type (K, Φ) is called degenerate when the rank $t(\Phi)$, defined in §3, is less than the maximal value of $n + 1$. The problem of computing the rank was proposed by Kubota, and in the case $K/\mathbb{Q}$ Abelian, Lemma 2 of Kubota [**8**] gives a formulation in terms of group characters. The Theorem of §3.1.1 reduces the calculation of the rank to linear algebra when K has an imaginary quadratic subfield, a case emphasized in this context by Weil [**24**]. While several degenerate cases with A simple have been constructed, Ribet [**15**] proves that there are no such examples with $n = p$, for p a prime. In §3 a converse to Ribet's Theorem is proved by constructing simple degenerate Abelian varieties in every composite dimension.

Shimura's Theorem 2.5 of [**21**] gives $(n + 1 - t(\Phi))$ algebraic relations among certain transendental numbers arising as the periods of Abelian integrals, so the actual values of $t(\Phi)$ are of special interest (cf. [**1**]). The present result is that when n

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has a factorization $n = kl$, $k > 2$, $l \geqslant 2$, there is a type Φ with $t(\Phi) = n - l + 2$ for the cyclotomic fields K, with $\mathrm{Gal}(K/\mathbb{Q}) = \langle \rho \rangle \times \mathbb{Z}_n$, where $\mathbb{Z}_n$ denotes the cyclic group of order n, and ρ is the automorphism induced by complex conjugation. Further, when certain non-Abelian totally real fields exist, the same rank occurs for $K/\mathbb{Q}$ non-Galois, and smaller ranks are obtained when n is even or divisible by a square.

Suppose the Galois groups of the Galois closures of the totally real fields of degree n are known, and consider, for example, the problem of listing the values of n' that occur as $n' = \frac{1}{2}[K' : \mathbb{Q}]$ for the reflex field K' of a *CM*-type for a *CM*-field of degree $2n$; or the problem of listing the values of $t(\Phi)$, for primitive types Φ, when n is composite. Group theoretic techniques are introduced in §§1 and 2 to obtain such information. Generating permutations and defining relations for the distinct permutation groups of degree n that occur is sufficient information about the groups of totally real fields (cf. the Classification Theorem of §2.3). The relation of these calculations to the structure of the group of automorphisms of the Galois group $\mathrm{Gal}(K^c/\mathbb{Q})$ is described in §6.

To illustrate the present methods, complete information is given for the reflex fields when $n = 3, 4, 5$, and 7. The list for $n = 4$ uses the methods of group cohomology introduced in §2, while results for the other values of n are obtained from the classical lists of permutation groups of degree $2n = 6$, 10 and 14. The distribution of the values of $t(\Phi)$ is given for $n = 4$ by the Theorem of §3.3.2, and a procedure for the composite values with $n < 16$ is sketched in §5.3 with $n = 6$.

A related arithmetic topic is taken up for the example of the dihedral group $G = D_{2n}$, n odd. The equivalence classes of reflex types introduced in Shimura [**21**, Remark 2.4] are also determined for this case. A formula relating the relative class numbers of Shimura [**17**, **20**] is given in §4. Professor Shimura's suggestion of a more general formula involving the equivalence classes of reflex types initiated the author's interest in the present investigation.

The author wishes to acknowledge and express his appreciation for Professor Shimura's comments and encouragement. In particular, Professor Shimura patiently explained the reflex principle to the author; brought the work of Ribet to the author's attention; and suggested that the reflex fields for the dihedral group and the corresponding chracters be examined. At several points in the investigation the author was influenced by the comments of Professor Weil. The author also wishes to thank Dr. Yoshida for his comments on an earlier manuscript and for providing several corrections. Finally, the author wishes to thank the referee for the observation that the results deserved to be rewritten.

1. The reflex degree theorem.

1.1. A *CM-field* is a totally imaginary quadratic extension of a totally real field K_0. Let K_0^c be the Galois closure of K_0 over $\mathbb{Q}$.

PROPOSITION. *The Galois group of a CM-field of degree* $2n$ *is given by an exact sequence*

$$0 \to (\mathbb{Z}_2)^v \to \mathrm{Gal}(K^c/\mathbb{Q}) \to \mathrm{Gal}(K_0^c/\mathbb{Q}) \to 1,$$

$1 \leqslant v \leqslant n$, *where* $(\mathbb{Z}_2)^v$ *is identified with the subgroup* $\mathrm{Gal}(K^c/K_0^c)$ *of* $\mathrm{Gal}(K^c/\mathbb{Q})$.

PROOF. By Galois theory only the identification of the normal subgroup $\mathrm{Gal}(K^c/K_0^c)$ with $(\mathbb{Z}_2)^v$ must be established. Write $K = K_0((-\delta)^{1/2})$, with $\delta \in K_0$, let $\delta_1, \ldots, \delta_n$ be the conjugates of δ over $\mathbb{Q}$, and observe that $K^c = K_0^c((-\delta_1)^{1/2}, \ldots, (-\delta_n)^{1/2})$. Every automorphism ε of K^c fixing K_0^c is determined by the images $(-\delta_1)^{1/2} \to \varepsilon_1(-\delta_1)^{1/2}, \ldots, (-\delta_n)^{1/2} \to \varepsilon_n(-\delta_n)^{1/2}$, with $\varepsilon_j = \pm 1$, $j = 1, \ldots, n$. Then $\mathrm{Gal}(K^c/K_0^c)$ may be identified with the image in $(\mathbb{Z}_2)^n$ of the map $\varepsilon \to (e_1, \ldots, e_n) \in (\mathbb{Z}_2)^n$, with e_j defined by $\varepsilon_j = (-1)^{e_j}$, $j = 1, \ldots, n$. Note that the automorphism ρ of K^c induced by the complex conjugation has image $(1, \ldots, 1)$, so $v \geqslant 1$. □

Let $\mathrm{Im}(n, 2) \subset S_{2n}$ denote the maximal imprimitive subgroup of the symmetric group S_{2n} admitting a specified n sets of order 2 as sets of imprimitivity. More explicit Galois theoretic information will be obtained from the following:

IMPRIMITIVITY THEOREM. *Let G be an abstract group with $G \cong \mathrm{Gal}(K^c/\mathbb{Q})$ for K a CM-field of degree $2n$. Then G may be represented as an imprimitive permutation group of degree $2n$ with n sets of imprimitivity of order 2 so that*

$$G = \bigcup_{\sigma \in G_0} (\mathbb{Z}_2)^v (s(\sigma), \sigma),$$

where:

(1) $G_0 \cong \mathrm{Gal}(K_0^c/\mathbb{Q})$ *is given as a transitive permutation group of degree n and may be identified with the group of permutations of the sets of imprimitivity in the representation of G;*

(2) $(\mathbb{Z}_2)^v \cong \mathrm{Gal}(K^c/K_0^c)$ *is identified with the group of permutations preserving the sets of imprimitivity and is acted upon by G_0 by permutation of coordinates under an inclusion i: $(\mathbb{Z}_2)^v \to (\mathbb{Z}_2)^n$; and*

(3) $(s(\sigma), \sigma) \in \mathrm{Im}(n, 2) \cong (\mathbb{Z}_2)^n \times_s S_n$, *the semidirect product, is a lift of $\sigma \in G_0$, so that, for the mapping s: $G_0 \to (\mathbb{Z}_2)^n$, the mapping $j \circ s$: $G_0 \to (\mathbb{Z}_2)^n/(\mathbb{Z}_2)^v$ is a 1-cocycle, j being the projection mapping.*

PROOF. Fix the isomorphism $G \cong \mathrm{Gal}(K^c/\mathbb{Q})$ and let $H \subset G$ be the subgroup corresponding to $\mathrm{Gal}(K^c/K)$. Note that, since K^c is the minimal Galois closure, the action of G on the coset space $H \backslash G$ is effective. Since K/K_0 is a quadratic extension there is a subgroup S of G with $H \subset S$ and indices $(S : H) = 2, (G : S) = n$, giving that the action of G on $H \backslash G$ admits the imprimitivity as asserted. Next observe that the kernel of the action of G on the coset space $S \backslash G$ is the subgroup of G fixing all conjugates of S, so that the identifications of (1) and (2) hold.

Identify G with the image of G under permutation representation on the cosets of H, and let $\mathrm{Im}(n, 2)$ be the maximal subgroup of S_{2n} admitting the sets of imprimitivity given by using the n cosets of S to specify the n sets of cosets of H. First note that $(\mathbb{Z}_2)^v$ is a subgroup of the maximal subgroup $(\mathbb{Z}_2)^n$ of $\mathrm{Im}(n, 2)$ preserving the n sets of imprimitivity. Next observe that the sequence $0 \to (\mathbb{Z}_2)^n \to \mathrm{Im}(n, 2) \to S_n \to 1$ has a splitting that may be explicitly given by lettering the sets of imprimitivity as $\{\pm 1\}, \ldots, \{\pm n\}$ and lifting $\sigma \in S_n$ by the mapping $\sigma \to \sigma_+\sigma_-$ defined by $+j \to +\sigma(j)$, $-j \to -\sigma(j)$, $j = 1, \ldots, n$. The permutation action of G_0 then follows from

the inclusion $G_0 \subset S_n$ and the S_n-action on $(\mathbb{Z}_2)^n$. Now, with

$$G \subset (\mathbb{Z}_2)^n \times_s G_0 \subset (\mathbb{Z}_2)^n \times_s S_n \cong \mathrm{Im}(n,2),$$

define s: $G_0 \to (\mathbb{Z}_2)^n$ by picking an arbitrary lift $(s(\sigma),\sigma) \in G$, for each $\sigma \in G_0$; and observe that the subset of $(\mathbb{Z}_2)^n \times_s G_0$ determined by $(\mathbb{Z}_2)^v \subset (\mathbb{Z}_2)^n$ and s is closed under multiplication if and only if $j \circ s$ is a crossed homomorphism. □

In view of the identifications made in the Imprimitivity Theorem, the exact sequence of the proposition will be referred to as an *imprimitivity sequence* for $\mathrm{Gal}(K^c/\mathbb{Q})$. For any (transitive) $G \subset \mathrm{Im}(n,2)$, the projection $G \to G_0$ will be denoted by $\mathrm{Proj}_{\mathrm{Sets}}$, or by Proj_{G_0}, and the exact sequence will be referred to as an *imprimitivity sequence* for G, regardless of the existence of a *CM*-field K and an isomorphism $G \cong \mathrm{Gal}(K^c/\mathbb{Q})$ inducing $G_0 \cong \mathrm{Gal}(K_0^c/\mathbb{Q})$.

1.2. Let $\mathbb{C}$ denote the field of complex numbers, and let t_j: $K \to \mathbb{C}$, $j = 1,\ldots,n$, be field embeddings so that every embedding t: $K \to \mathbb{C}$ is among the collection $\{t_1, \bar{t}_1, \ldots, t_n, \bar{t}_n\}$, $\bar{t}_j$ being the embedding given by complex conjugation. Then K has 2^n *CM-types* (K,Φ), where each type Φ on K may be viewed as giving n choices, picking one embedding from each of the sets $\{t_j, \bar{t}_j\}$. Recall that $\mathrm{Gal}(K^c/\mathbb{Q})$ acts on the set of types on K by sending $\Phi = \{\phi_1,\ldots,\phi_n\}$ to the type Φ^g with entries ϕ_j^g given by applying $g \in \mathrm{Gal}(K^c/\mathbb{Q})$ to the image of ϕ_j. Let 1^{st} denote the projection of $(\mathbb{Z}_2)^n$ onto the first coordinate.

PROPOSITION. *Let $G \cong \mathrm{Gal}(K^c/\mathbb{Q})$ be given by an imprimitive permutation representation as in the Imprimitivity Theorem. Then a CM-type Φ may be specified by giving $\mathbf{f} \in (\mathbb{Z}_2)^n$ in such a way that the G-set structure on the set of types is given by writing $g \in G$ as $g = (es(\sigma),\sigma)$, and then sending $\Phi = \Phi^{\mathbf{f}}$ to $\Phi^g = \Phi^{\mathbf{h}}$, with $\mathbf{h} = \sigma^{-1} * (\mathbf{f}es(\sigma))$, the product $\mathbf{f}es(\sigma)$ being taken formally in $(\mathbb{Z}_2)^n$, with G_0 acting by permutation of coordinates.*

PROOF. Let $H_0 \subset G_0$ be the stabilizer of a letter under the inclusion $G_0 \subset S_n$. Give coset representatives τ_j, $j = 1,\ldots,n$, for H_0 in G_0, and let H be the subgroup of G fixing the letters in the set of imprimitivity corresponding to the letter fixed by H_0. Observe that taking $g_j = (s(\tau_j),\tau_j)$, $\rho g_j = (\rho s(\tau_j),\tau_j)$ gives $\{g_1, \rho g_1,\ldots, g_n, \rho g_n\}$ as coset representatives for H in G, where $\rho = ((1,\ldots,1),(1)) \in (\mathbb{Z}_2)^v \times \langle(1)\rangle \subset G$ is the element of G corresponding to the automorphism ρ of K^c over $\mathbb{Q}$ induced by the restriction of complex conjugation. Then use $\mathbf{f} \in (\mathbb{Z}_2)^n$ to specify the type given by the embeddings corresponding to the cosets $\{H\rho^{f_1}g_1,\ldots, H\rho^{f_n}g_n\}$ for $\mathbf{f} = (f_1,\ldots,f_n)$, $f_j \in \mathbb{Z}_2$ being written as 0 or 1.

To verify the G-action, let H_0 stabilize the letter 1, H stabilize the letters in the set $\{\pm 1\}$, and observe that $\rho \in G$ may be used to alter the map s: $G_0 \to (\mathbb{Z}_2)^n$ so that $s(\sigma)$ has first coordinate 0 for all $\sigma \in G_0$. Consider the jth entry $H\rho^{f_j}g_j$ of $\Phi^{\mathbf{f}}$, and the translated coset $H\rho^{f_j}g_jg$. Write $g = (e',\sigma)$, with $e' = es(\sigma)$. Then compute $\rho^{f_j}g_jg = (\rho^{f_j}s(\tau_j)\tau_j * e', \tau_j\sigma)$. The coset of H_0 in G_0 determines which coordinate of $\mathbf{h}$ is being given, and $\tau_j\sigma$ takes l to $\sigma(j)$ (permutation multiplication being read left-to-right), so $\tau_j\sigma$ belongs to the coset $H_0\tau_{\sigma(j)}$.

The entry $h_{\sigma(j)}$ of $\mathbf{h} = (h_1, \ldots, h_n)$ is then determined by deciding whether $\rho^{f_j} g_j g$ belongs to $Hg_{\sigma(j)}$ or to $H\rho g_{\sigma(j)}$. Noting that $1^{st}(s(\sigma)) = 0$ for all $\sigma \in G_0$, the calculation

$$1^{st}\left(\rho^{f_j} s(\tau_j)\tau_j * e'\right) = f_j 1^{st}\left(\tau_j * e'\right) = f_j e'_j,$$

for $e' = (e'_1, \ldots, e'_n)$, gives $h_{\sigma(j)} = f_j e'_j$, where the addition in $\mathbb{Z}_2$ is written multiplicatively. But then the G_0-action allows the entries to be given by $\sigma * \mathbf{h} = \mathbf{f}e'$, so $\mathbf{h} = \sigma^{-1} * (\mathbf{f}es(\sigma))$, recalling $e' = es(\sigma)$.

1.3. The action of the proposition may be temporarily extended to give an action $\mathbf{f} \to \sigma^{-1} * (\mathbf{f}e)$ for all $(e, \sigma) \in (\mathbb{Z}_2)^n \times_s G_0$. For the extended action, observe that requiring (e, σ) to fix $\mathbf{f}$ determines $e = e(\sigma, \mathbf{f})$ uniquely, since $\mathbf{f} = \sigma^{-1} * (\mathbf{f}e)$ implies $e = \mathbf{f}\sigma * \mathbf{f}$. Restricting the action back to $G \subset (\mathbb{Z}_2)^n \times_s G_0$ provides the following:

DEFINITION. The *splitting subgroup* of $G \subset (\mathbb{Z}_2)^n \times_s G_0$ and $\mathbf{f} \in (\mathbb{Z}_2)^n$ is the subgroup $S_0 = S_0(\mathbf{f})$ of G_0 defined by $S_0(\mathbf{f}) = \{\sigma \in G_0 / \mathbf{f}\sigma * \mathbf{f} \in (\mathbb{Z}_2)^v s(\sigma)\}$, where G is given by $(\mathbb{Z}_2)^v$ and s as in the Imprimitivity Theorem.

THE REFLEX DEGREE THEOREM. *Let $K/\mathbb{Q}$ be a CM-field with maximal totally real subfield K_0. Let $0 \to (\mathbb{Z}_2)^v \to G \to G_0 \to 1$ be the imprimivity sequence of $G \cong \mathrm{Gal}(K^c/\mathbb{Q})$ and represent G as in the Imprimitivity Theorem. Let (K, Φ) be a CM-type and $K' = K'(\Phi)$ the reflex field of (K, Φ). Then, when Φ is written in the form $\Phi = \Phi^{\mathbf{f}}$,*

$$[K' : \mathbb{Q}] = 2^v (G_0 : S_0),$$

where $2^v = [K^c : K_0^c]$, and $(G_0 : S_0)$ is the index in G_0 of the splitting subgroup $S_0 = S_0(\mathbf{f})$.

