TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 284, Number 1, July 1984

ON THE UNIVERSAL THEORY OF CLASSES OF FINITE MODELS[1]

BY
S. TULIPANI

ABSTRACT. First order theories for which the truth of a universal sentence on their finite models implies the truth on all models are investigated. It is proved that an equational theory has such a property if and only if every finitely presented model is residually finite. The most common classes of algebraic structures are discussed.

0. Introduction. John T. Baldwin in a review of the book *Selected papers of Abraham Robinson. Volume* 1, posed the following problem: "For what first order theories T does the truth of a universal sentence σ on the finite models of T imply that σ is consequence of T?" Throughout this paper we will call such theories universally-finite. Baldwin's problem is suggested by Robinson's paper [**19**] where it is proved that the theory of Abelian groups and the theory of fields are universally-finite. See also Kueker [**15**].

We recall from well-known results [**5**] that a theory T is universally-finite iff every model of T can be embedded in an ultraproduct of finite models of T. However, such a characterization is difficult to handle even in simple cases. In this paper we look rather for a more useful characterization.

First, we confine our attention to equational theories (§2). Then, we show that to every theory T in any language we can associate an equational theory E in a language without relation symbols in such a way that Baldwin's problem for T is reducible to Baldwin's problem of E (Proposition 3).

The main theorem of this paper (see §2) proves that several statements are equivalent to the assertion that an equational theory T is universally-finite. The most important are: "Every finitely presented model of T is residually finite" and "Every quasi-identity true in all finite subdirectly irreducible models of T is true in all subdirectly irreducible models of T".

We are convinced that our theorem provides a satisfactory characterization. In fact we get as an immediate consequence of it that the following varieties are universally-finite: Commutative unitary rings, commutative von Neumann regular rings with quasi-inverse as operation, lattices, R-modules, where R is a finitely generated commutative unitary ring (this generalizes Robinson's result). On the other hand groups and unitary rings (cf. [**6**]) are examples of non-universally-finite varieties. Moreover, we characterize (§4) the varieties of R-modules which are

Received by the editors June 20, 1983.
1980 *Mathematics Subject Classification.* Primary 03C13, 03C05; Secondary 03C60, 08C10.
Key words and phrases. Universal sentence, finite models, residually finite, finitely presented.
[1]This research is part of the program partially supported by CNR and MPI (60%).

universally-finite when R is Noetherian, when R is Artinian and when R is a von Neumann commutative regular ring.

ACKNOWLEDGEMENTS. Thanks to some helpful comments of Professor G. Sabbagh and to a kind letter of Ch. Berline we could avoid a second lemma for Theorem 1.

The method of reduction to varieties in §3 was inspired by a very similar method used in [**22**, §9]. However, as I learned from the Referee's report, I have to acknowledge that such a general method goes back to McKenzie (see [**23**]).

1. Notation and preliminary definitions. Theory always means first order theory in a language with equality. For notational simplicity we use capital italic letters $A, B, \dots$ for structures as well as their basic sets. A structure for a language without relation symbols will be called an algebra. All members of a class of structures are assumed to be similar, i.e. of the same type.

If K is a class of structures, we denote the class of finite members of K by K_{fin}. We say that a class K is *universally-finite* if every universal sentence in the language of K which is true in all members of K_{fin} is also true in all members of K. We say that a *theory is universally-finite* if the class of its models is universally-finite.

For the definitions of equational theories, of varieties, of identities and quasi-identities see [**4** and **9**]. We denote the usual operators of taking cartesian product, subalgebras, homomorphic images, reduced products, ultraproducts and finite direct product by $P, S, H, P_R, P_U, P_{\text{fin}}$, respectively. If V is a variety, we denote the class of subdirectly irreducible members of V by V_{SI}. An algebra of V is called *finitely presented* if it is isomorphic to a quotient by a finitely generated congruence of a free algebra in V of finite rank (cf. [**17**]). An algebra is called residually finite if it is the subdirect product of finite algebras.

