TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 284, Number 1, July 1984

UNICELLULAR OPERATORS

BY

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ABSTRACT. An operator is unicellular if its lattice of invariant subspaces is totally ordered by inclusion. The list of nests which are known to be the set of invariant subspaces of a unicellular operator is surprisingly short. We construct unicellular operators on l^p, $1 \leqslant p < \infty$, and on c_0 with lattices isomorphic to $\alpha + X + \beta^*$ where α and β are countable (finite or zero) ordinals, and X is in this short list. Certain other nests are attained as well.

An operator T is *unicellular* if its lattice of invariant subspaces is totally ordered by inclusion (i.e. $\operatorname{Lat} T$ is a nest). The purpose of this paper is to construct unicellular operators with prescribed lattices. These lattices include nests of order type $\alpha + 1 + \beta^*$ for any countable ordinals α and β.

Halmos (see [**13**]) first asked for a description of those lattices which may be the invariant subspace lattice of a single operator acting on a separable Hilbert space. Which nests are *attainable* is an important special case, and it has been investigated by many authors. The best known unicellular operator is the Volterra integral operator on $L^p(0,1)$, which has a lattice order isomorphic to the unit interval [**4**]. Donoghue [**6**] showed that certain weighted unilateral shifts have only the obvious invariant spaces, a nest of order type $1 + \omega^*$. The weighted shifts have been extensively studied, especially by Nikol'skii [**10–12**], as well as [**7, 15**].

Construction of unicellular operators for more complicated nests has resisted attack, and only a few have been found. Notably, a unicellular bilateral weighted shift (with lattice order isomorphic to $1 + \mathbf{Z} + 1$) has only recently been constructed by Domar [**5**]. More general unicellular Volterra operators have been found in [**2, 14**]. Harrison and Longstaff [**8**] constructed a unicellular operator with lattice of order type $\omega + \omega + 1$ by putting two weighted shifts together. Their computation was tricky, and became very complex in a generalization by the first author [**1**] to the case $\omega m + 1 + (\omega n)^*$. In this paper, we simplify and generalize the method of [**8**]. This will form the major building block in our constructions.

Let B be an operator such that $\|B^n\| = O((n!)^{-d})$ for some $d > 0$ which has a cyclic vector and a cocyclic vector. We show (Corollary 4.6) that $\alpha + \operatorname{Lat} B + \beta^*$ is attainable for any countable ordinals α and β (they may be finite or zero, also). Since unicellular operators always have cyclic vectors [**13**, Theorem 4.4], these

Received by the editors April 20, 1983.
1980 *Mathematics Subject Classification*. Primary 47A15, 47B37.
Key words and phrases. Unicellular operator, invariant subspaces, attainable lattices.
[1]Research partially supported by an NSERC grant.

hypotheses are readily seen to include most of the known examples. In particular, $\alpha + 1 + \beta^*$ is attainable. It also includes the unicellular operators of order type $(1 + \mathbf{Z} + 1)n$ and $(1 + \mathbf{Z} + 1)\omega + 1$ which are constructed in §5.

There is no (maximal) nest on separable Hilbert space which is known to be unattainable. It seems plausible that all are attainable. We conjecture that, at least, all countable nests are attainable. This problem may be somewhat simplified by the results of [**3, 9**] which imply that a maximal nest is determined up to similarity by its order type. This reduces considerably the collection of nests that need be considered.

Although our main interest is in operators on Hilbert space, our methods are also valid in l^p or c_0, so we work in this more general setting.

1. Preliminaries. The constructions will take place using Banach spaces of a fixed type, l^p for $1 \leqslant p < \infty$ or c_0. These spaces will be denoted by letters $\mathscr{H}$, $\mathscr{K}$, $\mathscr{X}$, The standard basis will be denoted by $\{e_n, n \geqslant 0\}$ or $\{e_n, n \in \mathbf{Z}\}$. We will use the notation e_n^* for the corresponding functionals in $\mathscr{H}^*$. Given copies $\mathscr{H}_\beta$ of the space for all β less than a countable ordinal α, $\sum_{\beta<\alpha} \oplus \mathscr{H}_\beta$ will denote the l^p or c_0 direct sum, which is also isomorphic to l^p or c_0 itself.

Also note that there is a contractive projection P_β onto the β-summand. The dual space is the l^q or l^1 direct sum of $\sum_{\beta<\alpha} \oplus \mathscr{H}_\beta^*$. When $p = 2$, we make the usual antilinear identification of the dual with the original Hilbert space.

By the weighted shift on $\mathscr{H}$ with weights a_n, we mean the operator $Ae_n = a_n e_{n+1}$, $n \geqslant 0$. There is no loss of generality in assuming that $a_n \geqslant 0$ [**15**], so we shall always do so. If a_n are monotone decreasing (nonincreasing) and belong to l^p for some $p < \infty$, then A is unicellular [**10**]. In this case, $\|A^k\| = \prod_{n=0}^{k-1} a_n = \|A^k e_0\|$. Also

$$\|A^{*k} e_m\| = \prod_{n=m-k}^{m-1} a_n = \|A^m\| / \|A^{m-k}\|, \tag{0}$$

and for fixed k, this is decreasing as m increases.

The ordinal sum $\mathscr{L} + \mathscr{M}$ of two lattices of (closed) subspaces $\mathscr{L}$ and $\mathscr{M}$ consists of all elements $L \oplus 0$, $L \in \mathscr{L}$ and $\mathscr{H} \oplus M$, $M \in \mathscr{M}$ with the obvious lattice operations (cf. [**13**, p. 76]). Note that the unit of $\mathscr{L}$ is identified with the zero of $\mathscr{M}$. All lattices will be complete with respect to intersection (= inf) and closed linear span (= join), and contain a zero $\{0\}$ and a unit $\mathscr{H}$. If α is an ordinal, α^* denotes the dual lattice of α.

Suppose A and B are unicellular operators. It would be desirable to have a general procedure to produce an operator T with $\operatorname{Lat} T \cong \operatorname{Lat} A + \operatorname{Lat} B$. One might attempt to build T as follows. Pick a cyclic vector x for A and a cyclic vector y^* for B^*. Let $C = x \otimes y^*$ be the rank one operator $Cz = \langle z, y^* \rangle x$, and define $T = \left[\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right]$. It is easy to see that $\operatorname{Lat} T$ contains $\operatorname{Lat} A + \operatorname{Lat} B$. However, in general this is a proper subset because Domar [**5**] has shown that many bilateral weighted shifts have nonstandard invariant subspaces. Nonetheless, control of the growth rate of $\|A^n\|$ and $\|B^n\|$ often enables one to prove that T is unicellular [**8, 1, 5**]. The key step in our construction is to do this whenever B is a weighted shift and $\|A^n\|$ dies off rapidly relative to $\|B^n\|$. Transfinite induction and some technicalities allow us to soup this up to B's with $\operatorname{Lat} B \cong (\alpha + 1)^*$ for any countable ordinal α.

2. The key construction. The idea in [8] is to show that for two carefully chosen weighted shifts A and B, there are monomials in the operator T constructed above which converge to an operator D of the form $\left[\begin{smallmatrix}0 & *\\ 0 & 0\end{smallmatrix}\right]$. We are able to do this in much greater generality, which will allow us to use induction to build up our operators.