PROOF. Recall that $K'(\Phi)$ is the subfield of K^c fixed by the stabilizer of Φ under the $\mathrm{Gal}(K^c/\mathbb{Q})$-action, so that $[K' : \mathbb{Q}] = (G : H'(\Phi))$, with $H'(\Phi)$ the corresponding subgroup of G. Observe that $H'(\Phi) = \{(\mathbf{f}\sigma * \mathbf{f}, \sigma) / \sigma \in S_0(\mathbf{f})\}$, since $(\mathbf{f}\sigma * \mathbf{f}, \sigma)$ must belong to G. Then $|G| = 2^v |G_0|$ gives

$$(G : H'(\Phi)) = |G| / |H'(\Phi)| = 2^v |G_0| / |S_0| = 2^v (G_0 : S_0),$$

since Proj_{G_0} gives an isomorphism of $H'(\Phi)$ onto S_0.

REMARK. Note that $[K' : \mathbb{Q}]$ is also the order of the orbit of Φ under the G-action. Also observe that $H'(\Phi)$ is explicitly given, at least when $s(\sigma)$ is explicitly given.

2. ρ-structures.

2.1.1. To study the G-set structure on the collection of *CM*-types, a refinement of permutation structure on G is introduced by the following:

DEFINITIONS. Let G and G_1 be abstract groups with central involutions. Let $\rho \in G$ and $\rho_1 \in G_1$ be central involutions, and let H be a subgroup of G and H_1 a subgroup of G_1, with $(G : H) = (G_1 : H_1) = 2n$, so G and G_1 act effectively on the coset spaces $H \backslash G$ and $H_1 \backslash G_1$. Then the triple (G, H, ρ) will be said to be *ρ-equivalent* to the triple (G_1, H_1, ρ_1) if there is an isomorphism $\psi: G \to G_1$, so that $\psi(H) = H_1$, and $\psi(\rho) = \rho_1$. A *ρ-structure* of degree $2n$ will refer to an equivalence class of triples (G, H, ρ) under ρ-equivalence.

REMARK 1. Shimura [**17**] observed that the automorphism ρ of K^c over $\mathbb{Q}$ induced by complex conjugation is central in $\mathrm{Gal}(K^c/\mathbb{Q})$. A verification in the present context may be obtained by noting that ρ is identified with the central element $((1,\ldots,1),(1))$ of $\mathrm{Im}(n,2)$ in which $\mathrm{Gal}(K^c/\mathbb{Q})$ has been effectively represented.

REMARK 2. The above definition is a special case of an "imprimitivity equivalence" defined on triples (G, H, S), with $(G:S)=n$, $(S:H)=k$, and G effective on $H\backslash G$. Note the special circumstances $k=2$ and $S=H\times\langle\rho\rangle$, a direct product with ρ central in G, in the present study. The general case of these "imprimitivity structures" may also be of some Galois theoretic interest.

The present interest in ρ-structures is established by the following:

PROPOSITION. *Let K and K_1 be CM-fields of degree $2n$, and suppose G is an abstract group with $G\cong\mathrm{Gal}(K^c/\mathbb{Q})\cong\mathrm{Gal}(K_1^c/\mathbb{Q})$. Let ρ and ρ_1 be the central involutions of G corresponding to the restriction of complex conjugation to K^c and K_1^c, respectively, and let H and H_1 be the subgroups of G corresponding to* $\mathrm{Gal}(K^c/K)$ *and* $\mathrm{Gal}(K_1^c/\mathbb{Q})$. *Then if (G, H, ρ) is ρ-equivalent to (G, H_1, ρ_1), the collection of CM-types for K and K_1 are equivalent G-sets.*

PROOF. Observe that a ρ-equivalence ψ takes the pairing of cosets of H in G under ρ to the pairing of cosets of H_1 in G under ρ_1, so that the action of G on types commutes with ψ.

REMARK 3. In the case $K^c=K_1^c$, a ρ-equivalence is an automorphism of $\mathrm{Gal}(K^c/\mathbb{Q})$ preserving the distinguished central element ρ. The subgroup of $\mathrm{Aut}(G)$ preserving ρ will be denoted by $\mathrm{Aut}(G,\rho)$. Note that $n=2$, $G=D_4$ dihedral of order 8, gives an example with K and K_1 nonconjugate over $\mathbb{Q}$. An identification of the types for all such K and K_1 may be observed in the treatment of $n=2$ in Shimura and Taniyama [**23**, §8.4(2)].

2.1.2. The technique of imprimitive permutation representations may be introduced into the study of ρ-structures by the following:

PROPOSITION. *Let $G_0\subset S_n$ be a transitive group of degree n and let $(\mathbb{Z}_2)^v$ be a sub-G_0-module of $(\mathbb{Z}_2)^n$, with permutation action, so that $\rho\in(\mathbb{Z}_2)^v$ for $\rho=(1,\ldots,1)$. Then $\bar{s}\in Z^1(G_0,(\mathbb{Z}_2)^n/(\mathbb{Z}_2)^v)$ specifies a unique ρ-structure.*

PROOF. The ρ-structure will be defined on the abstract group G given as a group extension by $c^1([\bar{s}])$, c^1 being the connecting homomorphism

$$c^1\colon H^1\big(G_0,(\mathbb{Z}_2)^n/(\mathbb{Z}_2)^v\big)\to H^2\big(G_0,(\mathbb{Z}_2)^v\big).$$

Note that $G_0\subset S_n$ specifies a subgroup H_0 up to conjugacy, and then a subgroup S of G as the inverse image of H_0 under $G\to G_0$. Note that by hypothesis $\rho\in(\mathbb{Z}_2)^v\subset S$, and ρ is uniquely determined in S. A ρ-structure on G is then determined by using $\bar{s}$ to specify H of index 2 in S as the stabilizer of the two letters in the set fixed by H_0 in the imprimitive permutation representation of G defined by $\bar{s}$.

As a first indication of the relation between group extension equivalence and ρ-equivalence, consider the following:

COROLLARY OF THE REFLEX DEGREE THEOREM. *Let K be a CM-field. Then the ρ-structure determined by an imprimitive permutation representation of $G \cong \mathrm{Gal}(K^c/\mathbb{Q})$ is given by the trivial class $[\bar{0}] \in H^1(G_0, (\mathbb{Z}_2)^n/(\mathbb{Z}_2)^v)$ if and only if K has a reflex field K' with $[K' : \mathbb{Q}] = [K^c : K_0^c] = 2^v$, in which case $\mathrm{Gal}(K^c/K') \cong \mathrm{Gal}(K_0^c/\mathbb{Q})$ gives a splitting of the imprimitivity sequence of $\mathrm{Gal}(K^c/\mathbb{Q})$ (respectively, $H' \cong G_0$ for G).*

PROOF. If (G, H, ρ) is given by $\bar{s} \in [\bar{0}]$, then there is $b \in (\mathbb{Z}_2)^n$, so $\bar{s}(\sigma) = b\sigma * b(\mathbb{Z}_2)^v$. Take $s(\sigma) = b\sigma * b$ and note that G is given as

$$G = \bigcup_{\sigma \in G_0} (\mathbb{Z}_2)^v (b\sigma * b, \sigma).$$

But then the type defined by $\Phi = \Phi^{\mathbf{f}}$, with $\mathbf{f} = b$, has $H'(\Phi) = \{(b\sigma * b, \sigma) / \sigma \in G_0\} \cong G_0$ giving a splitting and $[K' : \mathbb{Q}] = |G|/|H'| = 2^v|G_0|/|G_0| = 2^v$. Conversely, if K has a type Φ with $[K'(\Phi) : \mathbb{Q}] = 2^v$, represent $\mathrm{Gal}(K^c/\mathbb{Q})$ as in the Imprimitivity Theorem and write the type as $\Phi = \Phi^{\mathbf{f}}$, $\mathbf{f} \in (\mathbb{Z}_2)^n$. Then observe $\sigma \to (\mathbf{f}\sigma * \mathbf{f}, \sigma)$ splits the imprimitivity sequence of $\mathrm{Gal}(K^c/\mathbb{Q})$, and $\bar{s} = j \circ s$, for $s(\sigma) = \mathbf{f}\sigma * \mathbf{f}$, is a coboundary.

REMARK. In the Classification Theorem of §2.3, $[\bar{0}]$ is shown to define a unique ρ-structure of degree $2n$ on $G = (\mathbb{Z}_2)^v \times_s G_0$. The condition on the degree of the reflex field in the corollary will therefore uniquely determine a ρ-structure for each $G_0 \subset S_n$ and $(\mathbb{Z}_2)^v \subset (\mathbb{Z}_2)^n$. This method of specifying the G-set structure on the collection of CM-types is featured in §§3 and 4.

2.2.0. A phenomenon that occurs for permutation groups, and may be observed in the analysis of Miller [**11**] for $n = 6, 7$, forces a distinction between ρ-equivalence and group extension equivalence. The existence of CM-fields whose imprimitivity sequences require this distinction will be established before proceeding with the algebraic definitions. One method for establishing the existence of a CM-field K so that $\mathrm{Gal}(K^c/\mathbb{Q})$ has a given ρ-structure is to give a subgroup analysis of the group $\mathrm{Gal}(L/\mathbb{Q})$, where L is a (non-Abelian) CM-field that is known to exist. Since interesting fields K with K^c proper in L occur, the triple (G, H, ρ) will be said to define a *ρ-structure with kernel* when the action of G on $H \setminus G$ is not effective.

2.2.1 In §5 it will be shown that there are exactly four ρ-structures for $n = 3$.

THEOREM. *There exist CM-fields K_1 such that each of the 4 ρ-structures of degree $2n$ with $n = 3$ occurs as $\mathrm{Gal}(K_1^c/\mathbb{Q})$. The field with structure having $v = 1$ occur as subfields of the fields with structures having $v = 3$. The groups for $v = 3$ are $\mathbb{Z}_2 \times A_4$ and $\mathbb{Z}_2 \times S_4$, and the CM-subfields of L are classified by 5 ρ-structures for $\mathbb{Z}_2 \times A_4$ and 10 ρ-structures for $\mathbb{Z}_2 \times S_4$.*

PROOF. See §5 for the fact that the structures and the groups are as specified. The existence of the fields with $v = 3$ is provided by Shimura [**18**, §1] as an application of results from class field theory, with the general case giving the existence of a field L with $G = (\mathbb{Z}_2)^n \times_s G_0$ whenever there exists a totally real field K_0^c having group

G_0, and the present case resulting for every K_0 totally real cubic. An explicit example may be found in Pohlman [**12**] for $G = \mathbb{Z}_2 \times S_4$. The existence of fields with $v = 1$, if in doubt, will follow from the assertions below on ρ-structues with kernel.

Consider $G = G_0^* \times \mathbb{Z}_2$ for $G_0^* = A_4, S_4$. The splitting here is the splitting of $0 \to \langle \rho \rangle \to G \to G_0 \to 1$, $G_0 \cong \mathrm{Gal}(L^{\langle \rho \rangle}/\mathbb{Q})$, given by a type Φ of L as in the corollary of §2.1.2. Thus, $G_0^* = H'(\Phi) \cong \mathrm{Gal}(L/\mathbb{Q}(\sqrt{-d}))$, with $\mathbb{Q}(\sqrt{-d})$ imaginary quadratic, as may always be arranged whenever (G, ρ) has a ρ-structure with $(G : H \times \langle \rho \rangle) = 2n$ for n odd.

First obtain the permutation structures $(G, H \times \langle \rho \rangle)$ which classify the totally real fields of L admitting CM-subfields of L as quadratic extensions. These may be written as $H \times \langle \rho \rangle = H_0^* \times \langle \rho \rangle$ with $H_0^* \subset G_0^*$. Then for $G_0^* = A_4$, $H_0^* = 1, \mathbb{Z}_2, \mathbb{Z}_3, (\mathbb{Z}_2)^2$, or A_4, and for $G_0^* = S_4$, $H_0^* = 1$, $\langle (12) \rangle$, $\langle (12)(34) \rangle$, $\mathbb{Z}_3$, $(\mathbb{Z}_2)_t^2$, $(\mathbb{Z}_2)_i^2$, $\mathbb{Z}_4$, S_3, D_4, A_4 or S_4, where $(\mathbb{Z}_2)_t^2$ is transitive and $(\mathbb{Z}_2)_i^2$ is intransitive. Then observe that since ρ is unique, the permutation structures (G, H), with H of index two in these $\langle \rho \rangle \times H_0^*$, determine the ρ-structures (as in Proposition 1 of §5.1). Immediate conclusions are that $H = H_0^*$ in $\langle \rho \rangle \times H_0^*$ gives ρ-structures for $n = 4, 6$, and 12 for $G_0^* = A_4$, and for $n = 4, 6, 6, 8, 12, 12$, and 24 for $G_0^* = S_4$. For the omitted cases, H_0^* gives ρ-structures with kernel, the kernel $(\mathbb{Z}_2)_t^2$, for each G_0^*, giving the required Galois cases with $n = 3$, $v = 1$.

The remaining ρ-structures have H of index 2 in $\langle \rho \rangle \times H_0^*$, $H \neq H_0^*$. For $n = 3$, take

$$H = \langle (\rho, (12)(34)), (0, (13)(24)) \rangle$$

for $G_0^* = A_4$, and

$$H = \langle (\rho, (12)(34)), (0, (1234)) \rangle$$

for $G_0^* = S_4$. Note that the present notation is not in conflict with the previous notation used for $\mathbb{Z}_2 \times G_0 = G$, provided that G is given its $n = 4$ structure with G_0^* identified with G_0. The first nontrivial structures are obtained, one for each group, by observing that $\langle (0, (12)(34)) \rangle$ is inequivalent to $\langle (\rho, (12)(34)) \rangle$, since only $(0, (12)(34))$ is a commutator. The final ρ-structure has $n = 6$, $G_0^* = S_4$ and $S = \langle \rho \rangle \times (\mathbb{Z}_2)_i^2$, and is specified by the index two subgroup $H^{nt} \subset S$ given as $H^{nt} = \langle (\rho, (34)), (0, (12)) \rangle$. The inequivalence of this ρ-structure from the above structure with the same S will be established below, while the other ρ-structure with $n = 6$ has the distinct permutation structure of degree 6 on $G_0 \cong S_4$ (cf. §5.3). To check that no further ρ-structures are allowed, either compute H^1 as suggested in §6, or use generators and relations directly, to produce an outer automorphism taking, e.g., $(0, (34))$ to $(\rho, (34))$ for S_4. The remaining index two H are either conjugate to one of the above or to the image of such a conjugate under this automorphism.

2.2.2. DEFINITIONS. Let $G_0 \subset S_n$ and $(\mathbb{Z}_2)^v \subset (\mathbb{Z}_2)^n$ be as usual. The *trivial ρ-structure* will refer to the ρ-structure defined on the split group $G = (\mathbb{Z}_2)^v \times_s G_0$ defined by the 1-cocycle $\bar{s} = \bar{0} \in Z^1(G_0, (\mathbb{Z}_2)^n/(\mathbb{Z}_2)^v)$ as in the Proposition of §2.1.2. A ρ-structure will be said to be *nonsplit* if it is defined by $\bar{s} \in Z^1(G_0, (\mathbb{Z}_2)^n/(\mathbb{Z}_2)^v)$, so that $c^1([\bar{s}]) \neq [0]$ in $H^2(G_0, (\mathbb{Z}_2)^v)$, c^1 being the map c^1: $H^1(G_0, (\mathbb{Z}_2)^n/(\mathbb{Z}_2)^v) \to H^2(G_0, (\mathbb{Z}_2)^v)$. For $\bar{s} \in Z^1(G_0, (\mathbb{Z}_2)^n/(\mathbb{Z}_2)^v)$, a *nontrival*

ρ-structure relative to $\bar{s}$ will refer to a ρ-structure defined by $\bar{s}_1 \in Z^1(G_0, (\mathbb{Z}_2)^n/(\mathbb{Z}_2)^v)$ with $c^1([\bar{s}_1]) = c^1([\bar{s}])$, but inequivalent to the ρ-structure defined by $\bar{s}$. A *nontrivial split ρ-structure* will refer to a nontrivial ρ-structure relative to the initial structure given by $\bar{0}$.

PROPOSITION. *Let L be a CM-field with* $\mathrm{Gal}(L/\mathbb{Q}) \cong \mathbb{Z}_2 \times S_4$, *and let K and K_{nt} be the two CM-fields having K_0 given as the fixed field of $S = \langle \rho \rangle \times (\mathbb{Z}_2)^2_i$ as a common maximal totally real subfield, K and K_{nt} being given by the subgroups H and H^{nt}, of index 2 in S, as specified in the proof of the above theorem. Then K_{nt} defines a nontrivial split ρ-structure inequivalent to the ρ-structure defined by K.*

PROOF. Observe that both fields have $v = 1$, and K has an imaginary quadratic subfield, which therefore occurs as a reflex field K' for a type on K. By contrast, K_{nt} may be seen to have no imaginary quadratic subfield and, therefore, no G-orbit of types containing just two elements. In fact, L has only two imaginary quadratic subfields, the second being the fixed field of the image of $1 \times G_0^*$ under the outer automorphism used in the proof of the theorem. But $(\rho, (12(34)) \in H_{nt}$, so H_{nt} is not a subgroup of either $1 \times G_0^*$ or its image. Thus K and K_{nt} are inequivalent, but may be observed to have identical imprimitivity sequences.