All nonexplained notations will be standard. The reader is referred to [**5**, **9** and **17**] for notions of model-theory and universal algebra and to [**1** and **21**] for notions of algebra.

2. Universally-finite varieties.

LEMMA 1. *Let V be a variety and let σ be an existential sentence in the language of V which is consistent with V. Then, σ holds in some finitely presented member of V.*

Moreover, if σ holds in A, where A is a subdirect product of some family $\mathcal{F}$, then σ holds in a direct product of a finite subfamily of $\mathcal{F}$.

PROOF. Without loss of generality we may assume that σ is of the form $\exists x_1 \cdots \exists x_n \psi$, where $\psi(x_1, \dots, x_n)$ is a finite conjunction of some atomic formulas, say $t_1 = s_1, \dots, t_h = s_h$, and some negations of atomic formulas, say $\psi_1, \psi_2, \dots, \psi_k$. Assume, now, that ψ is satisfied by $a_1, \dots, a_n$ in some member A of V.

Let $F(y_1, \dots, y_n)$ be the free algebra of rank n in V and $f\colon F(y_1 \cdots y_n) \to A$ be the homomorphism determined by $f(y_i) = a_i$, $i = 1, \dots, n$. Consider the congruence θ of $F(y_1, \dots, y_n)$ generated by h pairs $(u_1, v_1), \dots, (u_h, v_h)$, where $f(u_j) = t_j^A(\vec{a})$, $f(v_j) = s_j^A(\vec{a})$, $j = 1, \dots, h$. Since $A \models \psi(\vec{a})$ it follows that $\theta \subseteq \ker f$. Then, there exists a homomorphism $g\colon F(y_1 \cdots y_n)/\theta \to A$ such that $g(y_i/\theta) = a_i$, $i = 1, \dots, n$. Hence, $\psi(x_1, \dots, x_n)$ is true in the finitely presented algebra $F(y_1, \dots, y_n)/\theta$ by interpreting the variables x_i with y_i/θ.

When A is the subdirect product of a family $\{A_i : i \in I\}$ there are $i_1, \dots, i_k \in I$ such that A_{i_s} satisfies $\psi_s(a_1(i_s), \dots, a_n(i_s))$, $s = 1, \dots, k$. Then, σ holds in $A_{i_1} \times \cdots \times A_{i_k}$.

The following is the main theorem of this paper.

THEOREM 1. *For every variety V the following are equivalent.*

(i) *Every quasi-identity true in all finite members of V is true in all members of V.*

(ii) $V = SPP_U(V_{\text{fin}} \cap V_{SI})$.

(iii) $V_{SI} \subseteq SP_U(V_{\text{fin}} \cap V_{SI})$.

(iv) *V is universally-finite.*

(v) *Every finitely presented member of V is residually finite.*

(vi) *V_{SI} is universally-finite.*

(vii) *Every quasi-identity true in all finite members of V_{SI} is true in all members of V_{SI}.*

PROOF. (i)→(ii). By well-known facts (cf. [**4** and **16**, Chapter 31](i) implies that $V = SPP_U(V_{\text{fin}})$. Hence $V = SPP_U SP(V_{\text{fin}} \cap V_{SI}) = SPP_U(V_{\text{fin}} \cap V_{SI})$.

(ii)→(iii), (iii)→(vi). Simple.

(iv)→(i), (vi)→(vii). Trivial.

(vii)→(i). Since $V = SP(V_{SI})$.

(v)→(iv). From Lemma 1.

(i)→(v). Let A be a finitely presented algebra in V. Then, A is isomorphic to $F(x_1, \dots, x_n)/\theta$, where $F(x_1, \dots, x_n)$ is the free algebra on some system of generators $x_1, \dots, x_n$ and θ is a finitely generated congruence.

