TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 284, Number 1, July 1984

ON THE WEIL-PETERSSON METRIC ON TEICHMÜLLER SPACE

BY

A. E. FISCHER AND A. J. TROMBA[1]

ABSTRACT. Teichmüller space for a compact oriented surface M without boundary is described as the quotient $\mathscr{A}/\mathscr{D}_0$, where $\mathscr{A}$ is the space of almost complex structures on M (compatible with a given orientation) and $\mathscr{D}_0$ are those C^∞ diffeomorphisms homotopic to the identity. There is a natural $\mathscr{D}_0$ invariant L_2 Riemannian structure on $\mathscr{A}$ which induces a Riemannian structure on $\mathscr{A}/\mathscr{D}_0$. Infinitesimally this is the bilinear pairing suggested by Andre Weil—the Weil-Petersson Riemannian structure. The structure is shown to be Kähler with respect to a naturally induced complex structure on $\mathscr{A}/\mathscr{D}_0$.

0. Introduction and statement of main results. In [7] the authors, using only concepts from Riemannian geometry and global nonlinear analysis, develop an "a priori" approach to Teichmüller theory. Teichmüller's theorem states (roughly) that the space of conformally inequivalent Riemann surfaces of genus p, $p > 1$ (with some topological restrictions) is homeomorphic to Euclidean $\mathbf{R}^{6p-6}$ space. In proving "homeomorphism", Teichmüller had put a complete Finsler (but not Riemannian) metric on Teichmüller space.

Since this original work [13] (circa 1939), other metrics have been put on Teichmüller space. Kobayashi [9] introduced a metric on a general complex manifold. By a theorem of Ahlfors [3], Teichmüller space has a natural complex structure. Using this fact Royden was able to prove that the metric of Kobayashi coincided with Teichmüller's metric.

In 1956 Weil suggested another metric, and in [2] Ahlfors proved that this metric was Kähler. Somewhat later he showed that it had nonpositive Ricci curvature.

In this work we show how, in the context of the approach to Teichmüller theory developed in [7 and 8], the Weil-Petersson metric arises naturally. We shall provide a simple and direct proof that the metric is Kähler.

We now present a more detailed description of our results. We begin first, however, with a review of our basic approach to the development of a Teichmüller theory and of some of the results in [7 and 8].

Let $\mathscr{M}$ be a C^∞ finite-dimensional manifold without boundary. Let $T_1^1(\mathscr{M})$ be the vector bundle over $\mathscr{M}$ of tensors of type $(1,1)$, and $C^\infty(T_1^1(\mathscr{M}))$ the space of C^∞ sections.

Received by the editors September 1, 1983.

1980 *Mathematics Subject Classification*. Primary 58B20; Secondary 30F20.

[1] Diese Arbeit ist mit Unterstützung des von der Deutschen Forschungsgemeinschaft getragenen Sonderforschungsbereiches 72 an der Universität Bonn entstanden und als Manuskript vervielfältigt worden.

An *almost complex structure* $\mathcal{J}$ on $\mathcal{M}$ is an element $\mathcal{J} \in C^\infty(T_1^1(\mathcal{M}))$ such that $\mathcal{J}^2 = -I$; i.e., at each $x \in \mathcal{M}$, $\mathcal{J}(x) \circ \mathcal{J}(x) = -I(x)$, where $I(x)$: $T_x\mathcal{M} \to T_x\mathcal{M}$ is the identity map. If such a $\mathcal{J}$ exists, it must be even dimensional, say $\dim \mathcal{M} = 2m$.

Moreover, in this case $\mathcal{J}$ defines an orientation for $\mathcal{M}$ by choosing in each tangent space $T_x\mathcal{M}$ a basis of the form

$$X_1, \ldots, X_m, \mathcal{J}X_1, \ldots, \mathcal{J}X_m. \tag{0.1}$$

The orientation of $\mathcal{M}$ determined by such a basis is called the *natural orientation* induced by $\mathcal{J}$.

Now let $\mathcal{M}$ be a compact connected oriented C^∞ 2-manifold without boundary. Let the space $\mathcal{A}$ of oriented complex structures on $\mathcal{M}$ be defined by

(0.2)

$$\mathcal{A} = \left\{ \mathcal{J} \in C^\infty\left(T_1^1(\mathcal{M})\right) \middle| \mathcal{J}^2 = -I, \text{ and the natural orientation induced by } \mathcal{J} \text{ is that of } \mathcal{M} \right\}.$$

In [7] we proved

THEOREM (0.3). *Let $\mathcal{M}$ be an oriented, compact connected C^∞ 2-manifold without boundary of genus greater than one. Then $\mathcal{A}$ carries the structure of a C^∞ strong ILH (inverse limit of Hilbert; see* [7, 11]) *manifold. The tangent space $T_{\mathcal{J}}\mathcal{A}$ is characterized by*

$$T_{\mathcal{J}}\mathcal{A} = \left\{ \dot{\mathcal{J}} \in C^\infty\left(T_1^1(\mathcal{M})\right) \middle| \dot{\mathcal{J}} \circ \mathcal{J} + \mathcal{J} \circ \dot{\mathcal{J}} = 0 \right\},$$

where "$\circ$" denotes the composition of (1, 1) *tensor fields.*

A complex structure on a $2m$-manifold $\mathcal{M}$ is an atlas of coordinate charts $(\varphi_i, U_i)_{i \in I}$ such that, when defined, $\varphi_i \circ \varphi_j^{-1}$ are holomorphic when viewed as maps on a neighbourhood of $\mathbf{C}^m$. Every complex structure on $\mathcal{M}$ naturally induces an almost complex structure in the following manner.

Denote by $(z_1, \ldots, z_m)$ the coordinates of $\mathbf{C}^m$, with $z_j = x_j + iy_j$, $i = \sqrt{-1}$. With respect to the $\mathbf{R}^{2m}$ coordinate system, $\mathbf{C}^m$ is represented as $(x_1, \ldots, x_m, y_1, \ldots, y_m)$. In each coordinate neighbourhood we have a basis of the tangent space given by $\{\partial/\partial x_i\}$ and $\{\partial/\partial y_j\}$. Thus, in a coordinate neighbourhood one can define an almost complex structure $\mathcal{J}$ by setting

$$\mathcal{J}(\partial/\partial x_j) = \partial/\partial y_j, \qquad \mathcal{J}(\partial/\partial y_j) = -\partial/\partial x_j.$$

This definition of $\mathcal{J}$ is independent of coordinate charts and thus defines a C^∞ almost complex structure on $\mathcal{M}$.