THEOREM 2.1. *Suppose S has a cyclic vector g and $\|S^n\| = O((n!)^{-s})$ for some $s > 0$. Let A be any unilateral weighted shift with weights decreasing monotonely to zero and belonging to l^p for some $p > 0$ such that*

$$\|S^n\| = O\big(\|A^{n+k}\|\big) \quad \textit{for all } k \geqslant 0. \tag{1}$$

Then there is an operator T of the form $\left[\begin{smallmatrix}S & C\\ 0 & A\end{smallmatrix}\right]$ such that $\operatorname{Lat} T = \operatorname{Lat} S + (\omega + 1)^*$. *$T$ also has a cyclic vector.*

REMARK 2.2. Condition (1) can always be achieved. If $0 < r < s$, the shift with weights $(n + 1)^{-r}$ will do. Condition (1) implies the following stronger condition:

$$\|S^n\| = o\big(n^{-1}\|A^{n+k}\|\big) \quad \text{for all } k \geqslant 0. \tag{1$'$}$$

To see this, first notice that because a_n^p is *monotone decreasing* and summable, $\lim_{n\to\infty} na_n^p = 0$. Hence,

$$\|A^{n+k+p}\| \leqslant a_n^p\|A^{n+k}\| = o\big(n^{-1}\|A^{n+k}\|\big).$$

Consequently,

$$\|S^n\| = O\big(\|A^{n+k+p}\|\big) = o\big(n^{-1}\|A^{n+k}\|\big).$$

Furthermore, $\|S^n\|/\|A^{n+k}\| = O(a_n^p)$, hence

$$\sum_{n=1}^{\infty} \|S^n\|/\|A^{n+k}\| < \infty \quad \text{for all } k \geqslant 0. \tag{1$''$}$$

PROOF OF THEOREM 2.1. For convenience, multiply S and A by a scalar so that they both have norm at most one. First we construct C. Choose a sequence c_n of positive real numbers in l^1 such that

$$\lim_{n\to\infty} c_n^{-1}\Big(\sum_{m>n} c_m\Big) = 0. \tag{2}$$

Next, we choose an increasing sequence μ_n of positive integers such that

$$\lim_{n\to\infty} c_n^{-1}a_{\mu_n} = 0 \tag{3}$$

and

$$\lim_{n\to\infty} c_n^{-1}\mu_n\frac{\|S^{\mu_n-\mu_{n-1}}\|}{\|A^{\mu_n}\|} = 0. \tag{4}$$

This choice is made inductively. (3) can be achieved by taking μ_n sufficiently large. If μ_{n-1} is defined, then (1$'$) implies that for t big enough, $\|S^{t-\mu_{n-1}}\| < (c_n/n)\|A^t\|/t$. Thus (4) can be achieved by taking μ_n large enough as well. The operator C is defined as

$$C = g \otimes \sum_{m=1}^{\infty} c_m e_{\mu_m}^*.$$

The following lemma is the main ingredient in the proof of Theorem 2.1.

LEMMA 2.3. *Let $T = \left(\begin{smallmatrix} S & C \\ 0 & A \end{smallmatrix}\right)$ where S and A are contractions satisfying the hypotheses of Theorem 2.1, and C is defined as above. Let $\lambda_n = (c_n \| A^{\mu_n} \|)^{-1}$, and let*

$$D = \sum_{n=0}^{\infty} \frac{S^j g}{\|A^j\|} \otimes e_j^*.$$

Then

$$\lim_{n \to \infty} \lambda_n T^{\mu_n + 1} = \begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix}.$$

PROOF. Write

$$\lambda_n T^{\mu_n + 1} = \begin{bmatrix} \lambda_n S^{\mu_n + 1} & X_n \\ 0 & \lambda_n A^{\mu_n + 1} \end{bmatrix}.$$

Then

$$\|\lambda_n A^{\mu_n + 1}\| = \|A^{\mu_n + 1}\| / c_n \|A^{\mu_n}\| = a_{\mu_n + 1} / c_n$$

which tends to zero by (3). By (1), $\lim_{n \to \infty} \|\lambda_n S^{\mu_n + 1}\| = 0$ also. Now

$$X_n = \lambda_n \sum_{j=0}^{\mu_n} S^j C A^{\mu_n - j}.$$

Set $C_m = g \otimes c_m e_{\mu_m}^*$, and $X_{n,m} = \lambda_n \sum_{j=0}^{\mu_n} S^j C_m A^{\mu_n - j}$. Clearly, $C = \sum_{m=1}^{\infty} C_m$ and $X_n = \sum_{m=1}^{\infty} X_{n,m}$. Also,

$$S^j C_m A^{\mu_n - j} = c_m S^j g \otimes A^{*\mu_n - j} e_{\mu_m}^*.$$

If $m < n$, $A^{*j} e_{\mu_m}^* = 0$ for $j > \mu_m$. Thus

$$\begin{aligned} \|X_{n,m}\| &= \lambda_n c_m \left\| \sum_{j=0}^{\mu_m} S^{\mu_n - j} g \otimes A^{*j} e_{\mu_m}^* \right\| \\ &\leqslant \lambda_n c_m (\mu_m + 1) \|S^{\mu_n - \mu_m}\| \, \|g\| \\ &\leqslant c_m \|g\| c_n^{-1} \mu_n \frac{\|S^{\mu_n - \mu_{n-1}}\|}{\|A^{\mu_n}\|}. \end{aligned}$$

Hence by (4),

$$\lim_{n \to \infty} \sum_{m=1}^{n-1} \|X_{n,m}\| \leqslant \|g\| \left(\sum_{m=1}^{\infty} c_m \right) \lim_{n \to \infty} c_n^{-1} \mu_n \frac{\|S^{\mu_n - \mu_{n-1}}\|}{\|A^{\mu_n}\|} = 0.$$

If $m > n$, we obtain

$$\begin{aligned} \|X_{n,m}\| &\leqslant \lambda_n c_m \sum_{j=0}^{\mu_n} \|S^j\| \, \|g\| \, \|A^{*\mu_n - j} e_{\mu_m}^*\| \\ &\leqslant \frac{c_m}{c_n} \|g\| \, \|A^{\mu_n}\|^{-1} \sum_{j=0}^{\mu_n} \|S^j\| \, \|A^{*\mu_n - j} e_{\mu_n}^*\| \\ &= \frac{c_m}{c_n} \|g\| \sum_{j=0}^{\mu_n} \frac{\|S^j\|}{\|A^j\|}. \end{aligned}$$

The last equality follows from (0). Hence by (2),

$$\lim_{n\to\infty} \sum_{m=n+1}^{\infty} \|X_{n,m}\| \leqslant \|g\| \left(\sum_{j=0}^{\infty} \frac{\|S^j\|}{\|A^j\|} \right) \lim_{n\to\infty} c_n^{-1} \left(\sum_{m=n+1}^{\infty} c_m \right) = 0.$$

Finally, using (0) again, $m = n$ gives

$$X_{n,n} = \lambda_n c_n \sum_{j=0}^{\mu_n} S^j g \otimes A^{*\mu_n - j} e_{\mu_n}^* = \sum_{j=0}^{\mu_n} \frac{S^j g}{\|A^j\|} \otimes e_j^*.$$

Since $\sum_{j=0}^{\infty} \|S^j g\|/\|A^j\|$ converges,

$$\lim_{n\to\infty} \lambda_n T^{\mu_n+1} = \lim_{n\to\infty} X_{n,n} = \sum_{j=0}^{\infty} \frac{S^j g}{\|A^j\|} \otimes e_j^*. \quad \blacksquare$$

PROOF OF THEOREM 2.1 (CONTINUED). Let $\mathscr{M}$ be an invariant subspace for T. If $\mathscr{M}$ is contained in $\mathscr{H}_1 \oplus 0$, it belongs to Lat S and hence to Lat $S + (\omega + 1)^*$. Otherwise, let $\mathscr{M}_0 = \mathscr{M} \cap (\mathscr{H}_1 \oplus 0)$, which is clearly also invariant for T. We need to show that $\mathscr{M}_0 = \mathscr{H}_1 \oplus 0$, for then $\mathscr{M} = \mathscr{H}_1 \oplus \mathscr{N}$ for some invariant subspace $\mathscr{N}$ of A. The hypotheses on A force $\mathscr{N}$ to be standard.

In any case, the closure $\mathscr{N}$ of the range of the contractive projection of $\mathscr{M}$ onto $\mathscr{H}_2$ is always invariant for A. Since it is nonzero, it equals span$\{e_n, n \geqslant n_0\}$ for some integer n_0. By Lemma 2.3, monomials in T converge in norm to an operator $D = \sum_{j=0}^{\infty} S^j g/\|A^j\| \otimes e_j^*$. Thus $\mathscr{M}$ is invariant for D, and so $\mathscr{M}$, and indeed $\mathscr{M}_0$, contains $S^j g$ for $j \geqslant n_0$. But g is cyclic for S, so $\mathscr{M}_0$ together with $S^j g$, $0 \leqslant j < n_0$, spans $\mathscr{H}_1$. Thus $\mathscr{M}_0$ has finite codimension in $\mathscr{H}_1$. As $\mathscr{N}$ has finite codimension in $\mathscr{H}_2$, it follows that $\mathscr{M}$ has finite codimension also.