2.3.1. The material of this subsection is not used until §5, and, in particular, is not required for the arithmetic results of §§3 and 4. Representatives for the distinct ρ-structures are provided by the following:

CLASSIFICATION THEOREM. *The ρ-structures of degree $2n$ are classified by*:

(A) *picking representatives $G_0 \subset S_n$ for the distinct permutation structures of degree n*;

(B) *picking representatives for the class of G_0-submodules $(\mathbb{Z}_2)^v \subset (\mathbb{Z}_2)^n$ with $\rho \in (\mathbb{Z}_2)^v$ under identifications by the normalizer $N_{S_n}(G_0)$ of G_0 in S_n; and then,*

(C) *finding the $\overline{N_{S_n}(G_0)}$-orbits of $H^1(G_0, (\mathbb{Z}_2)^n/(\mathbb{Z}_2)^v)$ for G_0 and $(\mathbb{Z}_2)^v$ ranging over the choices in* (A) *and* (B), *where $\overline{N_{S_n}(G_0)}$ is the subgroup of $N_{S_n}(G_0)$ preserving $(\mathbb{Z}_2)^v$.*

PROOF. Let ψ: $G \to G_1$ be a ρ-equivalence. First observe that ψ induces an equivalence on the group actions of the groups obtained from $\mathrm{Proj}_{\mathrm{Sets}}$, so that $G_0 = \mathrm{Proj}_{\mathrm{Sets}}(G)$ and $G_0^1 = \mathrm{Proj}_{\mathrm{Sets}}(G_1)$ may be regarded as equivalent permutation groups. Then the two ρ-structures may be given by imprimitive permutation representations in a common subgroup $(\mathbb{Z}_2)^n \times_s G_0 \subset \mathrm{Im}(n, 2)$, with a single $G_0 \subset S_n$ for each permutation structure of degree n. Next observe that $(\mathbb{Z}_2)^n \times_s N_{S_n}(G_0) \subset \mathrm{Im}(n, 2)$ acts on $(\mathbb{Z}_2)^n \times_s G_0$ by conjugation and is the maximal group having an action that corresponds to a reindexing of the elements $\{\pm 1, \ldots, \pm n\}$ preserving the given sets of imprimitivity $\{\pm 1\}, \ldots, \{\pm n\}$ specified in the choice of $\mathrm{Im}(n, 2)$ and preserving the choice $G_0 \subset S_n$. Note that the action of $0 \times_s N_{S_n}(G_0)$ gives the identifications of (B). Finally, the action of $(\mathbb{Z}_2)^n \times_s 1$ gives $B^1(G_0, -)$-equivalence on the 1-cocycle defining the image of the permutation representation of G, and then only the above $\overline{N_{S_n}(G_0)}$-equivalence is allowed on $H^1(G_0, (\mathbb{Z}_2)^n/(\mathbb{Z}_2)^v)$.

PROPOSITION. *The abstract groups G in the above classification are given as group extensions by fixing the choices in* (A) *and* (B) *and computing*

$$H^1\big(G_0, (\mathbb{Z}_2)^n/(\mathbb{Z}_2)^v\big)/j^1\big(H^1\big(G_0, (\mathbb{Z}_2)^n\big)\big),$$

which is isomorphic to

$$Z^1\big(G_0, (\mathbb{Z}_2)^n/(\mathbb{Z}_2)^v\big)/j^1\big(Z^1\big(G_0, (\mathbb{Z}_2)^n\big)\big),$$

where j^1 is used for the cocycle mapping induced by the projection j: $(\mathbb{Z}_2)^n \to (\mathbb{Z}_2)^n/(\mathbb{Z}_2)^v$, and also for the induced mapping on cohomology classes.

PROOF. In fact, the condition $G \subset (\mathbb{Z}_2)^n \times_s G_0$ assures that $i^2(f) = 0$, for f the 2-cocycle defining the extension, and the inclusion map i^2: $H^2(G_0, (\mathbb{Z}_2)^v) \to H^2(G_0, (\mathbb{Z}_2)^n)$. The assertion follows by exactness of the long exact cohomology sequence. The cocycle formulation, $Z^1/j^1(Z^1)$, is obtained by observing that j^1: $B^1(G_0, (\mathbb{Z}_2)^n) \to B^1(G_0, (\mathbb{Z}_2)^n/(\mathbb{Z}_2)^v)$ is surjective.

2.3.2. A choice of representatives for (A), (B) and (C) in the Classification Theorem will be referred to as a *normalization*, or "normal form", for the elements of a ρ-structure. Some examples will be given here, with further cohomological considerations being postponed to §6.

EXAMPLE 1. For $n = p$, p a prime, any G_0 may be normalized to contain the p-cycle $(1\ 2\ \cdots\ p)$. Note that recent results on 2-transitivity supply a complete list of these G_0, as in [**4**].

EXAMPLE 2. For $n = 4$, the transitive permutation groups with $G_0 \cong \mathbb{Z}_4$, $G_0 \cong D_4$ may be normalized as $\langle(1234)\rangle$, and $\langle(1234), (12)(34)\rangle$. By contrast $G_0 \cong (\mathbb{Z}_2)^2$, G_0 transitive, is uniquely determined as a subgroup of S_4, and allows arbitrary reindexing: $N_{S_4}((\mathbb{Z}_2)^2) = S_4$.

$G_0 = (\mathbb{Z}_2)^2$ also has three submodules of the form $(\mathbb{Z}_2)^v \subset (\mathbb{Z}_2)^n$ for $n = 4$, $v = 2$. A 3-cycle may be used to establish an equivalence between these three submodules, and also among the permutation groups G with $G_0 = (\mathbb{Z}_2)^2$ and $v = 2$. The choice $(\mathbb{Z}_2)^v = \langle \rho, \xi' \rangle$, for $v = 2$, with $\xi' = (0101) \in (\mathbb{Z}_2)^4$, is preferred, since that choice fits the nesting $(\mathbb{Z}_2)^2 \subset D_4$, for D_4 normalized as above.

EXAMPLE 3. Consider $G = \mathbb{Z}_2 \times \mathbb{Z}_4$. First observe that G has 3 central order two elements in two $\mathrm{Aut}(G)$ orbits. The imprimitivity sequence of $\mathrm{Gal}(K^c/\mathbb{Q})$ may be observed to depend upon the isomorphism $G \cong \mathrm{Gal}(K^c/\mathbb{Q})$, since these two classes correspond to K_0 having group $(\mathbb{Z}_2)^2$ or $\mathbb{Z}_4$. The two noncharacteristic central order two elements define a single ρ-structure, although the imprimitivity sequences are inequivalent. The characteristic central order two element gives a ρ-structure that is supplied by a single cocycle group, $Z^1((\mathbb{Z}_2)^2, (\mathbb{Z}_2)^4/\langle\rho\rangle)$. Observe that $|Z^1/j^1(Z^1)| = 8$, while only four abstract groups are obtained. A reindexing of the form $(0, \sigma_\psi)$, with σ_ψ a 3-cycle, gives a ρ-equivalence on the three group extensions with $G \cong \mathbb{Z}_2 \times \mathbb{Z}_4$, and on the three group extensions with $G \cong D_4$, so that the identifications made in (C) of the Classification Theorem are nontrivial. Note that $Z^1/j^1(Z^1)$ classifies subgroups of $(\mathbb{Z}_2)^n \times_s G_0$ up to extension equivalence, which is therefore too strong, in addition to being too weak, as shown in §2.2.

EXAMPLE 4. Nonsplit extensions are supplied by the group

$$G^+ = \{(e, \sigma) \in (\mathbb{Z}_2)^w \times_s G_0 = G/\mathrm{sgn}(e)\mathrm{sgn}(\sigma) = +1\},$$

with $\operatorname{sgn}(e)$ from $e \in (\mathbb{Z}_2)^w \subset (\mathbb{Z}_2)^n \subset S_{2n}$ and $\operatorname{sgn}(\sigma)$ from $\sigma \in G_0 \subset S_n$ (cf. Coxeter and Moser [3]). Whenever G_0 and $(\mathbb{Z}_2)^w$ have odd elements, G^+ will be a nonsplit extension over G_0, with $v = w - 1$, and therefore distinct from the trivial structure on the split group $[(\mathbb{Z}_2)^w]^+ \times_s G_0$, where $[(\mathbb{Z}_2)^w]^+$ is the even subgroup of $(\mathbb{Z}_2)^w$, common to both groups. When n is even both groups contain ρ, while when n is odd, extra structures for G_0^1 of degree $n_1 = 2n$ are obtained.

Note especially the case n even, $w = n$, $v = n - 1$. By the Reflex Degree Theorem, $v = n - 1$ implies that either there are two G-orbits of types, each of order 2^{n-1}, in which case the ρ-structure is trivial, or else all types form a single G-orbit. Further examples with a single G-orbit of types are found for $n = 4$, one with $v = 1$, two with $v = 2$ and several cases with $v = 3$ that are nonsplit but not of the form G^+. For complete results when $n = 4$, see §5.2.

3. Degeneracy in composite dimensions.

3.1.0. Recall that for a *CM*-type (K, Φ), an Abelian variety A with complex multiplication of type (K, Φ) is obtained by the analytic construction of Shimura and Taniyama [23]. The existence of a *CM*-field with a given type therefore assures the existence of Abelian varieties of that type. The *rank* of A, $t(A) = t(\Phi)$, is the rank of the free $\mathbb{Z}$-module M spanned by the $\operatorname{Gal}(K^c/\mathbb{Q})$ orbit of Φ inside the $\mathbb{Z}$-module spanned by the $2n$ embeddings of K into $\mathbb{C}$. Observe that the sum of the coefficients of the embeddings τ and $\bar{\tau}$ is a constant for each element of M, so that the maximal rank is obtained when coefficients for n embeddings, pairwise nonconjugate, and the common value of the sum of the coefficients are given independently. Then the rank of A is $n + 1$ and A is said to be *nondegenerate*. The variety A is said to be *degenerate* when $\operatorname{rank}(A) = \operatorname{rank}(K, \Phi) < n + 1$.

Let (K', Φ') be the reflex type of (K, Φ). Recall that $\operatorname{rank}(K, \Phi) = \operatorname{rank}(K', \Phi')$ (Kubota [8] or Shimura [22]), so $[K' : \mathbb{Q}] \geqslant [K : \mathbb{Q}]$ is a necessary condition for (K, Φ) to be nondegenerate. Also, (K, Φ) is said to be *primitive*, and A is simple, when $((K, \Phi)')' = (K, \Phi)$ (Shimura [19]); and $(K')' \subseteqq K$, with $(K')' = K$ if and only if (K, Φ) is primitive. Observe that if A is reducible,

$$\operatorname{rank}(K, \Phi) = \operatorname{rank}(K', \Phi') = \operatorname{rank}((K', \Phi')') < \tfrac{1}{2}[K : \mathbb{Q}] + 1,$$

so that the distinction between degeneracy and nondegeneracy is meaningful only when (K, Φ) is primitive.

The proposition of §1.2 provides a method for computing the G-orbit, and therefore, the rank $t(\Phi)$ of (K, Φ), subject to explicit information on $G_0 \subset S_n$, $(\mathbb{Z}_2)^v \subset (\mathbb{Z}_2)^n$, and s: $G_0 \to (\mathbb{Z}_2)^n$. Also, the Classification Theorem of §2.3 provides a reduction to the distinct cases. In the following, the *weight* of $\mathbf{f} \in (\mathbb{Z}_2)^n$, $\mathbf{f} = (f_1, \ldots, f_n)$ will refer to the sum $f_1 + \cdots + f_n$, where each f_j is regarded as being given by an integer, $f_j = 0$ or 1, $j = 1, \ldots, n$.

3.1.1. The following Theorem will be referred to as the "constant weight criterion" for degeneracy.

THEOREM. *Suppose the CM-field K has an imaginary quadratic subfield, and let $G_0 \cong \operatorname{Gal}(K_0^c/\mathbb{Q})$ be given by $G_0 \subset S_n$. Then $G \cong \operatorname{Gal}(K^c/\mathbb{Q})$ has the trivial ρ-structure with $v = 1$ on $\langle \rho \rangle \times G_0$. Let $\Phi = \Phi^{\mathbf{f}}$, $\mathbf{f} \in (\mathbb{Z}_2)^n$, be a CM-type on K and let r be*

the rank of the $\mathbb{Z}$-span of the orbit $G_{0*}(\mathbf{f})$, regarded as a subset of $\mathbb{Z}^n$. Then $t(\Phi) = r + 1$, unless the weight of $\mathbf{f}$ is $n/2$, in which case $t(\Phi) = r$ may also occur.

PROOF. Let D be an imaginary quadratic subfield of K. Then K^c is the composite $K^c = DK_0^c$, so that $[K^c : K_0^c] = 2$, i.e., $v = 1$. But then K has two types having D as a reflex field of degree $2 = 2^v$, so that K determines the trivial ρ-structure by the corollary of the Reflex Degree Theorem and the fact that $B^1(G_0, -)$ gives ρ-equivalences (cf. the Classification Theorem). The ρ-structure is then given by $s(\sigma)$ identically 0, so that the G-orbit of $\mathbf{f}$ is $G_{0*}(\mathbf{f}) \cup G_{0*}(\rho\mathbf{f})$.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_r \in G_{0*}(\mathbf{f})$ span $G_{0*}(\mathbf{f}) \subset \mathbb{Z}^n$, still identifying the elements of $\mathbb{Z}_2$ with $0, 1 \in \mathbb{Z}$. Observe that $t(\Phi)$ is the rank of the $\mathbb{Z}$-module spanned by $W = \{(\mathbf{v}, \rho\mathbf{v}) \in \mathbb{Z}^{2n} / \mathbf{v} \in G_*(\mathbf{f}) \subset \mathbb{Z}^n\}$, where $\rho\mathbf{v}$ denotes the multiplication in $(\mathbb{Z}_2)^n$, the result regarded as an element of $\mathbb{Z}^n$. Next, note that the weight $w = \text{weight}(\mathbf{v})$ is constant for $\mathbf{v} \in G_{0*}(\mathbf{f})$, and, in particular, weight $(\mathbf{v}_j) = w$, for $j = 1, \ldots, r$. The assertion of the Theorem will follow from the lemma below, which will be used to show that $(\mathbf{v}_1, \rho\mathbf{v}_1), \ldots, (\mathbf{v}_r, \rho\mathbf{v}_r), (\rho, \rho)$ is a spanning set for W.

LEMMA. *Let $\mathbf{v}_1, \ldots, \mathbf{v}_r \in (\mathbb{Z}_2)^n$ have constant weight w. Suppose every entry of $\mathbf{w} \in \mathbb{Z}^n$ is a 0 or a 1, and that* weight$(\mathbf{w}) = w$. *Then $\mathbf{w} = \sum n_j \mathbf{v}_j$ implies $\rho\mathbf{w} = \sum n_j(\rho\mathbf{v}_j)$.*

PROOF. Consider the kth entry, w_k, in $\mathbf{w}$. Then $w_k = \sum n_j v_{jk}$, for v_{jk} the kth entry of $\mathbf{v}_j$. But observe that the kth entry, ρv_{jk}, of $\rho\mathbf{v}_j$ is defined by $v_{jk} + \rho v_{jk} = 1$, with $v_{jk}, \rho v_{jk} \in \{0, 1\}$. Then

$$w = \text{weight}(\mathbf{w}) = \sum n_j \text{weight}(\mathbf{v}_j) = w\Big(\sum n_j\Big)$$
$$= w\Big(\sum n_j(v_{jk} + \rho v_{jk})\Big) = w\Big(w_k + \Big(\sum n_j \rho v_{jk}\Big)\Big),$$

so the kth entry $\sum n_j \rho v_{jk}$ and w_k satisfy the same defining condition.

Returning to the proof of the Theorem, the lemma gives that $\{(\mathbf{v}, \rho\mathbf{v})/\mathbf{v} \in G_{0*}(\mathbf{f})\}$ is contained in the $\mathbb{Z}$-span of the $(\mathbf{v}_j, \rho\mathbf{v}_j)$, while $(\rho\mathbf{v}, \mathbf{v}) = (\rho, \rho) - (\mathbf{v}, \rho\mathbf{v})$ accounts for $G_{0*}(\rho\mathbf{f})$. Finally, the rank of $\Phi = \Phi^{\mathbf{f}}$ can be r if and only if (ρ, ρ) belongs to the $\mathbb{Z}$-span of the $(\mathbf{v}_j, \rho\mathbf{v}_j)$, in which case weight$(\rho\mathbf{v}_j) = n - w$ gives that $(\sum n_j)w = (\sum n_j)(n - w)$, so $2w = n$.