Now given an arbitrary almost complex structure $\mathcal{J}$ on $\mathcal{M}$ there is a (1, 2) tensor $N(\mathcal{J})$, called the Nijenhuis tensor, which has the property that if $\mathcal{J}$ arises from a complex structure then $N(\mathcal{J}) \equiv 0$. However, by the now celebrated theorem of Newlander and Nirenberg, the converse is also true: if $N(\mathcal{J}) = 0$, then $\mathcal{J}$ arises from a complex structure in the manner described above. If $N(\mathcal{J}) = 0$, we say that $\mathcal{J}$ is *integrable*. Thus, $\mathcal{J}$ is integrable if and only if $\mathcal{J}$ arises from a complex structure.

If $\mathcal{M}$ is a C^∞ 2-manifold, then every almost complex structure $\mathcal{J}$ on $\mathcal{M}$ is integrable. Thus, if $\mathcal{C}$ denotes the space of complex structures on $\mathcal{M}$, then there is a bijective

correspondence $\mathscr{A} \leftrightarrow \mathscr{C}$. Let $\mathscr{D}$ be the group of C^∞ diffeomorphisms of $\mathscr{M}$, and let $\mathscr{D}_0 \subset \mathscr{D}$ denote the component of the identity. As a group, $\mathscr{D}$ (and hence $\mathscr{D}_0$) acts on $\mathscr{A}$ by pull-back; i.e., if $f \in \mathscr{D}$ then the action sends $\mathscr{J} \to f^*\mathscr{J}$, where, for $x \in \mathscr{M}$,

$$(f^*\mathscr{J})(x) = \mathscr{D}f(x)^{-1}\mathscr{J}f(x)\mathscr{D}f(x),$$

where $\mathscr{D}$ denotes derivatives.

$\mathscr{D}$ acts on $\mathscr{C}$ by sending each $\varphi_i\colon \mathscr{U} \to \mathbf{C}$ in a given atlas to $\varphi_i \circ f$. The bijective correspondence between $\mathscr{A}$ and $\mathscr{C}$ is $\mathscr{D}$-equivariant and thus establishes a bijective correspondence between the quotient spaces $\mathscr{A}/\mathscr{D} \leftrightarrow \mathscr{C}/\mathscr{D}$ and $\mathscr{A}/\mathscr{D}_0 \leftrightarrow \mathscr{C}/\mathscr{D}_0$. The space $\mathscr{R} = \mathscr{C}/\mathscr{D}$ is defined classically as the Riemann space of moduli. Riemann had conjectured (and, in fact, gave a heuristic argument) that in case $\mathscr{M}$ is an oriented surface of genus p, $p > 1$, then $\mathbf{R}$ had dimension $6p - 6$. The space $\mathscr{C}/\mathscr{D}_0 = \mathscr{T}$ is the Teichmüller space of $\mathscr{M}$. Since $\mathscr{D}_0 \subset \mathscr{D}$ is normal we can form the quotient group $\Gamma = \mathscr{D}/\mathscr{D}_0$, the *modular group* of $\mathscr{M}$. For a compact surface without boundary it is well known that this group is discrete and acts on $\mathscr{J}$ in the obvious way. Clearly

$$\mathscr{T}/\Gamma \simeq \frac{\mathscr{C}/\mathscr{D}_0}{\mathscr{D}/\mathscr{D}_0} \simeq \mathscr{R}.$$

Thus, the actual space of Riemann moduli for a compact surface without boundary is the quotient of Teichmüller space by the action of a discrete group. Identifying $\mathscr{T}$ with $\mathscr{A}/\mathscr{D}_0$ we have proven the following [7]

THEOREM (0.4). *Let $\mathscr{M}$ be a C^∞ compact oriented surface without boundary of genus p, $p > 1$. Then the space $\mathscr{T} = \mathscr{A}/\mathscr{D}_0$ carries the structure of a C^∞ finite-dimensional contractible manifold of dimension $6p - 6$. Moreover, the bundle $(\pi, \mathscr{A}, \mathscr{A}/\mathscr{D}_0)$ is a C^∞ principal strong ILH fibre bundle over $\mathscr{T}$ with structure group $\mathscr{D}_0$.*

We remark that a natural complex structure exists on $\mathscr{A}$ which induces a complex structure on $\mathscr{T}$ (see [8] for details).

In order to define the Weil-Petersson metric on $\mathscr{T}$ we need a few definitions. Let S_2 be the space of C^∞ symmetric $(0, 2)$ tensor fields on a compact oriented manifold $\mathscr{M}$ without boundary and let $\mathfrak{M} \subset S_2$ be the open subset of Riemannian metrics. Let $\mathfrak{M}_{-1}$ be those metrics with scalar curvature negative one. If $\dim \mathscr{M} = 2$ and genus $\mathscr{M} > 1$, $\mathfrak{M}_{-1}$ is a nonempty strong ILH manifold [7].

Let P be the space of C^∞ positive functions on $\mathscr{M}$. P acts on $\mathfrak{M}$ and we may form the quotient $\mathfrak{M}/P$. Then from [7] we have

THEOREM (0.5). *Let $\dim \mathscr{M} = 2$, $\mathscr{M}$ compact, oriented and without boundary. Then there exists a C^∞ ILH diffeomorphism $\psi\colon \mathfrak{M}/P \to \mathscr{A}$ induced by the map of $\tilde{\psi}\colon \mathfrak{M} \to \mathscr{A}$; $g \mapsto -g^{-1} \cdot \mu_g$ where μ_g is the unique volume element determined by the metric g and the given orientation of $\mathscr{M}$. Here $g^{-1}\mu_g$ is the contraction of a $(0, 2)$ tensor to a $(1,1)$ tensor using the metric g; formally, for all $X, Y \in T_x\mathscr{M}$,*

$$g(\tilde{\psi}(g)X, Y) = -\mu_g(X, Y).$$

REMARK. $\mathfrak{M}/P$ can be given the structure of a C^∞ strong ILH manifold with ψ an ILH diffeomorphism.

