CONTRACTION SEMIGROUPS FOR DIFFUSION WITH DRIFT

BY

R. SEELEY

Abstract. Recently Dodziuk, Karp and Li, and Strichartz have given results on existence and uniqueness of contraction semigroups generated by the Laplacian $\Delta$ on a manifold $M$; earlier, Yau gave related results for $L = \Delta + V$ for a vector field $V$. The present paper considers $L = \Delta - V - c$, with $c$ a real function, and gives conditions for (a) uniqueness of semigroups on the bounded continuous functions, (b) preservation of $C_0$ (functions vanishing at $\infty$) by the minimal semigroup, and (c) existence and uniqueness of contraction semigroups on $L^p(\mu)$, $1 \leq p < \infty$, for an arbitrary smooth density $\mu$ on $M$. The conditions concern $L p / p$, where $p$ is a smooth function, $p \to \infty$ as $x \to \infty$. They variously extend, strengthen, and complement the previous results mentioned above.

Introduction. Consider an operator

$$L u = \Delta u - Vu - cu,$$

where $\Delta$ is the Laplacian on a noncompact Riemannian manifold $M$, $V$ is a vector field, and $c$ a function. (All coefficients are assumed reasonably smooth.) $L$ represents a diffusion with “drift” $V$ and “local dissipation” $c$. The evolution in time of an initial distribution $f$ is a solution of the Cauchy problem for the heat equation:

$$\begin{cases}
\frac{\partial u}{\partial t} = Lu, & t > 0, \\
u = f & \text{when } t = 0, \\
u \text{ continuous for } t \geq 0.
\end{cases}$$

The sign of $V$ in (1) has a simple interpretation: if $V(x) \cdot du(x) > 0$ then the stuff brought toward $x$ by $V$ is at a lower temperature than the stuff being taken away, thus causing a drop in the temperature $u$ at $x$.

If $c \geq 0$, equation (2) always has a “minimal solution” which can be obtained as follows (see Dodziuk [D] for more details). Represent $M$ as the union of an increasing sequence of compact manifolds with boundary $M_n$. Let $u_n$ be the solution of (2) on $M_n$ with $u_n = 0$ on $\partial M_n$. This solution can be represented as

$$u_n(t, x) = \int_{M_n} p_n(t, x, y) f(y) \, dv_y,$$

where $p_n$ is positive, $\int_{M_n} p_n \, dv_y \leq 1, \lim_{t \to 0} \int p_n \, dv_y = 1$, and

$$\partial p_n / \partial t = L_x p_n = L^* p_n.$$
By the maximum principle and the monotone convergence theorem, the \( p_n \) converge up to a function \( p \) with \( \int p \, dv < 1 \), and (in the sense of distributions)

\[
\frac{\partial p}{\partial t} = L_x p = L_x^* p, \quad t > 0.
\]

Since \( p \) satisfies the parabolic equation \( 2p_t = L_x p + L_x^* p \), it is smooth in all variables for \( t > 0 \). If \( f \) is a bounded continuous function on \( M \) (\( f \in BC(M) \)), then the function

\[
(3) \quad u(x, t) = \int p(t, x, y)f(y) \, dv_y
\]

solves the Cauchy problem (2). The map \( f \to u(\cdot, t) \) defines a contraction semigroup on the space \( BC(M) \), with infinitesimal generator \( L \) defined on an appropriate domain containing \( C^2(M) \), the \( C^2 \) functions with compact support.

§1 considers uniqueness of the solution (3), and §2 concerns the preservation of \( C_0 \) (continuous functions vanishing at \( \infty \)). §3 considers the existence of contraction semigroups on \( L^p \), \( 1 \leq p < \infty \), generalizing some results of Strichartz [S]. The interesting condition there relates the divergence of \( V \) to the local dissipation \( c \). Specifically, Theorem 3 says: The operator \( L \) is dissipative on \( L^p \) if

\[
\frac{1}{p} \, \text{div} \, V \leq c.
\]

As \( p \to \infty \) this reduces to the familiar condition \( c \geq 0 \). The condition is also necessary when \( p = 1 \), or when \( L = -V - c \) (\( \Delta = 0 \)). In any case, if it is uniformly violated on a sufficiently large set, then \( L \) is not dissipative.

There are several results similar to those in §§1 and 2. A standard reference (particularly for counterexamples) is Azencott [A]. Generalizing results of Feller and Hille in the case \( M \subset R^1 \), he gives conditions based on integrals involving coefficients \( A, B, C \) in the expression

\[
Lu(p) = Au''(p) + Bu'(p) + cu(p),
\]

where \( p \to \infty \) as \( x \to \infty \). The conditions seem closely related to ours, yet not directly comparable.