Suppose $\mathscr{M}_0$ does not equal $\mathscr{H}_1 \oplus 0$. Consider the restriction B of the adjoint T^* to the invariant subspace $\mathscr{M}^\perp$. Since $\mathscr{M}^\perp$ is finite dimensional and T is quasinilpotent, B must be nilpotent. Let q be an integer such that $B^q = 0$. Then T^{*q} annihilates $\mathscr{M}^\perp$, and thus the range of T^q is contained in $\mathscr{M}$. Choose any $\mu_m \geqslant n_0$ such that $\mu_m > \mu_{m-1} + q$. Then $\mathscr{M}$ contains $T^q(0 \oplus e_{\mu_m - q + 1}) = ag \oplus be_{\mu_m + 1}$ where a and b are both nonzero. Thus $\mathscr{M}$ contains the vector g, which is cyclic for S, and hence all of $\mathscr{H}_1 \oplus 0$. $\blacksquare$

COROLLARY 2.4. *If S has a cyclic vector and $\|S^n\| = O((n!)^{-s})$ for $s > 0$, then* Lat $S + (\omega + 1)^*$ *is attainable.* $\blacksquare$

COROLLARY 2.5 [1]. *For any finite nonnegative integers m and n, $\omega m + 1 + (\omega n)^*$ is attainable.*

PROOF. $(\omega m + 1)^*$ is obtained by induction as the lattice of an operator B. Since B^* is unicellular (with lattice $\omega m + 1$), it has a cyclic vector. So by induction n times more, $\omega m + 1 + (\omega n)^*$ is obtained. One need only verify the norm estimate at each stage. These will be verified in the next section. $\blacksquare$

REMARK 2.6. Given S as in Theorem 2.1, the operators A and C can be chosen to have arbitrarily small norm. Lemma 2.3 holds for any vector g, even if it is not cyclic.

3. Norm estimates. We need a method for controlling the growth of $\|T^n\|$ for the operators we construct. For our purposes, the tidiest and sharpest results are obtained by using specific growth constants. However, in other contexts a more general result might be required. Such a result is stated without proof at the end of the section.

Let

$$M(k, \varepsilon) = \sup k^n (n!)^{-\varepsilon}.$$

This is finite, and achieves its maximum at $n = [k^{1/\varepsilon}]$. Since k^n is the sum of all the multinomial coefficients $\binom{n}{m_1 m_2 \cdots m_k}$ it follows that

$$\binom{n}{m_1 m_2 \cdots m_k} \leqslant M(k, \varepsilon)(n!)^{\varepsilon}.$$

Thus if m_i are nonnegative integers with $\sum_{i=1}^{k} m_i = n$, we obtain

$$\left(\prod_{i=1}^{k} m_i!\right)^{-1} \leqslant M(k, \varepsilon)(n!)^{\varepsilon - 1}.$$

LEMMA 3.1. *Let p_k be an increasing sequence of positive real numbers, and ε_k a sequence monotonely decreasing to zero. Suppose that $\|A_k^n\| \leqslant (n!)^{-p_k}$ for $n, k \geqslant 1$, and $\|C_k\| \leqslant [k!M(k+1, \varepsilon_k)M(2, \varepsilon_k)]^{-p_k}$. Let*

$$T = \begin{bmatrix} A_1 & & & \bigcirc \\ C_1 & A_2 & & \\ & C_2 & A_3 & \\ \bigcirc & & \ddots & \ddots \end{bmatrix}$$

act on $\mathscr{H} = \sum_{n=1}^{\infty} \oplus \mathscr{H}_n$. Then $\|T^n\| = o((n!)^{-p_1+\varepsilon})$ for all $\varepsilon > 0$.

PROOF. The operator T^n has nonzero entries only on the subdiagonals for $0 \leqslant r \leqslant n$. Let $T_{(r)}^n$ denote the rth subdiagonal, which has entries of the form

$$X_{r,i} = \sum A_i^{m_0} C_i A_{i+1}^{m_1} C_{i+1} \cdots C_{i+r-1} A_{i+r}^{m_r},$$

where the sum runs over all nonnegative integer solutions of $\sum_{j=0}^{r} m_j = n - r$. The number of solutions of $\sum_{j=0}^{r} m_n = n - r$ is exactly $\binom{n}{r}$.

Now $\|T_{(r)}^n\| = \sup \|X_{r,i}\|$. Although there is no condition on the relative sizes of the various $\|C_k\|$, the *estimates* are monotone decreasing. So it suffices to estimate $\|X_{r,1}\|$ in order to bound $\|T_{(r)}^n\|$. In particular, note that $\|C_k\| \leqslant 1$ for all k since $M(k, \varepsilon) \geqslant 1$.

Let $0 < \varepsilon \leqslant 1$ be given, and set $\delta = \varepsilon/2p_1$. We compute

$$\begin{aligned}
\|T^n_{(r)}\| &\leqslant \binom{n}{r}\prod_{j=1}^{r}\|C_j\| \max \prod_{j=0}^{r}\|A^{m_j}_{j+1}\| \\
&\leqslant \binom{n}{r}\|C_r\| \max \prod_{j=0}^{r}(m_j!)^{-p_1} \\
&\leqslant \binom{n}{r}\left[r!M(r+1,\varepsilon_r)M(2,\varepsilon_r)\right]^{-p_1}M(r+1,\delta)^{p_1}((n-r)!)^{-p_1(1-\delta)} \\
&\leqslant \binom{n}{r}\left[\frac{M(r+1,\varepsilon)}{M(r+1,\delta_r)}\right]^{p_1}(r!(n-r)!M(2,\varepsilon_r))^{-p_1(1-\delta)} \\
&\leqslant \binom{n}{r}\left[\frac{M(r+1,\delta)}{M(r+1,\varepsilon_r)}\right]^{p_1}\left[\frac{M(2,\delta)}{M(2,\varepsilon_r)}\right]^{p_1(1-\delta)}(n!)^{-p_1(1-\delta)^2}.
\end{aligned}$$

Let

$$K = \sup_{r\geqslant 0}\left[\frac{M(r+1,\delta)}{M(r+1,\varepsilon_r)}\right]^{p_1}\left[\frac{M(2,\delta)}{M(2,\varepsilon_r)}\right]^{p_1(1-\delta)},$$

which is finite since the terms are less than one if $\varepsilon_r \leqslant \delta$. Hence

$$\begin{aligned}
\|T^n\| &\leqslant \sum_{r=0}^{n}\|T^n_{(r)}\| \leqslant K\sum_{r=0}^{n}\binom{n}{r}(n!)^{-p_1(1-\delta)^2} \\
&= K2^n(n!)^{-p_1(1-\delta)^2} \\
&= o\left((n!)^{-p_1(1-2\delta)}\right) = o\left((n!)^{-p_1+\varepsilon}\right). \quad \blacksquare
\end{aligned}$$

COROLLARY 3.2. *Suppose*

$$T = \begin{bmatrix} A_1 & C \\ 0 & A_2 \end{bmatrix} \quad \textit{and} \quad \|A_i^n\| = O\left((n!)^{-p}\right)$$

for $i = 1, 2$, $p > 0$. *Then* $\|T^n\| = o((n!)^{-p+\varepsilon})$ *for all* $\varepsilon > 0$. $\blacksquare$

COROLLARY 3.3. *Suppose*

$$T = \begin{bmatrix} A_1 & & & * \\ & A_2 & & \\ & & \ddots & \\ \bigcirc & & & A_n \end{bmatrix}$$

and $\|A_i^n\| = O((n!)^{-p})$ *for* $1 = 1, \ldots, n$, *and* $p > 0$. *Then* $\|T^n\| = o((n!)^{-p+\varepsilon})$ *for all* $\varepsilon > 0$. $\blacksquare$

REMARK 3.4. Suppose the numbers p_k are strictly increasing, and set

$$T_k = \begin{bmatrix} A_{k+1} & & \bigcirc \\ C_{k+1} & A_{k+2} & \\ & C_{k+2} & \ddots \\ \bigcirc & & \ddots \end{bmatrix}.$$

It follows from Lemma 3.1 that $\|T_k^n\| = o((n!)^{-p_{k+1}+\varepsilon})$ for all $\varepsilon > 0$.