We also have

THEOREM (0.6). *Let* $\pi\colon \mathfrak{M} \to \mathfrak{M}/P$ *be the natural projection map and* $\pi_{-1} = \pi|\mathfrak{M}_{-1}$. *Then* π_{-1} *is a* C^∞ *ILH diffeomorphism.*

Combining the last two results we obtain a C^∞ ILH diffeomorphism $\theta\colon \mathscr{A} \to \mathfrak{M}_{-1}$ ($\theta = (\psi \circ \pi_{-1})^{-1}$). We now define the L_2-Riemannian structure on $\mathscr{A}$. Let $\dot{\mathscr{J}}_1, \dot{\mathscr{J}}_2 \in T_{\mathscr{J}}\mathscr{A}$. We identify $T_{\mathscr{J}}\mathscr{A}$ with the space of those $\dot{\mathscr{J}}$ which satisfy $\{\dot{\mathscr{J}} \in C^\infty(T_1^1(M)) \mid \dot{\mathscr{J}}\cdot\mathscr{J} + \mathscr{J}\cdot\dot{\mathscr{J}} = 0\}$. Then define $(\ ,\ )_{\mathscr{J}}\colon T_{\mathscr{J}}\mathscr{A} \times T_{\mathscr{J}}\mathscr{A} \to R$ by

$$(0.7) \qquad (\dot{\mathscr{J}}_1, \dot{\mathscr{J}}_2)_{\mathscr{J}} = \int_M \operatorname{tr}(\dot{\mathscr{J}}_1 \cdot \dot{\mathscr{J}}_2^*)\, d\mu_{g(\mathscr{J})},$$

where $g(\mathscr{J}) = \theta(\mathscr{J})$, tr denotes the trace of a $(1,1)$ tensor, and $\mathscr{J}_2^*$ denotes the adjoint of $\mathscr{J}_2$ with respect to $g(\mathscr{J})$. Thus our "weak" Riemannian structure is the integral of the C^∞ function $\operatorname{tr}(\dot{\mathscr{J}}_1 \cdot \dot{\mathscr{J}}_2^*)$ over $\mathscr{M}$ with respect to the negative scalar curvature metric $g(\mathscr{J})$. The reader might be interested in comparing this with the L_2-Riemannian structure on $\mathfrak{M}$ defined in 1 of [7]. The term "weak" is justified by the fact that the topology associated with, or induced by, this Riemannian structure is not the C^∞ topology.

Our main result is then

THEOREM (0.8). $(\ ,\ )$ *is a* C^∞ $\mathscr{D}_0$*-invariant Riemannian structure on* $\mathscr{A}$, *and hence passes to a* C^∞*-Riemannian structure* $\langle\ ,\ \rangle$ *on* $\mathscr{T} = \mathscr{A}/\mathscr{D}_0$. *This Riemannian metric is Hermitian with respect to the natural complex structure on* $\mathscr{T}$, *is Kähler, and is, in fact, the Weil-Petersson metric.*

1. Weak-Riemannian principal fibre bundles. In this section we shall assume that (π, P, Σ), $\Sigma = P/\mathscr{G}$ is a Hilbert or strong ILH principal bundle with structure group $\mathscr{G}$ and with $\dim \Sigma < \infty$. In the first case P and Σ are Hilbert manifolds and A is a C^∞ Lie group; in the second case P and Σ are C^∞ strong ILH manifolds and A is a C^∞ strong ILH Lie group in the sense of Omori [**11**]. In the Hilbert case all morphisms (maps, vector fields, tensor fields) are C^∞ in the Frechét sense, and in the ILH case they are C^∞ ILH smooth. A brief discussion of this category is given in [7].

DEFINITION (1.1). A weak-Riemannian principal fibre bundle (π, P, Σ) is a Hilbert or ILH principal bundle with a C^∞ $\mathscr{G}$-invariant Riemannian structure G on P.

Thus, for each $p \in P$, $G(p)\colon T_pP \times T_pP \to \mathbf{R}$ and $p \mapsto G(p)$ is C^∞. We make no assumption on completeness.

THEOREM (1.2.). *Let* (π, P, Σ) *be a* C^∞ *weak-Riemannian principal fibre bundle with Riemannian structure* G. *If* $\dim \Sigma < \infty$ *then* G *naturally induces a Riemannian structure* G_Σ *on* Σ.

PROOF. For each $p \in P$ there is a "vertical" subspace $V_p \subset T_pP$ defined by $V_p = \ker D\pi(p)$. Since $\dim \Sigma < \infty$, V_p is of finite codimension. The metric G then defines a horizontal subspace $H_p \subset T_pP$, defined as the orthogonal complement to V_p with respect to $G(p)$ {it is here where we need $\dim \Sigma < \infty$}. Thus $H_p = V_p^\perp$. Then the induced structure G_Σ on Σ is defined as follows. For $x \in \Sigma$, $X_x, Y_x \in T_x\Sigma$

we set

$$G_\Sigma(x)(X_x, Y_x) = G(p)(\tilde{X}_p, \tilde{Y}_p),$$

where p is any point with $\pi(p) = x$, and $\tilde{X}_p$, $\tilde{Y}_p \in H_p$ are the unique horizontal lifts of X_x, Y_x respectively. Thus, $D\pi(p)\tilde{X}_p = X_x$, and similarly for $\tilde{Y}_p$ and Y_x.

One now readily checks that the equivariance of G under the group $\mathscr{G}$ implies that G_Σ is well defined and is Riemannian metric for Σ. □

If X and Y are C^∞ vector fields on $\mathscr{M}$ with unique horizontal lifts $\tilde{X}$, $\tilde{Y}$, then $\tilde{X}$ and $\tilde{Y}$ are $\mathscr{G}$-invariant vector fields on P and $p \mapsto G(p)(\tilde{X}_p, \tilde{Y}_p)$ is a C^∞ $\mathscr{G}$-variant real-valued function on P. Moreover,

THEOREM (1.3). $\{G_\Sigma(X, Y)\} \circ \pi = G(\tilde{X}, \tilde{Y})$.

DEFINITION (1.4.). The principal bundle (π, P, Σ) is an almost complex principal bundle if the manifold P admits an almost complex structure $\mathscr{J} \in C^\infty(T_1^1(P))$ satisfying:

(i) $\mathscr{J}^2 = -I$;

(ii) $\mathscr{J}$ is $\mathscr{G}$-invariant;

(iii) for each $p \in P$, $\mathscr{J}(p)\colon V_p \to V_p$ ($\mathscr{J}$ preserves vertical subspaces).