Let $R(v_1, \dots, v_n)$ be the conjunction of all the defining relations of A, i.e. the conjunction of all equalities $t(v_1, \dots, v_n) = s(v_1, \dots, v_n)$, where every pair $(t(x_1, \dots, x_n),\ s(x_1, \dots, x_n))$ belongs to a fixed finite system of generators for θ. Take elements $a, b \in A$ with $a \neq b$ and consider the sentence

$$\Phi : \exists v_1 \cdots \exists v_n (R(v_1, \dots, v_n) \wedge p(v_1, \dots, v_n) \neq q(v_1, \dots, v_n)),$$

where $p(x_1, \dots, x_n)/\theta = a$, $q(x_1, \dots, x_n)/\theta = b$. But, sentence Φ is the negation of a quasi-identity which is true in A. Therefore, from (i) Φ must be true in a finite member B of V. Hence, there exists a homomorphism $f_{ab} : A \to B$ with $f_{ab}(a) \neq f_{ab}(b)$. This proves (v).

COROLLARY 1. *Let V be a variety such that every finitely generated subdirectly irreducible member of V is finite, then every subvariety of V is universally-finite.*

We make some remarks on the results obtained and we discuss the most natural examples.

REMARKS 1.1. Let A be a subdirectly irreducible commutative unitary ring generated by n elements. Then, A is the quotient of the polynomial ring $\mathbf{Z}[x_1, \dots, x_n]$ by a completely irreducible, hence primary ideal I. We take a maximal ideal M which contains I and hence the prime ideal belonging to I. Then, from Theorem 12′ of [**21**, p. 217], we can show that there exists some positive integer k such that $M^k \subseteq I$. Therefore, A must be finite since $\mathbf{Z}[x_1, \dots, x_n]/M^k$ is finite.

Then, from Corollary 1, any variety of commutative unitary rings is universally-finite.

1.2. If an element in a unitary ring has more than one left inverse, then it has infinitely many (I. Kaplansky). Therefore, the universal formula

$$\forall x_1 \forall x_2 \forall x (x_1 x = 1 \wedge x_2 x = 1 \rightarrow x_1 = x_2)$$

is true in all finite unitary rings. But, such a formula is false in some rings with unity.

1.3. Locally finite or residually countable (cf. [**20**]) varieties are universally-finite by Corollary 1.

1.4. Let V be the variety of von Neumann commutative regular rings with quasi-inverse as operation. V_{SI} is the class of all fields (cf. [**18**]) which is (cf. [**19**] a universally-finite class. Therefore, from Theorem 1, V is universally-finite.

1.5. An algebra A is called Hopfian iff every onto endomorphism of A is an automorphism. From Theorem 1, every finitely presented algebra of a universally-finite variety is Hopfian (see [**17**, Lemma 6, p. 287]).

1.6. Every universally-finite variety is determined by its finite members. However, the converse is false.

Since every free group is residually finite (see [**13**]) the variety $\mathcal{G}$ of groups is determined by the class of finite groups. But, $\mathcal{G}$ is not universally-finite since a Higman's example (see [**11**]) provides a finitely presented group which is non-Hopfian.

From another Higman's example [**12**] it is possible to show that the universal-Horn sentence

$$\forall x \forall y \forall z \forall v ((y^{-1} x y = x^2 \wedge z^{-1} y z = y^2 \wedge v^{-1} z v = z^2 \wedge x^{-1} v x = v^2) \rightarrow x = 1)$$

is true in all finite groups but it is false in some groups (see also [**6**, Lemma 6.1, p. 177]).

1.7. Since every finitely presented lattice is residually finite (see [**8**, p. 298]) the variety $\mathcal{L}$ of lattices is universally-finite. However there are some subvarieties of $\mathcal{L}$ (cf. [**2**]) which are not even determined by their finite members. This shows that a subvariety of a universally-finite variety is not necessarily universally-finite.

3. Reduction to equational theories. Let T be a theory in any language. In this section using simple observations we show that Baldwin's problem for T is reducible to the same problem for an appropriate equational theory associated with T.

We state first two easy propositions without proof. For the proof of the first proposition recall that a structure is finite if and only if it is isomorphic to each of its ultrapowers.