3.1.2. The following Theorem, which suffices for the present applications, will be referred to as the "minimal weight criterion" for nondegeneracy.

THEOREM. *Let $G \subset (\mathbb{Z}_2)^n \times_s G_0 \subset \text{Im}(n, 2)$ be an imprimitive permutation representation of* Gal$(K^c/\mathbb{Q})$, *with $[K : \mathbb{Q}] = 2n$, and let the G-orbit of the type $\Phi = \Phi^{\mathbf{f}}$ be given by $G_*(\mathbf{f}) = \{\mathbf{w}_1, \ldots, \mathbf{w}_{n'}, \rho\mathbf{w}_1, \ldots, \rho\mathbf{w}_{n'}\}$. Then (K, Φ) is nondegenerate provided there are indices $i_1, \ldots, i_n$ so that when $\mathbf{v}_j \in \{\mathbf{w}_{i_j}, \rho\mathbf{w}_{i_j}\}$, for $j = 1, \ldots, n$, is selected to be of minimal weight, the collection $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is linearly independent (in $\mathbb{Q}^n$) and for at least one index j_0,* weight$(\mathbf{v}_{j_0}) < n/2$.

PROOF. The weight condition assures that there is no nonzero relation $\sum m_j(\mathbf{v}_j, \rho\mathbf{v}_j) + m_{n+1}(\rho, \rho) = 0$, since weight$(\mathbf{v}_j) \leqslant n/2$, weight$(\mathbf{v}_{j_0}) < n/2$, gives

$$\sum m_j \text{weight}(\mathbf{v}_j) < \sum m_j \text{weight}(\rho\mathbf{v}_j).$$

REMARK. When $\mathbf{0} = (0,\dots,0) \in (\mathbb{Z}_2)^n$ belongs to the orbit $G_*(\mathbf{f})$, $\mathbf{v}_j$ of maximal weight may be preferred. As only a single G-orbit is affected, the choice of $\mathbf{v}_j$ of minimal weight is suggested by the natural determinate calculations in low degrees. Note that $n' \geqslant n$ is necessary, as observed above in §3.1.0, and that in the applications $n' = n$, in which case the $\binom{n'}{n}$ possible choices of indices may be avoided.

3.2.1. THEOREM (A CONVERSE OF RIBET'S THEOREM [15]). *Let $n > 4$ be composite and factor n as $n = kl$, with $k > 2$, $l \geqslant 2$. Then there exist simple degenerate Abelian varieties of dimension n and rank $n - l + 2$.*

PROOF. Observe that there exist cyclic totally real fields for every n, so there exist CM-fields K with $K/\mathbb{Q}$ Abelian and $\mathrm{Gal}(K/\mathbb{Q}) \cong \langle\rho\rangle \times \mathbb{Z}_n = G$. Normalize the ρ-structure so that $G_0 \cong \mathbb{Z}_n$ is given by $G_0 = \langle\sigma\rangle$, $\sigma = (1\ 2\ \cdots\ n)$, and consider the type defined by $\mathbf{f} = ((1,\dots,1), \mathbf{0}_l, \dots, \mathbf{0}_l)$, written in k blocks, each with l entries. Let M be the $n \times n$ matrix with rows given by successive cyclic shifts under σ, to obtain the successive images of $\mathbf{f}$ under $g = (0, \sigma^{-1}) \in G$. The first $(k-1)l + 1 = n - l + 1$ rows of M are linearly independent, while the last l rows may be written in $l \times l$ blocks as $(I_1, 0, \dots, 0, I_2)$, with

$$I_1 = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & & 0 \\ \vdots & & & & \vdots \\ 1 & 1 & \dots & 1 & 0 \end{pmatrix},$$

and

$$I_2 = \begin{pmatrix} 1 & 1 & \dots & & 1 \\ 0 & 1 & \dots & & 1 \\ \vdots & & & & \vdots \\ 0 & & \dots & 0 & 1 \end{pmatrix}.$$

Subtract the first row of M from the last $l - 1$ rows and then add the jth row to the $((k-1)l + j)^{\text{th}}$, for $j = 2,\dots, l$ to obtain the last rows $(0, I_1, 0, \dots, 0, I_2)$. Repeating the procedure with the next l rows will shift I_1 again, until the rows $(0,\dots, 0, I_1 + I_2)$ are obtained. Then $r = (k-1)l + 1$ in the constant weight criterion, so $\mathrm{rank}(K, \Phi^{\mathbf{f}}) = n - l + 2$, with $\Phi^{\mathbf{f}}$ the type defined by $\mathbf{f}$. Finally, note that Φ is primitive since the orbit of $\mathbf{f}$ under G has order $2n$, with $K/\mathbb{Q}$ Abelian.

3.2.2. Existence of simple Abelian varieties of smaller rank, in special dimensions, is provided by the following:

THEOREM. *Suppose the prime ideal (2) decomposes in the cyclotomic field $\mathbb{Q}(\zeta_p)$, p a prime, into $g > 1$ factors of degree f, and note that $p \equiv \pm 1 \pmod 8$ is sufficient but not necessary to insure $g > 1$. Then there exist Abelian varieties with complex multiplication having dimension $2^l p$, $l = 0, 1, \dots, g - 1$, and rank $p + 1$. In particular, for $l > 0$, these Abelian varieties are simple, degenerate, and in fact are of CM-type (K, Φ), with Φ primitive, but with reflex field K', of degree $2p$, a proper subfield of K.*

PROOF. First observe that the condition $p \equiv \pm 1 \pmod 8$ is necessary and sufficient for (2) to decompose in the quadratic subfield of $\mathbb{Q}(\zeta_p)$, so $g > 1$. Note the case $p = 43$, where $f = 14$. Next observe that there exist cyclic totally real fields of degree p, e.g., as subfields of $\mathbb{Q}(\zeta_q)$, for q an auxiliary prime with $p/(q-1)$. Therefore by Shimura [**18**, §1], there exist *CM*-fields with Galois groups $(\mathbb{Z}_2)^p \times_s \mathbb{Z}_p$.

Viewing the decomposition of (2) in $\mathbb{Q}(\zeta_p)$ as corresponding to a factorization of $(x^p - 1)/(x - 1)$ into g irreducible polynomials of degree $f \pmod 2$, use the coefficients of these irreducible polynomials to construct irreducible representations V_j of $\mathbb{Z}_p$ with $\dim_{F_2}(V_j) = f$, and $V_j \subset \mathbb{F}_2[\mathbb{Z}_p]$, $\mathbb{F}_2$ being the field with two elements and $\mathbb{F}_2[\mathbb{Z}_p]$ being the regular representation. The corresponding decomposition of $(\mathbb{Z}_2)^p$ is then $(\mathbb{Z}_2)^p = \langle \rho \rangle \oplus V_1 \oplus \cdots \oplus V_g$, with $V_j \cong (\mathbb{Z}_2)^f$. Note that V_j and each sum $V_{j_1} \oplus \cdots \oplus V_{j_{g-1}}$ is normal in $(\mathbb{Z}_2)^p \times_s \mathbb{Z}_p$, and, since no such subgroup contains ρ, will fix a Galois *CM*-subfield of the *CM*-field with group $(\mathbb{Z}_2)^p \times_s \mathbb{Z}_p$. The fixed field K then has group

$$(\mathbb{Z}_2)^p \times_s \mathbb{Z}_p / (\mathbb{Z}_2)^{f(g-l)} \cong (\mathbb{Z}_2)^{fl+1} \times_s \mathbb{Z}_p,$$

with $(\mathbb{Z}_2)^{fl+1}$ being identifiable with the complementary summands of $V_{j_1} \oplus \cdots \oplus V_{j_{g-l}}$ in $(\mathbb{Z}_2)^p$. Next take $H' = \operatorname{Ker}(1^{\text{st}}|_{(\mathbf{Z}_2)^{fl+1}})$ and note that $\rho \notin H'$ so H' fixes a *CM*-field K' of degree $(G : H') = (2^{fl+1}p)/2^{fl} = 2p$, G being the group $(\mathbb{Z}_2)^{fl+1} \times_s \mathbb{Z}_p$. The properties asserted above will be shown to hold for this choice of K and K'.

Now observe that K' has a single class of types, which maybe represented by a type written as $\Phi^{\mathbf{0}}$, with an orbit of order $2^{fl+1} < 2^p$, and that all other types have orbits of order $2^{fl+1}p$, so have reflex field $(K')' = K$, the Galois closure of K'. In fact, let such a type be given as $\Phi^{\mathbf{f}}, (G, H', \rho)$ having structure by $s(\sigma) = 0$, for all $\sigma \in \mathbb{Z}_p$. Then $\mathbb{Z}_p$ fixes $\mathbf{f} \in (\mathbb{Z}_2)^p \pmod{(\mathbb{Z}_2)^v}$, $v = fl + 1$, if and only if $\mathbf{f} \in (\mathbb{Z}_2)^v$, so $[(K')' : \mathbb{Q}] = 2^v(\mathbb{Z}_p : (1)) = 2^{fl+1}p$. Since K' has no *CM*-subfields, K' will be the reflex field of $(K, (\Phi^{\mathbf{f}})')$, for such an $\mathbf{f}$, $\mathbf{f} \notin (\mathbb{Z}_2)^v$, so the notation K' is not misleading, and the primitivity of the type on K, with corresponding simplicity of Abelian varieties of type $(K, (\Phi^{\mathbf{f}})')$, is assured, since the reflex type is always primitive. Finally, observe that rank $t((\Phi^{\mathbf{f}})') = t(\Phi^{\mathbf{f}}) \leqslant p + 1$, with the exact value $p + 1$ being the result of Ribet's Theorem in [**15**].

REMARK 1. The lowest-dimensional cases have dimension 56, from $n' = 7$, $v = 4$, and $2^5 \cdot 31 = 992$, from $n' = 31$, $v = 6$. The method described for obtaining $V_j \subset (\mathbb{Z}_2)^p$ is standard in algebraic coding theory, and a proof for the assertions required in the above proof may be found in [**9**, p. 277], a reference kindly supplied by E. F. Assmus, Jr. Professor Assmus also emphasized that the cases $p = 7$, $v = 4$, and $p = 23$, $v = 12$ are extremely exceptional among the cases $p \equiv \pm 1 \pmod 8$, $v = (p + 1)/2$, where $(\mathbb{Z}_2)^v \subset (\mathbb{Z}_2)^p$ is one of the two choices given as quadratic reciprocity codes. The automorphism groups of these linear codes, which correspond to a maximal G_0 in the present terminology, are $\mathbb{Z}_p \times_s \mathbb{Z}_{(p-1)/2}$ except in the above two cases, $p = 23$ admitting the Mathiew group M_{23}. As a consequence, the other codes for this v presumably having smaller automorphism groups, *CM*-fields of degree $2p$ having reflex fields K' of degree $2n'$ in the range $2^{(p+1)/2}p(p-1)/2 < 2n' < 2^p$ may be expected to give Abelian varieties with interesting properties.

REMARK 2. The first example of G with a type for which K could have a reflex field with $K \subsetneqq K'$ was obtained by Shimura in [**20**] for $G = D_{14}$. Such types predominate for D_{2n}, n odd, as will follow from the formula of the proposition of §4.1. Other examples provided by the present investigation include a case with $n = 4$ and $|G| = 16$ (minimal), and examples with $n = 7$, $v = 4$, the cases with $G_0 = \mathbb{Z}_7 \times_s Z_3$ and PSL(2, 7) having $K \subsetneqq K' \subsetneqq K^c$.

REMARK 3. Yoshida has brought to the author's attention an elementary proof that there exist totally real fields with group S_n for every n. The existence of simple degenerate Abelian varieties in the dimensions

$$\frac{1}{2}\binom{n}{n/2} \quad \text{and} \quad \binom{n}{k}, \qquad k \neq \frac{n}{2},$$

with rank $\leqslant n + 1$ follows from Yoshida's comment and the present methods.

3.3.1. While the $K/\mathbb{Q}$ Abelian case avoids existence problems, the special properties that result from the existence of various non-Abelian groups as the Galois groups of totally real fields may be formulated as the following:

THEOREM. (A) *Suppose there exists a totally real field K_0^c with Galois group* $\mathrm{Gal}(K_0^c/\mathbb{Q}) = \mathbb{Z}_k \wr \mathbb{Z}_l$, $k > 2$, $l \neq 1$, *the wreath product. Then there exist simple degenerate Abelian varieties with complex multiplication by a non-Galois CM-field such that the varieties have dimension $n = kl$ and rank $n - l + 2$.*

(B) *Under the same hypothesis, except that $k = 2$ is allowed, there exist simple Abelian varieties with dimension $n' = k^2 l$ having rank $\leqslant kl + 1$.*

(C) *Further, in the even case, the existence of totally real fields with Galois groups* $[(\mathbb{Z}_2)^m]^+ \times_s \mathbb{Z}_m$ ($m \geqslant 3$) *and D_m for m odd ($m > 5$) supplies simple Abelian varieties of CM-type (K, Φ), and reflex field K', with dimension $n = 4m$, and with dimension $n = 2m$ (m odd, $m > 5$), respectively, in each case with $K' \subset K$, $[K : K'] = 2$, so that* $\mathrm{rank}(\Phi) \leqslant \frac{1}{2}n + 1$.

PROOF. (A) Write

$$\mathbb{Z}_k \wr \mathbb{Z}_l = \langle \sigma_1, \sigma_2, \ldots, \sigma_l, \sigma_{l+1} \rangle,$$

with

$$\sigma_1 = (12 \cdots k), \sigma_2 = (k + 1 \cdots 2k), \ldots, \sigma_l = (k(l - 1) + 1 \cdots kl),$$

and

$$\sigma_{l+1} = (1\, k + 1 \cdots k(l - 1) + 1)(2 \cdots) \cdots (k \cdots kl).$$

Then take K_0 to be the subfield fixed by $H_0 = \langle \sigma_2, \ldots, \sigma_l \rangle$. Note that a transitive degree $n = kl$ structure, $G_0 = \mathbb{Z}_k \wr \mathbb{Z}_l \subset \mathrm{Im}(l, k)$, has been given, with $H_0 = \mathrm{Stab}_{G_0}(1)$. Take K to be the composite of K_0 with an imaginary quadratic field, and then observe that $K^c/\mathbb{Q}$ has group $G = \mathbb{Z}_2 \times G_0$, with K determining the ρ-structure that may be given by $s(\sigma) = 0$ for all $\sigma \in G_0$. Taking any τ_j with $\tau_j(1) = j$, let $\Phi^{\mathbf{f}}$ be the *CM*-type with $\mathbf{f} = ((1, 0, \ldots, 0)_k, \ldots, (1, 0, \ldots, 0)_k)$, with $\mathbf{f}$ being written in l blocks, $(1, 0, \ldots, 0)_k$ having k entries, with σ_r, $r = 1, \ldots, l$, transitive on the rth block. Then since $\sigma_1, \ldots, \sigma_l$ act independently and σ_{l+1} stabilizes, $\mathbf{f}$ has a G-orbit of order $2k^l$.

To check that $(K, \Phi^{\mathbf{f}})$ is primitive, and establish that the Abelian varieties of type $(K, \Phi^{\mathbf{f}})$ are simple, apply the method of Shimura and Taniyama [**23**, Proposition 26]. Observe that explicit coset representatives τ_j can be given as $\sigma_{l+1}^{u}\sigma_{1+u}^{w}$, taking 1 to $uk + 1 + w$. Then write $S = \cup_{j=1}^{n} H\rho^{f_j}\tau_j$, with the notation there. Since $k > 2$, observe that $(\rho, \sigma) \notin H_1 = \{g \in G/gS = S\}$, the number of elements with first entry ρ being changed. Next observe that $\sigma_1^{a_1} \cdots \sigma_l^{a_l} \in H_1$ implies $a_1 = 0$, and, finally, that $\sigma_1^{a_1} \cdots \sigma_{l+1}^{a_{l+1}} \in H_1$ implies $f_{k+1} = \cdots = f_{2k}$, which is also false for the present **f**.

To determine the rank, use the constant weight criterion and observe that **f** and the elements $((\sigma_a)^b) * \mathbf{f}$, $a = 1, \ldots, l$, $b = 1, \ldots, (k-1)$, supply $1 + l(k-1)$ independent elements of $G_{0*}(\mathbf{f})$ that span $G_{0*}(\mathbf{f})$. Then

$$\text{rank } t(\Phi^{\mathbf{f}}) = 1 + l(k-1) + 1 = lk - l + 2 = n - l + 2 < n + 1,$$

as asserted.

(B) With the same subgroups as above, consider the type with $\mathbf{f} = ((1\,0 \cdots 0)_k, (1\,0 \cdots 0)_k, \mathbf{0}_k, \ldots, \mathbf{0}_k)$. Then $\sigma_3, \ldots, \sigma_l$ stabilize this type, and no element of the form $\sigma_1^{a_1} \cdots \sigma_{l+1}^{a_{l+1}}$, with any one of a_1, a_2, or a_{l+1} nonzero, can fix this type, so an orbit of order $k^2 l$ is obtained. Then the reflex type of this type may be used to construct Abelian varieties with the required properties.