More comparable are the results of Yau [Y]. He assumes \( c = 0 \) in (1) and assumes a function \( \rho \) such that \( \rho(x) \to \infty \) as \( x \to \infty \), while \( L\rho \leq k, |d\rho| = o(\rho) \), and then constructs a kernel \( p \) with \( \int p = 1 \). (According to Theorem 1, this is the same as the minimal solution if \( \rho \) is \( C^2 \).) He shows further that if \( V = 0 \), then the solution preserves \( C_0 \) (continuous functions vanishing at \( \infty \)). His conditions on the coefficients of \( L \) are slight, and \( \rho \) may be just Hölder continuous, with \( L\rho \leq k \) valid in the sense of distributions. He notes that the result applies when \( M \) is complete and Ric (the Ricci curvature) is bounded below.

Doc's [D] gives very similar results by more elementary methods. Karp and Li [KL] get nearly optimal results of this type for the case \( L = \Delta \). If \( r \) denotes distance from a fixed point, they show that

\[
\text{Ric} \geq -k(r^2 + 1) \Rightarrow \text{vol} \{ r \leq R \} \leq e^{c(R^2)} \Rightarrow \text{uniqueness}
\]

and

\[
\text{Ric} \geq -k(r^2 + 1) \Rightarrow \text{G is preserved}.
\]
Our conditions are of the same type as Yau’s. For uniqueness of solutions of (2), we require a positive $C^2$ function $p$ such that $p(x) \to \infty$ as $x \to \infty$, with $Lp \leq k \rho$ (Theorem 1). If $\rho$ is the distance $r_p$ from a fixed point $P$ and $M$ is complete, then the inequality need hold only within the cut locus of $p$. As Li mentions in a letter, this condition follows from $\text{Ric} \geq -C(1 + r_p^2)$; see the Appendix.

For preservation of $C_0$ we assume $\Delta \rho - V \rho \geq -k \rho$ and $|d \rho| \leq k \rho$ (which implies $M$ is complete). This allows us to recover Karp’s and Li’s result involving the Ricci curvature, but only on a manifold $M$ with a “pole”, a point $p$ where the exponential map is a diffeomorphism: $M_p \to M$.

We are indebted to Peter Li and Robert Strichartz for some very helpful correspondence; to C. L. Terng for explaining the relation between Ric and bounds on $\Delta r$; to Walter Rosenkrantz for several discussions, and for bringing up the Ornstein-Uhlenbeck example (the original motivation of the paper); and to Freda Bennett for sharing her proof of a uniqueness theorem, which suggested a considerable simplification of Theorem 1.

1. Uniqueness. First we consider whether (3) is the only solution of (2).

**Theorem 1.** Let $L u = \Delta u - Vu - cu$ (where $c$ has any sign) and suppose $M$ carries a $C^2$ function $\rho > 0$ such that $\rho(x) \to \infty$ as $x \to \infty$ (in the one-point compactification) and $L \rho \leq k \rho$ for some constant $k$. Then

$$ \begin{align*}
    u_t &= Lu, \\
    u|_{t=0} &= 0, \\
    |u| &\leq e^{at} \quad \text{for some } a
\end{align*} $$

implies that $u \equiv 0$.

**Proof.** Let $w = e^{-Kt} u \rho^{-1}$ with $K > \max(a, k)$. Then $w \to 0$ uniformly as $(x, t) \to \infty$ and

$$ w_t = \Delta w - Vw + 2 \frac{d \omega \cdot d \rho}{\rho} - \left( K - \frac{L \rho}{\rho} \right) w. $$

Since $K - (L \rho)^{-1} > 0$, the maximum principle shows $w = 0$. In fact, if $w \neq 0$ then there is a positive maximum or a negative minimum. Suppose, say, $w(x_0, t_0)$ is a positive maximum. Then at $(x_0, t_0)$, $0 = w_t = d \omega - Vw$, while $\Delta w \leq 0$ and $-(K - L \rho/\rho)w < 0$, contradicting (1.1).

**Remarks.** In case $c \geq 0$, the existence of a $\rho$ as in Theorem 1 guarantees that the minimal solution (3) is the only solution of (2).

In case $c = 0$, then $\rho \equiv 1$ admits the solution $u \equiv 1$. When Theorem 1 applies, this is the only bounded solution; hence

$$ u(x, t) = \int p(t, x, y) \, dy = 1 \quad \text{for } t > 0. $$

This implies that the adjoint problem is conservative, i.e. solutions of $L^* u = u_t$ have $\int_M u$ independent of $t$. (For the adjoint problem, $u$ is heat per unit volume.)

