THE EQUIVARIANT DOLBEAULT LEMMA

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ABSTRACT. A form of the Dolbeault Lemma is obtained for circular domains $D \subset \mathbb{C}^n$, which is equivariant for the subgroup of $GL(n; \mathbb{C})$ that stabilizes $D$.

1. Introduction and statement. In our work on representation theory we have had occasion to use holomorphic triviality arguments in an equivariant context. Since the appropriate form of the Dolbeault Lemma is generally useful but is not in the literature, we decided to write it up separately.

Let $D$ be a domain in $\mathbb{C}^n$ that is circular in the sense $tD \subset D$ whenever $t$ is a complex number with $|t| \leq 1$. $G = \{ g \in GL(n; \mathbb{C}) : gD = D \}$ is its stabilizer in the general linear group. Fix a Frechet space $F$ and denote

- $C^q(D; F)$: the space of all $F$-valued $C^\infty$ $(0, q)$-forms on $D$,
- $Z^q(D; F)$: the subspace of $C^q(D; F)$ given by $\bar{\partial}\omega = 0$.

Here, of course, $C^q(D; F)$ is a Frechet space in the $C^\infty$ topology and $Z^q(D; F)$ is a closed subspace. Our result is the

EQUIVARIANT DOLBEAULT LEMMA. There exist continuous operators

$$H: C^{q+1}(D; F) \to C^q(D; F), \quad 0 \leq q < n,$$

such that

(i) $H\bar{\partial}\omega + \bar{\partial}H\omega = \omega$ for every $\omega \in C^q(D; F), q \geq 1$;
(ii) if $g \in G$ then $g^*H\omega = Hg^*\omega$ for every $\omega \in C^q(D; F)$;
(iii) if $b: F \to F$ is a continuous linear operator then $b \cdot H\omega = H(b \cdot \omega)$ for every $\omega \in C^q(D; F)$.

In particular, $H: Z^{q+1}(D; F) \to C^q(D; F)$ is a continuous $G \times BL(F)$ equivariant inverse to $\overline{\partial}$: $C^q(D; F) \to Z^{q+1}(D; F)$ for $q \geq 1$.

2. The one variable formula. Suppose $n = 1$, so either $D = \mathbb{C}$ or $D$ is a disk ($z \in \mathbb{C}$: $|z| < \rho$) in $\mathbb{C}$. If $\phi: F \to \mathbb{C}$ is a continuous linear functional, we define

$$h_\phi: C^1(D; F) \to C^0(D; \mathbb{C})$$

as follows. On the $\phi^{-1}$-image of the polynomial $(0, 1)$-forms we define

$$\text{if } \phi \cdot \omega = \sum a_{rs}z^r\bar{z}^s \, d\bar{z} \text{ then } h_\phi(\omega) = \sum \frac{a_{rs}}{s+1} z^r\bar{z}^{s+1}.$$
Now $h_\phi$ extends by continuity to $C^1(D; F)$. By construction, $\overline{\partial} h_\phi(\omega) = \phi \cdot \omega$. A short calculation with monomials gives the formal properties

if $a \in \mathbb{C}$ then $aD \subset D$, and $m_a z \mapsto az$ satisfies
$$m_a^* h_\phi(\omega) = h_\phi(m_a^* \omega);$$

if $m$ is a positive integer and $f \in C^0(D; \mathbb{C})$ then
$$h_\phi(\overline{\partial}(z^m f)) = z^m \phi(f).$$

Let $F'_1$ and $F'_2$ be subspaces of the continuous dual $F'$ of $F$, and suppose that we have continuous linear maps
$$h_i: C^1(D; F) \to C^0(D; F/(F'_i + F'_2)^\perp), \quad i = 1, 2,$$
such that
if $f \in F'_i$ then $\phi \cdot h_i(f) = h_\phi(\omega)$ for all $\omega \in C^1(D; F)$.

Then there is a unique continuous linear map
$$h_3: C^1(D; F) \to C^0(D; F/(F'_1 + F'_2)^\perp)$$
that yields $h_1$ and $h_2$ on projection of function values from $F/(F'_1 + F'_2)^\perp$ to $F/(F'_i)^\perp$ and $F/(F'_2)^\perp$, and that satisfies $\phi \cdot h_3(\omega) = h_\phi(\omega)$ for all $\phi \in F'_1 + F'_2$.

Now an easy transfinite induction give us

**Lemma.** There is a unique continuous linear map $h: C^1(D; F) \to C^0(D; F)$ such that

(2.1) if $f \in F'$ and $\phi \cdot \omega = \sum a_{rs} z^r \overline{z}^s d\overline{z}$, then $\phi \cdot h(\omega) = \sum \frac{d_{rs}}{s + 1} z^r \overline{z}^{s+1}$.