(C) The case of D_{2n}, n odd, is deferred until §4.1. For $G_0 = [(\mathbb{Z}_2)^m]^+ \times_s \mathbb{Z}_m$, take $H_0 = \text{Ker}(1^{\text{st}}|_{[(\mathbb{Z}_2)^m]^+})$, $G = \mathbb{Z}_2 \times G_0$, and $H = 0 \times H_0$, G corresponding to the composite with an imaginary quadratic field. Then $(G : H \times \langle \rho \rangle) = 2m$. Using

$$G_0 = \langle (1\,m+1)(2\,m+2), \ldots, (m-1\,2m-1)(m\,2m), (1\,2 \cdots m)(m+1 \cdots 2m) \rangle = \langle \sigma_1, \ldots, \sigma_m \rangle,$$

and $s(\sigma)$ identically zero, let $\Phi^{\mathbf{f}}$ be given by $\mathbf{f} = (1\,1\,0 \cdots 0)$. Then $H'(\Phi^{\mathbf{f}}) = \langle (0, \sigma_3), \ldots, (0, \sigma_{m-1}) \rangle$ with $H' \subset H$, $(H : H') = 2$, so we take K to be the fixed field of H'.

REMARKS. In case (C) observe that $m = 3$ gives $G_0 = A_4$ in degree $2m = 6$, so the degenerate reflex type gives an Abelian variety of dimension $4m = 12$ with rank 7, which exists by the theorem of §2.2.1. Under the hypothesis on the existence of the totally real fields, (B) combined with (C) gives the existence of pairs (K, Φ), (K', Φ') with $[K' : \mathbb{Q}] > [K : \mathbb{Q}]$, except when $[K' : \mathbb{Q}] = 2n'$ with n' odd and square free. Note the exceptional case $n' = 6$, which is even, but does not allow this strong form of degeneracy. Likewise, case (B) and the proof of case (A) give much smaller ranks than those obtained in the Abelian case of §3.2.1 for dimensions divisible by a large square, or of the form k^l.

3.3.2. As a final result on degeneracy, the results of §5.2 clarify the number of cases involved and provide a method of proof for a result stated without proof in Ribet's paper [**14**]. One formulation of the result is the following:

THEOREM. *Let A be a simple Abelian variety of CM-type (K, Φ), with $\dim_{\mathbb{C}}(A) = 4$. Then A is degenerate if and only if $\text{Gal}(K^c/\mathbb{Q}) \cong \mathbb{Z}_2 \times A_4$, or $\mathbb{Z}_2 \times S_4$ and (K, Φ) is the reflex of a type on a CM-field of degree 6.*

PROOF. The essential observation will be made in the proposition of §5.2.2, where the cocycle analysis establishes that there are only six orbits in question, the case $[(\mathbb{Z}_2)^4]^+ - \langle\rho\rangle$ being the degenerate type known to occur. The minimal weight criterion establishes nondegeneracy for the other five cases.

4. A relative class number formula.

4.0. For cases of the form $G = (\mathbb{Z}_2)^v \times_s \mathbb{Z}_p$, as in the theorem of §3.2.2, the general partition formula for the degrees of the reflex fields,

$$2^p = 2^v + (2^v p)\Big(\sum_c 1\Big),$$

where $\Sigma_c 1$, the sum of $c = (1/p)(2^{p-v} - 1)$ ones, has been established in the proof. Other cases with $K \subsetneqq K'$ have been remarked upon, but the dihedral group $G = D_{2n}$, n odd, and $\langle\rho\rangle \times S_n$ are the cases for which a general partition formula has been obtained. More information is available for the case $G = D_{2n}$, n odd, where the elementary character theory of G may be used to give explicit relations among certain characters $\chi_{K'/K_0'}$ and χ_{K/K_0}, where K' is a reflex field of K and $K_0' = (K')^{\langle\rho\rangle}$. The previously obtained example of Shimura is of the form $\chi_{K/K_0} = \chi_{K'/K_0'}$ and gives the relative class number relation of Proposition A.7 in [**20**, p. 84].

4.1. In the case of $G = D_{2n}$, n odd, G determines a unique ρ-structure (G, H, ρ) with $(G : H) = 2n$ (D_{2n} of order $4n$). Giving G the usual split structure $s(\sigma) = 0$ for all $\sigma \in G_0 = D_n$, the partition is given by the following:

PROPOSITION. *For $G = D_{2n}$, n odd, the partition giving the degrees of the reflex fields is given as*

$$2^n = 2 + \sum_{l/n} (2ls_{l,1} + 4ls_{l,2}),$$

where $s_{l,1}$, $s_{l,2}$ are summations of $s_{l,1}$ and $s_{l,2}$ ones, with $s_{l,1}$ defined recursively by

$$s_{l,1} = 2^{(l-1)/2} - \sum_{l'/l} s_{l',1},$$

and $s_{l,2}$ defined recursively as

$$s_{l,2} = \frac{1}{4l}\left[2^l - 2 - \sum_{l'/l} (2l's_{l',1} + 4l's_{l',2}) - 2ls_{l,1}\right],$$

both summations being over divisors $l' < l$. For the particular case of primes,

$$s_{p,1} = 2^{(p-1)/2} - 1 \quad \textit{and} \quad s_{p,2} = (1/4p)\left[2^p - 2 - p(2^{(p+1)/2} - 2)\right].$$

$s_{l,1} \neq 0$ for all l, and $s_{l,2} \neq 0$, except for $l = 3, 5$.

PROOF. The term 2 corresponds to the type $\Phi^{\mathbf{0}}$. For $l < n$, the term $2ls_{l,1}$ counts the types $\Phi^{\mathbf{f}}$ for which $\mathbf{f} \in (\mathbb{Z}_2)^n$ is fixed by $D_{n/l} \subset D_n$ but no larger subgroup, and the term $4ls_{l,2}$ counts $\mathbf{f}$ fixedby $\mathbb{Z}_{n/l}$ but no larger subgroup. The factors $2l$ and $4l$ give the orbit of such types under G. Finally, $2ns_{n,1}$ counts the types fixed only by a reflection (in $G_0 = D_n$), $2n$ being the order of such orbits, and $4ns_{n,2}$ counts the types fixed by no element of G.

The cases $n = 3, 5, 7, 15$, and 45 give illustrations of the formula and its recursive nature. Note that all of the partition terms $s_{n,2}$ correspond to types for which the reflex fields K' of K have $K \subsetneqq K' = K^c$, the existence of such types supplying the remaining portion of the proof of part (C) of the theorem in §3.3.1. The case D_{2^n}, n even, appears to present somewhat contrasting phenomena, which has not yet been fully developed. Note that D_{2^k}, like D_{2^p}, allows few *CM*-subfields. The partition $64 = 4 + 12 + 12 + 12 + 24$ from $n = 6$, $G = D_{12}$, shows that $K \subsetneqq K' = K^c$ can also occur for n even.

4.2. A concern for the ρ-structure identifications made by Outer(G) will be supported by the introduction of character theory to the description of a more arithmetic structure. The relative class numbers below retain geometric significance, as is established, for example, in Shimura [**17**] (cf. also Weil's review in Math. Reviews), [**20**], via the field of moduli.

DEFINITION. Let (G, H, ρ) be an element of a ρ-structure, possibly having kernel. Then $\chi_{H\times\langle\rho\rangle/H}$ denotes the character of G induced from the character of $H \times \langle\rho\rangle$ trivial on H, but nontrivial on $H \times \langle\rho\rangle$. The notation χ_{K/K_0} will be used when the case $G \cong \mathrm{Gal}(K^c/\mathbb{Q})$, with H corresponding to $\mathrm{Gal}(K^c/K)$ and ρ to complex conjugation, is intended.

PROPOSITION. *Suppose $G = D_{2^n}$, n odd, occurs for a CM-field $L = K^c$, $[L : K] = 2$. Let K^1 be a CM-subfield of L, $[L : K^1] = 2$, nonconjugate to K over $\mathbb{Q}$, and let L_0, K_0, and K_0^1 denote the maximal totally real subfields. Then the character relation*

$$\chi_{L/L_0} = \chi_{K/K_0} + \chi_{K^1/K_0^1}$$

holds.

PROOF. Observe that χ_{L/L_0} may be decomposed into irreducible characters, with the decomposition written in the notation of Serre [**16**] as

$$\chi_{L/L_0} = \bigoplus_{h \text{ odd}} 2\chi^h \oplus \psi_3 \oplus \psi_4,$$

the last two being characters of degree 1, the others of degree 2. The above relation then follows from the decompositions of χ_{K/K_0} and χ_{K^1/K_0^1}, one giving $\oplus_{h \text{ odd}}\chi^h \oplus \psi_3$, the other $\oplus_{h \text{ odd}} \chi^h \oplus \psi_4$.

Note that which character has which decomposition will depend upon which Outer(G) class is represented by (G, H, ρ) in the ρ-structure defined by (G, H, ρ). For an algebraic number field F, let h_F be the class number of F, d_F its discriminate, and E_F the group of units. Then applying the method of Shimura [**20**] provides the following:

THEOREM. *Suppose $L/\mathbb{Q}$ is a CM-field, $\mathrm{Gal}(L/\mathbb{Q}) = D_{2^n}$, n odd, and let K, K^1, L_0, K_0, K_0^1 be as in the above proposition. Then the following relations hold.*

(i) $d_L/d_{L_0} = (d_K/d_{K_0})(d_{K^1}/d_{K_0^1})$ *and*

(ii) $2\left[E_L : E_{L_0}\right]^{-1} h_L/h_{L_0} = \left[E_K : E_{K_0}\right]^{-1} h_K/h_{K_0'}\left[E_{K^1} : E_{K_0^1}\right]^{-1} h_{K^1}/h_{K_0^1}$.

PROOF. By the formula of Hecke [7], E/E_0 Abelian and quadratic gives $\xi_E(s) = \zeta_{E_0}(s)L(s, \chi_0)$, ζ being the Dedekind Zeta function and $L(s, \chi_0)$ being the L-function for the character of the extension E/E_0. Apply the formula for the residue of

the Zeta function. Then the discriminate terms are isolated, and the regulator terms are absorbed as in Shimura [**20**, p. 84]. But the L-function terms are equal by standard properties of the Artin L-series and the above proposition.

5. Examples for $n < 8$.

5.0. In this section ρ-structures are not allowed to have kernels. In §5.1 all cases for the values $n = 3$, 5, and 7 will be obtained with some general properties of n odd, and $n = p$, p a prime, included. All of the ρ-structures for these values are split, with $n = 7$ giving two cases having nontrivial structure. The case $n = 4$ is treated in 5.2, with many nonsplit cases obtained. Preliminary results for $n = 6$, especially the examples generalized in §3, are presented in 5.3.

For all of the ρ-structures with $n = 3, 4, 5$ and 7, the G-orbit structure of the collection of types is given. Recall from §1 that the G-orbit structure is a partition of the set of types, i.e., of $\{\mathbf{f} \mid \mathbf{f} \in (\mathbb{Z}_2)^n\}$. Pick Φ_j, $j = 1, \ldots, l$, to represent each G-orbit. Then, in the *CM*-field case, a numerical partition $2^n = \sum_{j=1}^{l}[J_j : \mathbb{Q}]$ results, J_j being the reflex field of (K, Φ_j). Every other reflex type of K is then given as $(J_j, \Phi_j')^g = (J_j^g, (\Phi_j')^g)$, for $g \in \mathrm{Gal}(K^c/\mathbb{Q})$, so that the (J_j, Φ_j'), $j = 1, \ldots, l$, represent all of the reflex types of K. The partition $2^n = \sum_{j=1}^{l}(G : H'(\Phi_j))$ is then given for each ρ-structure.

5.1.1. First observe that when G has a ρ-structure with n odd the classifications of ρ-structures and permutation structures coincide. The result is the following:

PROPOSITION. *Suppose G has a central order two element ρ. Then for a permutation structure (G, H) to support distinct ρ-structures, every permutation structure (G, H_1), G effective on $H_1 \setminus G$, for which $(G : H_1) = 2n_1$, must have n_1 even. Further, (G, H_1, ρ) must have $\mathrm{Proj}_{\mathrm{Sets}}(G)$ with a central order two element.*

PROOF. For (G, H) to support distinct ρ-structures, G must have at least two central order two elements, else ρ is characteristic, and $\mathrm{Aut}(G, \rho) = \mathrm{Aut}(G)$. Consider the imprimitivity sequence $0 \to (\mathbb{Z}_2)^{v_1} \to G \to G_0' \to 1$ for (G, H_1, ρ). Since G_0' is transitive, ρ is the unique central order two element of G in $(\mathbb{Z}_2)^{v_1}$, so a second central order two element, ρ_1, would be of the form $\rho_1 = (e', \sigma')$, $\sigma' \neq (1)$. But then ρ_1 of order two implies σ' of order two, and ρ_1 central in G implies σ' central in G_0'. Finally, since H_1 is without normal subgroups of G, $\rho_1 \notin H_1$, and G has the subgroup lattice $G \supset H_1 \times \langle \rho \rangle \times \langle \rho_1 \rangle \supset H_1 \times \langle \rho \rangle \supset H_1$, with $2 = (H_1 \times \langle \rho \rangle \times \langle \rho_1 \rangle : H_1 \times \langle \rho_2 \rangle)$, so $2/n_1$, for $n_1 = (G : H_1 \times \langle \rho \rangle)$.

The ρ-structures for the values $n = 3, 5$ and 7 will then be obtained by determining which of the permutation structures of degree $2n = 6$, 10 and 14 have abstract groups containing a central order two element. The analysis is further simplified by the fact that $n = p$, p a prime, strongly restricts the values of v, and therefore of $|G| = 2^v|G_0|$. In fact, observe the following:

PROPOSITION. *Let $n = p$ be an odd prime. Then for any G_0 of degree p, and any imprimitivity sequence $0 \to (\mathbb{Z}_2)^v \to G \to G_0 \to 1$ for a ρ-structure (G, H, ρ) ($\rho \in (\mathbb{Z}_2)^v$), the divisibility condition $p/2^{v-1} - 1$ holds.*

PROOF. G_0 must contain a p-cycle σ, so $(\mathbb{Z}_2)^v$ may be decomposed into $\mathbb{Z}_p$-orbits for $\mathbb{Z}_p = \langle\sigma\rangle$. Since p is odd, ρ gives a pairing of these orbits, so

$$(\mathbb{Z}_2)^v = \{0 \cup \rho\} \cup \{O_1 \cup \rho O_1\} \cup \cdots \cup \{O_s \cup \rho O_s\},$$

with each O_j of order p.

Thus for $p = 3$ and 5, $v = 1$ or p; while for $p = 7$, $v = 1$, 4 or 7. The general case of $p \equiv \pm 1 \pmod 8$ allows $(\mathbb{Z}_2)^v$ with $1 < v < p$ by quadratic reciprocity and the construction in §3. Recall that $p = 43 \equiv 3 \pmod 8$, but $v = 15$ occurs.

5.1.2. Now recall the transitive G_0 for $n = 3, 5$ and 7. For $n = 3$, there are only $G_0 = \mathbb{Z}_3$ and S_3; for $n = 5$, $G_0 = \mathbb{Z}_5$, D_5, $\mathbb{Z}_5 \times_s \mathbb{Z}_4$, A_5 and S_5; and, for $n = 7$, $G_0 = \mathbb{Z}_7$, D_7, $\mathbb{Z}_7 \times_s \mathbb{Z}_3$, $\mathbb{Z}_7 \times_s \mathbb{Z}_6$, PSL(2, 7), A_7 and S_7. The ρ-structures for these values are then described by the following:

THEOREM. *For $n = 3, 5$ and 7 there are a total of 33 ρ-structures. Of these, 28 are the split structures $\mathbb{Z}_2 \times G_0$ and $(\mathbb{Z}_2)^n \times_s G_0$ for the above 14 G_0. Three more are the $n = 7$, $v = 4$ split structures $(\mathbb{Z}_2)^4 \times_s G_0$, for $G_0 = \mathbb{Z}_7$, $\mathbb{Z}_7 \times_s \mathbb{Z}_3$, and* PSL(2, 7). *All but two of these split groups have unique permutation structure, the two nontrivial split ρ-structures resulting from distinct permutation structures on $\mathbb{Z}_2 \times$ PSL(2, 7) and $(\mathbb{Z}_2)^4 \times_s$ PSL(2, 7).*

PROOF. See the lists of Miller [**10**] or Zassenhaus [**25**] for $n = 3$ ($2n = 6$), the list of Cole [**2**] for $n = 5$ ($2n = 10$), and the list of Miller [**11**] for $n = 7$ ($2n = 14$). Note that the corrections to Cole's list in Miller [**11**] do not affect the present case.

Only the list for $n = 7$, $2n = 14$ presents any problem. Since $n = 7$ is odd, observe that the only cases admitting 2 sets of order 7 are split and have $v = 1$. The other cases listed above account for 12 of the 27 permutation groups that admit 7 sets of order 2 but not 2 sets of order 7. The remaining 15 such permutation groups fail to have a central order two element, since they occur as $[(\mathbb{Z}_2)^v]^+ \times_s G_0$, as $[(\mathbb{Z}_2)^v \times_s G_0]^+$, or as G_1 with $(\mathbb{Z}_2)^v \times_s G_0 = \langle\rho\rangle \times G_1$, the $[\ ]^+$ cases failing to contain ρ since $n = 7$ is odd.