The condition $L \rho \leq k \rho$ cannot be replaced by $L \rho \leq k \rho^{1+\varepsilon}$ for any $\varepsilon > 0$. For example, with $M = (-1, 1)$, $u = u''$, and $\rho(x) = (1 - x^2)^{-2/\varepsilon}$, we have $L \rho \leq k \rho^{1+\varepsilon}$. But in this case there are two-well known distinct solutions of (2), one with $u(x, t) \to 0$ and another with $u(x, t) \to 0$.  

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Thinking in terms of heat, the condition $\Delta \rho - V \rho - c \rho \leq k \rho$ means that what happens at $\infty$ (the boundary of $M$) cannot diffuse to the finite part of $M$; note that the term $-V \rho$ gives the rate of drift in from $\infty$.

Theorem 1 applies to the generalized Ornstein-Uhlenbeck process in $\mathbb{R}^n$, where

\[(1.2) \quad L = \Delta - l(x) \cdot \nabla\]

with $l(x)$ linear in $x$. Here $V = l(x)$ could represent a rotation, expansion, dilation, shear, etc. Taking $\rho(x) = 1 + |x|^2$, Theorem 1 guarantees uniqueness of solutions, and conservation.

In geometric applications the function $\rho$ in Theorem 1 is generally taken to be (essentially) the distance $r_p$ from a fixed point $p$ in $M$. If $\rho$ has a “cut locus” there are difficulties which can be avoided by a device due to Calabi [C]; see also [D].

**Theorem 1a.** Suppose $M$ is complete and, for some point $p$, $Lr_p \leq kr_p$ inside the cut locus of $p$ and for $r_p \geq \delta$. Then the conclusion of Theorem 1 holds.

**Proof.** Let $\rho = r_p$ for $r_p \geq \delta$, with $\rho > 0$ and smooth except on the cut locus of $p$. Then $L \rho \leq k \rho$, perhaps with a new $k$. Let $w = e^{-Kt}u^{-1}$ as before, with $K > k$. If $u \neq 0$ then $w$ has, say, a positive maximum $w(x_0, t_0)$. If $x_0$ is not on the cut locus then $r_p$ is smooth at $x_0$, and (1.1) gives a contradiction as before. If the maximum is achieved only with $x_0$ on the cut locus, let $y$ be a minimal geodesic from $p$ to $x_0$. Let $q$ be on $y$ at a small distance $\varepsilon > 0$ from $p$. On the segment of $y$ between $q$ and $x_0$, $r_p = r_q + \varepsilon$; and everywhere else $r_p \leq r_q + \varepsilon$ by the triangle inequality. So the function

\[w_q = e^{-Kt}u(r_q + \varepsilon)^{-1}\]

has a maximum at $(x_0, t_0)$. At this point $r_q$ is smooth, and when $\varepsilon$ is small then

\[L(r_q + \varepsilon)/(r_q + \varepsilon) < K\]

in a neighborhood of $x_0$. So again (1.1) gives a contradiction, with $\rho = r_q + \varepsilon$.

2. **Vanishing at $\infty$.**

**Theorem 2.** Let $L = \Delta - V - c$ with $c \geq 0$. Then $L$ is the generator of a contraction semigroup on $C_0$ if there is a Hölder continuous function $\rho$ such that as $x \to \infty$ then $\rho(x) \to \infty$ and

\[(2.1) \quad |d\rho| \leq k \rho \quad (k \text{ constant}),\]

\[(2.2) \quad V \rho \leq \Delta \rho + k \rho \quad \text{in the sense of distributions.}\]

**Remarks.** (2.1) implies $M$ is complete, and (2.2) says the drift to $\infty$ is not too rapid. A simple application of Theorem 2 is the Ornstein-Uhlenbeck process (1.2).

The proof uses a version of the Lumer-Phillips Theorem [Yö]: A closed operator $L$ on a Banach space $B$ is the generator of a contraction semigroup $e^{Lt}$ if and only if $L$ is dissipative and $L^{*}$ has no positive eigenvalue. Dissipative means that for every $u$ in the domain of $B$ there is a $\tilde{u}$ in $B^{*}$ with

\[\|\tilde{u}\| = 1, \quad \langle u, \tilde{u} \rangle = \|u\|, \quad \text{and} \quad \langle Lu, \tilde{u} \rangle \leq 0.\]
This implies that the range of $\lambda I - L$ is closed for $\lambda > 0$, and the lack of positive eigenvalues for $L^*$ then implies $\lambda I - L$ is surjective.

**Lemma 2.1.** $L$ is dissipative on the domain $\{ u \in C_0 : Lu \in C_0 \}$ iff $c \geq 0$.

This is more or less well known, but for completeness we prove the “if” part.