It may sometimes be desirable to require $\|A_i^k\|$ to die off much more rapidly than $(n!)^{-s}$. We state the following result without proof. It will not be used in our constructions.

PROPOSITION 3.5. *Let $\varepsilon(k)$ be monotone decreasing to zero, and let A_0 be the weighted shift with weights $\varepsilon(k)$. Suppose that $\|A_k^n\| \leqslant \|A_0^{8^k n}\|$ and $\|C_k\| \leqslant \|A_0^{4k}\|$. Then*

$$T = \begin{bmatrix} A_1 & & \bigcirc \\ C_1 & A_2 & \\ & C_2 & \\ & & \ddots \\ \bigcirc & & \ddots \end{bmatrix}$$

satisfies $\|T^n\| = o(\|A_0^{2n}\|)$. ∎

4. Countable ordinals. We are ready to construct unicellular operators for arbitrary countable (successor) ordinals. It is an elementary fact that the restriction of an operator T to an invariant subspace $\mathscr{M}$ has $\operatorname{Lat} T|\mathscr{M} = \{\mathscr{L} \in \operatorname{Lat} T\colon \mathscr{L} \subseteq \mathscr{M}\}$. Thus it will suffice to construct operators T with $\operatorname{Lat} T \cong \omega\alpha + 1$ for countable ordinals α. In fact, we will construct T with $\operatorname{Lat} T \cong (\omega\alpha + 1)^*$. The dual T^* will have the desired lattice. This does not quite work for c_0, but the nature of the operators constructed is such that those T acting on l^1 are of the form S^* for an operator S on c_0. Hence, $\operatorname{Lat} S \cong \omega\alpha + 1$. The details will be left to the reader.

First, we describe the nature of the operators to be constructed. For convenience, we will call these operators *shifts of class α with weight interval* $[a, b]$. Let $[a, b]$ be an interval in $\mathbf{R}^+$. Let p be a function from α into $[a, b]$ such that

$$p_-(\beta) = \sup_{\gamma<\beta} p(\gamma) < p(\beta)$$

for all $\beta < \alpha$. Let p_β denote $p(\beta)$. Set

$$\varepsilon_\beta = \min\left\{\tfrac{1}{3}(p_{\beta+1} - p_\beta), 1\right\}$$

for $\beta < \alpha$. The operator T to be constructed will act on the space $\mathscr{H} = \sum_{\beta<\alpha} \oplus \mathscr{H}_\beta$. For $\beta < \alpha$, let A_β be the weighted unilateral shift on $\mathscr{H}_\beta$ with weights $\varepsilon_\beta(n+1)^{-p_\beta}$. Then

$$\|A_\beta^n\| \leqslant \varepsilon_\beta^n (n!)^{-p_\beta} \leqslant (n!)^{-p_\beta} \quad \text{for } n \geqslant 1,$$

and

$$\|A_\beta^n\| \geqslant (n!)^{-p_\beta - \varepsilon_\beta} \quad \text{for } n \text{ sufficiently large.}$$

For $\beta + 1 < \alpha$, we define rank one operators $C_\beta = \phi_\beta \otimes \psi_\beta$ from $\mathscr{H}_\beta$ to $\mathscr{X}_\beta = \sum_{\beta<\gamma<\alpha} \oplus \mathscr{H}_\gamma$, with $\|\phi_\beta\| < 1$, and $\psi_\beta = \sum_{n=1}^\infty c_n^\beta e_{\mu_n^\beta}^*$ satisfying the following conditions:

$$(2') \qquad \lim_{n\to\infty} \left(c_n^\beta\right)^{-1} \sum_{m>n} c_m^\beta = 0, \qquad \sum_{n=1}^\infty c_n^\beta < \varepsilon_\beta;$$

$$(3') \qquad \lim_{n\to\infty} \left(c_n^\beta\right)^{-1} \left(\mu_n^\beta !\right)^{-p_\beta} = 0;$$

$$(4') \qquad \lim_{n\to\infty} \left(c_n^\beta\right)^{-1} \mu_n^\beta \frac{\left[\left(\mu_n^\beta - \mu_{n-1}^\beta\right)!\right]^{-p_{\beta+1}+\varepsilon_\beta}}{\|A_\beta^{\mu_n^\beta}\|} = 0.$$

Define $T = \sum_{\beta<\alpha} A_\beta + \sum_{\beta+1<\alpha} C_\beta$ and $T_\beta = T|\mathscr{X}_\beta$. The vectors ϕ_β are chosen so that the following conditions are satisfied:

$$(5') \qquad \|T^n\| = O\left((n!)^{-p_1+\varepsilon}\right) \quad \text{for some } \varepsilon > 0, \quad \text{and}$$

$$\|T_\beta^n\| = O\left((n!)^{-p_{\beta+1}+\varepsilon}\right) \quad \text{for all } \beta + 1 < \alpha, \text{ and some } \varepsilon < \varepsilon_p;$$

$$(6') \qquad \lim_{j\to\infty} \|\phi_{\beta+j}\| = 0 \quad \text{for all } \beta < \alpha \text{ such that } \beta + \omega \leqslant \alpha$$

and

$$(7') \qquad \phi_\beta \quad \text{is cyclic for } T_\beta.$$

The operator T so defined is what we call a shift of class α with weight interval $[a, b]$.

REMARK 4.1. The main concern of this section is to prove the existence of the operator T described above. This is important because these operators will give all the nests which are order isomorphic to countable ordinals. The existence of T is a nontrivial problem because (5′)–(7′) are very restrictive conditions. The construction of ψ_β satisfying (2′)–(4′) can be done easily. Indeed, this follows from the proof of Theorem 2.1 because the condition (1) there is satisfied by $A = \varepsilon_\beta^{-1} A_\beta$ and the weighted shift S with weights $(n+1)^{-p_{\beta+1}+\varepsilon_\beta}$.

LEMMA 4.2. *Suppose B acts on a space $\mathscr{X}$, has a cyclic vector f, and $\|B^n\| = O((n!)^{-d})$ for some $d > 0$. Let T be a shift of class α with weight interval $[a, b]$ and $0 < a < b < d$.*

(i) *Given $\delta > 0$, there is a rank one operator C with $\|C\| < \delta$ so that $S = \left(\begin{smallmatrix} B & C \\ 0 & T \end{smallmatrix}\right)$ has* $\operatorname{Lat} S \cong \operatorname{Lat} B + \operatorname{Lat} T$.

(ii) *If B is a shift of class β with weight interval $[d_1, d_2]$ and $b < d_1$, then S is a shift of class $\alpha + \beta$ with weight interval $[a, d_2]$.*

(iii) *T is unicellular.*

PROOF. We assume that $\|f\| = 1$. First, suppose α is a successor ordinal $\alpha = \alpha_0 + n$ with α_0 a limit ordinal or 1. Define $C = C_{\alpha_0+n-1} = f \otimes \psi_{\alpha_0+n-1}$ where ψ_{α_0+n-1}

satisfies (2′)–(4′) with $p_{\alpha_0+n} = d$ and $\varepsilon_{\alpha_0+n-1} = \frac{1}{3}(d-b)$. Then n applications of Theorem 2.1 and Corollary 3.2 show that

$$B' = \begin{bmatrix} B & C_{\alpha_0+n-1} & & \bigcirc & \\ & A_{\alpha_0+n-1} & C_{\alpha_0+n-2} & & \\ & & A_{\alpha_0+n-2} & \ddots & \\ & & & & C_{\alpha_0} \\ & \bigcirc & & \ddots & \\ & & & & A_{\alpha_0} \end{bmatrix}$$

has $\operatorname{Lat} B' \cong \operatorname{Lat} B + (1 + \omega^* n)$. If B was a shift of class β for $[d_1, d_2]$ $(d_1 > d)$, then B' is a shift of class $n + \beta$ for $[p_{\alpha_0}, d_2]$. By Corollary 3.2,

$$\|B'^n\| = o\left((n!)^{-p_{\alpha_0}+\delta_0}\right)$$

with $\delta_0 = \frac{1}{2}(p_{\alpha_0} - p_-(\alpha_0))$. Clearly $T' = \sum_{\beta<\alpha_0}(A_\beta + C_\beta)$ is a shift of class α_0 with the weight interval $[a, p_-(\alpha_0)]$.