5.1.3. PROPOSITION 1. *When $n = 3$, $v = 1$, the partition $2^3 = 2 + 6$ holds for both $G = \mathbb{Z}_2 \times G_0$, and $2^3 = 8$ holds for both $G = (\mathbb{Z}_2)^3 \times_s G_0$. When $n = 5$ there are two cases with $v = 1$. For $G = \mathbb{Z}_2 \times G_0$ with $G_0 = \mathbb{Z}_5$ or D_5, the partition is $2^5 = 2 + 10 + 10 + 10$, while for $G_0 = \mathbb{Z}_5 \times_s \mathbb{Z}_4$, A_5, or S_5, $2^5 = 2 + 10 + 20$. The case $G = (\mathbb{Z}_2)^5 \times_s G_0$ gives a single orbit, $2^5 = 32$.*

PROOF. Use $\mathbb{Z}_p \subset G_0$ to give coset representatives τ_j, and take $s(\sigma) = 0$ for all $\sigma \in G_0$. Then the type $\Phi^{\mathbf{0}}$, $\mathbf{0} \in (\mathbb{Z}_2)^n$, accounts for the term 2 when $v = 1$. The first term of $2n$ is then represented by (001) or (00001). For $n = 5$, the first two G_0 have (11000) and (10100) in distinct orbits, while the other three G_0, being 2-transitive, identify these to give a single orbit. That the case $v = n$ gives a single orbit has been previously distinguished by Shimura [**18**, §1].

While the same priniciples apply to $n = 7$, $v = 1$, $s(\sigma)$ identically 0, there are several more cases. The notation $[\ ]_{k,n-k}$ is used to denote the preliminary decomposition into orbits having **f** with k or $n - k$ entries of 1. Also the notation $\Sigma_r 1$, for the sum of r 1's is used. Then the $n = 7$ partitions are given by the following:

PROPOSITION 2. *For* $n = 7$, $v = 1$ $s(\sigma)$ *identically* 0, *the partitions are as follows*:
(1) $2^7 = 2 + 14(\Sigma_9 1)$, *for* $G_0 = \mathbb{Z}_7$;
(2) $2^7 = 2 + 14 + [14(\Sigma_3 1)]_{2,5} + [14(\Sigma_3 1) + 28]_{3,4}$, *for* D_7;
(3) $2^7 = 2 + 14 + [42]_{2,5} + [14 + 14 + 42]_{3,4}$, *for* $\mathbb{Z}_7 \times_s \mathbb{Z}_3$ *and* PSL(2, 7);
(4) $2^7 = 2 + 14 + [42]_{2,5} + [28 + 42]_{3,4}$, *for* $\mathbb{Z}_7 \times_s \mathbb{Z}_6$; *and*
(5) $2^7 = 2 + 14 + [42]_{2,5} + [70]_{3,4}$ *for* A_7 *and* S_7.
Representatives for the orbits are as given below.

Further, the other cases have the partitions $2^7 = 16 + 112$ *for the three split cases with* $v = 4$ *and* $s = 0$, *and the nontrivial split case with* $v = 1$, *and a single orbit,* $2^7 = 128$, *for the nontrivial split case with* $v = 4$, *in addition to the cases with* $v = 7$.

PROOF. For the cases with $v = 1$, $s = 0$ use $\mathbb{Z}_7 \subset G_0$ to specify $\Phi^{\mathbf{f}}$, and obtain the preliminary decomposition

$$2^7 = 2 + [14]_{1,6} + \left[14\left(\sum_3 1\right)\right]_{2,5} + \left[14\left(\sum_5 1\right)\right]_{3,4}.$$

The case $G_0 = \mathbb{Z}_7$ is completed by specifying the representatives

$$(1100000), \quad (1010000) \quad \text{and} \quad (10001000)$$

for $[\ ]_{2,5}$; and

$$(0000111), (0001011), (0001101), (0010011), \quad \text{and} \quad (0010101)$$

for $[\ ]_{3,4}$. All other cases except for D_7 have $G_0 = \mathbb{Z}_7 \times_s \mathbb{Z}_3$ as a subgroup, which identifies the three orbits from $[\ ]_{2,5}$ to give the $[42]_{2,5}$ terms. $G_0 = D_7$ has an element stabilizing each of the representatives, so the orbits remain distinct.

For $[\ ]_{3,4}$, note that the second and third representatives give the exceptional orbits for $\mathbb{Z}_7$, and the representatives chosen give the coefficients in the factorization

$$(x^7 - 1)/(x - 1) = (x^3 + x + 1)(x^3 + x^2 + 1) \pmod 2.$$

These exceptional orbits give terms of 14 + 14 for the G_0 admitting $v = 4$, and 28 for $G_0 = D_7 \subset G_0' = \mathbb{Z}_7 \times_s \mathbb{Z}_6$. Again D_7 has a reflection fixing the other three representatives, giving distinct orbits, while $G_0 = \mathbb{Z}_7 \times_s \mathbb{Z}_3 \subset G_0'$ identifies these. Finally, $G_0 = A_7$ and S_7 are 3-transitive, so that $[\ ]_{3,4}$ is a single orbit.

For the cases with $v = 4$ and $s = 0$, a term of 16 is required by the corollary of the Reflex Degree Theorem and can be represented by **0**. Then $(\mathbb{Z}_2)^4 \times_s \mathbb{Z}_7$ is a subgroup of the other two, and $(\mathbb{Z}_2)^4$ gives identifications of the other orbits for $\mathbb{Z}_2 \times \mathbb{Z}_7$, the orbit used to construct $(\mathbb{Z}_2)^4$ being excepted, and odd = (0000001) is a representative in any case. The nontrivial case with $v = 4$ and $G_0 = \mathrm{PSL}(2, 7)$ has a

trivial subgroup with $G_0 = \mathbb{Z}_7$, $v = 4$, but must not allow a term of 16 by the same corollary, so must identify the two $(\mathbb{Z}_2)^4 \times_s \mathbb{Z}_7$ orbits, giving a single orbit. Finally, the nontrivial structure with $v = 1$ may be written as $G = \mathbb{Z}_2 \times \mathrm{PSL}(2,7)^*$, $\mathrm{PSL}(2,7)^*$ being a "head = 1" subgroup. Explicitly,

$$((0000000), (1234567)) \quad \text{and} \quad ((0011101), (12)(47))$$

satisfy Burnside's relations for PSL(2, 7), and make identifications on the oribits of the trivial $\mathbb{Z}_7 \times_s \mathbb{Z}_3$ subgroup given by the above 7-cycle and (253)(467).

5.2.1. For $n = 4$, $2n = 8$, several cases allow nonunique central order two elements, in distinct Aut(G)-orbits, so permutation structures are not sufficient. The cocycle method, having been established in general, provides the following:

THEOREM. *There are* 38 *ρ-structures for* $n = 4$, *which may be arranged according to* $(G_0, (\mathbb{Z}_2)^v)$, *and have partitions as on the table of* $n = 4$ *results below.*

PROOF. Observe that for each of the five transitive G_0, $v = 1, 3$ and 4 determines $(\mathbb{Z}_2)^v$ uniquely, $v = 1$ giving $\langle \rho \rangle$ and $v = 3$ giving $[(\mathbb{Z}_2)^4]^+$. Neither $G_0 = A_4$, nor $G_0 = S_4$ allow $v = 2$, and recall that the three G_0-modules for $G_0 = (\mathbb{Z}_2)^2$ have been found to be equivalent in §2.3.2, where the normalizations for $G_0 = \mathbb{Z}_4$, $G_0 = D_4$ determine $(\mathbb{Z}_2)^v$, $v = 2$, uniquely.

Now compute $H^1(G_0, (\mathbb{Z}_2)^4)$ to get $\langle 0 \rangle$ for $G_0 = (\mathbb{Z}_2)^2$, $\mathbb{Z}_4$ and A_4; and $\langle \rho \rangle$ for $G_0 = D_4$ and S_4. In the latter case, note that $H^1(G_0, (\mathbb{Z}_2)^v) = \langle \rho \rangle$, so the inclusion i^1 is an isomorphism, giving $j^1 = 0$ in all cases. Note that Outer(G) makes nontrivial identifications if and only if $G_0 = D_4$ or S_4, as in the Structures Theorem of §6.

Since $j^1 = 0$ we have $H^1(G_0, (\mathbb{Z}_2)^4/(\mathbb{Z}_2)^v) \cong Z^1/j^1(Z^1)$, so we compute the group $Z^1/j^1(Z^1)$ of extension classes. For the cases $G_0 = (\mathbb{Z}_2)^2, \mathbb{Z}_4$ and D_4, pick s: $G_0 \to (\mathbb{Z}_2)^4$ for each class and compute the order structure to see that nonisomorphic groups result from each extension class with $(G_0, (\mathbb{Z}_2)^v)$ held constant, except in the two cases $G_0 = (\mathbb{Z}_2)^2$, $v = 1$ and $v = 3$. The case with $v = 1$ has appeared in §2.3.2, and the case with $v = 3$ gives $|Z^1/j^1(Z^1)| = 4$, with two $N_{S_4}(\mathbb{Z}_2)^2 = S_4$ orbits. For $G_0 = A_4$ and S_4 note that the groups listed on the following table for $n = 4$ are nonisomorphic and belong to the appropriate cohomology spaces. Since nonisomorphic representatives have been obtained, and equivalence between distinct $(G_0, (\mathbb{Z}_2)^v)$ is not allowed, the number of structures is given by $|Z^1/j^1(Z^1)|$, with the above two exceptions.

The identifications of the abstract 2-groups, while not essential, may be easily obtained by comparing the structure $(\mathbb{Z}_2)^4 \times_s G_0$ with successive index two subgroups, as provided on the lattice diagrams of Hall and Senior [**5**]. The partition results follow from the methods of §§1 and 2, explicit calculation with $(s(\sigma), \sigma)$ being rarely required. The primitive cases will be described below.

Note that in each case the split structure is given first in the following:

TABLE OF $n = 4$ RESULTS.

Structures		Partition giving degrees of reflexes
$G_0 = (\mathbb{Z}_2)^2$		
$v = 1$	$(\mathbb{Z}_2)^3$	$2 + 2 + 2 + 2 + 8$
	$(\mathbb{Z}_2) \times \mathbb{Z}_4$ $(\rho \in \mathbb{Z}_4)$	$4 + 4 + 8$
	D_4	$4 + 4 + 4 + 4$
	Q	$8 + 8$
$v = 2$	$\mathbb{Z}_2 \times D_4$ $(\rho \in D_4)$	$4 + 4 + 8$
	$16\Gamma_2 b$	$8 + 8$
$v = 3$	$32\Gamma_5 a_1$	$8 + 8$
	$32\Gamma_7 a_1$	16
$v = 4$	$64\Gamma_{25} a_1$	16
$G_0 = \mathbb{Z}_4$		
$v = 1$	$\mathbb{Z}_2 \times \mathbb{Z}_4$ $(\rho \notin \mathbb{Z}_4)$	$2 + 2 + 4 + 8$
	$\mathbb{Z}_8$	$8 + 8$
$v = 2$	$16\Gamma_2 c_1$	$4 + 4 + 8$
	$16\Gamma_2 d$	16
$v = 3$	$32\Gamma_7 a_1$	$8 + 8$
	$32\Gamma_7 a_2$	16
$v = 4$	$64\Gamma_{22} a_1$	16
$G_0 = D_4$		
$v = 1$	$\mathbb{Z}_2 \times D_4$ $(\rho \notin D_4)$	$2 + 2 + 4 + 8$
	$16\Gamma_2 c_1$	$4 + 4 + 8$
	$16\Gamma_3 a_1$ $(= D_8)$	$8 + 8$
	$16\Gamma_3 a_2$	$8 + 8$
$v = 2$	$32\Gamma_4 a_1$	$4 + 4 + 8$
	$32\Gamma_7 a_1$	$8 + 8$
	$32\Gamma_3 e$	16
	$32\Gamma_6 a_1$	16
$v = 3$	$64\Gamma_{25} a_1$	$8 + 8$
	$64\Gamma_{23} a_1$	16
	$64\Gamma_{22} a_1$	16
	$64\Gamma_{26} a_1$ ($\cong$ 2-Sylow of $W(D_4)$)	16
$v = 4$	$(\mathbb{Z}_2)^4 \times_s D_4$ ($\cong$ 2-Sylow of $W(C_4)$)	16
$G_0 = A_4$		
$v = 1$	$\mathbb{Z}_2 \times A_4$	$2 + 6 + 8$
	SL(2, 3)	$8 + 8$
$v = 3$	$[(\mathbb{Z}_2)^4]^+ \times_s A_4$	$8 + 8$
$v = 4$	$(\mathbb{Z}_2)^4 \times_s A_4$	16
$G_0 = S_4$		
$v = 1$	$\mathbb{Z}_2 \times S_4$	$2 + 6 + 8$
	GL(2, 3)	16
$v = 3$	$[(\mathbb{Z}_2)^4]^+ \times_s S_4$	$8 + 8$
	$[(\mathbb{Z}_2)^4 \times_s S_4]^+$ $(\cong W(D_4))$	16
$v = 4$	$(\mathbb{Z}_2)^4 \times_s S_4$ $(\cong W(C_4))$	16

REMARK 1. For 2-groups, the notation is that of Hall and Senior [5], in which the first number is the order of the group, the symbol Γ_k denotes the family to which the group belongs in the sense of P. Hall [6], and the remaining portion of the notation specifies the group within that family. The symbol Q denotes the quaternion group. The $[\ ]^+$ notation is from Coxeter and Moser [3], as recalled in §2.3.2; likewise, $W(-)$ denotes the Weyl group of the reduced root system of the simple Lie algebra of that type. Note that $\mathrm{Im}(n,2) \cong W(B_n) = W(C_n)$ is the "hyperoctahedral group".

REMARK 2. Twenty of these structures are nonsplit, some of which belong to more general cases. The three cases with two distinct ρ-structures on a single permutation group are $G = \mathbb{Z}_2 \times \mathbb{Z}_4$, $G = \mathbb{Z}_2 \times D_4$, and $G = 16\ \Gamma_2 c_1$. In particular, 35 degree 8 permutation groups admit 4 sets of order 2 and the central order two element ρ. The groups $64\ \Gamma_{22} a_1$ and $64\ \Gamma_{25} a_1$ which occur twice have distinct permutation structures in each appearance, while $32\ \Gamma_7 a_1$ occurs three times, each time with distinct permutation structure. Thus 31 nonisomorphic groups occur, 22 of which are 2-groups.

5.2.2. PROPOSITION. *Up to normalization there are just six subsets of* $(\mathbb{Z}_2)^4$ *that occur as the G-orbits of primitive types. Four of these arise from the split structures, and are given as* $(\mathbb{Z}_2)^4$, $[(\mathbb{Z}_2)^4]^+$, $[(\mathbb{Z}_2)^4]^+$ *(odd), and* $[(\mathbb{Z}_2)^4]^+ - \langle\rho\rangle$. *The other two are associated with the cyclic case* $G = \mathbb{Z}_8$, *and are both of order* 8, *containing even and odd elements.*

PROOF. First observe that normalization by B^1 does not disturb $s(\sigma)$ odd or even, so that specific $\Phi^{\mathbf{f}}$ need not be given. Checking cocycle calculations, observe that, for $v < 3$, there is $\sigma \in G_0$ so that $s(\sigma)$ is odd if and only if G_0 has a 4-cycle τ with $s(\tau)$ odd. But then G has a subgroup $\mathbb{Z}_8 = \langle(s(\tau), \tau)\rangle$, which may easily be observed to have the partition $2^4 = 8 + 8$, each orbit being primitive. For primitivity, observe that $H = \langle 1\rangle$ with every $H \subset H_1$ having $\rho \in H_1$, so no proper containment is allowed. For explicit representatives, take $\mathbb{Z}_8 = \langle((0001),(1234))\rangle$, and observe the elements $\{\mathbf{0}, (1000), (1100), (1110)\}$ in one orbit, $\{(0010), (1001), (0100), (1010)\}$ in the other. As to the orbits of G, either G must preserve these orbits, or else G has $(\mathbb{Z}_2)^4$ as a single G-orbit with partition $2^4 = 16$.

Next observe that the cases with $v = 3$ give $(\mathbb{Z}_2)^4$, $[(\mathbb{Z}_2)^4]^+$ and $[(\mathbb{Z}_2)^4]^+$ (odd), the latter two if and only if G is split, as analyzed in §2.3.2, Example 4. The remaining cases all have $s(\sigma)$ even for all $\sigma \in G_0$, and also have $v \leqslant 2$. But when $s(\sigma)$ is always even, $[(\mathbb{Z}_2)^4]^+$ and $[(\mathbb{Z}_2)^4]^+$ (odd) must be preserved by the G-action. Now if $3 \nmid |G_0|$, no orbit of order 6 occurs. As the reflex of a type with an orbit of order 4 is of degree 2 or 4, no orbit of order 4 contains primitive types, and no further primitive types are obtained. For $3/|G_0|$, the two split cases may be easily computed and give

$$(\mathbb{Z}_2)^4 = \langle\rho\rangle \cup \left(\left[(\mathbb{Z}_2)^4\right]^+ - \langle\rho\rangle\right) \cup \left[(\mathbb{Z}_2)^4\right]^+ (\text{odd}).$$

Otherwise there is no term of order two, so no term of order 6 occurs, and only the two "split" orbits of order 8 occur for $s(\sigma)$ even. As the required cocycle calculations may be easily supplied, the present description is complete.