Then it follows from [**8**] that $\mathscr{J}$ induces a natural almost complex structure $\mathscr{J}_\Sigma$ on Σ via

$$\mathscr{J}_\Sigma(x)X_x = D\pi(p)\mathscr{J}(p)\tilde{X}_p,$$

where p is any point with $\pi(p) = x$, and $\tilde{X}_p$ is *any* vector in T_pP with $D\pi(p)\tilde{X}_p = X_x$. Then we have

THEOREM (1.5). *Let (π, P, Σ) be a weak-Riemannian almost complex principal fibre bundle with Riemannian structure G and almost complex structure $\mathscr{J}$. If* $\dim \Sigma < \infty$ *and if G is Hermitian with respect to $\mathscr{J}$, then G is Hermitian with respect to $\mathscr{J}_\Sigma$.*

PROOF.

$$G_\Sigma(x)(\mathscr{J}_\Sigma(x)X_x, \mathscr{J}_\Sigma(x)Y_x) = G_\Sigma(x)(D\pi(p)\mathscr{J}(p)\tilde{X}_p, D\pi(p)\mathscr{J}(p)\tilde{Y}_p),$$

where $\tilde{X}_p$, $\tilde{Y}_p$ are the unique horizontal lifts of X_x, Y_x to $p \in \pi^{-1}(x)$. But this, by definition, is equal to

$$G(p)(\mathscr{J}(p)\tilde{X}_p, \mathscr{J}(p)\tilde{Y}_p) = G(p)(\tilde{X}_p, \tilde{Y}_p) = G_\Sigma(x)(X_x, Y_x). \quad \square$$

2. The natural L_2-metric on $\mathscr{A}$. For the remainder of this paper $\mathscr{M}$ shall denote a compact oriented C^∞ 2-manifold without boundary with genus greater than one. Let $\mathscr{A}$ be the space of oriented almost complex structures on $\mathscr{M}$ defined on (0.2). Recall that

$$\mathscr{A} = \{\mathscr{J} \in C^\infty(T_1^1(\mathscr{M})) \mid \mathscr{J}^2 = -I, \text{ and the natural orientation induced by } \mathscr{J} \text{ is that of } \mathscr{M}\},$$

and, for $\mathscr{J} \in \mathscr{A}$,

$$T_{\mathscr{J}}\mathscr{A} \approx \{ \dot{\mathscr{J}} \in C^{\infty}(T_1^1(\mathscr{M})) \mid \dot{\mathscr{J}}\mathscr{J} = -\mathscr{J}\dot{\mathscr{J}} \}.$$

Recall also that we have defined an L_2 weak-Riemannian structure on the manifold $\mathscr{A}$ as follows. For $\dot{\mathscr{J}}_1, \dot{\mathscr{J}}_2 \in T_{\mathscr{J}}\mathscr{A}$,

$$(\dot{\mathscr{J}}_1, \dot{\mathscr{J}}_2)_{\mathscr{J}} = \int_M \operatorname{tr}(\dot{\mathscr{J}}_1 \circ \dot{\mathscr{J}}_2^*)\, d\mu_{g(\mathscr{J})}, \tag{2.1}$$

where $g(\mathscr{J})$ is the unique scalar curvature minus one metric associated to $\mathscr{J} \in \mathscr{A}$, and * denotes adjoint with respect to $g(\mathscr{J})$.

Our first observation is that we can give a simpler description of this Riemannian structure.

THEOREM (2.2). $(\dot{\mathscr{J}}_1, \dot{\mathscr{J}}_2)_{\mathscr{J}} = \int_M \operatorname{tr}(\dot{\mathscr{J}}_1 \circ \dot{\mathscr{J}}_2)\, d\mu_{g(\mathscr{J})}$.

PROOF. Let us work in conformal coordinates for the metric $g(\mathscr{J}) = g$. Then locally we can assume that the metric $g_{ij} = \lambda\delta_{ij}$, λ a C^{∞} positive function. Then for each $x \in \mathscr{M}$ in this coordinate system

$$\mathscr{J}(x) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Moreover, since $\dot{\mathscr{J}}\mathscr{J} = -\mathscr{J}\dot{\mathscr{J}}$, it follows that each $\dot{\mathscr{J}}$ must then have the form

$$\dot{\mathscr{J}}(x) = \begin{pmatrix} a_x & b_x \\ b_x & -a_x \end{pmatrix}.$$

Thus, $\dot{\mathscr{J}}(x)^* = \dot{\mathscr{J}}(x)$.

THEOREM (2.3). *The weak-Riemannian structure on $\mathscr{A}$ is $\mathscr{D}_0$-invariant and hence gives $(\pi, \mathscr{A}, \mathscr{A}/\mathscr{D}_0)$ the structure of a weak-Riemannian principal fibre bundle.*

PROOF. Let $f \in \mathscr{D}_0$. Then we must show that

$$f^*\{(\dot{\mathscr{J}}_1, \dot{\mathscr{J}}_2)_{\mathscr{J}}\} = (\dot{\mathscr{J}}_1, \dot{\mathscr{J}}_2)_{\mathscr{J}}.$$

But

$$f^*\{(\dot{\mathscr{J}}_1, \dot{\mathscr{J}}_2)_{\mathscr{J}}\} = (f^*\dot{\mathscr{J}}_1, f^*\dot{\mathscr{J}}_2)_{f^*\mathscr{J}} = \int_M \operatorname{tr}(f^*\dot{\mathscr{J}}, f^*\dot{\mathscr{J}}_2)\, d\mu_{g(f^*\mathscr{J})}.$$

However, from [7] it follows that the map $\mathscr{J} \to g(\mathscr{J})$ is $\mathscr{D}_0$-equivariant, so $g(f^*\mathscr{J}) = f^*g(\mathscr{J})$. Continuing the equality, the above is equal to

$$\int_M f^*\{\operatorname{tr}(\dot{\mathscr{J}}_1 \circ \dot{\mathscr{J}}_2)\}\, d\mu_{f^*g(\mathscr{J})}.$$

Now, the change of variables theorem implies that this is equal to

$$\int_M \operatorname{tr}(\dot{\mathscr{J}}_1 \circ \dot{\mathscr{J}}_2)\, d\mu_{g(\mathscr{J})}.$$

3. The metric is Hermitian. Let $(\pi, \mathscr{A}, \mathscr{A}/\mathscr{D}_0)$ be as in the last section. In [**8**] we show that this bundle has the structure of an almost complex principal bundle. The almost complex structure $\Phi \in C^\infty(T_1^1(\mathscr{A}))$ is defined by $\Phi(\mathscr{J}) \circ \dot{\mathscr{J}} = \mathscr{J} \circ \dot{\mathscr{J}}$. If $\dot{\mathscr{J}} \in T_{\mathscr{J}}\mathscr{A}$ then $\dot{\mathscr{J}} \circ \mathscr{J} = -\mathscr{J} \circ \dot{\mathscr{J}}$. Thus

$$(\mathscr{J} \circ \dot{\mathscr{J}}) \circ \mathscr{J} = \mathscr{J} \circ (\dot{\mathscr{J}} \circ \mathscr{J}) = -\mathscr{J}^2 \circ \dot{\mathscr{J}} = -\mathscr{J} \circ (\mathscr{J} \circ \dot{\mathscr{J}}),$$

so $\Phi(\mathscr{J})$: $T_{\mathscr{J}}\mathscr{A} \to T_{\mathscr{J}}\mathscr{A}$. Clearly, $\Phi^2 = -I$ and one checks easily that Φ is $\mathscr{D}_0$-equivariant. Hence Φ induces an almost complex structure $\Phi_{\mathscr{J}}$ on $\mathscr{T} = \mathscr{A}/\mathscr{D}_0$.