On $C_0$ the vector $\hat{u}$ is a unit measure concentrated at a maximum point $x_0$ for $u$, or minus a unit measure concentrated at a minimum point. Suppose the first case. Since $Lu \in C_0$, the elliptic regularity theorem shows that $u$ is $C^{2+\varepsilon}$ for every $\varepsilon > 0$; hence $Vu$ and, with it, $\Delta u = Lu + Vu + cu$ are continuous. Since $x_0$ is a maximum point, $du(x_0) = 0$ and

$$\langle Lu, \hat{u} \rangle = (Lu)(x_0) = \Delta u(x_0) - c(x_0)u(x_0) \leq \Delta u(x_0),$$

since we assume $c \geq 0$, and $u(x_0) \geq 0$ at the maximum of any function in $C_0$. By an argument as in Courant and Hilbert [CH, leading up to p. 286], any continuous function $u$ such that $\Delta u$ is also continuous satisfies

$$\Delta u(x_0) = \lim_{R \to 0} \frac{2n}{|S^{n-1}|} R^{-n} \int_{d(x, x_0) = R} \left[ u(x) - u(x_0) \right] d\sigma,$$

where $|S^{n-1}|$ is the area of the $n - 1$ sphere, $d(x, x_0)$ is geodesic distance, and $d\sigma$ is the induced measure on the sphere $\{ d(x, x_0) = R \}$. Since $u(x_0)$ is a maximum, it follows that $\Delta u(x_0) \leq 0$, and $L = \Delta - V - c$ is dissipative, proving Lemma 2.1.

It remains to prove that $L^*$ has no positive eigenvalue. Suppose $\mu$ is a finite measure such that $L^*\mu = \lambda \mu$, $\lambda > 0$. By regularity, $d\mu = u\, dv$, where $dv$ is the Riemann volume element, and $u$ is a $C^2$ function, in $L^1(dv)$ since $\mu$ is a finite measure, satisfying $L^*u = \lambda u$. Let $\psi$ be a $C^2$ function on the line, $\psi \geq 0$, $\psi(\rho) = 1$ for $0 \leq \rho < 1$, $\psi(\rho) = 0$ for $\rho > 2$, $\psi' \leq 0$. Set

$$\phi_m(x) = \psi(\rho/m).$$

Since $L^*u = \lambda u$ with $\lambda > 0$,

$$0 \leq \lambda \int \phi_m u \, \text{sgn} u = \int \phi_m(L^*u) \, \text{sgn} u$$

$$\leq \int \phi_m(\Delta u - V^*u) \, \text{sgn} u \quad [\text{since } c \geq 0]$$

$$\leq \int (\Delta \phi_m - V\phi_m) |u|$$

by Kato’s inequality [K]. Further,

$$\Delta \phi_m - V\phi_m = \frac{\rho^2}{m^2} \psi' \left( \frac{\rho}{m} \right) \frac{|d\rho|^2}{\rho^2} + \frac{\rho}{m} \psi' \left( \frac{\rho}{m} \right) \frac{\Delta \rho - V\rho}{\rho} \leq K$$

in view of (2.1) and (2.2), since $\psi' \leq 0$. The integrand tends to 0 and is dominated by a constant times $|u|$, so $\lambda \int |u| = 0$. Hence, $u = 0$ and Theorem 2 is proved.
3. The $L^p$ case. It is fair to ask: $L^p$ with respect to what measure? We allow a general $C^1$ measure on $M$ which, in local coordinates, is $m\,dx$, with $m$ in $C^1$. Our operator can be written

\begin{equation}
L = \frac{1}{m} \sum \frac{\partial}{\partial x^j} g^{jk} m \frac{\partial}{\partial x^k} - \sum b_j \frac{\partial}{\partial x^j} - c
\end{equation}

where $g^{jk}, b_j, c,$ are sufficiently smooth, and the matrix $(g^{jk})$ is positive definite, defining a metric on the cotangent bundle. Define inner products

$$\langle u, v \rangle = \int uv m\,dx$$

and, for 1-forms,

$$\langle u, v \rangle = \int (u, v) m\,dx,$$

where $(u, v)$ is the inner product in the cotangent bundle. Define the "$m$-divergence" of the vector field

$$V = \sum b_j \frac{\partial}{\partial x^j}$$

as

$$\nabla_m V = \frac{1}{m} \sum \frac{\partial b_j m}{\partial x^j}.$$

Then for $u, w$ in $C^2_c$,

$$\langle Au, w \rangle = -\langle du, dw \rangle = \langle u, Aw \rangle,$nabla_m V = \frac{1}{m} \sum \frac{\partial b_j m}{\partial x^j}.$$

These formulas show that the representation of $A$, and the "$m$-divergence", are coordinate invariant.

**Theorem 3.** For $1 < p < \infty$, $L$ is dissipative on the domain $C^2_c \subset L^p$ if

$$(D_p) \quad (1/p) \nabla_m V_m \leq c.$$nabla_m V = \frac{1}{m} \sum \frac{\partial b_j m}{\partial x^j}.$$

Hence, $L$ is also dissipative on the closure of this domain in graph norm.