If $f \in C^0(D; F)$ and $\omega \in C^1(D; F)$ then, from the corresponding properties of the $h_\phi$,

(2.2) $\overline{\partial} h(\omega) = \omega$;

(2.3) $m_a^* h(\omega) = h(m_a^* \omega)$ when $a \in \mathbb{C}$ with $aD \subset D$; and

(2.4) $h \cdot \overline{\partial}(z^m f) = z^m f$ for all integers $m > 0$.

3. **Several variables.** Now $D \subset \mathbb{C}^n$, $n \geq 1$. If $1 \leq j \leq n$ then $i(\partial / \partial z_j)$ and $e(d\overline{z}_j)$ denote interior and exterior product, respectively. $D$ is assumed stable under the multiplications $m_t: z \mapsto tz$, $|t| \leq 1$, and of course if $z \in D$ we have $\epsilon > 0$ depending on $z$ such that $m_t$ maps a neighborhood of $z$ into $D$ whenever $|t| < 1 + \epsilon$. By (2.3) we may thus define continuous linear maps

$$H: C^{q+1}(D, F) \to C^q(D, F)$$

for $0 \leq q < n$

by the formula

(3.1) $H\omega = \sum_{j=1}^n \overline{z}_j h_i \left( m_t^* \left( \frac{\partial}{\partial z_j} \right) \omega \right) \wedge d\overline{t}$. $

Here $h_i$ is the operator of the one variable lemma in §2, for the complex variable $t$.
Compute

\begin{equation}
(3.2) \quad H \bar{\omega} + \bar{\delta} H \omega = \sum_{j=1}^{n} \bar{\partial}_j h_t \left( m^*_t \left( i \left( \frac{\partial}{\partial \bar{z}_j} \right) \bar{\delta}_j \omega \right) \wedge d\bar{t} \right) \bigg|_{t=1} + \sum_{j=1}^{n} e_j (d\bar{z}_j) \cdot h_t \left( m^*_t \left( i \left( \frac{\partial}{\partial \bar{z}_j} \right) \omega \right) \wedge d\bar{t} \right) \bigg|_{t=1} + \sum_{j=1}^{n} \bar{z}_j \delta_z \left[ h_t \left( m^*_t \left( i \left( \frac{\partial}{\partial \bar{z}_j} \right) \omega \right) \wedge d\bar{t} \right) \bigg|_{t=1} \right].
\end{equation}

To prove (3.3) below, it suffices to consider the case \( \omega = f d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_k \), \( k = q + 1 \geq 1 \). In (3.2) the second summand is equal to

\begin{align*}
& \sum_{j=1}^{n} h_t \left( \left( d\bar{z}_j \wedge \partial_j^{-1} i \left( \frac{\partial}{\partial \bar{z}_j} \right) f(tz) i^k \bar{z}_1 \wedge \cdots \wedge \bar{z}_k \right) \wedge d\bar{t} \right) \bigg|_{t=1} \\
& = \sum_{j=1}^{n} h_t \left( \partial_j^{-1} e_j (d\bar{z}_j) \cdot i \left( \frac{\partial}{\partial \bar{z}_j} \right) m^*_t \omega \wedge d\bar{z} \right) \bigg|_{t=1} \\
& = k h_t \left( \partial_j^{-1} (m^*_t \omega) \wedge d\bar{t} \right) \bigg|_{t=1}. 
\end{align*}

The first and third summands of (3.2) add up to

\begin{align*}
& \sum_{j=1}^{n} h_t \left( \bar{z}_j m^*_t \left( i \left( \frac{\partial}{\partial \bar{z}_j} \right) \bar{\delta}_j \omega \right) + \bar{\partial}_j m^*_t \left( i \left( \frac{\partial}{\partial \bar{z}_j} \right) \omega \right) \wedge d\bar{t} \right) \bigg|_{t=1} \\
& = \sum_{j=1}^{n} h_t \left( \bar{z}_j m^*_t \left( i \left( \frac{\partial}{\partial \bar{z}_j} \right) \omega \wedge d\bar{z} \right) \right) \bigg|_{t=1} \\
& = \sum_{j=1}^{n} h_t \left( \bar{z}_j m^*_t \left( \frac{\partial f}{\partial \bar{z}_j} \bar{z}_1 \wedge \cdots \wedge \bar{z}_k \right) \wedge d\bar{t} \right) \bigg|_{t=1} \\
& = h_t \left( \left( i^k \frac{\partial}{\partial \bar{t}} f(tz) \bar{z}_1 \wedge \cdots \wedge \bar{z}_k \right \wedge d\bar{t} \right) \bigg|_{t=1}. 
\end{align*}