Thus, all the assertions of the lemma involving B can be proved under the assumption that α is a limit ordinal. In this case, for $\beta < \alpha$, define

$$c_\beta = \min\left\{(b-a)^{-1}\delta\|\phi_\beta\|^{1/2}, \tfrac{1}{2}(1 - \|\phi_\beta\|)\right\}.$$

Define $C = f \otimes \sum_{\beta<\alpha} c_\beta \psi_\beta$. Then

$$\|C\| \leqslant (b-a)^{-1}\delta \sum_{\beta<\alpha} \|\psi_\beta\| < \delta(b-a)^{-1} \sum_{\beta<\alpha} \varepsilon_\beta < \delta.$$

Let $S = \left[\begin{smallmatrix} B & C \\ 0 & T \end{smallmatrix}\right]$. Let $C'_\beta = (I - P_\beta)SP_\beta$. Then

$$C'_\beta = C_\beta + CP_\beta = (\phi_\beta + c_\beta f) \otimes \psi_\beta.$$

Note that C'_β is still a rank one operator with the same initial functional ψ_β, with $\phi'_\beta = \phi_\beta + c_\beta f$ as the range and

$$\|\phi'_\beta\| \leqslant \|\phi_\beta\| + \tfrac{1}{2}(1 - \|\phi_\beta\|) < 1,$$

and $\lim_{j\to\infty}\|\phi'_{\beta+j}\| = 0$.

Now we are ready to show that $\operatorname{Lat} S \cong \operatorname{Lat} B + \operatorname{Lat} T$. Fix $\mathscr{M}$ in $\operatorname{Lat} S$. If $\mathscr{M}$ is contained in $\mathscr{K} \oplus 0$, there is nothing to prove. Otherwise, let β be the least ordinal so that $\overline{P_\beta \mathscr{M}} = \mathscr{H}'_\beta$ is not $\{0\}$, where P_β is the contractive projection onto $\mathscr{H}_\beta$. Then $\mathscr{H}'_\beta$ is invariant for A_β, so it has finite codimension in $\mathscr{H}_\beta$. Let A'_β be the restriction of A_β to $\mathscr{H}'_\beta$, let $S_\beta = S|\mathscr{K} \oplus \mathscr{X}_\beta$ for $\beta < \alpha$, and let

$$R_\beta = \begin{bmatrix} S_\beta & C'_\beta \\ 0 & A'_\beta \end{bmatrix}.$$

It is clear that $\mathscr{M}$ is also invariant for R_β (here $\mathscr{M}$ is considered as a subspace of $\mathscr{K} \oplus \mathscr{X}_\beta \oplus \mathscr{H}'_\beta$).

By Corollary 3.2, $\|S_\beta^n\| = o((n!)^{-p_{\beta+1}+\varepsilon_\beta})$ and so $\|S_\beta^n\| = o(\|(A'_\beta)^{n+k}\|)$ for $k \geqslant 0$. Conditions (2′)–(4′) allow us to apply Lemma 2.3 to get monomials in R_β converging to

$$D'_\beta = \sum_{j=0}^{\infty} \frac{S_\beta^j \phi'_\beta}{\|A_\beta'^j\|} \otimes e_j'^*,$$

where $\{e'_j\}_{j\geqslant 0}$ is the standard basis for $\mathscr{H}'_\beta$. Now $D'_\beta = D'_\beta P'_\beta$, and $P'_\beta \mathscr{M}$ is dense in $\mathscr{H}'_\beta$. Hence $D'_\beta e'_0 = \phi'_\beta$ belongs to $\mathscr{M}$. So $\mathscr{M}_1 = \mathscr{M} \cap (\mathscr{K} \oplus \mathscr{X}_\beta)$ contains $\phi'_\beta = \phi_\beta + c_\beta f$.

Now we repeat this procedure ω times. Since $P_{\beta+1}\mathscr{M}_1$ contains $P_{\beta+1}\phi_\beta$ which is cyclic for $A_{\beta+1}$, we obtain $\overline{P_{\beta+1}\mathscr{M}_1} = \mathscr{H}_{\beta+1}$. Repeating the process above with $R_{\beta+1}$, we conclude that $\phi'_{\beta+1} = \phi_{\beta+1} + c_{\beta+1} f$ belongs to $\mathscr{M}$. By induction, the vectors $\phi'_{\beta+j} = \phi_{\beta+j} + c_{\beta+j} f$ belong to $\mathscr{M}$. Since $c_{\beta+j}^{-1}\|\phi_{\beta+j}\|$ tends to 0 as j tends to ∞, it follows that f belongs to $\mathscr{M}$.

But f is cyclic for B, whence all of $\mathscr{K}$ belongs to $\mathscr{M}$. So $\mathscr{M} = \mathscr{K} \oplus \mathscr{N}$ where $\mathscr{N}$ is a subspace of $\mathscr{H}$ invariant for T. This concludes the proof that Lat $S \cong$ Lat B + Lat T.

Now we take up the unicellularity of T. Fix $\mathscr{M}$ in Lat T. Let β be the least ordinal so that $\overline{P_\beta \mathscr{M}} = \mathscr{H}'_\beta$ is not $\{0\}$. Then the first part of the induction above shows that ϕ_β belongs to $\mathscr{M}$. Since ϕ_β is cyclic for T_β we conclude that $\mathscr{X}_\beta \subseteq \mathscr{M}$ and so $\mathscr{M} = \mathscr{L} \oplus \mathscr{X}_\beta$ where $\mathscr{L}$ is an invariant subspace for A_β. This implies that T is unicellular.

Finally, we further assume that B is a shift of class β with weight interval $[d_1, d_2]$ where $d_1 > b$. To prove that the operator S constructed above is a shift of class $\alpha + \beta$ with weight interval $[a, d_2]$ we have still to verify conditions (5′) and (7′). Condition (5′) follows from Corollary 3.2. Since Lat $S \cong$ Lat B + Lat T, and since both B and T are unicellular, then S is unicellular. Now condition (7′) follows immediately. ■

THEOREM 4.3. *For every countable ordinal, the lattice* $\alpha + 1$ *is attainable.*

PROOF. By the remarks at the beginning of this section, it suffices to show that $(\omega\alpha + 1)^*$ is attainable for every countable ordinal α. This in turn will be achieved by proving the existence of shifts of class α for all such ordinals. We assume that shifts of class β with arbitrarily prescribed weight intervals exist for all ordinal $\beta < \alpha$.

First, suppose $\alpha = \alpha_0 + \alpha_1$ with α_0 and α_1 less than α. Given $[a, b]$, choose a_1 and b_1 with $a < a_1 < b_1 < b$, and let A_0 and A_1 be shifts of class α_0 and α_1 with weights in $[a, a_1]$ and $[b_1, b]$, respectively. By Lemma 4.2, there is a shift S of class $\alpha_0 + \alpha_1 = \alpha$ with weights in $[a, b]$.

In the second case, $\alpha > \alpha_0 + \alpha_1$ whenever α_0 and α_1 are less than α. Write $\alpha = \lim_{n\to\infty} \alpha_n$, where each α_n is less than α. Then $\alpha = \sum_{n=1}^{\infty} \alpha_n$. Given $[a, b]$, let $[a_n, b_n]$ be disjoint intervals contained in $[a, b]$ with $a_{n+1} > b_n$ for all $n \geqslant 1$. Let B_n be shifts of class α_n with weights in $[a_n, b_n]$. Define E_n using Lemma 4.2 so that

$$\operatorname{Lat}\begin{pmatrix} B_{n+1} & E_n \\ 0 & B_n \end{pmatrix} \cong \operatorname{Lat} B_{n+1} + \operatorname{Lat} B_n,$$

and so that $\|E_n\|$ is small enough to satisfy the hypothesis of Lemma 3.1 with $\varepsilon_n = \frac{1}{3}(a_{n+1} - b_n)$. Let

$$T = \begin{bmatrix} B_1 & & & \bigcirc \\ E_1 & B_2 & & \\ & E_2 & B_3 & \\ \bigcirc & & \ddots & \ddots \end{bmatrix} \quad \text{and} \quad T_n = \begin{bmatrix} B_{n+1} & & \bigcirc \\ E_{n+1} & B_{n+2} & \\ & E_{n+2} & \ddots \\ \bigcirc & & \ddots \end{bmatrix}.$$

Then T has the form of a shift of class $\sum_{n=1}^{\infty}\alpha_n = \alpha$. All the properties are automatic except for (5′) and (7′). The first of these conditions follows from Lemma 3.1. The second will follow after we prove that T is unicellular. Let $\mathcal{M}$ be invariant for T, and let n be the least integer for which $\overline{P_n\mathcal{M}} = \mathcal{N}$ is not $\{0\}$. As in the proof of Lemma 4.2, it follows that the vector g_{n+1} in the range of E_n must belong to $\mathcal{M}$. The intersection of $\mathcal{M}$ with $\mathcal{X}_n = \sum_{k=n+1} \oplus \mathcal{H}_k$ is invariant for T, so by repeating the argument, we obtain that g_{n+2} belongs to $\mathcal{M}$ as well. Inductively, one finds that g_{n+k} belongs to $\mathcal{M}$ for all $k \geqslant 1$.

