THE HEAT EQUATION WITH A SINGULAR POTENTIAL

BY

PIERRE BARAS AND JEROME A. GOLDSTEIN

ABSTRACT. Of concern is the singular problem \( \frac{\partial u}{\partial t} = \Delta u + \left( \frac{c}{|x|^2} \right) u + f(t,x) \), \( u(x,0) = u_0(x) \), and its generalizations. Here \( c > 0, x \in \mathbb{R}^n, t > 0 \), and \( f \) and \( u_0 \) are nonnegative and not both identically zero. There is a dimension dependent constant \( C_*(N) \) such that the problem has no solution for \( c > C_*(N) \). For \( c \leq C_*(N) \) necessary and sufficient conditions are found for \( f \) and \( u_0 \) so that a nonnegative solution exists.

1. Introduction. Of concern is the heat equation with a potential

\[ \frac{\partial u}{\partial t} - \Delta u = V(x)u + f(x,t) \]

for \( t > 0 \) and \( x \in \Omega \subset \mathbb{R}^n \). Take either \( \Omega = \mathbb{R}^n \) or else \( \Omega \) to be a bounded domain containing \( B_1 = \{ x \in \mathbb{R}^n : |x| < 1 \} \), in which case we impose the Dirichlet boundary condition

\[ u(x,t) = 0 \quad \text{for} \ x \in \partial \Omega. \]

The initial condition is

\[ u(x,0) = u_0(x), \quad x \in \Omega. \]

We take \( u_0, f \geq 0 \) and \( 0 \leq V \in L^\infty(\Omega \setminus B_\varepsilon) \) (where \( B_\varepsilon = \{ x : |x| < \varepsilon \} \)) for each \( \varepsilon > 0 \), but \( V \) is singular at the origin. The question is: How singular must \( V \) be to prevent a solution \( u \) from existing? The answer, informally stated, is that \( V \) is too singular if \( V(x) > C_*(N)/|x|^2 \) near \( x = 0 \), while \( V \) is not too singular if \( V(x) \leq C_*(N)/|x|^2 \) near \( x = 0 \), where \( C_*(N) = ((N - 2)/2)^2 \), for \( N = 1, 2, 3, \ldots \).

Let us be more precise. Let \( \Omega \) be a domain in \( \mathbb{R}^n \) with \( B_1 \subset \Omega \subset \mathbb{R}^n \). If \( \Omega \neq \mathbb{R}^n \) let the boundary of \( \Omega \) be nice enough. Let

\[ V_*(x) = c/|x|^2, \quad x \in \Omega, \]

where \( c > 0 \). Consider the problem

\[
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u &= V_*u \quad \text{in} \ \Omega \times ]0, T[, \\
u &= 0 \quad \text{on} \ \partial \Omega \times ]0, T[, \\
u(0, x) &= u_0(x) \quad \text{in} \ \Omega
\end{align*}
\]

(Q)
with \( u_0 \geq 0 \) and \( u_0 \equiv 0 \) (i.e. \( u_0 \) not equal to zero a.e.). Here \( 0 < T \leq \infty \). Because \( V \) is so singular at the origin it is not clear if a solution \( u \) of (Q) exists. So we approximate \( V_\ast \) by

\[
V_n(x) = \min\{ V_\ast(x), n \},
\]

and we let \( u_n \) be the unique nonnegative solution of

\[
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u &= V_n u_n \quad \text{in } \Omega \times ]0, T[, \\
 u_n &= 0 \quad \text{on } \partial \Omega \times ]0, T[, \\
u_n(0, x) &= u_0(x) \quad \text{in } \Omega.
\end{align*}
\]

(Q_n)

Note that \( u_n \) exists if \( u_0 \) satisfies some innocuous conditions, which we assume. Let

\[
C_\ast(N) = ((N - 2)/2)^2.
\]

It will follow from the results of this paper that, if \( u_n \) is the solution of (Q_n), then:

(I) If \( 0 \leq c \leq C_\ast(N) \), then \( \lim_{n \to \infty} u_n(x, t) = u(x, t) \) exists and \( u \) is a solution of (Q).

(II) If \( c > C_\ast(N) \), then \( \lim_{n \to \infty} u_n(x, t) = \infty \) for all \( (x, t) \in \Omega \times ]0, T[ \).

In the existence result (I), by an argument using the maximum principle we can replace \( V_n, V_\ast \) by \( \tilde{V}_n, \tilde{V}_\ast \), respectively, where \( \tilde{V}_n(x) \leq V_n(x) \) a.e. for each \( n \). Similarly, in the nonexistence result (II) we can replace \( V_n, V_\ast \) by \( \hat{V}_n, \hat{V}_\ast \) where \( \hat{V}_n \geq V_n \) a.e. for each \( n \) (or at least \( \hat{V}_n \geq V_n \) a.e. in a fixed neighborhood of the origin).

Two proofs will be given of the nonexistence result. One, based on the techniques of the theory of partial differential equations, is closely related to the existence results such as (I). The other, for the case of \( \Omega = \mathbb{R}^N \), is probabilistic and is based on the Feynman-Kac integral formula. This represents, to our knowledge, the first such application of the Feynman-Kac formula to a nonexistence question in partial differential equations.

In the next section the main results are stated. §§3–5 contain the proofs based on the techniques of partial differential equations. The probabilistic proof is given in §6. §7 contains complements and remarks.

It is a pleasure to thank Haim Brezis and J.-L. Lions for posing the questions considered here and Brezis for his continued interest in the work. One of us (J. G.) also thanks John Liukkonen and Steve Rosencrans for discussions about Brownian motion.

2. Statements of the main results. Let \( \Omega \) be a domain satisfying \( B_1 \subset \Omega \subset \mathbb{R}^N \). If \( N = 1 \) we delete the origin so that \( 0 \notin \Omega \); for simplicity we may take \( \Omega = ]0, R[ \) where \( 1 \leq R \leq \infty \). Let \( 0 \leq V \in L^1(\Omega) \), let \( 0 \leq f \in L^1(\Omega \times ]0, T[) \), and let \( u_0 \) be a nonnegative function in \( L^1(\Omega) \), or, more generally, let \( u_0 \) be a finite (positive) Radon measure on \( \Omega \). Consider the problem of finding a function \( u \) such that

\[
\begin{align*}
u &> 0 \text{ on } \Omega \times (0, T), \\
u V &\in L^1_{\text{loc}}(\Omega \times ]0, T[), \\
\frac{\partial u}{\partial t} - \Delta u &= Vu + f \quad \text{in } C^0(\Omega \times ]0, T[), \\
\text{ess lim}_{t \to 0^+} \int_{\Omega} u(t) \phi &= \int_{\Omega} \phi u_0 \quad \text{for all } \phi \in C^0(\Omega).
\end{align*}
\]

(P)
Here $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ with the usual topology; $\mathcal{D}'$, the dual space of $\mathcal{D}$, is a space of distributions; and for typographical convenience, arguments of functions and differentials in integrals will be omitted when no confusion can arise.

We shall attack (P) by studying the approximate problem

\begin{align*}
\frac{\partial u_n}{\partial t} - \Delta u_n &= V_n u_n + f_n \quad \text{in } \mathcal{D}'(\Omega \times ]0, T[), \\
u_n &= 0 \quad \text{on } \partial \Omega \text{ for all } t \in ]0, T[, \\
\lim_{t \to 0} \int u_n(t) \phi &= \int \phi u_0 \quad \text{for all } \phi \in \mathcal{D}(\Omega),
\end{align*}

\[(P_n)\]

where $f_n = \min(f, n)$ and where $V_n \in L^\infty(\Omega)$, $0 \leq V_n \leq V$, and $V_n \uparrow V$ a.e. in $\Omega$. Of course, the Dirichlet boundary condition will be absent if $\Omega = \mathbb{R}^N$.

The problem $(P_n)$ has a unique bounded nonnegative solution which satisfies the integral equation

\[(2.1) \quad u_n(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} V_n u_n(s) \, ds + \int_0^t e^{(t-s)\Delta} f_n(s) \, ds,
\]

where $\{e^{t\Delta} : t \geq 0\}$ denotes the semigroup generated by $\Delta$ with Dirichlet boundary conditions; note that the perturbation $V_n$ defines a bounded multiplication operator on $L^p(\Omega)$ for all $p$. Also,

\[(2.2) \quad (e^{t\Delta} u)(x) = \int_{\Omega} e^{t\Delta} \delta_x(y) u(y) \, dy.
\]

The sequence of nonnegative functions $(u_n)$ is clearly increasing.