**Proof.** Let $u \in C^2_c$, $p > 1$. Then $ud|u| = |u|\,du$ so

$$d(|u|^{p-2}u) = (p-1)|u|^{p-2}du.$$nabla_m V = \frac{1}{m} \sum \frac{\partial b_j m}{\partial x^j}.$$

Take $\tilde{u} = |u|^{p-2}u$, and compute

$$\langle Au - \frac{1}{p} V_m u - cu, |u|^{p-2} u \rangle = -(p-1)\langle du, |u|^{p-2} du \rangle + \frac{1}{p} \langle u, V_m (u|u|^{p-2}) \rangle$$

$$+ \frac{1}{p} \langle u, (\nabla_m V_m) u|u|^{p-2} \rangle - \langle cu, u|u|^{p-2} \rangle.$$nabla_m V = \frac{1}{m} \sum \frac{\partial b_j m}{\partial x^j}.$$
But
\[ \frac{1}{p} \langle u, V_m(u|u|^{p-2}) \rangle = \frac{p-1}{p} \langle u, (Vu)|u|^{p-2} \rangle, \]
so canceling and transposing gives
\[ \langle Au - V_m u - cu, |u|^{p-2} u \rangle \]
\[ = -(p-1) \langle du, |u|^{p-2} du \rangle + \left( \frac{1}{p} \nabla_m V_m - c \right) u, u|u|^{p-2} \rangle \]
\[ \leq 0 \quad \text{if (D)} \]
For \( p = 1 \), take the limit in the last equality as \( p \to 1 \). The first term remains \( \leq 0 \),
and the limit of the second term is \( \leq 0 \) by (D).

**Remark.** The vector field \( V_m \) represents drift. The condition (D) says that the
divergence of the drift must be compensated by the dissipation \( c \). Examples below
show that this condition is sharp in simple cases where \( \nabla_m \cdot V_m \) and \( c \) are constant.
When \( p = 1 \), condition (D) is also necessary, as the last equality in the proof shows.

Finally, we give conditions eliminating positive eigenvalues for \( L^* \), thus guarantee-
ing a contraction semigroup on \( L^p \).

**Theorem 4.** There are no nonzero solutions of \( L^* u = \lambda u \) in \( L^q \) (\( 1/p + 1/q = 1 \)) if
\[ \lambda \geq \left( \frac{1}{p} \right) \nabla_m V_m - c \]
and there is a function \( \rho \) on \( M \) such that
\[ \rho(\infty) = \infty, \quad |d\rho| = o(\rho), \quad |V_m \rho| \leq k \rho. \]

**Note.** If \( V_m = 0 \) this is in Strichartz [S], who refers also to Yau.

**Proof.** It suffices to take \( \lambda = 0 \) (replace \( c \) by \( c - \lambda \)), so the hypothesis is
\[ \frac{1}{p} \nabla_m V_m - c \leq 0. \]
Let \( H(u) = u|u|^{q-2} \). For \( \phi \) in \( C^2 \) consider
\[ I = \langle \phi^2 H(u), Au - \frac{1}{q} V_m^* u - cu \rangle \]
\[ = -\langle d(\phi^2 H), du \rangle - \frac{1}{q} \langle V_m(\phi^2 H), u \rangle - \langle c\phi^2 H, u \rangle \]
\[ = -2 \langle \phi H d\phi, du \rangle - \langle \phi^2 H', du \rangle - \frac{1}{q} \langle \phi^2 (V_m u) H', u \rangle - \frac{2}{q} \langle \phi HV_m \phi, u \rangle \]
\[ - \frac{1}{q} \langle \phi^2 (\nabla_m V_m) u H', u \rangle + \frac{1}{q} \langle \phi^2 (\nabla_m V_m) u H', u \rangle - \langle \phi^2 c H, u \rangle. \]
But \( u H' = (q - 1) H \) and \( V_m^* u = -V_m u - \nabla_m V_m u \), so
\[ I = -2 \langle \phi H d\phi, du \rangle - \langle \phi^2 H', du \rangle + \frac{1}{p} \langle \phi^2 H, V_m^* u \rangle \]
\[ - \frac{2}{q} \langle \phi HV_m \phi, u \rangle + \frac{1}{p} \langle \phi^2 (\nabla_m V_m) H, u \rangle - \langle \phi^2 c H, u \rangle. \]
Transpose \((1/p)\langle \phi^2 H, V_n^* u \rangle\), use \(L^* u = 0\) and (3.2) to get
\[
0 = \langle \phi^2 H, L^* u \rangle \leq -2 \langle \phi H d\phi, du \rangle - \langle \phi^2 H' du, du \rangle - \frac{2}{q} \langle \phi HV_n \phi, u \rangle.
\]
But \(H' = (q - 1)|u|^{q-2}\), so
\[
(q - 1)\|\phi u\|^{q/2-1} du \leq 2 \sup |d\phi| \cdot \|\phi u\|^{q/2-1} du \cdot \|u^{q/2}\|
\]
where all norms are in \(L^2\). Now let \(\phi = \psi(\rho/j)\), with \(\psi\) as in the proof of Theorem 2. Then
\[
u \phi HV_n \phi = |u|^q \psi \left( \frac{\rho}{j} \right) \left( \frac{\rho}{j} \right) \psi \left( \frac{\rho}{j} \right) \left( \frac{V_n \rho}{\rho} \right).
\]
By hypothesis, \(|V_n \rho| \leq k \rho\) and \(u \in L^q\), so by dominated convergence,
\[
\langle \phi HV_n \phi, u \rangle \to 0 \quad \text{as } j \to \infty.
\]
Also, as \(j \to \infty\),
\[
|d\phi| = \frac{\rho}{j} \psi \left( \frac{\rho}{j} \right) \left( \left| \frac{d\rho}{\rho} \right| \right) \to 0
\]
uniformly, since \(|d\rho|/\rho \to 0\) as \(\rho \to \infty\). Hence, from (3.3),
\[
\|\phi u\|^{q/2-1} du \to 0.
\]
It follows that \(u\) is constant on any open set where \(u \neq 0\). Since \(u\) is continuous and in \(L^q\), \(u \equiv 0\). Q.E.D.