Now g_{n+k} is cyclic for B_{n+k} on $\mathcal{H}_{n+k}$. We wish to show that $B_{n+k}^j g_{n+k}$ belongs to $\mathcal{M}$ for all $j \geqslant 0$. Suppose $B_{n+k}^j g_{n+k}$ belongs to $\mathcal{M}$. Then

$$TB_{n+k}^j g_{n+k} = B_{n+k}^{j+1} g_{n+k} + E_{n+k} B_{n+k}^j g_{n+k} = B_{n+k}^{j+1} g_{n+k} + \lambda g_{n+k+1}.$$

Hence $B_{n+k}^{j+1} g_{n+k}$ belongs to $\mathcal{M}$. Therefore, $\mathcal{X}_n$ belongs to $\mathcal{M}$, so $\mathcal{M} = \mathcal{X}_n \oplus \mathcal{N}$, and $\mathcal{N}$ belongs to Lat B_n. By Lemma 4.2, each B_n is unicellular, so T is unicellular also. ■

COROLLARY 4.4. *If B has a cyclic vector and $\|B^n\| = o((n!)^{-d})$ for some $d > 0$, then* Lat $B + \alpha^*$ *is attainable for every countable ordinal α.* ■

COROLLARY 4.5. *If α and β are countable ordinals, then $\alpha + 1 + \beta^*$ is an attainable lattice on c_0 and l^p for $1 \leqslant p < \infty$.* ■

COROLLARY 4.6. *If B is acting on l^p, $1 < p < \infty$, B and B^* have cyclic vectors, $\|B^n\| = o((n!)^{-d})$ for some $d > 0$, and countable ordinals α and β are given, then α +* Lat B *+ β^* is attainable.*

PROOF. Since B^* has a cyclic vector, there is an operator A on l^q with Lat $A \cong$ Lat $B^* + \alpha^*$ and $\|A^n\| = o((n!)^{-d_1})$ for some $d_1 > 0$. So Lat $A^* \cong \alpha +$ Lat B and A^* has a cyclic vector. Apply Corollary 4.4. ■

Specific examples of B's which are interesting are the Volterra type operators of [**2**] and the bilateral weighted shift of [**5**]. A generalization of the latter is obtained in the next section. By [**13**, Theorem 4.4], if B is unicellular, then B and B^* have cyclic vectors. So in this case, only the growth condition of Corollary 4.6 need be verified.

REMARK 4.7. If T is a shift of class α with weights in $[a, b]$ and $a > 1$, then T is nuclear (trace class). In general, on Hilbert space, T is of class $\mathcal{C}_p$ if $pa > 1$.

REMARK 4.8. If may be desirable to have unicellular operators with lattice order isomorphic to $\alpha + 1$ for which $\|T^n\|$ dies off at a prescribed (very fast) rate. A similar construction to the preceding one can achieve this if one uses Proposition 3.5

in place of Lemma 3.1. One needs the easy fact that given any countable collection of sequences, there is a sequence which decays much quicker than any of them. It seems inevitable that in general the weights of A_β must die off at an incredibly fast pace.

5. Unicellular operators of integer type. In **[5]**, Domar shows that certain bilateral weighted shifts are unicellular, and thus have a lattice order isomorphic to $1 + \mathbf{Z} + 1$. We show that methods analogous to those of §§2–4 allow us to put finitely many or ω of these shifts together.

THEOREM 5.1. *For each $m \leqslant \omega$, there is a nuclear operator T on l^p or c_0, $1 \leqslant p < \infty$, with* Lat T *order isomorphic to* $(1 + \mathbf{Z} + 1)m$ *(also* $(1 + \mathbf{Z} + 1)\omega + 1$), *and* $\|T^n\| = o((n!)^{-1})$.

To facilitate the exposition, say that an operator B is of type Z_p, for $p > 0$, if:

(a) $\|B^n\| = o((n!)^{-p})$.

(b) There exist vectors g_n, $n \geqslant 0$, such that $\|g_0\| = 1$, $Bg_n = g_{n-1}$, $n \geqslant 1$, and $b_n = \|g_n\|/\|g_{n+1}\|$ is decreasing.

(c) The smallest B invariant subspace containing $\{g_n, n \geqslant 0\}$ is $\mathscr{H}$.

LEMMA 5.2. *Given B of type Z_p, and $0 < s < p$, there is an operator T of type Z_s such that* Lat $T \cong$ Lat $B + (1 + \mathbf{Z} + 1)$.

PROOF. The desired operator will have the form $T = \left[\begin{smallmatrix} B & C \\ 0 & A \end{smallmatrix}\right]$ where A is a bilateral, unicellular weighted shift and C is a compact operator taking certain vectors e_{μ_n} to multiples of g_{ν_n}, where μ_n and ν_n both tend to infinity with n. As before, we will show that certain monomials in T converge to operators of the form $\left(\begin{smallmatrix} 0 & D_N \\ 0 & 0 \end{smallmatrix}\right)$. However, in order to make certain norm estimates, we have been obliged to make a very delicate choice in our construction.

Fix r such that $s < r < p$. Let A be a bilateral weighted shift with positive weights a_n. Set

$$\Omega_n = \|A^n e_0\| = \prod_{i=0}^{n-1} a_i \quad \text{and} \quad \Omega_{-n} = \|A^{*n} e_0^*\| = \prod_{i=1}^{n} a_{-i}$$

for $n \geqslant 0$. Choose the a_n so that

(i) $a_n = (n+1)^{-r}$ for $n \geqslant 0$, so $\Omega_n = (n!)^{-r}$;

(ii) a_{-n} is monotone decreasing and $\Omega_{-n} = O(4^{-n^2})$; and

(iii) $\sum_{n=1}^\infty \Omega_{-n}\|g_{2n}\| < \infty$.

Conditions (i) and (ii), together with Corollary 3.2 imply that $\|T^n\| = O((n!)^{-s})$, regardless of the choice of C. The monotonicity and the fact that Ω_{-n} decreases rapidly imply **[5]** that A is unicellular. Condition (iii) will be used to control certain terms that occur in the computation of T^n.

Fix a decreasing sequence c_n such that $\sum_{n=1}^\infty c_n < \infty$. We will define sequences of positive integers ν_n and μ_n, and then C will be defined as $C = \sum_{n=1}^\infty c_n \|g_{\nu_n}\|^{-1} g_{\nu_n} \otimes e_{\mu_n}^*$. Also, set $t_n = \nu_n + \mu_n$ and $R = [r^{-1} + 1]$. The numbers ν_n and μ_n will be defined inductively so that

(iv) $b_{\nu_{n+1} - t_n} < 2^{-1} c_{n+1} t_n^{-r}$;

(v) $\nu_{n+1} > 4t_n$; and

(vi) $b_{2k}b_{2k+1} < a_{\mu_{n+1}+R-1} \leqslant b_{2k-2}b_{2k-1}$ where $k = \nu_{n+1} - 2t_n$.

To achieve this, first choose any ν large enough to satisfy (iv) and (v). Let μ_{n+1} be the least integer so that $a_{\mu_{n+1}+R-1} \leqslant b_{2k'-2}b_{2k'-1}$ for $k' = \nu - 2t_n$. Then take ν_{n+1} to be the unique integer $\geqslant \nu$ satisfying (vi).