**Proposition 2.1.** (i) Suppose there is an $(x_0, t_0) \in \Omega \times ]0, T[\text{ such that } \lim_{n \to \infty} u_n(x_0, t_0) < \infty. \text{ Then (P) has a nonnegative solution on } \Omega \times ]0, T_0[ \text{ for all } T_0 \in ]0, t_0[. \text{ This solution is given by}

\[(2.3) \quad u(x, t) = \lim_{n \to \infty} u_n(x, t)
\]

a.e. in $\Omega \times ]0, T_0[.$

(ii) If (P) has a nonnegative solution in $\Omega \times ]0, T[\text{, then}

\[
\lim_{n \to \infty} u_n(x, t) < \infty
\]

a.e. in $\Omega \times ]0, T[.$

Thus the existence of a nonnegative solution of (P) depends only on the limiting behavior of the solution of $(P_n)$. The borderline case concerns the potential given by

\[
V_0(x) = \begin{cases} 
c/|x|^2 & \text{if } x \in B_1, \\
0 & \text{if } x \in \Omega \setminus B_1.
\end{cases}
\]

Let

\[
C_\ast(N) = ((N - 2)/2)^2.
\]
Theorem 2.2. (i) Let $0 \leq c \leq C_*(N)$ and let the (measurable) potential $V \geq 0$ satisfy $V \in L^\infty(\Omega \setminus B_1)$. If $V \leq V_0$ in $B_1$, then (P) has a solution $u$ if

$$
\int \Omega |x|^{-\alpha} u_0 < \infty, \quad \int_0^T \int \Omega |x|^{-\alpha} f(x,s) \, dx \, ds < \infty
$$

where $\alpha$ is the smallest root of $(N-2-\alpha)\alpha = c$. If $V \geq V_0$ in $B_1$ and if (P) has a solution $u$, then

$$
\int \Omega |x|^{-\alpha} u_0 < \infty, \quad \int_0^T \int \Omega |x|^{-\alpha} f(x,s) \, dx \, ds < \infty
$$

for each $\epsilon \in [0,T]$ and each $\Omega' \subseteq \Omega$, where $\alpha$ is as above. If either $u_0 \not\equiv 0$ or $f \not\equiv 0$ in $\Omega \times [0,T]$, then given $\Omega' \subseteq \Omega$ there is a constant $C = C(\epsilon, \Omega') > 0$ such that

$$
u(x,t) \geq C |x|^{\alpha} \quad \text{if} \quad (x,t) \in \Omega' \times [\epsilon, T].
$$

(ii) If $c > C_*(N)$, $V \geq V_0$ and either $u_0 \not\equiv 0$ or $f \not\equiv 0$, then (P) does not have a solution.

Note that if we set $\phi(x) = |x|^{-\alpha}$, then

$$
\Delta \phi = \phi_{rr} + (N-1) \phi/r = \alpha (\alpha + 2 - N) |x|^{-\alpha-2},
$$

so that $-\Delta \phi/\phi = c/|x|^2$, where $c = (N-2-\alpha)\alpha$ is as in the above theorem. Theorem 2.2 can be extended to cover potentials of the form $V = -\Delta \phi/\phi$ where $\phi > 0$, $\Delta \phi \in L^1_{\text{loc}}(\Omega)$, and a supplementary hypothesis is made (cf. Remark 7.3). In this case the conditions on the data should be

$$
\int \Omega \phi u_0 < \infty, \quad \int_0^T \int \Omega f(x,s) \phi(x) \, dx \, ds < \infty.
$$

When $\phi(x) = |x|^{-\alpha}$, the condition $\Delta \phi \in L^1(B_1)$ becomes $N-2-\alpha > 0$, which holds if $c > 0$ (and $\alpha > 0$).

We finally note that the smaller root $\alpha$ of $(N-2-\alpha)\alpha = c$ is given by

$$
\alpha = \frac{N-2}{2} - \left( \left( \frac{N-2}{2} \right)^2 - c \right)^{1/2}
$$

when $0 \leq c \leq C_*(N)$. This is one of the technical ways in which $C_*(N)$ arises.

3. Proof of Proposition 2.1. We begin with the proof of Proposition 2.1. Part (ii) is easy. If $u$ is a nonnegative solution of (P), then $u_n \leq u$ holds for all $n$, whence

$$
\lim_{n \to \infty} u_n(x,t) \leq u(x,t)
$$

a.e. in $\Omega \times [0,T]$.

Part (i) is more difficult. To start the proof, let $v_n = e^n u_n$. Then

$$
\partial v_n/\partial t - \Delta v_n = (V_n + 1)v_n + e^n f_n.
$$

Applying (2.1) and (2.2) to $v_n$ gives

$$
e^{\epsilon \sigma} u_n(x_0, t_0) \geq \int_0^t \int \Omega \left( e^{(t_0-s)\sigma} \delta_{x_0}(y)(V_n(y) + 1) u_n(y, s) e^s \, dy \, ds.\right.
$$
If $\Omega' \subset \subset \Omega$ and $0 < \varepsilon < T$,
\[
\inf \left\{ (e^{s \Delta} \delta_{x_0})(y) : (y, s) \in \Omega' \times [\varepsilon, T] \right\} = c_0 > 0.
\]
Therefore
\[
(3.2) \quad c_0 \int_{t_0}^{t_0 - \varepsilon} \int_{\Omega'} V_n(y) u_n(y, s) \, dy \, ds + c_0 \int_{t_0}^{t_0 - \varepsilon} \int_{\Omega'} u_n(y, s) \, dy \, ds \leq e^{c_0} u_n(x_0, t_0).
\]
By hypothesis (cf. (2.3)), $u_n$ increases and the right-hand side of (3.2) is bounded, thus $u_n$ increases to $u$ and $V_n u_n$ increases to $V u$ in $L^1(\Omega' \times ]0, t_0 - \varepsilon[)$, and $u$ is a solution of (P) in the sense of distributions. \qed

Remark. This solution $u$ satisfies the integral equation
\[
u(x, t) = \int_{\Omega} \left( e^{-\Delta} \delta(y) u_0(y) + \int_0^t e^{s \Delta} \delta(y) V(y) u(y, s) \, ds \right) \, dy
\]
\[
+ \int_0^t \int_{\Omega} e^{s \Delta} \delta(y) f(y, s) \, dy \, ds
\]
a.e. in $\Omega \times ]0, t_0[$. By (3.1),
\[
(\nu, s) \mapsto e^{(t_0 - s) \Delta} \delta(y) V(y) u(y, s) \in L^1(\Omega \times ]0, t_0[)
\]
since $\lim_{n \to \infty} u_n(x, t) = u(x, t) < \infty$ a.e. in $\Omega \times ]0, t_0[$.

4. Proof of Theorem 2.2(i). We first show that assumption (2.4) on the data implies the existence of a solution.

Let $\phi(x) = |x|^{-a}$ and let $p \in C^2(\mathbb{R})$ be a convex function satisfying $p(0) = p'(0) = 0$. Multiply the equation (cf. (P'')) satisfied by $u_n$ by $p'(u_n) \phi$ and integrate over $\Omega \times [\delta, t[$ for $0 < \delta < t < T$. One gets, using integration by parts,
\[
\int_{\Omega} p(u_n(t)) \phi + \int_{\delta}^{t} \int_{\Omega} \nabla u_n \cdot \nabla (p'(u_n) \phi) = \int_{\delta}^{t} \int_{\Omega} (V_n u_n + f) p'(u_n) \phi + \int_{\Omega} p(u_n(\delta)) \phi,
\]
whence, since $p$ is convex,
\[
\int_{\Omega} p(u_n(t)) \phi + \int_{\delta}^{t} \int_{\Omega} p(u_n)(-\Delta \phi) \leq \int_{\delta}^{t} \int_{\Omega} (V_n u_n + f) p'(u_n) \phi + \int_{\Omega} p(u_n(\delta)) \phi.
\]
Replace $p(r)$ by a sequence $p_m(r)$ satisfying the hypotheses for $p$ and converging to $|r|$ as $m \to \infty$. We obtain the limiting inequality
\[
u(\delta) \phi + \int_{\delta}^{t} \int_{\Omega} u_n(-\Delta \phi) \leq \int_{\delta}^{t} \int_{\Omega} (V_n u_n + f) \phi + \int_{\Omega} (u_n(\delta)) \phi.
\]