When the conditions in Theorems 3 and 4 are met for two values \(p_1 < p_2\), the resulting semigroups agree on \(L^{p_1} \cap L^{p_2}\). The proof follows Strichartz [S], showing that the resolvent \((L - \lambda)^{-1}\) is the same in \(L^{p_1}\) as in \(L^{p_2}\) for large \(\lambda\). If the resolvents are different, there is a \(u_1 \) in \(L^{p_1}\) and a \(u_2 \) in \(L^{p_2}\) with \(u_1 \neq u_2\) and
\[
(L - \lambda) u_1 = (L - \lambda) u_2
\]
so \(u_1 - u_2 \) is an eigenvalue of \(L\) in \(L^{p_1} + L^{p_2}\). We prove that the difference \(u = u_1 - u_2 = 0\) if
\[
\frac{1}{p} \nabla_m V_m - c \leq \lambda, \quad \text{for } p = p_1 > 1, \text{ and } p = p_2 > p_1.
\]
The proof imitates Theorem 4 with an appropriate choice of \(H\). Choose an increasing smooth function \(\theta\) with
\[
\theta(u) = \begin{cases} 
1/p_2, & 0 \leq u \leq 1, \\
1/p_1, & 2 \leq u.
\end{cases}
\]
Define an even function \(G\) with
\[
G(u) = \frac{1}{u} \int_0^u \theta = \int_0^1 \theta(u) \, dt, \quad u > 0,
\]
\[
= \begin{cases} 
1/p_2, & 0 \leq u \leq 1, \\
1/p_1, & 2 \leq u.
\end{cases}
\]
$G'(u) = \int_0^1 t \theta'(tu) \, dt \geq 0$, so $1/p_2 \leq G \leq 1/p_1$. Hence (3.4) gives

\[ G(u) \nabla_m V_m - c \leq \lambda. \]  

(3.4a)

Next define $H$ odd with

\[ H(u) = (p_2 Gu)^{-1} \exp \left( \int_1^u (Gs)^{-1} \, ds \right), \quad u > 0, \]

\[ = \begin{cases} 
  u^{p_2-1}, & 0 \leq u \leq 1, \\
  c_2 (u - p_1 c_1)^{p_1-1}, & 2 \leq u.
\end{cases} \]

Then

\[ GH + (GH)'u = H \]

so $H' = [1 - (Gu)]H/Gu = [1 - \theta]H/Gu > 0$. Hence further

\[ (3.4b) \]

\[ \sqrt{H/u} \leq \text{const} \sqrt{H'} . \]

Now suppose that $Lu = \lambda u$ with $u \in L^{p_1} + L^{p_2}$. Then $uH(u)$ is in $L^1$. Consider

\[ I = \langle \phi^2 H(u), Au - G(u)Vmu - (c + \lambda)u \rangle \]

and calculate as before, using (3.4a, b), to get

\[ \langle \phi^2 H' du, du \rangle \leq -2 \langle \phi H d\phi, du \rangle + 2 \langle GH\phi Vm\phi, u \rangle. \]

Thus

\[ \|\phi\sqrt{H'} du\|^2 \leq 2 \sup|d\phi| \cdot \|\phi\sqrt{H}/u\| \cdot \|\sqrt{Hu}\| + 2 \langle GH\phi Vm\phi, u \rangle. \]

Applying (3.4c) and assuming a $\rho$ as in Theorem 4, we find as before that $u \equiv 0$. This proves that the semigroups in $L^{p_1}$ and in $L^{p_2}$ agree on $L^{p_1} \cap L^{p_2}$. [Note: The above result is a revision in the galley proofs of a weaker result in the original article.]