Let $K = \max\{\Omega_{-k}\|g_{2k}\|, \|B^k\|/\Omega_k \colon k \geqslant 0\}$, which is finite by (a), (i) and (iii). For each $m \geqslant 0$,

$$\|A^m\| = \max_{0 \leqslant k \leqslant m} \Omega_{-k}\Omega_{m-k} \leqslant K \max_{0 \leqslant k \leqslant m} \|g_{2k}\|^{-1}\Omega_{m-k} = \eta(m)$$

and $\|B^m\| \leqslant K\Omega_m \leqslant \eta(m)$ as well. The sequence η is decreasing. To calculate $\eta(m)$, consider the ratios

$$\beta_k = \frac{\|g_{2k+2}\|^{-1}\Omega_{m-k-1}}{\|g_{2k}\|^{-1}\Omega_{m-k}} = \frac{b_{2k}b_{2k+1}}{a_{m-k-1}}.$$

These β_k decrease monotonely since, by (b), $b_{2k}b_{2k+1}$ is decreasing, and by (i), a_{m-k} is increasing with k. So the maximum $\eta(m)$ occurs when $\beta_k < 1 \leqslant \beta_{k-1}$. That means, when

$$b_{2k}b_{2k+1} < a_{m-k-1} \quad \text{and} \quad a_{m-k} \leqslant b_{2k-2}b_{2k-1}.$$

For $m_{n+1} = t_{n+1} + R - 2t_n$, condition (vi) shows that this holds exactly for $k = \nu_{n+1} - 2t_n$. Thus,

$$\begin{aligned}\eta(m_{n+1}) &= K\frac{\Omega_{\mu_{n+1}+R}}{\|g_{2\nu_{n+1}-4t_n}\|} \leqslant K\frac{\Omega_{\mu_{n+1}}\mu_{n+1}^{-rR}}{\|g_{\nu_{n+1}}\|} b_{\nu_{n+1}}^{\nu_{n+1}-4t_{n+1}} \\ &< \frac{K}{2\mu_{n+1}}\left[\frac{c_{n+1}\Omega_{\mu_{n+1}}}{\|g_{\nu_{n+1}}\|}\right] t_n^{-r} < \frac{Kt_n^{-r}}{t_{n+1}\lambda_{n+1}}.\end{aligned}$$

The last line follows because of (i), (iv), and the fact that (a), (i) and (vi) easily imply that $t_{n+1} < 2\mu_{n+1}$.

Fix $N \geqslant 1$, and consider $\lambda_n T^{t_n+1-N}$, where

$$\lambda_n = c_n^{-1}\Omega_{\mu_n}^{-1}\|g_{\nu_n}\|.$$

For n large enough, $N - 1 \leqslant 2t_{n-1} - R$ and hence $m_n \leqslant t_n + 1 - N$. We obtain that $\|\lambda_n A^{t_n+1-N}\|$ and $\|\lambda_n B^{t_n+1-N}\|$ are both less than $Kt_n^{-1}t_{n-1}^{-r}$, which tends to zero as n increases. Let X_n be the $(1,2)$ entry of $\lambda_n T^{t_n+1-N}$. Set

$$X_{n,m} = \lambda_n \sum_{j=0}^{t_n-N} c_m\|g_{\nu_m}\|^{-1}B^j g_{\nu_m} \otimes A^{*t_n-N-j}e_{\mu_m}^*.$$

Note that $X_n = \sum_{m=1}^{\infty} X_{n,m}$.

First, consider $m = n$.

$$\begin{aligned}X_{n,n} &= \lambda_n c_n\|g_{\nu_n}\|^{-1}\sum_{j=-\nu_n}^{\mu_n-N} g_{-j} \otimes A^{*\mu_n-N-j}e_{\mu_n}^* \\ &= \sum_{j=-\nu_n}^{\mu_n-N} g_{-j} \otimes \alpha_j e_{N+j}^*,\end{aligned}$$

where $g_{-j} = B^j g_0$ if $j \geqslant 1$, $\alpha_j = \Omega_{N+j}^{-1}$ if $N + j \geqslant 0$, and $\alpha_j = \Omega_{N+j}$ if $N + j < 0$. One has

$$\sum_{j=0}^{\infty} \|g_{-j}\| \Omega_{N+j}^{-1} \leqslant \sum_{j=0}^{\infty} \|B^j\| \Omega_{N+j}^{-1} \leqslant \sum_{j=0}^{\infty} (j!)^{-p} ((N+j)!)^r < \infty$$

and

$$\sum_{j=-2N}^{-\infty} \|g_{-j}\| \Omega_{N+j} = \sum_{n=N}^{\infty} \|g_{N+n}\| \Omega_{-n} \leqslant \sum_{n=N}^{\infty} \|g_{2n}\| \Omega_{-n} < \infty.$$

Hence, $X_{n,n}$ converges to $D_N = \sum_{j=-\infty}^{\infty} g_{-j} \otimes \alpha_j e_{N+j}^*$.

Next, consider $m < n$:

$$\begin{aligned}
\|X_{n,m}\| &\leqslant \lambda_n c_m (t_n + 1 - N) \max_{0 \leqslant j \leqslant t_n - N} \frac{\|B^j g_{\nu_m}\|}{\|g_{\nu_m}\|} \|A^{*t_n - N - j} e_{\mu_m}^*\| \\
&\leqslant \lambda_n c_m t_n K \max_{0 \leqslant j \leqslant t_n - N - \mu_m} \Omega_j \|A^{*t_n - \mu_m - N - j} e_0^*\| \\
&\leqslant c_m \lambda_n t_n K \max_{0 \leqslant j \leqslant t_n - \mu_{n-1} - N} \Omega_j \Omega_{-(t_n - \mu_{n-1} - N - j)} \\
&\leqslant c_m \lambda_n t_n \eta(m_n) < c_m t_{n-1}^{-r}.
\end{aligned}$$

Finally, we estimate $\|X_{n,m}\|$ for $m > n$.

$$\begin{aligned}
\|X_{n,m}\| &\leqslant c_m \lambda_n t_n \max_{0 \leqslant j \leqslant t_n - N} \frac{\|B^j g_{\nu_m}\|}{\|g_{\nu_m}\|} \|A^{*t_n - N - j} e_{\mu_m}^*\| \\
&= c_m \lambda_n t_n \max_{0 \leqslant j \leqslant t_n - N} \frac{\|g_{\nu_m - j}\|}{\|g_{\nu_m}\|} \frac{\Omega_{\mu_m}}{\Omega_{\mu_m + N + j - t_n}}.
\end{aligned}$$

Compute the ratio of the $(j + 1)$st term to the jth term

$$\begin{aligned}
\frac{\|g_{\nu_m - j - 1}\|}{\|g_{\nu_m - j}\|} \frac{\Omega_{\mu_m + N + j - t_n}}{\Omega_{\mu_m + N + j + 1 - t_n}} &> \frac{b_{\nu_m}}{a_{\mu_m - t_n}} = \frac{b_{\nu_m}}{a_{\mu_m + R}} \left[\frac{\mu_m + R + 1}{\mu_m - t_n + 1} \right]^{-r} \\
&> 2^{-r} \frac{b_{\nu_m}}{b\nu_m^2} = 2^{-r} b_{\nu_m}^{-1} > 2^{-r} b_{\nu_n}^{-1}.
\end{aligned}$$

For n sufficiently large, the ratio is greater than 1 for all j. So the maximum is attained at $j = t_n - N$. Thus

$$\begin{aligned}
\|X_{n,m}\| &\leqslant c_m \lambda_n t_n \frac{\|g_{\nu_m + N - t_n}\|}{\|g_{\nu_m}\|} \\
&\leqslant c_m \lambda_n t_n \left(b_{\nu_{n+1} - t_n}\right)^{\nu_n} \left(b_{\nu_{n+1} - t_n}\right)^{\mu_n - N} \\
&\leqslant c_m \lambda_n t_m \|g_{\nu_n}\|^{-1} \left(c_{n+1} t_n^{-r}/2\right)^{\mu_n - N} \\
&< c_m \left[(\mu_n!)^r \left(t_n^{\mu_n - N}\right)^{-r} \right] \left[t_n 2^{N - \mu_n} \right].
\end{aligned}$$

Because $\mu_n < t_n < 2\mu_n$, it follows that both bracketed expressions tend to zero as n tends to infinity.