THEOREM (3.1). *The L_2-metric $(\ ,\ )$ on $\mathscr{A}$ is Hermitian with respect to the almost complex structure Φ on $\mathscr{A}$.*

PROOF.

$$\begin{aligned}(\Phi(\mathscr{J})\dot{\mathscr{J}}_1, \Phi(\mathscr{J})\dot{\mathscr{J}}_2)_{\mathscr{J}} &= (\mathscr{J} \circ \dot{\mathscr{J}}_1, \mathscr{J} \circ \dot{\mathscr{J}}_2)_{\mathscr{J}} \\ &= \int_M \operatorname{tr}(\{\mathscr{J} \circ \dot{\mathscr{J}}_1\} \circ \{\mathscr{J} \circ \dot{\mathscr{J}}_2\})\, d\mu_{g(\mathscr{J})} \\ &= \int_M \operatorname{tr}(-\mathscr{J}^2 \circ \dot{\mathscr{J}}_1 \circ \dot{\mathscr{J}}_2)\, d\mu_{g(\mathscr{J})} = \int_M \operatorname{tr}(\dot{\mathscr{J}}_1 \circ \dot{\mathscr{J}}_2)\, d\mu_{g(\mathscr{J})} = (\dot{\mathscr{J}}_1, \dot{\mathscr{J}})_{\mathscr{J}}. \quad \square\end{aligned}$$

As a consequence of this result and (1.3), we immediately obtain

THEOREM (3.2). *The almost complex structure Φ and weak-Riemannian structure $(\ ,\ )$ on $(\pi, \mathscr{A}, \mathscr{A}/\mathscr{D}_0)$ induce an almost complex structure $\Phi_{\mathscr{T}}$ and a Riemannian structure $\langle\ ,\ \rangle$ on Teichmüller space such that $\langle\ ,\ \rangle$ is Hermitian with respect to $\Phi_{\mathscr{T}}$.*

REMARK. In [**8**] the authors show that $\Phi_{\mathscr{T}}$ is integrable; that is, it arises from a complex structure on $\mathscr{J}$. In the next section we show that $\langle\ ,\ \rangle$ is a Kähler metric.

4. The induced metric on Teichmüller space is Kähler. We begin this section with an abstract theorem on principal fibre bundles which we later apply to Teichmüller space. Let (π, P, Σ) be an almost complex weak-Riemannian principal $\mathscr{A}$-bundle with $\dim \mathscr{M} < \infty$, weak-Riemannian structure G and almost complex structure Φ. Then we know from §1 that Φ induces an almost complex structure Φ_Σ on Σ and G induces a Riemannian structure G_Σ on Σ. Moreover, if G is Hermitian with respect to Φ then G_Σ is Hermitian with respect to Φ_Σ. We would like to have conditions on Φ and G to insure that the Hermitian metric G_Σ is a Kähler metric.

DEFINITION (4.1). Let Σ be an almost complex Riemannian manifold with almost complex structure Φ_Σ and Hermitian metric G_Σ. We say that G_Σ is Kähler if the Kähler two-form Ω_Σ, defined by

$$\Omega_\Sigma\colon T\Sigma \times T\Sigma \to R; \qquad \Omega_\Sigma(X_x, Y_x) = G_\Sigma(\Phi_\Sigma X_x, Y_x),$$

is closed, i.e. $d\Omega_\Sigma = 0$.

We now have the following result:

THEOREM (4.2). *Let (π, P, Σ) be an almost complex weak-Riemannian principal $\mathscr{G}$-bundle with $\dim \Sigma < \infty$ and almost complex structure Φ and weak-Riemannian structure G. Let G_Σ and Φ_Σ denote the induced structures on Σ with Ω and Ω_Σ the*

corresponding Kähler forms. For $X, Y, Z \in \mathfrak{X}(\Sigma)$ vector fields on Σ we have

$$d\Omega_\Sigma(X, Y, Z) \circ \pi = d\Omega(\tilde{X}, \tilde{Y}, \tilde{Z}),$$

where $\tilde{X}, \tilde{Y}, \tilde{Z}$ are the unique horizontal lifts of X, Y, Z to vector fields on P.

PROOF. For this we use the well-known formula for $d\Omega$ and $d\Omega_\Sigma$ (see [**10**]):

$$(4.3)\quad d\Omega(\tilde{X}, \tilde{Y}, \tilde{Z}) = \tfrac{1}{3}\{\tilde{X}(\Omega(\tilde{Y}, Z)) + \tilde{Y}(\Omega(\tilde{Z}, \tilde{X})) + \tilde{Z}(\Omega(\tilde{X}, \tilde{Y})) - \Omega([\tilde{X}, \tilde{Y}], \tilde{Z}) - \Omega([\tilde{Y}, \tilde{Z}], \tilde{X}) - \Omega([\tilde{Z}, \tilde{X}], \tilde{Y})\},$$

where [,] denotes the Lie bracket and, for $\varphi \in C^\infty(P, \mathbf{R})$, $X(\varphi) = D\varphi(X)$.

From the construction of G_Σ and Φ_Σ it follows that

$$\{\Omega_\Sigma(Y, Z)\} \circ \pi = \Omega(\tilde{Y}, \tilde{Z}).$$

Thus

$$\begin{aligned}\tilde{X}(\Omega(\tilde{Y}, \tilde{Z})) &= \tilde{X}(\Omega_\Sigma(Y, Z) \circ \pi) = D(\Omega_\Sigma(Y, Z)) \circ D\pi(\tilde{X}) \\ &= \{D(\Omega_\Sigma(Y, Z)) \circ X\} \circ \pi = \{X(\Omega_\Sigma(Y, Z))\} \circ \pi,\end{aligned}$$

and similarly, for the second two terms in (4.3). For a $\mathscr{G}$-invariant vector field $\tilde{X}$ on P, denote by $\pi_*(\tilde{X})$ the push-down vector field on Σ,

$$\pi_*(\tilde{X})(x) = D\pi(\tilde{X}) \circ \pi^{-1}(x).$$

If $\tilde{X}$ is the horizontal lift of X, then $\pi_* \tilde{X} = X$. Since

$$\pi_*[\tilde{X}, \tilde{Y}] = [\pi_* \tilde{X}, \pi_* \tilde{Y}] = [X, Y],$$

it follows that

$$[X, Y] = [\tilde{X}, \tilde{Y}]_H,$$

the horizontal part of $[\tilde{X}, \tilde{Y}]$.