PROPOSITION 1. *Let K and K' be classes of structures in languages L and L', respectively. Suppose that K and K' are closed under isomorphisms and ultraproducts. Consider K and K' as categories with morphisms all monomorphisms. If there exist functors $F: K \rightarrow K'$ and $G: K' \rightarrow K$ which preserve ultraproducts and define an equivalence of categories, then, K is universally-finite if and only if K' is universally-finite.*

PROPOSITION 2. *Let T be a theory in a language L. Then there exist a language without relation symbols L' and a theory T' in L' such that the category K of models* *of T and the category K' of models of T' are equivalent by functors which preserve ultraproducts.*

For the remainder of this section we will assume that L is a language without relation symbols and d is a new symbol of a ternary operation; we denote the expanded language by L^d. If T is a theory in L, we define the theory T^d in L^d by the axioms of T plus

$$\forall x \forall y \forall z (d(x,x,z) = z \wedge (x = y \vee d(x,y,z) = x)).$$

The models of T^d are the models of T with addition of a ternary discriminator. (For a comprehensive treatment of discriminator varieties see [**4**], Chapter IV, §9.) Finally, we denote the equational part of the theory T^d by $\mathrm{Eq}(T^d)$.

We need also the following

DEFINITION. We say that a theory T has the property of Embedding Reducts of Finite Submodels (ERFS) if every reduct A to a finite type τ of a *finite submodel* of T can be embedded in a reduct to τ of a *finite model* of T.

REMARK 3.1. Note that every universal theory has the property ERFS, but not the converse. The theory of fields presented in the language $\{+, \cdot, 0, 1\}$ has the ERFS, but it is not universal.

PROPOSITION 3. *Let T be a first order theory in a language L without relation symbols.* $\mathrm{Eq}(T^d)$ *denotes, as above, the set of equations in the language L^d which are deducible by the theory T plus axioms which guarantee that the new ternary operation symbol d becomes a discriminator on models of T. Then, the following are equivalent.*

(i) *T is universally-finite.*

(ii) *T has the property ERFS and the equational theory* $\mathrm{Eq}(T^d)$ *is universally-finite.*

PROOF. K and K^d denote the classes of models of T and T^d, respectively. If V is the variety of models of $\mathrm{Eq}(T^d)$, then our proposition follows by Theorem 1 and the following claims.

Claim 1. T universally-finite implies that T has the property ERFS.

Claim 2. K is universally-finite iff K^d is universally-finite.

Claim 3. K^d universally-finite implies that SK^d is universally-finite.

Claim 4. SK^d universally-finite and the property ERFS for T imply that K^d is universally-finite.

Claim 5. SK^d is universally-finite iff the class V_{SI} is universally-finite.

PROOF OF CLAIM 1. Let τ be a finite subtype of the type for the language L. Suppose that A is a reduct to τ of a finite submodel of T. Replace the constants in the conjunction of the diagram of A by variables. We get a formula whose existential quantification is a sentence σ that is true in a structure B for the language L if and only if A can be embedded in the reduct of B to τ. But, when T is universally-finite σ must be true in a finite model of T. The proof of Claim 2 is immediate by Proposition 1; proofs of Claims 3 and 4 are simple.

PROOF OF CLAIM 5. The discriminator variety V is generated by the class K^d. Then, by well-known facts (cf. [**4**, Theorem 9.4, p. 165]) which follow from Jonsson's celebrated lemma (see [**14**]) the class V_{SI} consists of the class SK^d plus the one element algebra. To finish our proof observe that if the sentence

$\forall z_1 \cdots \forall z_n \psi(z_1, \ldots, z_n)$, where ψ is an open formula, is true in all finite members of SK^d, then the universal sentence $\forall x \forall y \forall \vec{z}(x = y \vee \psi(\vec{z}))$ is true in all finite members of V_{SI}.

4. Universally-finite varieties of R-modules. In this section we derive other consequences of the main theorem about the variety R-Mod of left R-modules, where R is a unitary ring.