Combining these last two estimates and $\Sigma_{m=1}^{\infty} c_m < \infty$, we obtain $\lim_{n\to\infty}\Sigma_{m\neq n}\|X_{n,m}\| = 0$. Altogether, we have

$$\lim_{n\to\infty} \lambda_n T^{t_n+1-N} = D_N.$$

Let $\mathcal{M}$ be an invariant subspace for T. If $\mathcal{M}$ is contained in $\mathcal{H}_1 \oplus 0$, it must belong to Lat B. Otherwise, the projection $\overline{P_{\mathcal{H}_2}\mathcal{M}} = \mathcal{N}$ is invariant for A, and thus contains some e_{n_0}. Since $\mathcal{M}$ is also invariant for each D_N, $\mathcal{M}$ contains $D_N e_{n_0} = \alpha_{n_0-N} g_{N-n_0}$ for all $N \geqslant 1$. By (c), $\mathcal{M}$ contains all of $\mathcal{H}_1 \oplus 0$. Hence $\mathcal{M} = \mathcal{H}_1 \oplus \mathcal{N}$.

This shows that Lat $T \cong$ Lat $B + (1 + \mathbf{Z} + 1)$. The vectors $0 \oplus e_{-n}$ satisfy conditions (b) and (c). Thus T is of class Z_s. ■

COROLLARY 5.3. *There are nuclear operators T on l^p and c_0, $1 \leqslant p < \infty$, which are unicellular with lattice order isomorphic to n copies of* $(1 + \mathbf{Z} + 1)$. ■

COROLLARY 5.4. *There is a nuclear operator T with* Lat T *order isomorphic to* $1 + (1 + \mathbf{Z} + 1)\omega^*$.

PROOF. Pick constants $1 < r_1 < r_2 < \cdots < 2, 2 > p_1 > p_2 > \cdots > 1$ and ε_i such that $\Sigma_{i=1}^{\infty}\varepsilon_i < \infty$. Let A_i be bilateral weighted shifts with weights $a_n^{(i)} = \varepsilon_i n^{-r_i}$, $n \geqslant 0$,

$$a_{-n}^{(i)} = \varepsilon_i 4^{-n^{p_i}}, \qquad n \geqslant 0.$$

Then

$$\Omega_{-n}^{(i)} = \prod_{k=1}^{n} a_{-k}^{(i)} = \varepsilon_i^n \exp\left(-\log 4 \sum_{k=1}^{n} k^{p_i}\right) = o(4^{-n^2})$$

since $p_i > 1$. Also

$$\frac{\Omega_{-n-1}^{(i)}/\Omega_{-2(n+1)}^{(i+1)}}{\Omega_{-n}^{(i)}/\Omega_{-2n}^{(i+1)}} = \frac{\varepsilon_i 4^{-(n+1)^{p_i}}}{\varepsilon_{i+1}^2 4^{-(2n+1)^{p_{i+1}}-(2n+2)^{p_{i+1}}}}$$
$$< \frac{\varepsilon_i}{\varepsilon_{i+1}^2} 4^{-(n+1)^{p_i}[1-2^{p_{i+1}+1}(n+1)^{p_{i+1}-p_i}]}.$$

This is less than 1 for n large, thus $\Sigma_{n=1}^{\infty}\Omega_{-n}^{(i)}/\Omega_{-2n}^{(i+1)} < \infty$.

Use Lemma 5.2 to define C_i from $\mathcal{H}_i$ to $\mathcal{H}_{i+1}$ using $g_n = e_{-n}^{(i+1)}$ and thus $b_n = a_{-n}^{(i+1)}$, and choose $\{c_n\}$ so that $\|C_i\|$ conform to the requirements of Lemma 3.1. Then let

$$T = \begin{bmatrix} A_1 & & & \bigcirc \\ C_1 & A_2 & & \\ & C_2 & A_3 & \\ \bigcirc & & \ddots & \ddots \end{bmatrix} \quad \text{and} \quad T_i = \begin{bmatrix} A_{i+1} & & & \bigcirc \\ C_{i+1} & A_{i+2} & & \\ & C_{i+2} & A_{i+3} & \\ \bigcirc & & \ddots & \ddots \end{bmatrix}.$$

By Lemma 3.1, $\|T_i^n\| = O((n!)^{-(r_i+r_{i+1})/2})$. Using $g_n = e_{-n}^{(i+1)}$, the operator T_i satisfies (a) and (b) of the definition of type Z_r for $r = \frac{1}{2}(r_i + r_{i+1})$. By the proof of Lemma 5.2, there are monomials in

$$T_{i-1} = \begin{bmatrix} T_i & C_i \\ 0 & A_i \end{bmatrix}$$

which converge to

$$D_N^{(i)} = \sum_{j=-\infty}^{0} e_j^{(i+1)} \otimes \alpha_j^{(i)} e_{N+j}^{(i)*} + \sum_{j=1}^{\infty} T_{i+1}^j e_0^{(i+1)} \otimes \alpha_j^{(i)} e_{N+j}^{(i)*}$$

where $\alpha_j^{(i)}$ are constants.

Let $\mathcal{M}$ be invariant for T. If $\mathcal{M}$ is not zero, let n_0 be the least integer such that $\overline{P_{n_0}\mathcal{M}} = \mathcal{N}$ is not zero. Then $\mathcal{M}$ is invariant for T_{i-1} and thus for $D_N^{(i)}$, $N \geqslant 1$. Since $\mathcal{N}$ is standard, it contains some $e_m^{(i)}$. So $\mathcal{M}$ contains all $e_j^{(i+1)}$ for $j \leqslant 0$. By induction, $\mathcal{M}_n = \mathcal{M} \cap \sum_{i=n}^{\infty} \oplus \mathcal{H}_i$ contains $e_j^{(n)}$, $j \leqslant 0$ for all $n > n_0$. Finally, since C_n takes each $\mathcal{H}_n$ into span$\{e_j^{(n+1)}, j < 0\}$, it follows that whenever $e_j^{(n)}$ belong to $\mathcal{M}$,

$$a_j^{(n)} e_{j+1}^{(n)} = A_n e_j^{(n)} = T e_j^{(n)} - c_n e_j^{(n)}$$

also belongs to $\mathcal{M}$. Hence $\mathcal{M}$ contains each $\mathcal{H}_n$ for $n > n_0$. Thus $\mathcal{M} = \mathcal{N} \oplus \sum_{i=n_0+1}^{\infty} \mathcal{H}_i$ is of the desired form.

Finally, a duality argument can be used to obtain an operator T with Lat $T \cong (1 + \mathbf{Z} + 1)\omega + 1$ on l^p, $p \geqslant 1$. It is also easy to see that the T constructed above on l^1 is the dual of a c_0 operator, which gives one on c_0 as well. ■

6. Concluding remarks. There are basic limitations to the method described here. Since we rely on finding monomials in $T = \left[\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right]$ which converge to a nonzero operator $\left[\begin{smallmatrix} 0 & D \\ 0 & 0 \end{smallmatrix}\right]$, it must follow that, in particular, $AD = DB$. In general, there will be no nonzero solutions of this equation. A case in point occurs when A and B^* are both weighted shifts. This is because A is one-to-one and span$\{\ker B^n, n \geqslant 0\}$ is dense in $\mathcal{H}$. Thus there is no hope of proving Domar's theorem by these methods. It also explains the requirements on the operators constructed in the last section. However, we believe that this method has not reached its limits, and that it may help to attain all countable nests.

The other method which will likely prove to be important is the use of integral operators. Given an order type of a nest, it is easy to write down integral operators whose lattice contains the desired nest. Some reasonably general criterion for when such an operator is unicellular would be desirable. An interesting test case would be to try to obtain Larson's nest order equivalent to the Cantor set. Let μ be counting measure on $\mathbf{Q}$, and let c_{rs} be positive constants for $r < s$ in $\mathbf{Q}$ such that $\sum_r \sum_s c_{rs} < \infty$. Define $Tf(r) = \sum_{r<s} c_{rs} f(s)$ for f in $L^2(\mu)$. Find sufficient conditions on $\{c_{rs}\}$ for T to be unicellular.

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