Thus, if $[\tilde{X}, \tilde{Y}]_\nu$ denotes the vertical part of $[\tilde{X}, \tilde{Y}]$,

$$[\tilde{X}, \tilde{Y}] = [X, Y] + [\tilde{X}, \tilde{Y}]_\nu.$$

Consequently,

$$\Omega([\tilde{X}, \tilde{Y}], \tilde{Z}) = \Omega([X, Y], \tilde{Z}) + \Omega([\tilde{X}, \tilde{Y}]_\nu, \tilde{Z}).$$

It is easy to see that, since $\tilde{Z}$ is horizontal, $\Omega([\tilde{X}, \tilde{Y}]_\nu, \tilde{Z}) = 0$. Thus

$$\Omega([\tilde{X}, \tilde{Y}], \tilde{Z}) = \Omega_\Sigma([X, Y], Z) \circ \pi,$$

and similarly, for the last three terms in (4.3). Using exactly the same formula for $d\Omega_\Sigma$ as for $d\Omega$ and substituting these results in (4.3) we obtain the conclusion of Theorem (4.2). □

We immediately have

COROLLARY (4.4). *Let (π, P, Σ) be an almost complex weak-Riemannian principal $\mathscr{G}$-bundle with* $\dim \Sigma < \infty$, *and Ω, Ω_Σ the Kähler forms on P, and Σ induced by the almost complex and weak-Riemannian structures on P. Then Σ is Kähler if $d\Omega(\tilde{X}, \tilde{Y}, \tilde{Z}) = 0$ for horizontal vectors $\tilde{X}, \tilde{Y}, \tilde{Z}$.*

PROOF. By (4.2),

$$d\Omega_\Sigma(X, Y, Z) \circ \pi = d\Omega(\tilde{X}, \tilde{Y}, \tilde{Z}). \quad \square$$

REMARK. Thus, it may be that a given almost complex weak-Riemannian principal $\mathscr{G}$-bundle P is not Kähler, yet the induced metric on the almost complex base manifold Σ is. This is precisely the situation for the principal $\mathscr{D}_0$ bundle $(\pi, \mathscr{A}, \mathscr{A}/\mathscr{D}_0)$ of almost complex structures on a Riemannian surface of genus greater than one.

We now return to our study of the almost complex weak-Riemannian principal bundle $(\pi, \mathscr{A}, \mathscr{A}/\mathscr{D}_0)$. Again let Φ be the almost complex structure on $\mathscr{A}$, which at $\mathscr{J} \in \mathscr{A}$ is multiplication by $\mathscr{J}$ on $T_{\mathscr{J}}\mathscr{A}$, let $(\, , \,)$ be the L_2-metric on $\mathscr{A}$ defined in (2.1) and $\langle \, , \, \rangle$ the induced metric on $\mathscr{A}/\mathscr{D}_0 = \mathscr{T}$.

Denote by Ω and $\Omega_{\mathscr{T}}$ the corresponding Kähler forms on $\mathscr{A}$ and $\mathscr{T}$. In order to show that $\langle \, , \, \rangle$ is Kähler we need to show (by (4.4)) that $d\Omega(\tilde{X}, \tilde{Y}, \tilde{Z}) = 0$ for all horizontal $\tilde{X}, \tilde{Y}, \tilde{Z}$. Therefore, our first step will be to identify the horizontal subspace of $T_{\mathscr{J}}\mathscr{A}$. We already know that $X \in T_{\mathscr{J}}\mathscr{A}$ means that $X \in C^\infty(T_1^1(\mathscr{M}))$ and $X\mathscr{J} = -\mathscr{J}X$. Moreover, X is vertical if $X = L_W\mathscr{J}$, the Lie derivative of $\mathscr{J}$ with respect to some $W \in \mathfrak{X}(\mathscr{M})$. We now have the following result:

THEOREM (4.5). *$\tilde{X} \in T_{\mathscr{J}}\mathscr{A}$ is* horizontal *if and only if the divergence of $\tilde{X}$ (as a* (1, 1) *tensor on $\mathscr{M}$) with respect to the metric $g(\mathscr{J})$ is zero.*

PROOF. For $\mathscr{J} \in \mathscr{A}$ consider the map

$$\alpha_{\mathscr{J}}: \mathfrak{X}(\mathscr{M}) \to C^\infty(T_1^1(\mathscr{M})), \qquad X \mapsto \alpha_{\mathscr{J}}(x) = L_x\mathscr{J}.$$

LEMMA. *For $\mathscr{J} \in \mathscr{A}$, the map $\alpha_{\mathscr{J}}$ has injective symbol.*

PROOF. A straightforward calculation.

Now let g be an arbitrary metric on $\mathscr{M}$ and consider the L_2 inner product on $C^\infty(T_1^1(\mathscr{M}))$ given by

$$(A, B) = \int_M (\operatorname{tr} AB^*)\, d\mu_g,$$

as described earlier. Since $\alpha_{\mathscr{J}}$ has an injective symbol, $\alpha_{\mathscr{J}}$ has an L_2 adjoint

$$\alpha_{\mathscr{J}}^*: C^\infty(T_1^1(\mathscr{M})) \to \mathfrak{X}(\mathscr{M}).$$

By results of Berger-Ebin [5] (or e.g. see Fischer-Marsden [6]) one has the following

PROPOSITION (4.6). *$C^\infty(T_1^1(\mathscr{M}))$ splits L_2-orthogonally as*

$$C^\infty(T_1^1(\mathscr{M})) = \ker \alpha_{\mathscr{J}}^* \otimes \operatorname{Range} \alpha_{\mathscr{J}}.$$

We now present the calculation of $\alpha_{\mathscr{J}}^*$.