We want to let $\delta \to 0$. First we claim that
\[
\int_{\Omega} u_n(\delta) \phi \to \int_{\Omega} \phi u_0.
\]
To see why this is so, note that
\[
e^{\delta \Delta} u_0 \leq u_n(\delta) = e^{\delta(\Delta + V_n)} u_0 + \int_0^\delta e^{(\delta - s)(\Delta + V_n)} f_n(s) \, ds
\]
\[
\leq e^{\delta \lambda} e^{\delta \Delta} u_0 + \int_0^\delta e^{\lambda(\delta - s) \Delta} f_n(s) \, ds.
\]
if \(|V_n|_\infty \leq \lambda\), since
\[ e^{\delta(\Delta + V_n)}v_0 = \lim_{m \to -\infty} \left( e^{\delta \lambda/m}e^{(\delta/m)V_n} \right)^m V_0 \leq e^{\delta \lambda}e^{\delta \lambda} V_0 \]
by the positivity preserving property of \((e^{\delta \lambda})\). Thus
\[ \int_\Omega \left( e^{\delta \lambda} u_0 \right) \phi \leq \int_\Omega u_n(\delta) \phi \leq e^{\delta \lambda} \int_\Omega \left( e^{\delta \lambda} u_0 \right) \phi + e^{\delta \lambda} \delta \|f_n\|_\infty \left( \int_\Omega \phi \right) , \]
whence
\[ \int_\Omega \left( e^{\delta \lambda} u_0 \right) \phi = \int_\Omega \left( e^{\delta \lambda} \phi \right) u_0 \rightarrow \int_\Omega \phi u_0 \]
as \(\delta \to 0\), as asserted. Letting \(\delta \to 0\) in (4.1), we deduce
\[ \int_\Omega u_n(t) \phi + \int_0^t \int_\Omega u_n(-\Delta \phi) = \int_0^t \int_\Omega V_n u_n \phi + \int_0^t \int_\Omega f_n \phi + \int_\Omega \phi u_0 . \]
But
\[ -\Delta \phi = \left( N - 2 - a \right) \alpha / |x|^{n+2} \geq V_n \phi \]
since \(c = (N - 2 - a)\alpha\). Consequently
\[ \int_\Omega u_n(t) \phi \leq \int_0^t \int_\Omega f_n \phi + \int_\Omega \phi u_0 , \]
and therefore if
\[ \int_0^t \int_\Omega f_n \phi + \int_\Omega \phi u_0 < \infty \]
we conclude that \(u_n(x, t)\) increases to a finite limit \(u(x, t)\) as \(n \to \infty\), for all \(t \in [0, T]\) and for a.e. \(x \in \Omega\). By Proposition 2.1, this proves the first part of Theorem 2.2(i).

Thus the problem (P) has a solution for \(u_0 = \phi(x)^{-1}\delta_x\) and \(f \equiv 0\), where \(x \in \Omega \setminus \{0\}\) is fixed. Let \(u_x\) denote this solution and let \(h_x(y, t) = u_x(y, t)/\phi(y)\). Let also \(h = u/\phi\), \(h_n = u_n/\phi\) where \(u\) and \(u_n\) are the solutions of (P) and \((P_n)\) constructed above. We claim that
\[ h(x, t) = \int_\Omega h_x (y, t) \phi(y) u_0(y) + \int_0^t \int_\Omega h_x (y, t - s) f(y, s) \phi(y) . \]
For the proof, let \(u_n\) and \(v_n\) satisfy
\[ \begin{align*}
\partial u_n/\partial t - \Delta u_n &= V_n u_n + f_n, \quad u_n = 0 \text{ on } \partial \Omega, \quad u_n(0) = u_0; \\
\partial v_n/\partial t - \Delta v_n &= V_n v_n, \quad v_n = 0 \text{ on } \partial \Omega, \quad v_n(0) = \phi(x)^{-1}\delta_x .
\end{align*} \]
Then
\[ \frac{\partial}{\partial s} \int_\Omega u_n(s) v_n(t - s) \, dx = \int_\Omega f_n(s) v_n(t - s) \, dx , \]
whence
\[ \int_\Omega u_n(t - \delta) v_n(\delta) \, dx = \int_\Omega u_n(t - \delta) v_n(\delta) \, dx + \int_0^t \int_\Omega f_n(s) v_n(t - s) \, dx \, ds . \]
As $\delta \to 0$,
\[ u_n(t - \delta) \to u_n(t) \quad \text{and} \quad v_n(t - \delta) \to v_n(t) \quad \text{in} \quad C(\Omega), \]
\[ v_n(\delta) \to \phi(x)^{-1} \delta_x \quad \text{and} \quad u_n(\delta) \to u_0 \]
where $\to$ denotes weak convergence. Thus when $\delta \to 0$ we obtain, from (4.4),
\[ (4.5) \quad u_n(x, t) \phi(x)^{-1} = \int_\Omega v_n(y, t) u_0(y) + \int_0^t \int_\Omega f_n(y, s) v_n(y, t - s). \]
When $n \to \infty$,
\[ u_n(x, t) \uparrow u(x, t) \phi(x), \quad v_n(y, t) \uparrow h_x(y, t) \phi(y), \]
and taking the limit in (4.5) gives (4.3).

Our next assertion is that if $V \geq V_0$ and $u_0 \equiv 0$, for $\varepsilon > 0$ and $\Omega' \subset \subset \Omega$ with $0 \in \Omega'$, there is a $C > 0$ such that
\[ (4.6) \quad h(x, t) \geq C \]
for all $x \in \Omega'$ and $t \in [\varepsilon, T[$. For the proof we first recall that if $u_0 \equiv 0$, there is a positive constant $C_0$ such that $e^{tA}u_0(y) \geq C_0$ if $x \in \Omega'$ and $t \in [\varepsilon/2, T[$. Next $u$ is bounded below by the solution $w$ of
\[ \frac{\partial w}{\partial t} - \Delta w = V_0 w \quad \text{in} \quad \Omega'(\Omega \times [\varepsilon/2, T[), \]
\[ w = 0 \quad \text{on} \quad \partial \Omega, \quad w(y, \varepsilon/2) = C_0 \chi_{\Omega}(y) \quad \text{in} \quad \Omega. \]
and $w$ is the (increasing) limit of the unique nonnegative solution $w_n$ of
\[ \frac{\partial w_n}{\partial t} - \Delta w_n = V_n w_n \quad \text{in} \quad \Omega'(\Omega \times [\varepsilon/2, T[), \]
\[ w_n = 0 \quad \text{on} \quad \partial \Omega, \quad w_n(y, \varepsilon/2) = C_0 \chi_{\Omega}(y) \quad \text{in} \quad \Omega. \]
Choose a ball $B$ in $\Omega'$, centered at the origin and of radius $r_0$. Then $w_n \equiv v_n$ where
\[ (4.7) \quad \frac{\partial v_n}{\partial t} - \Delta v_n = V_n v_n \quad \text{in} \quad \Omega'(B \times [\varepsilon/2, T[), \]
\[ v_n = 0 \quad \text{on} \quad \partial B, \quad v_n(y, \varepsilon/2) = C_0 \quad \text{in} \quad B, \]
where here and in the sequel, $V_n = \inf(V_0, n)$. But $v_n$ is a radial function, i.e. a function of $|x| = r$ alone. Thus
\[ \frac{\partial v_n}{\partial t} - \frac{\partial^2 v_n}{\partial r^2} - \frac{(N - 1)}{r} \frac{\partial v_n}{\partial r} = V_n v_n, \]
\[ v_n(r_0, t) = 0 = \frac{\partial v_n}{\partial r}(0, t), \quad v_n(r, \varepsilon/2) = C_0. \]
Multiply (4.7) by $(v_n)^{p-1} \phi^{2-p}$ for $p > 1$ and integrate to obtain
\[ \frac{1}{p} \int_B \left( p^{-1} \int_B \frac{v_n}{\phi} \right)^p \phi^2 + \int_B \nabla v_n \cdot \nabla \left( v_n^{p-1} \phi^{2-p} \right) = \int_B V_n (v_n/\phi)^p \phi^2. \]
Setting $k_n = v_n/\phi$ we get
\[ \frac{1}{p} \int_B \left( p^{-1} \int_B k_n \phi^2 \right) + 4(p - 1) p^{-2} \int_B \nabla k_n^{p/2} \phi^2 + \int_B k_n^p (-\Delta \phi) \phi = \int_B V_n k_n^p \phi^2. \]
Recall (cf. §2) that $V_n(x) \leq V_0(x) \leq -\Delta \phi/\phi$. Thus $V_n \phi^2 \leq (-\Delta \phi) \phi$ and consequently

$$\frac{\partial}{\partial t} \left( p^{-1} \int_B k_n \phi^2 \right) \leq 0,$$

whence for $\epsilon/2 \leq t < T$,

$$\left( \int_B v_n^p \phi^2 - p \right)^{1/p} (t) \leq C_0 \left( \int_B \phi^2 - p \right)^{1/p},$$

the right side being the value of the left side for $t = \epsilon/2$. Letting $p \to \infty$ it follows that $k_n \leq C_0$ a.e. in $B$, which is equivalent to $v_n \leq C_0 \phi$ a.e. in $B$. We are now justified in setting

$$v = \lim_{n \to \infty} v_n, \quad k = \lim_{n \to \infty} k_n.$$

We will show that

(4.9) $C_0 \geq k(x, t) \geq C_1 > 0$ for $\epsilon < t < T$ and a.e. $x \in \frac{1}{2} B$.