We conclude with two examples. The first shows that the condition $(1/p)^{\nabla_m V_m} - c < \lambda$ in Theorem 4 is essential, as is $c - \nabla_m V_m > 0$ in Theorem 3.

**Example 1.** $Lu = u'' - bxu'$, where $b$ is constant; take $m \, dx = dx$. (When $b < 0$, this is the Uhlenbeck process.)

The Fourier transform with respect to $x$ converts $Lu = u$, into the first order equation

\[ -\xi^2 \hat{u} + b \frac{\partial}{\partial \xi} (\xi \hat{u}) = \frac{\partial \hat{u}}{\partial t}. \]

The solution with $\hat{u}(0, \xi) = \hat{f}(\xi)$ is

\[ \hat{u}(t, \xi) = \hat{f}(\xi e^{bt}) \exp \left( bt - (\xi^2/2b)(e^{2bt} - 1) \right). \]

Taking a limit as $b \to 0$ gives the usual solution of $u'' = u_r$.

If this is a contraction on $L^p$, then the adjoint $L^*$ has no positive eigenvalue in $L^q$.

The eigenvalue equation $L^*u = \lambda u$ is solved by taking the Fourier transform:

\[ \hat{u}(\xi) = |\xi|^{-\lambda/b} e^{-\xi^2/2b}. \]

If $b < 0$ there is no solution in $L^p$. If $b > 0$ we have
which is in $L^q$ iff $q(-1 + \lambda/b) < -1$, that is iff $\lambda < b/p$, where $1/p + 1/q = 1$.

Combined with Theorems 3 and 4, this shows that $L - c$ (where $c$ is constant) generates a contraction semigroup on $L$ iff $c - b/p \geq 0$. Note that in this case $b = \nabla'_m V_m$, so Theorems 3 and 4 are sharp.

Example 2. $Lu = \Delta u - b(xu_y - yu_x)$; $m$ is Lebesgue measure on $R^2$. Here $\nabla'_m V_m = 0$, so there is a contraction semigroup on $L^p$ for $1 < p < \infty$, and on $C_0$. We analyze the spectrum of $L$ on the space $L^2$ by taking the Fourier transform:

$$\hat{L}v = -(\xi^2 + \eta^2)v + b\eta \frac{\partial v}{\partial \xi} - b\xi \frac{\partial v}{\partial \eta}.$$ 

The first order equation

$$\dot{\phi} = \lambda \phi$$

(3.5)

gives the rotation vector field

$$\dot{\xi} = b\eta, \quad \dot{\eta} = -b\xi,$$

where $\dot{\xi}$ is the derivative of $\xi$ with respect to a parameter $\tau$. The flow is

$$\xi = r \cos b\tau, \quad \eta = -r \sin b\tau, \quad r \text{ constant}.$$ 

The solution of (3.5) is, with $b\tau = \theta$,

$$\phi(r \cos \theta, r \sin \theta) = e^{-(\lambda + r^2)\theta/b}\phi(r, 0),$$

where we obviously need $e^{-(\lambda + r^2)2\pi/b} = 1$, or

$$\lambda + r^2 = ibk, \quad k = 0, \pm 1, \pm 2, \ldots.$$ 

The spectrum of $L$ consists of the union of half-lines $\{\lambda: \lambda = ibk - r^2, k = 0, \pm 1, \ldots, r > 0\}$. This comes right down to $\text{Im } \lambda = 0$, so $\|e^{itL}\| = 1$.

**Appendix.** We sketch a proof of some relations between Ricci curvature and $\Delta r$. The proof of Lemma 1 was given by C. L. Terng.

**Lemma 1.** Let $r$ denote distance from a fixed point $p$. If $\text{Ric} \geq -C(1 + r^2)$, then $(\Delta r) \leq -(\Delta r)^2/n + C(1 + r^2)$ inside the cut locus of $p$.