On the other hand, since \( k > 0 \) we can use (2.4) to see

\begin{align*}
h_t (\bar{\partial}_j m^*_t \omega) &= h_t (\bar{\partial}_j \left[ i^k f(tz) \bar{z}_1 \wedge \cdots \wedge \bar{z}_k \right]) \\
& = i^k f(tz) \bar{z}_1 \wedge \cdots \wedge \bar{z}_k \\
& = m^*_t \omega \\
\end{align*}

so

\begin{align*}
\omega &= m^*_t \bigg|_{t=1} \\
& = h_t (\bar{\partial}_j m^*_t \omega) \bigg|_{t=1} \\
& = h_t (\bar{\partial}_j \left( i^k f(tz) \right) \bar{z}_1 \wedge \cdots \wedge \bar{z}_k) \bigg|_{t=1} \\
& = kh_t \left( \partial_j^{-1} (m^*_t \omega) \wedge d\bar{t} \right) \bigg|_{t=1} + h_t \left( \left( i^k \frac{\partial}{\partial \bar{t}} f(tz) \bar{z}_1 \wedge \cdots \wedge \bar{z}_k \right \wedge d\bar{t} \right) \bigg|_{t=1}. 
\end{align*}
In conclusion,

\[(3.3) \quad H \bar{\partial} \omega + \partial H \omega = \omega \quad \text{for all } \omega \in C^k(D; F), k \geq 1. \]

In the formula \((3.1)\) for \(H\), we can replace the multipliers \(z_j\) by any independent set of conjugate linear functionals \(l_1, \ldots, l_n\) on \(C^n\), provided that we replace the \(\partial/\partial z_j\) by the dual basis of \(\overline{C}^n\). Thus, if \(g \in \text{GL}(n; C)\) preserves \(D\), then \((3.1)\) is equivalent to

\[
H\omega = \sum_{j=1}^n g^*(\bar{z}_j) h_t \left( m^*_t \left( i \left( g^* \left( \frac{\partial}{\partial z_j} \right) \omega \right) \wedge dt \right) \right)
\]

Now

\[
H(g^*\omega) = \sum_{j=1}^n g^*(\bar{z}_j) h_t \left( m^*_t \left( i \left( g^* \left( \frac{\partial}{\partial z_j} \right) \omega \right) \wedge dt \right) \right)
= g^* \sum_{j=1}^n \bar{z}_j h_t \left( m^*_t \left( i \left( \frac{\partial}{\partial z_j} \right) \omega \right) \wedge dt \right) \]

because \(g^*m^*_t = m^*_t g^*\) and \((2.3)\)

\[
= g^*H(\omega).
\]

We have just verified that

\[(3.4) \quad g^*H\omega = Hg^*\omega \quad \text{for all } g \in G, \omega \in C^{q+1}(D; F). \]

Finally, let \(b: F \to F\) be a continuous linear map. It acts on the values of \(\omega\) and \(H\omega\). If \(b\) has rank 1 then we have \(\phi \in F'\) and \(v \in F\) such that

\[
b(u) = \phi(u)v \quad \text{for all } u \in F
\]

and then

\[
b \cdot H(\omega) = \sum_{j=1}^n \bar{z}_j \phi \left( h_t \left( m^*_t \left( i \left( \frac{\partial}{\partial z_j} \right) \omega \right) \wedge dt \right) \right) v
= \sum_{j=1}^n z_j h_t \left( m^*_t \left( i \left( \frac{\partial}{\partial z_j} \right) (\phi \cdot \omega) \right) \wedge dt \right) \cdot v
= H(b \cdot \omega).
\]

Now an extension argument along the lines of that \S2 gives a maximal subspace of \(F\) such that \(b \cdot H(\omega) = H(b \cdot \omega)\) whenever \(b\) has range in the subspace, and that subspace must be all of \(F\) because we can always add the range of a rank 1 operator to it. Thus

\[(3.5) \quad b \cdot H(\omega) = H(b \cdot \omega) \quad \text{for all } b \in BL(F). \]

The Equivariant Dolbeault Lemma is obtained by combining \((3.1)\) and \((3.3)-(3.5)\).
Note that, given a net \((\phi_\alpha; \alpha \in A) \subset F'\) such that the \(\phi_\alpha\) are linearly independent and \(F'\) is their weak* closed span, the definition (3.1) is constructive. This applies in particular if \(\dim F < \infty\). This constructiveness is the key point at which our argument differs from the standard, somewhat combinatorial, proof of the Dolbeault Lemma. Note also that our proof is loosely modeled on the standard proof of the Poincaré Lemma.

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