(Here $B = B_{r_0}, \frac{1}{2} B = B_{r_0/2}$ and $k \leq C_0$ is already proven.) Since

$$u \geq \phi h \geq w \geq w_n \geq v_n \geq k_n \phi,$$

(4.9) implies (4.6) with $y \in \Omega' = \frac{1}{2} B$. And for $y \in \Omega \setminus \{\frac{1}{2} B\}$ we have (since $u \geq e^{t \Delta} u_0$)

$$h(y, t) \geq \phi(y)^{-1}(e^{t \Delta} u_0)(y) \geq C_2 > 0$$

for all $y \in \Omega'$,

$$\phi(y)^{-1} \geq C_3 > 0 \quad \text{in} \quad \Omega \setminus \frac{1}{2} B,$$

where $C_2$ and $C_3$ are suitable constants. We now establish (4.9), using ideas of J. Moser [4, 5].

Let $g: [0, \infty[ \to [0, \infty[ be convex and of class $C^2$. Multiply (4.7) by $g'(k_n)g(k_n)\phi^2$ where $\psi \in \mathcal{D}(B \times ]e/2, T[)$ and integrate over $Q = B \times ]e/2, T[$. We obtain

$$\int_Q \frac{1}{2} \left( \frac{\partial}{\partial t} (g(k_n)^2) \right) \phi^2 \psi^2 + \int_Q \nabla (k_n \phi) \cdot \nabla (g'(k_n)g(k_n)\phi^2)$$

$$= \int_Q V_n k_n \phi g'(k_n)g(k_n)\psi^2 \phi.$$

Straightforward computations give

$$\int_B \nabla (k_n \phi) \cdot \nabla (g'(k_n)g(k_n)\phi^2)$$

$$= \int_B \nabla (k_n) \cdot \nabla (g'(k_n)g(k_n)\phi^2) \phi + \int_B k_n \nabla \phi \cdot \nabla (g'(k_n)g(k_n)\phi^2)$$

$$= \int_B g''(k_n) \nabla k_n \phi^2 \phi^2 \psi^2 + \int_B |\nabla g(k_n)|^2 \phi^2 \psi^2$$

$$+ \int_B (\nabla g(k_n) \cdot \nabla \phi) \psi^2 g(k_n) + \int_B \nabla g(k_n) \cdot (\nabla \psi^2) g(k_n) \phi^2$$

$$+ \int_B (-\Delta \phi) k_n g'(k_n)g(k_n)\phi^2 - \int_B (\nabla \phi \cdot \nabla g(k_n)) g(k_n) \phi^2,$$
whence

\[
\int_Q \left( \frac{\partial}{\partial t} \left( g(k_n) \right)^2 \right) \phi^2 \psi^2 + \int_Q |\nabla g(k_n)|^2 \phi^2 \psi^2 \\
+ \int_Q g''(k_n) |\nabla k_n|^2 \left( g(k_n) \phi^2 \psi^2 \right) + \int_Q (\nabla g(k_n) \cdot \nabla \psi^2) g(k_n) \phi^2 \\
+ \int_Q (-\Delta \phi) k_n \phi' g'(k_n) g(k_n) \phi \psi^2 = \int_Q \nabla k_n \phi^2 g'(k_n) g(k_n) \psi^2.
\]

The third term on the left is nonnegative since \( g \) is convex and nonnegative; we will integrate the first term by parts and for the fourth term use the trivial inequality

\[
\left| 2 \int_B (\nabla g(k_n) \cdot \nabla \psi) g(k_n) \phi^2 \psi \right| \leq \frac{1}{2} \int_B |\nabla g(k_n)|^2 \phi^2 \psi^2 + 2 \int_B |\nabla \psi|^2 |g(k_n)|^2 \phi^2.
\]

We thus obtain

\[
2^{-1} \left( \int_B g(k_n)^2 \phi^2 \right)(t) + 2^{-1} \int_B |\nabla g(k_n)|^2 \phi^2 \psi^2 \leq \int_Q \phi (V_0 \phi + \Delta \phi) k_n g'(k_n) g(k_n) \psi^2 \\
+ \int_Q g(k_n)^2 \phi^2 \left( 2 |\nabla \psi|^2 + \phi \frac{\partial \psi}{\partial t} \right).
\]

Now suppose \( \phi \triangleleft \phi \in L'(B) \) (which is equivalent to \( \alpha < (N - 2)/2 \)). Take \( B \) to have sufficient by small radius. Since \( V_0 = -\Delta \phi/\phi \) the first term on the right side of the above inequality tends to zero as \( n \to \infty \) by Lebesgue’s dominated convergence theorem. (Here we are using \( ||k_n||_{L^2} \leq C_0 \) in \( B \) and the hypotheses on \( g \).) Thus when \( n \to \infty \) we obtain

(4.10)

\[
\int_B g(k(t))^2 \psi(t)^2 \phi^2 + \int_Q |\nabla g(k(t))|^2 \phi^2 \psi^2 \leq 2 \int_Q g(k)^2 \phi^2 \left( 2 |\nabla \psi|^2 + \phi \frac{\partial \psi}{\partial t} \right).
\]

Now choose \( \psi \) so that \( 0 \leq \psi \leq 1 \), \( \psi = 1 \) in \( B_{r-\delta} \times [s + \delta, T] \), \( \psi = 0 \) in \( (B \times [0, s]) \cup (B \setminus B_r) \times [0, T] \) where \( 0 < s, \delta \). We further suppose that

\[
|\partial \psi/\partial t| \leq C_4/\delta, \quad |\nabla \psi|^2 \leq C_4/\delta^2
\]

where the constant \( C_4 \) is independent of the pair \( (s, \delta) \). Inequality (4.10) then yields

(4.11)

\[
\int_B g(k(t))^2 \phi^2 + \int_{s+\delta}^T \int_{B_{r-\delta}} |\nabla g(k)|^2 \phi^2 \leq 6 C_4 \delta^{-2} \int_s^T \int_{B_r} g(k)^2 \phi^2
\]

for all \( t \in [s + \delta, T] \).

If \( 0 < r' \leq r \leq 1 \) and \( h \) is a nice function on \([0, r]\), then

(4.12)

\[
\left( \int_0^r |h(s)|^m s^{N-2\alpha -2} ds \right)^{2/m} \leq C_5 \left( \int_0^r \left[ |h'(s)|^2 + |h(s)|^2 \right] s^{N-2\alpha -2} ds \right)
\]

where \( 1/m > 1/2 - 1/(N - 2\alpha) \) (and \( m < \infty \) if \( N - 2\alpha = 2 \)). The constant \( C_5 \) depends on \( r' \) but not on \( r \).
Remark 4.1. If $M > 2$ is given and if
\[ 2 \leq m \leq \min\{M, 2(N - 2\alpha)(N - 2\alpha - 2)^{-1}\}, \]
then the constant $C_5$ in (4.12) is uniformly bounded for $\alpha \in [0, (N - 2)/2]$.

A proof of (4.12) which contains a description of the constant $C_5$ is given in the Appendix.