5.3.1. For $n = 6$ and larger composite values such as $n = 8, 9$ or 15, a nesting of the G_0 as with $\mathbb{Z}_p \subset G_0$ for p prime and with $(\mathbb{Z}_2)^2 \cdot \mathbb{Z}_4 = D_4 \subset S_4$, $(\mathbb{Z}_2)^2 \subset A_4$ for

$n = 4$ is not so readily apparent. For the general case certain G_0 with corresponding properties are distinguished by the following:

DEFINITION. A transitive group G_0 of degree n is called a *minimal group* of degree n, and written as $G_0^{\min}$ (when n is understood), provided that no proper subgroup of G_0 is transitive.

For $n = 6$, the minimal groups are provided by the following:

PROPOSITION. *The imprimitive transitive permutation groups of degree* 6 *may be normalized to have the inclusion relations indicated on the lattice diagram for* $n = 6$ *below, with the notations introduced in the proof. The primitive* G_0 *of degree* 6 *are never minimal, so there are four minimal groups* $G_0^{\min}$ *for* $n = 6$; *these are the groups* $\mathbb{Z}_6$, S_3, A_4 *and* $\mathrm{Im}(2,3)^+$ *indicated in the lattice diagram by boxes.*

PROOF. To fix the notation, the relevant structure of these well-known groups is recalled. The group $\mathrm{Im}(2,3)$ has the structure $S_3 \wr \mathbb{Z}_2$, the wreath product, and is also given as $(S_3 \times S_3) \times_s \mathbb{Z}_2$, semidirect. $\mathrm{Im}(2,3)^+$ is the subgroup by

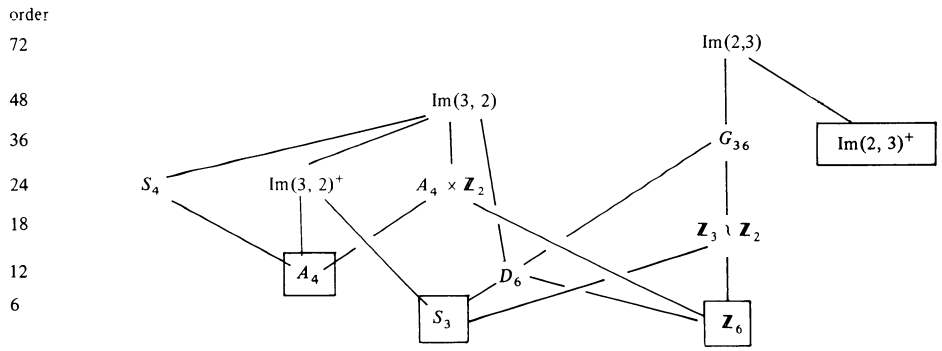
$$\mathrm{Im}(2,3)^+ = \{((\sigma_1, \sigma_2), \rho^e) \in (S_3 \times S_3) \times_s \mathbb{Z}_2 / \mathrm{sgn}(\sigma_1)\mathrm{sgn}(\sigma_2)(-1)^e = 1\}.$$

The group G_{36}, also of order 36, is $G_{36} = (S_3 \times S_3)^+ \times_s \mathbb{Z}_2$. Note that ρ is the usual product of 3 disjoint two cycles, but with nontrivial action in $\mathrm{Im}(2,3)$. Since (123) and (456) are even, the $\mathbb{Z}_6$ subgroup given by $(((123),(456)), \rho)$ does not belong to $\mathrm{Im}(2,3)^+$, but does to G_{36} as well as the wreath product $\mathbb{Z}_3 \wr \mathbb{Z}_2$. Further inclusions are obtained by observing that G_{36} is the holomorph of S_3, that is, $G_{36} \cong S_3 \times_s \mathrm{Aut}(S_3) \cong S_3 \times_s S_3$. A subgroup analysis may be used to establish that $\mathrm{Im}(2,3)^+$ is a minimal group.

For $\mathrm{Im}(3,2) = (\mathbb{Z}_2)^3 \times_s S_3$ as usual, observe that $\mathrm{Im}(3,2) \cap \mathrm{Im}(2,3) = D_6$, with D_6 also given as $D_6 = S_3 \cdot \mathbb{Z}_6$. A_4 is the subgroup $[(\mathbb{Z}_2)^3]^+ \times_s Z_3$ of $\mathrm{Im}(3,2)$, and contains no subgroups of order 6. Observe that $A_4 \cdot D_6 = A_4 \cdot S_3 \cdot \mathbb{Z}_6 = \mathrm{Im}(3,2)$. The permutation groups S_4 and $\mathrm{Im}(3,2)^+$ are both abstractly isomorphic to S_4, the notation "S_4" being reserved for $[(\mathbb{Z}_2)^3]^+ \times_s S_3$, which contains A_4 as given. $A_4 \times \mathbb{Z}_2 = (\mathbb{Z}_2)^3 \times_s \mathbb{Z}_3$ and is given as $A_4 \cdot \mathbb{Z}_6$. Likewise $\mathrm{Im}(3,2)^+ = A_4 \cdot S_3$, completing the description of the 12 imprimitive G_0.

The four primitive groups of degree 6 are given as $A_5 \cong \mathrm{PSL}(2,5)$, $S_5 \cong \mathrm{PGL}(2,5)$, A_6 and S_6. To see that none of these are minimal, observe $A_4 \subset A_5 \subset A_6 \subset S_6$ and $A_5 \subset S_5$.

Lattice Diagram for $n = 6$



The steps (A) and (B) of the Classification Theorem are given for $n = 6$ by the following:

THEOREM. *For $n = 6$ there are 66 inequivalent pairs $(G_0, (\mathbb{Z}_2)^v)$, with $\rho \in (\mathbb{Z}_2)^v$, and therefore 66 inequivalent ρ-structures with trivial cochain $s(\sigma) = 0$ for all $\sigma \in G_0$. All 16 G_0 admit $\langle \rho \rangle$ with $v = 1$, $[(\mathbb{Z}_2)^6]^+$ with $v = 5$, and $(\mathbb{Z}_2)^6$, accounting for 48 cases. G_0 admits $v = 2$ if and only if $G_0 \subset \mathrm{Im}(2,3)$, supplying 7 cases; G_0 admits $v = 3$ if and only if $G \subset \mathrm{Im}(3,2)$, supplying 8 cases; and, G_0 admits $v = 4$ if and only if $G_0 \subset D_6 = \mathrm{Im}(2,3) \cap \mathrm{Im}(3,2)$, for 3 cases.*

PROOF. Consider the 4 $G_0^{\min}$ cases, and decompose $(\mathbb{Z}_2)^6$ into $G_0^{\min}$ orbits, taking account of the pairing given by ρ. The action of G_0 preserves the number of 1's, allowing a preliminary decomposition $(\mathbb{Z}_2)^6 = \langle \rho \rangle \cup [\ ]_{1,5} \cup [\ ]_{2,4} \cup [\ ]_3$, subscripts denoting the number of 1's. Since G_0 is transitive, the pairing by ρ gives $[\ ]_{1,5}$ as a single orbit of order 12.

The remaining terms decompose as follows:

(1) $\quad |[\ ]_{2,4}| = 12 + 12 + 6, \quad |[\ ]_3| = 12 + 6 + 2, \quad \text{for } G_0 = \mathbb{Z}_6;$

(2) $\quad |[\ ]_{2,4}| = 6 + 6 + 6 + 12, \quad |[\ ]_3| = 2 + 6 + 6 + 6, \quad \text{for } G_0 = S_3;$

(3) $\quad |[\ ]_{2,4}| = 24 + 6, \quad |[\ ]_3| = 8 + 12, \quad \text{for } G_0 = A_4;$

and

(4) $\quad |[\ ]_{2,4}| = 12 + 18, \quad |[\ ]_3| = 2 + 18, \quad \text{for } G_0 = \mathrm{Im}(2,3)^+.$

The cases for $(G_0^{\min}, (\mathbb{Z}_2)^v)$ are then obtained by examining unions of orbits to see when sub-G_0-modules are obtained. First results are that $v = 3$ and $v = 4$ do not occur for $G_0^{\min} = \mathrm{Im}(2,3)^+$, nor does $v = 4$ for $G_0^{\min} = A_4$.

To obtain some of the pairs $(G_0^{\min}, (\mathbb{Z}_2)^v)$, aside from the cases with $v = 1, 5$ and 6, observe that $(\mathbb{Z}_2)^v = \{(e, e) \in (\mathbb{Z}_2)^6 / e \in (\mathbb{Z}_2)^3\}$, gives a case with $v = 3$ for $G_0 \subset \mathrm{Im}(3,2)$, and $(\mathbb{Z}_2)^v = \langle \rho, (\mathbf{0}, \rho_0) \rangle$ with $\rho_0 = (111) \in (\mathbb{Z}_2)^3$, gives a case with $v = 2$ for $G_0 \subset \mathrm{Im}(2,3)$. Observe that D_6 admits $(\mathbb{Z}_2)^v = \{(e, \rho_0^a e) \in (\mathbb{Z}_2)^6 / e \in (\mathbb{Z}_2)^3, a = 0, 1\}$ since both S_3 and $\mathbb{Z}_6$ do, and $D_6 = S_3 \cdot \mathbb{Z}_6$. Note that for G_0 containing either A_4 or $\mathrm{Im}(2,3)^+$, $[(\mathbb{Z}_2)^6]^+$ is unique for $v = 5$, since otherwise $[(\mathbb{Z}_2)^5]^+$ would give $v = 4$.

Next observe that each value $v = 1, 2, \ldots, 6$ occurs for $G_0 = \mathbb{Z}_6$, with $(\mathbb{Z}_2)^v$ uniquely determined for each value. For example, $v = 2$ gives odd elements, so $v = 3$ must have only even elements, else a second $v = 2$ would result. But then $(\mathbb{Z}_2)^v = \langle \rho \rangle \cup \mathcal{O}$, for $\mathcal{O}$ the orbit from $[\ ]_{2,4}$ with order 6. The remaining case $G_0^{\min} = S_3$ has 3 submodules $(\mathbb{Z}_2)^v$ for each of $v = 3$ and $v = 4$. The equivalence of these under $N_{S_6}(S_3) = G_{36}$ may be observed by noting that G_{36} gives the same decomposition of $(\mathbb{Z}_2)^6$ as $\mathrm{Im}(2,3)^+$, so that G_{36} does not preserve any $(\mathbb{Z}_2)^v$ for $v = 3$ or 4, as would be required for S_3 to have inequivalent submodules as in (B) of the Classification Theorem. No other nonunique $(\mathbb{Z}_2)^v$'s occur for S_3, as in the previous cases.

All 20 $(G_0^{\min}, (\mathbb{Z}_2)^v)$ cases having been obtained, most of the general cases follow from the inclusion relations in the above proposition. For example, in the case

$\mathbb{Z}_6 S_3 = \mathbb{Z}_3 \setminus \mathbb{Z}_2$, here with subgroups $\mathbb{Z}_6$ and S_3 so $\mathbb{Z}_6 \cap S_3 = \mathbb{Z}_2$, observe that S_3 does not preserve the unique submodule with $v = 3$ for $\mathbb{Z}_6$, and therefore does not preserve the submodule with $v = 4$ either. Finally, consider the primitive cases, all of which have been observed to contain A_4. The inclusion rules out 2 and 4, and gives uniqueness for $v = 5$. The remaining case, $v = 3$, may be eliminated since each of these G_0 is at least 2-transitive, so that $[\ ]_{2,4}$ is a single orbit or order 30.

COROLLARY OF PROOF. *Let* $G = \mathbb{Z}_2 \times G_0^{\min}$ *be the split* $v = 1$ *group with structure by* $s(\sigma) = 0$ *for all* $\sigma \in G_0$. *Then the types of* G, *i.e., of* $(G, \mathrm{Stab}_G(\pm 1), \rho)$, *have* G*-orbit structure giving the following partitions*:

(1) $2^6 = 2 + 12 + [12 + 12 + 6]_{2,4} + [12 + 6 + 2]_3$, *for* $G_0 = \mathbb{Z}_6$;

(2) $2^6 = 2 + 12 + [6 + 6 + 6 + 12]_{2,4} + [2 + 6 + 6 + 6]_3$, *for* $G_0 = S_3$;

(3) $2^6 = 2 + 12 + [24 + 6]_{2,4} + [8 + 12]_3$, *for* $G_0 = A_4$; *and*

(4) $2^6 = 2 + 12 + [12 + 18]_{2,4} + [2 + 18]_3$, *for* $G_0 = \mathrm{Im}(2, 3)^+$.

PROOF. The split structure $\mathbb{Z}_2 \times G_0$ with $s(\sigma) = 0$ for all $\sigma \in G_0$ exactly gives the G_0-orbits with pairing by multiplication by ρ, so the computations are the same.

The examples generalized in §3 occur for the cases $G_0 = \mathbb{Z}_6$, A_4, and $\mathrm{Im}(2, 3)^+$. The group $G_0 = \mathbb{Z}_3 \setminus \mathbb{Z}_2$ is adopted in place of $\mathrm{Im}(2, 3)^+$ as an orbit with similar properties is obtained.

6. The relation to group extensions.

6.0. The present section does not depend upon the material of §§3, 4 or 5. For the remainder of §6 the requirement $\rho \in (\mathbb{Z}_2)^v$ will be dropped, unless otherwise noted, to allow a general "imprimitivity structure", (G, H, S), with n sets of order 2, as in Remark 2 of §2.1. The imprimitivity sequence still has the form $0 \to (\mathbb{Z}_2)^v \to G \to G_0 \to 1$ and G is given by an imprimitive permutation representation as in the Imprimitivity Theorem of §1. While the case $v = 0$, where the action of G on $S \setminus G$ is effective, still allows such a representation, $v > 0$ will be supposed.

Let H be the subgroup fixing the elements of the set $\{\pm k\}$, and let k^{th} denote the projection of $(\mathbb{Z}_2)^n$ on the k^{th} coordinate. Then since $(S : H) = 2$, and the action of G on $S \setminus G$ has a nontrivial kernel, there exists $e_0 \in (\mathbb{Z}_2)^v$ such that $k^{\text{th}}(e_0) \neq 0$, else $(\mathbb{Z}_2)^v \subset H$, contradicting the requirment that G be effective on $H \setminus G$. Then e_0 may be used to arrange $k^{\text{th}}(s(\sigma)) = 0$, for all $\sigma \in G_0$, so H may still be given as $H = \{(es(\sigma), \sigma) \in G/\sigma \in H_0(k),\ k^{\text{th}}(e) = 0\}$, for $H_0 = H_0(k)$ the subgroup of G_0 fixing the set $\{\pm k\}$.

Fix $\bar{s} \in Z^1(G_0, (\mathbb{Z}_2)^n/(\mathbb{Z}_2)^v)$ and consider, in the case $\rho \in (\mathbb{Z}_2)^v$, the following:

PROPOSITION. *Let* $S = \mathrm{Proj}_{\mathrm{Sets}}^{-1}(H_0)$, *for* $\mathrm{Proj}_{\mathrm{Sets}}$ *defined by* $\bar{s}$; *and, for* $t \in Z^1(G_0, (\mathbb{Z}_2)^n)$, *let* $H_t = \{(es(\sigma), \sigma) \in S/k^{\text{th}}(e) = k^{\text{th}}(t(\sigma))\}$. *Then representatives for the nontrivial* ρ*-structures relative to* $\bar{s}$ *are among the* ρ*-structures* (G, H_t, S).

PROOF. Consider the factor $j^1(Z^1)$ in the proposition following the Classification Theorem in §2.3. Observe that H_t is the inverse image of the stabilizer of a letter in the image of G under the automorphism $(e, \sigma) \to (et(\sigma), \sigma)$ of $(\mathbb{Z}_2)^n \times_s G_0 \subset S_{2n}$ defined by t. Then (H_t, S) represents (under ρ-equivalence) the ρ-structure defined by $\bar{s}j^1(t)$.

6.1. To account for the structures (H_t, S) on G, the essential features are abstracted by the following:

DEFINITIONS. Let (G, H, S) define an imprimitivity structure with n sets of order 2, so that G acts effectively on $H \backslash G$, and suppose the action of G on $S \backslash G$ has nonzero kernel. Then (G, H, S) is said to have $H_0 = \mathrm{Proj}_{\mathrm{Sets}}(H)$ as its initial *structural subgroup*. Let N be a subgroup of H_0, normal in H_0. Then a pair (H_N, S), where H_N is a subgroup of S is said to define an imprimitivity structure with *structural subgroup* N provided that H_N is without normal subgroups (aside from (1)) of G, that the index $(S: H_N) = 2$, and that N is the largest among the normal subgroups K of H_0 for which $\mathrm{Proj}^{-1}_{\mathrm{Sets}}(K) \cap H = \mathrm{Proj}^{-1}_{\mathrm{Sets}}(K) \cap H_N$. The pair (H_N, S) is said to define a *nontrivial structure*, relative to the initial structure (H, S), provided that (H_N, S) defines an imprimitivity structure inequivalent to that defined by the initial structure (H, S). In the case $S = H \times \langle \rho \rangle$, where the imprimitivity structures are referred to as ρ-structures, the notation (H_N, ρ) is used.