COROLLARY 2. *Let R be a left Noetherian ring. Then, the following are equivalent.*

(i) R-Mod *is universally-finite.*

(ii) *Every cyclic subdirectly irreducible left R-module is finite.*

(iii) R-Mod *has a locally finite cogenerator.*

PROOF. (i)→(ii). Since R is left Noetherian, every cyclic left R-module is finitely presented. Therefore, from (i) and Theorem 1 we get (ii).

(ii)→(iii). If $\{T_i : i \in I\}$ is a system of representatives of all simple left R-modules, then $C = \bigoplus_{i \in I} E(T_i)$ is a cogenerator of R-Mod (see [**1**, p. 211]). Here, $E(T)$ is the injective hull of T. Since every finitely generated submodule of C is a finite sum of a finite number of cyclic subdirectly irreducible submodules of C, (ii) implies that C is locally finite.

(iii)→(i). If C is a cogenerator for R-Mod, then $R\text{-Mod} = SP\{C\}$. Therefore, from (iii) R-Mod satisfies (i) of Theorem 1. Hence, we have (i).

COROLLARY 3. *Let R be a left Artinian ring, then the following are equivalent.*

(i) R-Mod *is universally-finite.*

(ii) ${}_RR$ *as left R-module is residually finite.*

(iii) *R is a finite ring.*

PROOF. (i)→(ii). From Theorem 1.

(ii)→(iii). Since R is left Artinian, ${}_RR$ is finitely cogenerated (see [**1**, Proposition 10.18]). Hence, (ii)→(iii).

(iii)→(i). Obvious, since (iii) implies that R-Mod is locally finite.

COROLLARY 4. *Let R be a subring of the ring S such that R is a pure R-submodule of S. Then, S-*Mod *universally-finite implies that also R-*Mod *is so.*

PROOF. Let A be a finitely presented R-module. Then, there exists an exact sequence $0 \to K \to R^n \to 0$ with K finitely generated. Let H be the S-submodule of S^n generated by the generators of K. Since R is a pure R-submodule of S, there exists an R-embedding of A into $B = S^n/H$. Therefore, A is residually finite in R-Mod because B is residually finite in S-Mod.

COROLLARY 5. *For every von Neumann commutative regular ring R the following are equivalent.*

(i) R-Mod *is universally-finite.*

(ii) *R is a subdirect product of finite fields.*

PROOF. (i)→(ii). From (i) R is residually finite. Since every ideal in R is the intersection of maximal ideals, it follows (ii).

(ii)→(i). Since R is von Neumann regular, every finitely presented R-module A is embeddable in a free module (cf. [**7**, p. 10]). Therefore, from (ii) A is residually finite.

REMARKS. 4.1. Examples of rings R where R-Mod is universally-finite are the following. By Corollary 2: every finite ring; every finitely generated commutative ring (see Remark 1.1); the ring R of polynomial or formal power series in n indeterminates over a finite field (see [**21**, Theorem 12′, p. 217]); every ring R which is Noetherian and algebraically compact (see [**10**, Lemma 5.6, p. 70]). By Corollary 5: every Boolean ring.

4.2. For every ring R condition (ii) of Corollary 2 is sufficient for R-Mod to be universally-finite; however, it is not always necessary.

Let R be a cartesian product of infinitely many pairwise nonisomorphic finite fields. By Corollary 5 R-Mod is universally-finite. But, R has as quotient an infinite field F that is a simple R-module; hence, F is an infinite cyclic subdirectly irreducible R-module.

4.3. In Corollary 4 we cannot assert that R-Mod is universally-finite if and only if S-Mod is so. Take as a counterexample a finite field R and an infinite field S which extends R.

4.4. Recall that every ring R is Morita equivalent to the $n \times n$ matrices ring $M_n(R)$ for every n (see [**1**, Corollary 22.6, p. 265]), i.e. the categories R-Mod and $M_n(R)$-Mod are equivalent. Moreover, the category Mod-R of right R-modules is equivalent to the category R-Mod of left R-modules. Observe that functors which define an equivalence between varieties considered as categories must preserve ultraproducts. Then, from Proposition 1, Mod-R and $M_n(R)$-Mod for every natural number n are universally-finite when R-Mod is universally-finite.