Define $\beta$ by $\beta + 2/m = 1$ where $1 > 2/m \geq 1/2 - 1/(N - 2\alpha)$. By Hölder's inequality and (4.12) we get, for a nonnegative radial function $h$,

\[ \int_{B_r} h^{2 + 2\beta} \phi^2 \leq \left( \int_{B_r} h^m \phi^2 \right)^{2/m} \left( \int_{B_r} h^2 \phi^2 \right)^{\beta} \]

\[ \leq C_5 \left( \int_{B_r} |\nabla h|^2 \phi^2 + \int_{B_r} h^2 \phi^2 \right) \left( \int_{B_r} h^2 \phi^2 \right)^{\beta} \]

whence

\[ (4.13) \quad \int_a^b \int_{B_r} h^{2 + 2\beta} \phi^2 \leq C_5 \left( \int_a^b \int_{B_r} (|\nabla h|^2 + h^2) \phi^2 \right) \sup_{a \leq t \leq b} \left( \int_{B_r} h^2 \phi^2 \right)^{\beta} (t). \]

From (4.11) we deduce

\[ \sup_{t \in [s + \delta, T]} \int_{B_{r - \delta}} g(k(t))^2 \phi^2 \leq 6C_4 \delta^{-2} \int_s^T \int_{B_r} g(k)^2 \phi^2. \]

Note that $x \mapsto g(k(x, t))$ is, for fixed $t$, a radial function, so applying (4.13) with $[a, b] = [s + \delta, T]$ and with $B_{r - \delta}$ in place of $B_r$ we get

\[ \int_{s + \delta}^T \int_{B_{r - \delta}} g(k)^{2 + 2\beta} \phi^2 \leq C_5 (6C_4 \delta^{-2} + 1) \left( \int_s^T \int_{B_r} g(k)^2 \phi^2 \right)^{\beta} \cdot \left( 6C_4 \delta^{-2} \int_s^T \int_{B_r} g(k)^2 \phi^2 \right)^{\beta}, \]

whence

\[ (4.14) \quad \left( \int_{s + \delta}^T \int_{B_{r - \delta}} g(k)^{2 + 2\beta} \phi^2 \right)^{1/(2 + 2\beta)} \leq \left[ C_5^{1/2} (6C_4 + 1) \right]^{1/(1 + \beta)} \delta^{-\gamma} \left( \int_s^T \int_{B_r} g(k)^2 \phi^2 \right)^{1/2} \]

\[ = C_6 \delta^{-1} \left( \int_s^T \int_{B_r} g(k)^2 \phi^2 \right)^{1/2} \]

since the exponent of $\delta$ is $-\gamma = -(1 + \beta)^{-1}(1 + \beta) = -1$ and $0 < \delta \leq 1$.

Let $a > 0$ be a small number and let

\[ \delta = a/2^n, \quad r_{n+1} = r_n - a/2^n, \quad g_{n+1} = g_n^{1 + \beta}, \quad s_{n+1} = s_n + a/2^n, \]

\[ H_n = \left( \int_{s_n}^T \int_{B_{r_n}} g_n(k)^2 \phi^2 \right)^{1/2} \]
where \( g_1 = g \), and \( r_1 \) and \( s_1 \) are given positive numbers. With this notation the estimate (4.14) yields
\[
H_{n+1}^{1/(1+\beta)} \leq C_6 \cdot 2^n a^{-1} H_n,
\]
whence, by induction,
\[
H_{n}^{1/(1+\beta)} \leq (C_6 a^{-1})^{\alpha_n} 2^{\gamma_n} H_1^{(1+\beta)^{n-2}}
\]
where
\[
\alpha_n = (1 + \beta)^{n-2} \sum_{j=0}^{n-2} (1 + \beta)^{-j}, \quad \gamma_n = \sum_{j=0}^{n-1} (j + 1)(1 + \beta)^{n-2-j}.
\]
Now let \( n \to \infty \). Since \( g_n = g^{(1+\beta)^{n-1}} \) we get
\[
\sup_{B_{n-a} \times [s_1+a,T]} g(k(x,t)) \leq (C_6 a^{-1} \cdot 2^{(1+\beta)/\beta})^{(1+\beta)/\beta} \left( \int_{s_1}^{T} \int_{B_n} g(k(x))^2 \phi^2 \right)^{1/2}.
\]
Replace \( g \) by a sequence \( \{g_i\} \) satisfying the hypotheses and tending to \( 1/r^\gamma \) as \( l \to \infty \). We then obtain
\[
\sup_{B_{n-a} \times [s_1+a,T]} 1/k^\gamma \leq (C_6 a^{-1} \cdot 2^{(1+\beta)/\beta})^{(1+\beta)/\beta} \left( \int_{s_1}^{T} \int_{B_n} k^{-2\gamma} \phi^2 \right)^{1/2}.
\]
Now set
\[
s_1 = 3\varepsilon/4, \quad a = \varepsilon/4, \quad r_1 < r_0
\]
where \( \varepsilon > 0 \) is given (and \( r_0 \) is as before, cf. the sentence preceding (4.7)). Note that
\[
k(x,t) = v/\phi \equiv \phi |x|^{-1} \left( e^{\Delta_0} \phi \right)(x) \geq C_0 C_7 \phi(x)^{-1}
\]
for \( (x,t) \in B_1 \times [3\varepsilon/4, T] \) (where the constant \( C_7 \) is independent of \( r_1 \) and \( \varepsilon \) (but, of course, \( C_0 \) depends on \( \varepsilon \), as before). Thus we obtain
\[
\sup_{B_{1-\varepsilon/4} \times [\varepsilon,T]} k^{-\gamma} \leq C_0 C_7 e^{-1-1/\beta} \left( \int_{3\varepsilon/4}^{T} \int_{B_1} \phi^{2+2\gamma} \right)^{1/2},
\]
which implies the estimate
\[
(4.15) \quad k(x,t) \geq C_0 C_7 e^{(1+1/\beta)(1/\gamma)} \left( \int_{B_1} \phi^{2+2\gamma} \right)^{-1/2\gamma}
\]
for a.e. \( x \in B_{1-\varepsilon/4} \) and for all \( t \in [\varepsilon, T] \). Here the positive constant \( C_9 \) is independent of the pair \( (r_1, \varepsilon) \). The estimate (4.9) is therefore established for \( \alpha < (N-2)/2 \), and it also holds for the limiting case of \( \alpha = (N-2)/2 \).

[First of all, it is easy to see that \( \gamma > 1 \) can be chosen so that \( \int_{B_1} \phi^{2+2\gamma} \) for a.e. \( x \in B_1 \times [\varepsilon, T] \). Here the positive constant \( C_9 \) is independent of the pair \( (r_1, \varepsilon) \). The estimate (4.9) is therefore established for \( \alpha < (N-2)/2 \), and it also holds for the limiting case of \( \alpha = (N-2)/2 \).]
Remark 4.1 implies that for \( \beta = 1 - 1/2m \) with \( m = \min\{M, 2(N - 2\alpha)/(N - 2\alpha - 2)\} \), the constant \( C_9 \) in (4.15) is bounded below by a positive constant independent of \( \alpha \in [0, (N - 2)/2] \). Thus there is a \( C > 0 \) such that \( k_{\alpha} |x|^{-\alpha} \geq C|x|^{-\alpha} \) for \( x \in \frac{1}{2}B, t \geq \varepsilon \) and \( \alpha \in [0, (N - 2)/2] \). Hence \( k_{(N-2)/2} \geq C|x|^{(N-2)/2-\alpha} \) where \( C \) is independent of \( \alpha \). A passage to the limit now gives the result.

The estimate (2.5) is an immediate consequence of (4.6). To establish the rest of Theorem 2.2(i), apply (4.6) with \( u_0 = \delta_j \); we obtain the existence of a constant \( C > 0 \) such that

\[
h_x(y, t) \geq C/\phi(x) \quad \text{for all } (x, y) \in \Omega' \times \Omega', t \in [\varepsilon, T].
\]

Then (4.3) implies

\[
(4.16) \quad \phi(x)h(x, t) \geq C \int_{\Omega'} \phi u_0 + C \int_0^{T-\varepsilon} \int_{\Omega'} f\phi.
\]

If a solution \( u \) exists then necessarily \( h(x, t) < \infty \) for a.e. \( (x, t) \in \Omega \times [0, T] \). Consequently the condition

\[
\int_{\Omega'} \phi u_0 < \infty, \quad \int_0^{T-\varepsilon} \int_{\Omega'} f\phi < \infty
\]

are necessary for the existence of a solution \( u \). \( \Box \)

5. Proof of Theorem 2.2(ii). In this section we deduce it from the first part of the theorem. An independent proof based on probability theory will be given in the next section.

Let \( c > C_*(N) \). If (P) has a solution \( u \equiv 0 \), then one has

\[
\frac{\partial u}{\partial t} - \Delta u = C_*(N)|x|^{-2}u + (c - C_*(N))|x|^{-2}u
\]

in \( \varOmega'(\Omega \times [0, T]) \). From part (i) we know that the solution exists only if

\[
\left[ c - C_*(N) \right] u|x|^{-2}\phi \in L^1(\Omega' \times [0, T - \varepsilon])
\]

for \( \Omega' \subset \subset \Omega \) and \( \varepsilon > 0 \) (where we assume \( 0 \in \Omega' \)). By part (i) (see (2.5)) we have \( u \geq C_1|x|^{-c_1(N)} \) in \( \Omega' \times [\varepsilon, T] \), whence \( |x|^{-2 - 2c_1(N)} \in L'(\Omega') \), which is false. \( \Box \)

While this is a short proof it is not a simple one in view of the estimates that went into the proof of Theorem 2.2(ii).