PROPOSITION (4.7). *Let $A \in C^\infty(T_1^1(\mathscr{M}))$. Then $\alpha_{\mathscr{J}}^*(A) \in \mathfrak{X}(\mathscr{M})$ is given in local coordinates as follows:*

$$\left(\alpha_{\mathscr{J}}^*(A)\right)^a = g^{ab}\left\{(A^t)_i^j \mathscr{J}_{j|b}^i - \left((A^t \circ \mathscr{J})_b^j\right)_{|j} + \left((\mathscr{J} \circ A^t)_b^j\right)_{|j}\right\},$$

where the adjoint $(A^t)_i^j = g_{ik}g^{jl}A_l^k$, *and the vertical bar represents covariant differentiation with respect to* g.

COROLLARY (4.8). *Let* $A \circ \mathcal{J} + \mathcal{J} \circ A = 0$, *let* g *be Hermitian with respect to* $\mathcal{J}$, *and let* $\nabla \mathcal{J} = 0$. *Then*

$$\left(\alpha_{\mathcal{J}}^*(A)\right)^a = 2g^{ab}\mathcal{J}_b^k\left((A^t)_k^j\right)_{|j}.$$

If $A \in \ker \alpha_{\mathcal{J}}^*$, *then*

$$\operatorname{div}_g A = g^{ik}A_{|k}^j = 0.$$

PROOF (OF (4.8)). If g is Hermitian, $g(\mathcal{J}X, Y) = -g(X, \mathcal{J}Y)$ for all X, Y, so $\mathcal{J}^t = -\mathcal{J}$. Hence, if A anticommutes with $\mathcal{J}$, then A^t also anticommutes with $\mathcal{J}$. Thus $A^t \circ \mathcal{J} = -\mathcal{J} \circ A^t$, so

$$\left(\alpha_{\mathcal{J}}^*(A)\right)^a = g^{ab}\left\{(A^t)_i^j\mathcal{J}_{jib}^i + 2\left((\mathcal{J} \circ A^t)_b^j\right)_{ij}\right\}.$$

Since $\nabla \mathcal{J} = 0$,

$$\left(\alpha_{\mathcal{J}}^*(A)\right)^a = 2g^{ab}\mathcal{J}_b^k\left\{(A^t)_k^j\right\}_{ij}.$$

Thus, if $\alpha_{\mathcal{J}}^*(A) = 0$, multiplying by $g_{ca}\mathcal{J}_d^c$ gives $-2((A^t)_d^j)_{ij} = 0$, which means that $g^{jk}A_{jk}^i = \operatorname{div}_g A = 0$. □

REMARK. If $\dim \mathcal{M} = 2$, and g is Hermitian with respect to $\mathcal{J}$, then g is also Kähler, since the fundamental two-form is then closed. Since $N(\mathcal{J}) = 0$, it follows that $\nabla \mathcal{J} = 0$ (e.g. Kobayashi-Nomizu [**10**, Vol. II, p. 149]). Thus, in two dimensions we have

PROPOSITION (4.9). *Let* $\dim \mathcal{M} = 2$, $\mathcal{J} \in \mathcal{A}$, $\dot{\mathcal{J}} \in T_{\mathcal{J}}\mathcal{A} \cap \operatorname{ket} \alpha_{\mathcal{J}}^*$, g *Hermitian with respect to* $\mathcal{J}$. *Then* $\operatorname{div}_g \dot{\mathcal{J}} = 0$.

We denote $T_{\mathcal{J}}\mathcal{A} \cap \ker \alpha_{\mathcal{J}}^*$ by $(T_{\mathcal{J}}\mathcal{A})^0$.

Finally, we have the following splitting from which (4.5) follows immediately.

PROPOSITION (4.10). *Let* $\dim \mathcal{M} = 2$, $\mathcal{J} \in \mathcal{A}$, g *Hermitian with respect to* $\mathcal{J}$. *Then* $T_{\mathcal{J}}\mathcal{A}$ *splits* L_2*-orthogonally as*

$$T_{\mathcal{J}}\mathcal{A} = \left(T_{\mathcal{J}}\mathcal{A} \cap \ker \alpha_{\mathcal{J}}^*\right) \otimes \operatorname{Range} \alpha_{\mathcal{J}}.$$

PROOF. We must show that $\operatorname{Range} \alpha_{\mathcal{J}} \subset T_{\mathcal{J}}\mathcal{A}$. Let $\psi: \mathcal{A} \to C^\infty(T_1^1(\mathcal{M}))$ be defined by $A \to A^2$. Then $T_{\mathcal{J}}\mathcal{A} = \ker D\psi(\mathcal{J}) = \{\dot{\mathcal{J}} \mid \dot{\mathcal{J}}\mathcal{J} + \mathcal{J}\dot{\mathcal{J}} = 0\}$. We also have that $\psi(f^*A) = f^*\psi(A)$, which implies

$$D\psi(A)L_X A = L_X(\psi(A)).$$

Thus, if $\mathcal{J} \in \mathcal{A}$, $\psi(\mathcal{J}) = -IL_X(\psi(\mathcal{J})) = 0$. Thus if $\mathcal{J} \in \mathcal{A}$,

$$D\psi(\mathcal{J}) \cdot L_X\mathcal{J} = D\psi(\mathcal{J}) \cdot \alpha_{\mathcal{J}}(X) = 0$$

for all $X \in \mathfrak{X}(M)$. Thus

$$\operatorname{Range} \alpha_{\mathcal{J}} \subset \ker D\psi(\mathcal{J}) = T_{\mathcal{J}}\mathcal{A}. \quad \square$$

REMARK (4.11). If $\dot{\mathcal{J}} \in T_{\mathcal{J}}\mathcal{A}$, then $\operatorname{tr} \dot{\mathcal{J}} = 0$. Thus let

$$C^\infty(T_1^1(\mathcal{M}))^{TT} = \{A \in C^\infty(T_1^1(\mathcal{M})) | \operatorname{tr} A = 0, \operatorname{div}_g A = 0\}.$$

Furthermore, let

$$(T_{\mathcal{J}}\mathcal{A})^{TT} = T_{\mathcal{J}}\mathcal{A} \cap C^\infty(T_1^1(\mathcal{M}))^{TT}.$$

Then the decomposition above can be written as

$$T_{\mathcal{J}}\mathcal{A} = (T_{\mathcal{J}}\mathcal{A})^{TT} \otimes \operatorname{Range} \alpha_{\mathcal{J}}.$$

Hence, every $\dot{\mathcal{J}} \in T_{\mathcal{J}}\mathcal{A}$ decomposes L_2-orthogonally as

$$\dot{\mathcal{J}} = \dot{\mathcal{J}}^{TT} + L_X \mathcal{J},$$

where $\operatorname{tr} \dot{\mathcal{J}}^{TT} = 0$ and $\operatorname{div}_g \dot{\mathcal{J}}^{TT} = 0$.