PROPOSITION. *The subgroup H_N of S is uniquely determined by the normal subgroup $N \subset H_0$.*

PROOF. Let $(\mathbb{Z}_2)^{v-1}$ denote the subgroup $(\mathbb{Z}_2)^{v-1} = (\mathbb{Z}_2)^v \cap H = \ker(k^{\mathrm{th}}|_{(\mathbb{Z}_2)^v})$. Since $1 \in N$, the subgroup $(\mathbb{Z}_2)^{v-1}$ is common to both H and H_N. Write $H = \bigcup_{\sigma \in H_0} (\mathbb{Z}_2)^{v-1}(s(\sigma), \sigma)$, recalling that $s(\sigma)$ has been chosen so that $k^{\mathrm{th}}(s(\sigma)) = 0$ for all $\sigma \in G_0$. Then observe that

$$H_N = \bigcup_{\sigma \in H_0} (\mathbb{Z}_2)^{v-1}\left(e_0^{\mathrm{sgn}_N(\sigma)} s(\sigma), \sigma\right),$$

where e_0 has been fixed with $k^{\mathrm{th}}(e_0) \neq 0$, and $\mathrm{sgn}_N(\sigma) = \chi_{\bar{N}}(\sigma)$ is the characteristic function of $\bar{N} = H_0 - N$. To check, note that the above collection must be contained in H_N, but then equality holds by the requirement $(S: H_N) = 2$ in the definition of (H_N, S).

EXAMPLES. For the case $G = \mathbb{Z}_2 \times S_4$, $H \cong H_0 \cong S_3$, with $n = 4$, $(\mathbb{Z}_2)^v = \langle \rho \rangle$, and $e_0 = \rho$, observe that the subgroup $N = 1$ is not allowed as a structural subgroup. Let $\sigma \in H_0$ be a 3-cycle. Then $(\rho, \sigma) \in H_N$ gives $(\rho, \sigma)^3 = (\rho, (1)) \in H_N$, contradicting $(S: H_N) = 2$.

For the general subgroup H_t, observe that $H_t = H_N$ is the subgroup of S corresponding to the normal subgroup $N = \ker(k_t^{\mathrm{th}})$, for k_t^{th}: $H_0 \to \mathbb{Z}_2$ defined by $k_t^{\mathrm{th}}(\sigma) = k^{\mathrm{th}}(t(\sigma))$.

6.2. With the above preliminaries established, the nature of the ρ-structures classified by the Classification Theorem may be clarified by the following:

STRUCTURES THEOREM. *Let (G, H, S) define an imprimitivity structure, and suppose the imprimitivity sequence $0 \to (\mathbb{Z}_2)^v \to G \to G_0 \to 1$ has $v > 0$. Then the collection of structural subgroups N depends only on $(G_0, (\mathbb{Z}_2)^v)$. There is a collection $\mathcal{C}$ of structural subgroups with the following properties:*

(1) *the structures (H_N, S), with $N \in \mathcal{C}$, are equivalent to (H, S), independent of G;*

(2) *(H_N, S) is a nontrivial split structure for all $N \notin \mathcal{C}$, when G is split; and*

(3) *for G to have a smaller number of nontrivial structures than the split group,* $\mathrm{Aut}(G)$ *must contain elements as specified precisely below.*

Further, the collection $\mathcal{C}$ *is computable from* $H^1(G_0,(\mathbb{Z}_2)^v)$, *with the structural subgroups corresponding to nonzero classes requiring identifications by* Outer(G), *the group of outer automorphisms.*

Before beginning the proof, some technical results describing the effect of Aut(G) on the structures (H_N, S), will be given. In order that $\alpha \in \mathrm{Aut}(G)$ give an equivalence of the initial structure (H, S) with some structure (H_N, S), α must satisify $\alpha(S) = S$ and $\alpha(H) = H_N$, by definition. Then $\alpha((\mathbb{Z}_2)^v) = (\mathbb{Z}_2)^v$ so α defines an automorphism α_v: $(\mathbb{Z}_2)^v \to (\mathbb{Z}_2)^v$ by $(\alpha_v(e),(1)) = \alpha(e,(1))$ and induces an automorphism $\bar{\alpha}$: $G_0 \to G_0$. The automorphism α is then determined by these two maps and a third map α_s: $G_0 \to (\mathbb{Z}_2)^v$, with $\alpha_s(\sigma)$ defined by $\alpha(s(\sigma),\sigma) = (\alpha_s(\sigma)s(\bar{\alpha}(\sigma)),\bar{\alpha}(\sigma))$. Write $\alpha \in \mathrm{Aut}(G)$, under the assumption $\alpha(S) = S$, as $\alpha = (\alpha_v, \alpha_s, \bar{\alpha})$.

Next, S and H have $\mathrm{Proj}_{G_0}(S) = \mathrm{Proj}_{G_0}(H) = H_0$, since $v > 0$, so $\bar{\alpha}(H_0) = H_0$. Recall that the subgroup $(\mathbb{Z}_2)^{v-1} = (\mathbb{Z}_2)^v \cap H = \ker(k^{\mathrm{th}}|_{(\mathbb{Z}_2)^v})$, for $H = H(k)$, $H_0 = H_0(k)$, is common to all of the subgroups H_N so α_v is restricted by $\alpha_v((\mathbb{Z}_2)^{v-1}) = (\mathbb{Z}_2)^{v-1}$. Apply the requirement that α be multiplicative to the product $(e's(\sigma),\sigma)(e,(1)) = (e's(\sigma)\sigma * e,\sigma)$ to obtain the compatibility condition $\alpha_v(\sigma * e) = \bar{\alpha}(\sigma) * \alpha_v(e)$, for all $e \in (\mathbb{Z}_2)^v$, $\sigma \in G_0$.

The remaining condition for $\alpha = (\alpha_v, \alpha_s, \bar{\alpha})$ to define an element of Aut(G) taking (H, S) to some (H_N, S) is given by the following:

LEMMA 1. *Suppose* $\alpha_v \in \mathrm{Aut}((\mathbb{Z}_2)^v,(\mathbb{Z}_2)^{v-1})$, $\bar{\alpha} \in \mathrm{Aut}(G_0, H_0)$, *and the above compatibility condition relating* α_v *and* $\bar{\alpha}$ *holds. Then* $\alpha = (\alpha_v, \alpha_s, \bar{\alpha})$ *defines an automorphism if and only if the map* α_s: $G_0 \to (\mathbb{Z}_2)^v$ *satisfies the condition*

$$(*) \qquad (\delta_{\bar{\alpha}}(\alpha_s))\{\sigma,\sigma_1\} = \alpha_v((\delta s)\{\sigma,\sigma_1\})(\delta_{\bar{\alpha}} s)\{\bar{\alpha}(\sigma),\bar{\alpha}(\sigma_1)\},$$

where δ *is the coboundary map*

$$(\delta s)(\sigma,\sigma_1) = s(\sigma)\sigma * s(\sigma_1)s(\sigma\sigma_1),$$

and $\delta_{\bar{\alpha}}$ *is the coboundary*

$$(\delta_{\bar{\alpha}} t)(\tau,\tau_1) = t(\tau)\bar{\alpha}(\tau) * t(\tau_1)t(\tau\tau_1).$$

PROOF. The condition holds if and only if the map

$$\alpha(es(\sigma),\sigma) = (\alpha_v(e)\alpha_s(\sigma)s(\bar{\alpha}(\sigma)),\bar{\alpha}(\sigma))$$

is multiplicative, noting that the cochain map s has $(\delta s)(\sigma,\sigma_1) \in (\mathbb{Z}_2)^v$, since $(j \circ s) \in Z^1(G_0,(\mathbb{Z}_2)^n/(\mathbb{Z}_2)^v)$.

The above condition is also sufficient to insure that $\alpha(S) = S$ and $\alpha(H) = H_N$, so any such triple will produce a structural subgroup N, determining which H_N arise as $\alpha(H) = H_N$. Recall that $v > 0$ allows normalizing the k^{th} coordinate of $s(\sigma)$ to be 0, for all $\sigma \in G_0$, k depending upon $H = H(k)$. Then observe that N may be obtained by the following:

LEMMA 2. *The structural subgroup* N *determining* $\alpha(H) = H_N$ *is given as* $N = \ker(k^{\mathrm{th}}(\alpha_s(\bar{\alpha}^{-1}(\sigma)))|_{H_0})$.

PROOF. Recall $H = \cup_{\sigma \in H_0} (\mathbb{Z}_2)^{v-1}(s(\sigma), \sigma)$, and then observe that for $\alpha = (\alpha_v, \alpha_s, \bar{\alpha})$ as above,

$$\alpha(H) = \bigcup_{\sigma \in H_0} (\mathbb{Z}_2)^{v-1}(\alpha_s(\sigma)s(\bar{\alpha}(\sigma)), \bar{\alpha}(\sigma)).$$

Then

$$k^{\text{th}}(\alpha_s(\sigma)s(\bar{\alpha}(\sigma))) = k^{\text{th}}(\alpha_s(\sigma))$$

since $k^{\text{th}}(s(\sigma)) = 0$ for all $\sigma \in G_0$. But the map $\text{sgn}_N(\sigma)$ used to describe H_N gives the lift of σ, so $\sigma = \bar{\alpha}^{-1}(\sigma_1)$ gives $k^{\text{th}}(\alpha_s(\bar{\alpha}^{-1}(\sigma_1)))$ as $\text{sgn}_N(\sigma_1)$.

A complete account of the effect of $\text{Aut}(G)$ on the structures (H_N, S) has been reduced to understanding the maps α_s: $G_0 \to (\mathbb{Z}_2)^v$ allowed by $\text{Aut}(G)$. If the right-hand side, $\alpha_v(\delta s)\delta_{\bar{\alpha}}s$, of equation $(*)$ of Lemma 1 is nonzero, then the map α_s strictly depends on G, since the split extension with trivial structure $s(\sigma) = 0$ for all $\sigma \in G_0$ forces the right-hand side to be zero. Then the existence of such an automorphism of G supplies the promised precise specification in (3), as may be observed in the analysis of the case $\delta_{\bar{\alpha}}(\alpha_s) = 0$ in the following:

PROOF OF THE STRUCTURES THEOREM. For the collection of structural subgroups, observe that H_N a subgroup of index two in S gives that H_N is without normal subgroups of G. Only $(\mathbb{Z}_2)^k \subset (\mathbb{Z}_2)^{v-1}$ is possible, but if $e \in (\mathbb{Z}_2)^k$ has a nonzero entry, the transitive G_0-action will conjugate that entry to the k^{th} component, taking e outside H_N. To insure that $H_N = \cup_{\sigma \in H_0} (\mathbb{Z}_2)^{v-1}(\varepsilon_0^{\text{sgn}_N(\sigma)}s(\sigma), \sigma)$ is closed under products, consider a relation $\sigma_1\sigma_2 \cdots \sigma_r = 1$ among elements of H_0. Lifting the relation to H_N gives

$$\begin{aligned}
&\left(\varepsilon_0^{\text{sgn}_N(\sigma_1)}s(\sigma_1), \sigma_1\right)\left(\varepsilon_0^{\text{sgn}_N(\sigma_2)}s(\sigma_2), \sigma_2\right) \cdots \left(\varepsilon_0^{\text{sgn}_N(\sigma_r)}s(\sigma_r), \sigma_r\right) \\
&\quad = \left(\varepsilon_0^{\text{sgn}_N(\sigma_1)}s(\sigma_1)\sigma_1 * \varepsilon_0^{\text{sgn}_N(\sigma_2)}\sigma_1 * s(\sigma_2)\right. \\
&\qquad \left. \cdots (\sigma_1 \cdots \sigma_{r-1}) * \varepsilon_0^{\text{sgn}_N(\sigma_r)}(\sigma_1 \cdots \sigma_{r-1}) * s(\sigma_r), (1)\right).
\end{aligned}$$

Then closure requires the first entry to be in $(\mathbb{Z}_2)^{v-1} = \ker(k^{\text{th}}|_{(\mathbb{Z}_2)^v})$. But $k^{\text{th}}(s(\sigma)) = 0$ and sgn_N is only required on $H_0 = H_0(k)$, so that

$$k^{\text{th}}(-) = k^{\text{th}}\left(\varepsilon_0^{\text{sgn}_N(\sigma_1)}\sigma_1 * \varepsilon_0^{\text{sgn}_N(\sigma_2)} \cdots (\sigma_1 \cdots \sigma_{r-1}) * \varepsilon_0^{\text{sgn}_N(\sigma_r)}\right)$$

is independent of s. But lifting generators and relations for H_0 to H_N and taking account of the subgroup $(\mathbb{Z}_2)^{v-1}$, with action determined by H_0, suffices to determine H_N, regardless of s.

For $t_0 \in Z^1(G_0, (\mathbb{Z}_2)^v)$ the map $\alpha = (0, t_0, 1)$ is always an automorphism, as may be seen, in particular, from Lemma 1. But for $t_0 \in B^1(G_0, (\mathbb{Z}_2)^v)$, $k^{\text{th}}(t_0|_{H_0}) = 0$ since H_0 fixes k^{th} components, so $k^{\text{th}}([t_0])$ is well defined for $[t_0] \in H^1(G_0, (\mathbb{Z}_2)^v)$. Then $H^1(G_0, (\mathbb{Z}_2)^v)$ gives a collection of structural subgroups, independent of G. But $\alpha(H) = H_N$ for $N \neq H_0$ is never achieved by $\alpha \in \text{Inn}(G)$, since H_N is not one of the n conjugates $H(j)$, $j = 1, \ldots, n$, of $H = H(k)$. Therefore $H^1(G_0, (\mathbb{Z}_2)^v) \neq 0$, $t_0 \in Z^1(G_0, (\mathbb{Z}_2)^v)$ nonprinciple with $\ker(k^{\text{th}}(t_0|_{H_0})) = N \neq H_0$ always corresponds to an equivalence requiring $\text{Outer}(G)$.

Only the assertions on $G = \cup_{\sigma \in G_0} (\mathbb{Z}_2)^v(0, \sigma)$, by an argument depending only upon the right-hand side of the equation of Lemma 1 being zero for all $\alpha \in \text{Aut}(G)$,

remains. But under this assumption, $\alpha_s \in Z^1_{\bar{\alpha}}(G_0, (\mathbb{Z}_2)^v)$, the group of cocyles with action twisted by $\bar{\alpha}$, i.e., $\sigma_{\bar{\alpha}} * (e) = \bar{\alpha}(\sigma) * e$. Finally, observe that $t_0(\sigma) = \alpha_s(\bar{\alpha}^{-1}(\sigma))$, for $\alpha_s \in Z^1_{\bar{\alpha}}$, gives $t_0 \in Z^1$, untwisted, so that $k^{\text{th}}(\alpha_s(\bar{\alpha}^{-1}(\sigma))) = k^{\text{th}}(t_0(\sigma))$ adds no new structural subgroups.

REMARKS. In the low degree examples considered in the present investigation, nontrivial structures have been obtained only when the imprimitivity sequence of G is split. Observe that a nontrivial split imprimitivity structure, $\rho \notin (\mathbb{Z}_2)^v$, occurs for $n = 3$ and accounts for the fact that $G_0 \cong S_4$ has two distinct permutation structures of degree 6. Recall that the nontrivial structure on $\text{Gal}(K^c/\mathbb{Q})$ in §2.2 uses one of these.

The nontrivial structures obtained are only asserted to be distinct from the initial structure and might therefore allow further equivalences among themselves. A generalization of Miller's examples is the case $G = \mathbb{Z}_2 \times \text{PSL}(2, p)$, the direct product $G_0 = \text{PSL}(2, p)$ being allowed to range over all transitive permutation structures. For (H_N, ρ) with $H_N = H_l$, $t \in Z^1(G_0, (\mathbb{Z}_2)^{\deg(G_0)})$, a calculation establishes that the trivial cochain structure admits l such nontrivial structures for $l = 0, 1$ or 3, with $l = 3$ implying $H_0 \cong D_{2q_1}$, dihedral of order $4q_1$, $q_1 \geqslant 1$. Since ρ is characteristic, the ρ-structures $(\mathbb{Z}_2 \times \text{PSL}(2, p), \langle 0 \rangle \times D_{2q_1}, \rho)$ are in one-to-one correspondence with the permutation structures $(Z_2 \times \text{PSL}(2, p), \langle 0 \rangle \times D_{2q_1})$. The permutation structures do not determine the ρ-structures for each ρ only when ρ is noncharacteristic. The case where ρ is the unique element of order 2 strongly restricts the structure of G, as in [**13**].

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