6. The Feynman-Kac formula applied to nonexistence. We consider the problem

\[
(P') \quad \frac{\partial u}{\partial t} = \frac{1}{2} \Delta u + V(x)u \quad (x \in \mathbb{R}^N, t > 0),
\]

\[
u(x, 0) = u_0(x) \quad (x \in \mathbb{R}^N),
\]

where, as before, \( V \in L_{loc}^\infty(\mathbb{R}^N \setminus \{0\}) \). The normalization factor \( \frac{1}{2} \) has been introduced for the following reason: If \( V \) is bounded above, then the unique nonnegative solution of \( (P') \) is given by

\[
u(x, t) = \int_S \exp \left( \int_0^t V(\omega(s)) \, ds \right) u_0(\omega(t)) P_x(d\omega)
\]
where \( \omega \) is in the space of paths \( S = C([0, \infty[, \mathbb{R}^N) \) and where \( P_x \) is the Wiener probability measure starting at \( x \), i.e. \( P_x \) is supported on \( \{ \omega \in S : \omega(0) = x \} \).

**Theorem 6.1.** Suppose \( u_0 \) is measurable, \( u_0 \geq 0 \) and \( u_0 \equiv 0 \). Suppose further that \( V(x) \geq \nu(|x|) \) where \( \nu \) is a nonnegative measurable function on \( [0, \infty[ \) satisfying

\[
\liminf_{r \to 0^+} r^2 \nu(r) > \frac{N^2 \pi^2}{8}.
\]

Let \( (P'_n) \) be the problem \( (P') \) but with \( V \) replaced by \( V_n = \min(n, V) \), and let \( u_n \) be the unique nonnegative solution of \( (P'_n) \). Then \( \lim_{n \to \infty} u_n(x, t) = \infty \) for each \( (x, t) \in \mathbb{R}^N \times ]0, \infty[ \).

We shall give a direct probabilistic proof of this, independent of the previous sections. Before doing so we make some remarks. The change of variables \( x \mapsto \sqrt{2}x \) transforms the equation \( \frac{\partial u}{\partial t} = \frac{1}{2} \Delta u + V_n \) into \( \frac{\partial u}{\partial t} = \Delta u + \tilde{V}u \) where \( \tilde{V}(x) = V(x/ \sqrt{2}) \). Thus if \( V(x) > c/|x|^2 \), (6.1) implies that \( c > N^2 \pi^2/4 \) is sufficient condition for nonexistence. Of course, by Theorem 2.2, \( c > C^*(N) \) is the optimal sufficient condition and \( C^*(N) < N^2 \pi^2/4 \). The probabilistic proof given below by itself does not appear to be capable of giving the sharpest result. (But see the remark at the end of this section.) Probabilistic arguments can extend the result of Theorem 6.1 from the whole space case \( (x \in \mathbb{R}^N) \) to the bounded domain case \( (x \in \Omega \subset \mathbb{R}^N \) with the Dirichlet condition \( u = 0 \) on \( \partial \Omega \) imposed). We omit further discussion of this.

For the proof of the theorem, fix \( t > 0, x \in \mathbb{R}^N \). Since

\[
u_n(x, t) = \int_S \exp \left( \int_0^t V_n(\omega(s)) \, ds \right) u_0(\omega(t)) P_x(d\omega)
\]

it follows that

\[
u_n(x, t) \uparrow u(x, t) = \int_S \exp \left( \int_0^t V(\omega(s)) \, ds \right) u_0(\omega(t)) P_x(d\omega)
\]

by Lebesgue's monotone convergence theorem. What we must prove is that \( u(x, t) = \infty \).

To simplify the proof somewhat we assume \( u_0 \) is strictly positive in some ball, i.e.

\[
u_0(x) \geq \epsilon_0 \quad \text{for } |x - x_0| \leq \delta_0
\]

for some choice of \( \epsilon_0 > 0, \delta_0 > 0, x_0 \in \mathbb{R}^N \). This can be assumed without loss of generality since for \( 0 < \epsilon < t, u_n(x, t) (\leq u(x, t)) \) is the solution at \( (x, t - \epsilon) \) of \( (P'_n) \) having initial data \( u_0(x, \epsilon) \) which is everywhere continuous and positive.

Let \( 0 < \alpha < \frac{1}{2} \) and let

\[
S_n = \{ \omega \in S : \omega(0) = x, |\omega(t) - x_0| \leq \delta_0, \text{ and } |\omega(s)| < 1/n \text{ for } s \in J = [\alpha t, (1 - \alpha)t] \}.
\]

Our hypotheses imply

\[
u(x, t) \geq \int_{S_n} \exp \left( \int_{\alpha t}^{1 - \alpha t} V(\omega(s)) \, ds \right) u_0(\omega(t)) P_x(d\omega)
\]

\[
\geq \exp \left( \gamma \nu \left( \frac{1}{n} \right) \right) \epsilon_0 P_x(S_n)
\]
where $\gamma = 1 - 2\alpha$. The main estimate is

$$u(x, t) \geq \exp(\gamma n / (1 - e)) e_0 k_0 \exp\left\{-\pi^2 N^2 \gamma t n^2 / (8 - e)\right\} n^{-N}.$$ (6.3)

This implies $u(x, t) = \infty$ by (6.1) since $e > 0$ is arbitrary. The exact dependence of the constant $k_0$ on $(\gamma, t, x, x_0)$ will be made clear in the sequel (cf. (6.11), (6.12)).

The proof of the main estimate (6.3) is based on connections between Brownian motion and the heat equation.

To simplify matters further we work in $N = 1$ space dimension. The estimate for $N$ dimensions follows from the one-dimensional estimate (or rather its proof). The one-dimensional proof below could be done in $N$ dimensions, but the separation of variables part of the argument is clearest when $N = 1$.

Let $\beta(t, \omega) = \{\beta(t, u) : t \geq 0, \omega \in S\}$ be normalized one-dimensional Brownian motion (cf. [3]). Let $-\infty < a < b < \infty$ and let $t = \tau([a, b])$ be the exit time from $[a, b]$ for Brownian motion starting at $x$, i.e.,

$$\tau(\omega) = \inf\{t > 0 : \beta(t, \omega) = a \text{ or } b, \text{ given that } \beta(0, x) = x\}.$$

For typographical convenience we shall usually suppress $\omega$. Then $w$, defined by

$$w(x, t) = P_x\{\omega \in \Omega : \tau(\omega) > t\} = P_x\{\tau > t\},$$

satisfies

$$\frac{\partial w}{\partial t} = \frac{1}{2} \frac{\partial^2 w}{\partial x^2} \quad \text{for } a < x < b, t > 0,$$
$$w(t, a) = w(t, b) = 0 \quad \text{for } t > 0,$$
$$w(0, x) = 1 \quad \text{for } a < x < b.$$ (6.4)

For a nice discussion see [6, p. 207].

For $b, t_0 > 0$,

$$P_0\{|\beta(s)| < b \text{ for } 0 \leq s \leq t_0\} = P_0\{\tau([-b, b]) > t_0\} = P_b\{\tau([0, 2b]) > t_0\} = w(b, t_0)$$ (6.5)

where $w$ satisfies (6.4) with $[a, b]$ replaced by $[0, 2b]$. We calculate $w$ by separation of variables, obtaining

$$w(x, t) = \sum_{n \text{ odd}} \frac{4}{\pi} \exp\left(-\frac{n^2 \pi^2 t}{8b^2}\right) \sin\left(\frac{n \pi x}{2b}\right),$$

hence

$$w(b, t_0) = \sum_{n \text{ odd}} \frac{4}{\pi} \exp\left(-\frac{n^2 \pi^2 t_0}{8b^2}\right) \sin\left(\frac{n \pi}{2}\right) = \sum_{k=0}^{\infty} \frac{4}{\pi} (-1)^k \exp\left(-\left(2k + 1\right)^2 \frac{n^2 \pi^2 t_0}{8b^2}\right) \geq \frac{4}{\pi} \left\{\exp\left(-\frac{\pi^2 t_0}{8b^2}\right) - \exp\left(-\frac{9\pi^2 t_0}{8b^2}\right)\right\}.$$
because this alternating series exceeds the sum of its first two terms. Taking $b = 1/n$ and $t_0 = t$ gives
\[
w\left(\frac{1}{n}, t\right) \geq \frac{4}{\pi} \left\{ \exp\left(-\frac{\pi^2 n^2 t}{8}\right) - \exp\left(-\frac{9\pi^2 n^2 t}{8}\right) \right\}
\geq \frac{4 - e_1}{\pi} \exp\left(-\frac{\pi^2 n^2 t}{8}\right)
\]
for each $e_1 > 0$ and all $n > N_1(e_1, t)$. Thus, for large $n,$
\[
(6.6) \quad w\left(\frac{1}{n}, t\right) \geq c_1 \exp\left(-\pi^2 n^2 t/8\right)
\]
where $c_1$ is any number less than $4/\pi$.