PROBLEMS. 1. Let V be a congruence distributive variety. Is V universally-finite iff V is determined by its finite members?

2. Describe the class K of unitary rings R such that R-Mod is universally-finite. In particular, prove or disprove:

(a) K is closed under subrings.

(b) K is closed under cartesian products.

(c) $R \in K$, R commutative does imply $R[x_1, \dots, x_n]$, $R[[x_1, \dots, x_n]] \in K$.

(d) $R \in K$ iff R is residually finite.

Note that a positive answer to (d) would imply a positive answer to (a), (b), (c).

REFERENCES

1. F. W. Anderson and K. R. Fuller, *Rings and categories of modules,* Graduate Texts in Math., no. 13, Springer-Verlag, Berlin and New York, 1973.
2. K. A. Baker, *Equational classes of modular lattices*, Pacific J. Math. **28** (1969), 9–15.
3. J. T. Baldwin, *Review: Selected papers of Abraham Robinson. Volume* 1, J. Symbolic Logic **47** (1982).
4. S. Burris and H. P. Sankappanavar, *A course in universal algebra,* Graduate Texts in Math., no. 78, Springer-Verlag, Berlin and New York, 1981.
5. C. C. Chang and H. J. Keisler, *Model theory,* Studies in Logic, North-Holland, Amsterdam, 1973.
6. U. Felgner, *The model theory of FC-groups,* Mathematical Logic in Latin America (Arruda, Chuaqui and da Costa, Eds.), North-Holland, Amsterdam, 1980.
7. K. R. Goodearl, *Ring theory,* Pure and Appl. Math. 33, Marcel Dekker, New York, 1976.
8. G. Grätzer, *General lattice theory,* Mathematische Reihe 52, Birkhäuser-Verlag, Basel, 1978.
9. ——, *Universal algebra,* 2nd ed., Springer-Verlag, Berlin and New York, 1979.
10. D. K. Haley, *Equational compactness in rings,* Lecture Notes in Math., vol. 745, Springer-Verlag, Berlin and New York, 1979.

11. G. Higman, *A finitely related group with an isomorphic proper factor group*, J. London Math. Soc. **26** (1951), 59–61.
12. ——, *A finitely generated infinite simple group*, J. London Math. Soc. **26** (1951), 61–64.
13. Iwasawa Kenkiti, *Einige Sätze über freie Gruppen*, Proc. Imp. Acad. Tokyo **19** (1943), 272–274.
14. B. Jónsson, *Algebras whose congruence lattice are distributive*, Math. Scand. **21** (1967), 110–121.
15. D. W. Kueker, *A note on the elementary theory of finite abelian groups*, Algebra Universalis **3** (1973), 156–159.
16. A. I. Mal′cev, *The metamathematics of algebraic systems*, Collected Papers 1936–1967, North-Holland, Amsterdam, 1971.
17. ——, *Algebraic systems*, Die Grundlehren der Math. Wissenshaften, vol. 192, Springer-Verlag, Berlin and New York, 1973.
18. R. S. Pierce, *Modules over commutative regular rings*, Mem. Amer. Math. Soc. No. 70 (1967).
19. A. Robinson, *On axiomatic systems which possess finite models*, Methods **3** (1951), 140–149.
20. W. Taylor, *Residually small varieties*, Algebra Universalis **2** (1972), 33–53.
21. O. Zariski and P. Samuel, *Commutative algebra*, Vol. 1, Graduate Texts in Math., No. 28, Springer-Verlag, Berlin and New York, 1977.
22. S. Burris and H. Werner, *Sheaf constructions and their elementary properties*, Trans. Amer. Math. Soc. **248** (1979), 269–309.
23. R. McKenzie, *On spectra, and the negative solution of the decision problem for identities having a finite nontrivial model*, J. Symbolic Logic **40** (1975).

DEPARTMENT OF MATHEMATICS AND PHYSICS, UNIVERSITY OF CAMERINO, 62032 CAMERINO, ITALY