**Proof.** Choose an orthonormal local frame $e_1, \ldots, e_n$ with $e_1 = \partial/\partial r$. Set $r_i = \nabla e_i r$. Since $|\nabla r|^2 = 1$ and $r_{ij} = r_{ji}$,

$$0 = \Delta \left( \frac{1}{2} |\nabla r|^2 \right) = \sum_i \left( \sum_j r_{ij} r_{ji} \right) = \sum (r_{ij}^2 + r_{ij} r_{ji}) = \sum (r_{ij}^2 + r_{ji} r_{ij}).$$

The Ricci formula gives

$$r_{ijk} = r_{ikj} + \sum_j r_i R_{lijk},$$

so (now summing repeated indices)

$$r_{ijj} = r_{iij} + r_i R_{iji} = r_{iij} + r_i R_{ij},$$

with $R_{ij}$ the Ricci tensor. So (*) gives

$$0 = \sum r_{ijj}^2 + r_i (r_{iij}^2 + r_i R_{ij}).$$
Set $f = \Delta r = r_{ii}$, and get

$$r_j f_j = r_j r_{ii} = -\sum r_j^2 - r_j r_i R_{ij} \leq -\sum r_j^2 + C(1 + r^2),$$

Since $r_1 = 1, r_2 = \cdots = r_n = 0, f_1 = \partial f / \partial r$, and

$$\sum r_{ij}^2 \geq \sum r_{ii}^2 \geq \frac{1}{n} \left( \sum r_{ii} \right)^2 = \frac{1}{n} (\Delta r)^2,$$

the lemma follows.

**Lemma 2.** If

(1) \quad f'(r) < a^2 f^2 + b^2 + c^2 r^2, \quad r_0 > r > 0,

then

(2) \quad f \leq \frac{1}{2} (1 + ar + b + cr), \quad r_0 > r > 0.

If $r_0 = \infty$ then

(3) \quad f \geq -\frac{\rho}{a} \sqrt{b^2 + c^2 r^2}

as well, where $\rho$ is a constant $> 1$ satisfying

(4) \quad c \rho \leq ab^2 (\rho^2 - 1).

**Proof of (2).**

**Case 1.** Suppose that for small $r$,

(5) \quad \sqrt{b^2 + c^2 r^2} < a/\sqrt{2}.

Then $f'(r) \leq -a^2 f^2/2$, so $1/f' \geq a^2/2$, hence, by (5), $1/f \geq a^2 r/2$, or $f \leq 2/a^2 r$. Further, $f'(r) \leq 0$ as long as $f^2 \geq (b^2 + c^2 r^2)/a^2$. So $f$ must lie below the dashed line in Figure 1; it cannot cross the graph of $(1/a) (b^2 + c^2 r^2)^{1/2}$ from below, since (1) implies $f' \leq 0$ at such a crossing. So, being generous,

$$f \leq \frac{2}{a^2 r} + \frac{\sqrt{2}}{a} (b^2 + c^2 r^2)^{1/2} \leq \frac{2}{a} \left( \frac{1}{ar} + b + cr \right).$$

**Case 2.** This is even easier; $f$ starts out below the graph of $(\sqrt{2}/a) (b^2 + c^2 r^2)^{1/2}$. 

![Figure 1](http://www.ams.org/journal-terms-of-use)
Proof of (3). Compare $f$ to the function $g(r) = -(\rho/a)(b^2 + c^2r^2)^{1/2}$, with $\rho > 1$ satisfying (4). From (1), $f \leq g \Rightarrow f' < g'$, so $f$ cannot cross the graph of $g$ from below; if $f(r_1) \leq g(r_1)$, then $f(r) < g(r)$ for all $r > r_1$. But $f \leq g$ implies

$$f' \leq -a^2f^2 + b^2 + c^2r^2 \leq -a^2f^2 + a^2f^2/\rho^2 = -\alpha f^2, \quad \alpha > 0.$$ 

So $(1/f)' \geq \alpha > 0$, and this implies that $1/f > 0$ eventually, contradicting $f \leq g < 0$.

Combining the lemmas with $f = \Delta r$, we find from (2) that

$$\text{Ric} \geq -C(1 + r_1^2) \Rightarrow \Delta r_p \leq k(1/r_p + r_p)$$

inside the cut locus of $p$. (This also follows from the Laplacian Comparison Theorem of Greene and Wu (Lecture Notes in Math., vol. 699, Springer-Verlag, Berlin and New York) taking as model the space $R^n$ with metric $dr^2 + \exp(r^2)\,d\theta^2$ in polar coordinates.)

If $\exp: M_p \to M$ is a diffeomorphism, then (3) gives

$$\text{Ric} \geq -C(1 + r_p^2) \Rightarrow \Delta r_p \geq -kr_p$$

since $\Delta r \to +\infty$ as $r \to 0$. (This inequality does not seem to follow from the theorem of Greene and Wu.) Hence Theorem 2 applies: $C_0$ is preserved.

References

[KL] Leon Karp and Peter Li, The heat equation on complete Riemannian manifolds (preprint).

Department of Mathematics, University of Massachusetts, Boston, Massachusetts 02125