Let $x, x_0 \in \mathbb{R}^N$ and $t, \delta_0 > 0$ be as before. Then, using the strong Markov property repeatedly, we obtain, for $0 < \alpha < \frac{1}{2}, 0 < \alpha_1 < 1$,
\[
(6.7) \quad P_x\{\beta(s) < 1/n \text{ for all } s \in J = [\alpha t, (1 - \alpha) t], |\beta(t) - x_0| \leq \delta_0 \} \geq P_x\{\beta(\alpha t) < \alpha_1/n, |\beta(s) - x_0| < 1/n \text{ for } s \in J, |\beta(t) - x_0| \leq \delta_0 \}
\geq P_x\{\beta(\alpha t) < \alpha_1/n\} \cdot P\{\beta(s) < 1/n \text{ for } s \in J, |\beta(t) - x_0| \leq \delta_0 \} = P_x\{\beta(\alpha t) < \alpha_1/n\} \cdot P\{\beta(\alpha t) < 1/n \}\]
\[
(6.8) \quad \rho_1 = \frac{1}{\sqrt{2\pi t_1}} \int_{-\alpha_1/n}^{\alpha_1/n} \exp\left(-\frac{(s - \alpha_1/n)^2}{2t_1}\right) \exp\left(-\frac{(x - \alpha_1/n)^2}{2t_1}\right) e^{-y^2/2} dy
\geq \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{(|x| + \alpha_1/n)^2}{2t_1}\right) \left\{ \frac{2\alpha_1}{n/t_1} \right\}
\geq (2\pi \alpha t)^{-1/2} \left\{ \exp\left(-\frac{x^2}{2\alpha t}\right) - \varepsilon_2 \right\} \cdot 2\alpha_1/n
\]
for each $\varepsilon_2 > 0$ and all $n > N_2(\varepsilon_2, x, t, \alpha)$.

Similarly, $\rho_3$ is a weighted average of terms of the form
\[
(2\pi t_1)^{-1/2} \int_{x_0 - \delta_0}^{x_0 + \delta_0} \exp\left(-\frac{(s - \beta(0))^2}{2t_1}\right) ds
\]
where \( t_1 = at \) and \( \beta(0) \) ranges from \(-1/n\) to \(1/n\). Consequently,

\[
\rho_3 \geq (2\pi at)^{-1/2} \left\{ \exp \left( -\frac{(|x_0| + \delta_0)^2}{2at} \right) - \epsilon_3 \right\} \cdot 2\delta_0
\]

for each \( \epsilon_3 > 0 \) and all \( n > N_3(\epsilon_3, x, t, \alpha) \). Next,

\[
\rho_2 = P_0\{|\beta(s)| < (1 - \alpha_1)/n \text{ for } s \in [0, (1 - 2\alpha)t]\}
\]

\[
= \omega((1 - 2\alpha)t, (1 - \alpha_1)/n)
\]

by (6.5). Hence by (6.6),

\[
\rho_2 \geq \frac{(4 - \epsilon_3)}{\pi} \exp \left\{ -\pi^2 n^2 \frac{(1 - 2\alpha)t}{8(1 - \alpha_1)^2} \right\}
\]

for any \( \epsilon_3 > 0 \) and all \( n > N_2(\epsilon_3, x, t, \alpha, \alpha_1) \). By (6.6) and (6.8)–(6.10) we see that given \( \epsilon > 0 \), there is an \( \epsilon_4 < \epsilon \) and an \( N = N(\epsilon, x, t) \) such that for \( n > N \),

\[
P_x(S_n) \geq \rho_1 \rho_2 \rho_3 \geq (2\pi at)^{-1} \frac{4\pi \alpha_1 \delta_0}{n} \exp \left\{ -\frac{x^2}{2at} + \frac{(|x_0| + \delta_0)^2}{2at} + \epsilon_4 \right\}
\]

\[
\cdot \frac{4}{\pi} \exp \left\{ -\pi^2 n^2 \frac{(1 - 2\alpha)t}{8(1 - \alpha_1)^2} \right\}.
\]

Taking \( \alpha_1 \) so that \((1 - \alpha_1)^2 = 1 - \epsilon\) gives the estimate (6.3) with \( N = 1 \) where

\[
k_0 = \left( \frac{8\alpha_1 \delta_0}{\pi at} \right) \cdot \left( \frac{1}{n} \right) \cdot \exp \left\{ -\frac{x^2}{2at} - \frac{(|x_0| + \delta_0)^2}{2at} - \epsilon \right\}.
\]

Recall that the \( N \)-dimensional normalized Brownian motion \( \{\beta(t): t \geq 0\} \) consists of \( N \) independent one-dimensional normalized Brownian motions, i.e.

\[
\beta(t) = (\beta_1(t), \ldots, \beta_N(t))
\]

where \( \{\beta_j(t): t \geq 0\} \) is a one-dimensional normalized Brownian motion. Let \( x_0 = (x_{01}, \ldots, x_{0N}), x = (x_1, \ldots, x_N) \in \mathbb{R}^N \). Then

\[
P_x(S_n) = P_x\{|\beta(t) - x_0| \leq \delta_0, |\beta(s)| < 1/n \text{ for } s \in J = [at, (1 - \alpha)t]\}
\]

\[
\geq P_x\{\text{For } j = 1, \ldots, N, |\beta_j(t) - x_{0j}| \leq \delta_0/\sqrt{N}, |\beta_j(s)| < 1/n \sqrt{N} \text{ for } s \in J\}
\]

\[
= \left( P_x\{|\beta_1(t) - x_{01}| \leq \delta_0/\sqrt{N}, |\beta_1(s)| < 1/n \sqrt{N} \text{ for } s \in J\} \right)^N
\]

by independence

\[
\geq K_0 \exp\left\{ -\pi^2 n^2 (1 - 2\alpha) N^2 t/(8(1 - \epsilon)) \right\}
\]

by our one-dimensional results where

\[
K_0 = \left( \frac{8\alpha_1 \delta_0}{\sqrt{N} \pi at} \right)^N \left( \frac{1}{n} \right)^N \exp \left\{ -\frac{|x|^2}{2at} - \frac{N|x_0|^2 + \delta_0^2}{at} - \epsilon \right\}.
\]
In [2] we indicate how ideas from this section can be combined with the $L^2$ theory of $-\Delta + c/|x|^2$ developed by quantum theorists to give a complete proof of Theorem 2.2(ii) for the case of $f \equiv 0$ and $\Omega = \mathbb{R}^N$. Incidentally, [2] contains a list of some of the $L^2$ references.

7. Concluding remarks.

Remark 7.1. The potential $V(x) = c/|x|^2$ is of the form $V = -\Delta \phi/\phi$ where $\phi(x) = |x|^{-\alpha}$ and $c = (N - 2 - \alpha)\alpha$. Theorem 2.2 extends to cover potentials of the form $V = -\Delta \phi/\phi$ where $\phi > 0$, $\Delta \phi \in L^1_{\text{loc}}(\Omega)$, $\int \phi u_0 < \infty$, $\int_0^T \int \phi \phi < \infty$ and the weighted Sobolev estimate

$$(\int_\Omega |h|^m \phi)^{1/m} \leq C \left( \int_\Omega |\nabla h|^2 \phi^2 + \int_\Omega |h|^2 \phi^2 \right)^{1/2}$$

holds for some $m > 2$. (This last estimate reduces to (4.12) for $\phi(x) = |x|^{-\alpha}$ and for $h$ a radial function.)

Remark 7.2. Theorem 2.2(ii) extends to establish the nonexistence of solutions for certain nonlinear problems. For instance, consider (P) in which the term $V(x)u$ is replaced by the nonlinear term $W(x, u)u$, where $W(x, u) \geq c|x|^2$ holds for all $(x, u) \in \Omega \times \mathbb{R}$ and $c > C_*(N)$. Then no nonnegative solution to (P) can exist in that the approximating solutions $u_n$ of $(P_n)$ approach infinity on all of $\Omega \times [0, T]$.

Remark 7.3. It is well known that the solution of the Cauchy problem $\partial u/\partial t - \Delta u = 0$, $u(x, 0) \equiv 0$ is not unique (for $(x, t) \in \mathbb{R}^N \times [0, \infty]$). Nonuniqueness also holds for (P). This will be shown by one of us in a separate paper [1].

Appendix: Proof of inequality (4.12). If $0 \leq h \in C^1[0, 2r]$ and $h(2r) = 0$, then we claim that

$$\left( \int_0^{2r} h^{m}(s)s^{a-1} ds \right)^{1/m} \leq K_0 \left( \int_0^{2r} \left| \frac{dh}{ds} \right|^2 s^{a-1} ds \right)^{1/2}$$

where $a = N - 2\alpha > 2$, $m^{-1} = 2^{-1} - a^{-1}$, and $K_0$ is a constant, depending only on $\alpha$. For the proof, integration by parts gives

$$\int_0^{2r} h^{m}(s)s^{a-1} ds = -\frac{m}{a} \int_0^{2r} h^{m-1}(s) \frac{dh}{ds} s^{a} ds \leq \frac{m}{a} \left( \int_0^{2r} h^{2m-2}s^{a+1} ds \right)^{1/2} \left( \int_0^{2r} \left| \frac{dh}{ds} \right|^2 s^{a-1} ds \right)^{1/2} \leq \frac{m}{a} \sup_{s \in [0, 2r]} \left\{ s^2 h^{m-2}(s) \right\} \left( \int_0^{2r} h^{m}s^{a-1} ds \right)^{1/2} \left( \int_0^{2r} \left| \frac{dh}{ds} \right|^2 s^{a-1} ds \right)^{1/2}.$$

Thus to establish (A.1) it suffices to show

$$\sup_{[0, 2r]} \left\{ s^2 h^{m-2}(s) \right\} \leq K_1 \left( \int_0^{2r} \left| \frac{dh}{ds} \right|^2 s^{a-1} ds \right)^{m-2/2},$$

from which it follows that (A.1) holds with $K_0 = [K_1m^2/a^2]^{1/m}$ which depends only on $\alpha$.
For (A.2) we note that \( \frac{2}{m-2} = \frac{a}{m} \), and so

\[
\sup_{[0,2r]} \left\{ \frac{s^{2/(m-2)}h(s)}{s^{a/m}} \right\} = \sup_{[0,2r]} \left\{ \frac{s^{a/m}}{[h(s) - h(2r)]} \right\}
\]

\[
= \sup \left\{ \frac{s^{a/m}}{\int_{s}^{2r} (-h'(s)) ds} \right\}
\]

\[
\leq \sup \left\{ \frac{s^{a/m}}{\left( \int_{s}^{2r} \sigma^{a-a} d\sigma \right)^{1/2} \left( \int_{s}^{2r} h'(\sigma)^2 \sigma^{a-1} d\sigma \right)^{1/2}} \right\}
\]

\[
\leq \left\{ \sup_{[0,2r]} M(s) \right\} \left( \int_{0}^{2r} h'(\sigma)^2 \sigma^{a-1} d\sigma \right)^{1/2}
\]

where \( M(s) = \frac{s^{a/m}}{\left( \int_{s}^{2r} \sigma^{a-a} d\sigma \right)^{1/2}} \). On \((0,2r)\) we have

\[
M^2(s) = \frac{s^{2a/m}}{(2r)^{2-a} - s^{2-a}}(2-a)^{-1}
\]

\[
= \frac{((2r)^{2-a}s^{2a/m} - 1)(2-a)^{-1}}{}
\]

since \( 2a/m + 2 - a = 0 \); hence

\[
\frac{d}{ds} M^2(s) = (2r)^{2-a}(2-a)^{-1} \frac{2a}{m} s^{2a/m-1} < 0
\]

because \( a > 2 \). Thus \( \sup_{[0,2r]} M^2(s) = M^2(0) = (a-2)^{-1} \), and so (A.2) holds with

\[
K_1 = (a-2)^{-(m-2)/2}.
\]

Next we deduce (4.12). Fix \( \rho > 0 \) and let \( r \gg \rho \). Let \( h \in C^1(0, r) \). Let \( \xi \in C^1[r, 2r] \) satisfy \( 0 < \xi \leq 1 \), \( \xi \equiv 0 \) in \( [r + \rho/2, 2r] \), \( \xi \equiv 1 \) in \( [r, r + \rho/4] \), and \( 0 \gg \xi' \gg -5/\rho \) in \( [r, 2r] \). Let \( \psi(s) \) be \( h(s) \) or \( h(2r-s)\xi(s) \) according as \( s \in [0, r) \) or \( s \in [r, 2r] \). Then by (A.2),

\[
\left( \int_{0}^{r} h^m(s)s^{a-1} ds \right)^{2/m} \leq \left( \int_{0}^{r} \psi^m(s)s^{a-1} ds \right)^{2/m} \leq K_1^2 \left( \int_{r}^{2r} \psi'(s)^2 s^{a-1} ds \right)
\]

\[
\leq K_1^2 \left[ \int_{0}^{r} h'(s)^2 s^{a-1} ds + 2 \int_{r}^{2r} h'(2r-s)^2 \xi(s)^2 s^{a-1} ds \right]
\]

\[
+ 2 \int_{r}^{2r} h(2r-s)^2 \xi'(s)^2 s^{a-1} ds
\]

\[
\text{(let } \alpha = 2r-s\}
\]

\[
\leq K_1^2 \left[ \int_{0}^{r} h'(s)^2 s^{a-1} ds + 2 \int_{r}^{r+\rho/2} h'(\sigma)^2 (2r-\sigma)^{a-1} d\sigma \right]
\]

\[
+ 2 \int_{r+\rho/2}^{r+\rho/4} h(\sigma)^2 \xi'(2r-\sigma)^2 (2r-\sigma)^{a-1} d\sigma
\]

\[
\leq K_1^2 \left[ 1 + 2 \cdot \left( \frac{r+\rho/2}{r-\rho/2} \right)^{a-1} \right] \int_{0}^{r} h'(s)^2 s^{a-1} ds
\]

\[
+ 50\rho^{-2} K_1^2 \left( \frac{r+\rho/2}{r-\rho/2} \right)^{a-1} \int_{0}^{r} h(s)^2 s^{a-1} ds
\]

\[
\leq K_2 \int_{0}^{r} h'(s)^2 s^{a-1} ds + K_3 \int_{0}^{r} h(s)^2 s^{a-1} ds
\]
where

(A.3) \[ K_2 = \left[ \frac{(a - 2)^2 - m^2 / a^2}{2/m} \right]^{2/m} (1 + 2a) \rho^{-2} \equiv K_2(\alpha, \rho). \]

(A.4) \[ K_3 = 25 \left[ \frac{(a - 2)^2 - m^2 / a^2}{2/m} \right]^{2/m} 2a \rho^{-2} \equiv K_3(\alpha, \rho). \]

This proves (4.12). Now we study \( K_2 \) and \( K_3 \) as functions of \( \alpha \). (Note that these are dependent on \( \rho \).) Since \( m < \infty \) we have \( \alpha < (n - 2)/2 \). For \( \alpha \to (N - 2)/2 \) we observe the following. Write \( \alpha = (N - 2)/2 - \epsilon, \epsilon > 0 \). Then \( a = N - 2\alpha = 2 + 2\epsilon \), \( m = 2a/(a - 2) \approx 2/\epsilon \), and

\[ \left[ \frac{(a - 2)^2 - m^2 / a^2}{2/m} \right]^{2/m} \approx \left[ (2\epsilon)^{2-2/\epsilon} \epsilon^{-2} \right]^{\epsilon} \approx 2^{2\epsilon-2} \epsilon^{2\epsilon-2-2\epsilon} = \frac{1}{\epsilon^{-2}}. \]

Thus each of \( K_2 \) and \( K_3 \) behaves like

\[ \text{Const } \rho^{-2} \left[ \frac{(N - 2)/2 - \alpha}{\alpha} \right]^{-2} \]

as \( \alpha \to (N - 2)/2 \). Thus \( K_2 \) and \( K_3 \) blow-up in an inverse square manner in each variable.

**References**


LABORATOIRE IMAG, TOUR DES MATHEMATIQUES ANALYSE NUMERIQUE, BP 68, 38402 ST. MARTIN D’HERES CEDEX, FRANCE

DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LOUISIANA 70118