Let S_2 be the space of all C^∞ symmetric $(0, 2)$ tensors on $\mathcal{M}$. Then the tangent space $T_g\mathfrak{M} = \{g\} \times S_2 \approx S_2$. Let $S_2^{TT}(g) \subset S_2$ be defined by

$$S_2^{TT}(g) = \{h \in S_2 | \delta_g h = 0, \operatorname{tr}_g h = 0\}.$$

$S_2^{TT}(g)$ is the space of trace free, divergence free symmetric tensors on $\mathcal{M}$.

Let $\mathfrak{M}_{-1} \subset \mathfrak{M}$ be those metrics of constant scalar curvature negative one. Then it follows from [7] that the tangent space $T_g\mathfrak{M}_{-1}$ splits as a direct sum $S_2^{TT}(g) \oplus \operatorname{Range} \alpha_{\mathcal{J}}$. Thus every $h \in T_g\mathfrak{M}_{-1}$ can be written as $h = h^{TT} + L_X g$ for some (unique) $X \in \mathfrak{X}(\mathcal{M})$.

We shall need the following result.

THEOREM (4.12). *The C^∞ map $\mathcal{J} \to g(\Psi) = \theta(\mathcal{J})$ (see remarks following* (0.6)) *has a differential $D\theta$: $T_{\mathcal{J}}\mathcal{A} \to T_{g(\mathcal{J})}\mathfrak{M}_{-1}$ which restricts to an isomorphism between $(T_{\mathcal{J}}\mathcal{A})^{TT}$ and $S_2^{TT}(g(\mathcal{J})) \subset T_{g(\mathcal{J})}\mathfrak{M}_{-1}$.*

PROOF. Recall that θ is the inverse of the map Ξ: $\mathfrak{M}_{-1} \to \mathcal{A}$; $g \to -g^{-1}\mu_g$, μ_g the volume element of g determined by the metric g and the orientation of $\mathcal{M}$. Ξ is the restriction of the map Φ: $\mathfrak{M} \to \mathcal{A}$ given by $\Phi(g) = -g^{-1}\mu_g$ and, consequently, $D\Xi(g) = D\Phi(g)|T_g\mathfrak{M}_{-1}$. We know from [7] and an easy computation that

$$D\Phi(g)h = -\left(H - \left(\tfrac{1}{2}\operatorname{tr}_g h\right)I\right) \circ \mathcal{J},$$

where $H_j^i = g^{ik}h_{kj}$ is the contraction of a symmetric tensor field $h \in S_2$ by the metric g, and $\operatorname{tr}_g h = g^{ij}h_{ij}$ is the trace of h with respect to g. Note that $\ker D\Phi(g) = S_2^c(g)$, where

$$S_2^c(g) = \{\sigma g | \sigma \in C^\infty(\mathcal{M}, \mathbf{R})\}.$$

Let

$$h = h^T + \tfrac{1}{2}(\operatorname{tr}_g h)g, \qquad h^T = h - \tfrac{1}{2}(\operatorname{tr}_g h)g.$$

Then $D\Phi(g)h = D\Phi(g)h^T = H^T \circ \mathcal{J}$, H^T the contraction of h^T by g. Thus $\operatorname{tr} H^T = 0$ and $g(H^T X, Y) = h(X, Y)$ for all $X, Y \in \mathfrak{X}(\mathcal{M})$. Note that $H^T \circ \mathcal{J} \in T_{\mathcal{J}}\mathcal{A}$ since

$$(H^T \circ \mathcal{J}) \circ \mathcal{J} + \mathcal{J}(H^T \circ \mathcal{J}) = -H^T + \mathcal{J}H^T\mathcal{J}.$$

However, one easily checks that $\mathscr{J}H^T + H^T\mathscr{J} = 0$; in conformal coordinates

$$\mathscr{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}.$$

Thus

$$-H^T + \mathscr{J}H^T\mathscr{J} = -H^T = \mathscr{J}^2 H^T = 0.$$

Let

$$S_2^T(g) = \left\{ h \in S_2 | \operatorname{tr}_g h = 0 \right\} \quad \text{and}$$
$$D\Phi(g)^T = D\Phi(g)|S_2^T(g).$$

LEMMA (4.13). *$D\Phi(g)^T$: $S_2^T(g) \to T_{\mathscr{J}}\mathscr{A}$ is an isomorphism.*

PROOF. $D\Phi(g)$ is surjective, $\ker D\Phi(g) = S_2^c(g)$, and $S_2 = S_2^c(g) \otimes S_2^T(g)$.

REMARK. The inverse of $D\Phi(g)^T$, $(D\Phi(g)^T)^{-1}$: $T_{\mathscr{J}}\mathscr{A} \to S_2^T(g)$ is given by $\dot{\mathscr{J}} \mapsto -g^{-1}(\dot{\mathscr{J}} \circ \mathscr{J}) = g^{-1}(\mathscr{J} \circ \dot{\mathscr{J}})$. Note that if $h = -g^{-1}(\mathscr{J} \circ \dot{\mathscr{J}})$, then $\operatorname{tr}_g h = \operatorname{tr}(\mathscr{J} \circ \dot{\mathscr{J}}) = 0$, since $\operatorname{tr}(\dot{\mathscr{J}} \circ \mathscr{J}) = \operatorname{tr}(\mathscr{J} \circ \dot{\mathscr{J}}) = -\operatorname{tr}(\mathscr{J} \circ \dot{\mathscr{J}})$. Also, h is symmetric, as an easy calculation in conformal coordinates shows.

Let $D\Phi(g)^{TT} = D\Phi(g)|S_2^{TT}(g)$.

PROPOSITION (4.14). *$D\Phi(g)^{TT}$: $S_2^{TT}(g) \to (T_{\mathscr{J}}\mathscr{A})^0$, and $h \to H \circ \mathscr{J}$ is an isomorphism, where $(T_{\mathscr{J}}\mathscr{A})^0 = \{\dot{\mathscr{J}} \in T_{\mathscr{J}}\mathscr{A} | \operatorname{div}_g \dot{\mathscr{J}} = 0\}$, and $\operatorname{div}_g \dot{\mathscr{J}} = -g^{jk}\dot{\mathscr{J}}^i_{j|k}$.*

PROOF. If $h \in S_2^{TT}$ and, as we have already seen, $H \circ \mathscr{J} \in T_{\mathscr{J}}\mathscr{A}$, so $H \circ \mathscr{J} = -\mathscr{J} \circ H$, then

$$D\Phi(g)h = H \circ \mathscr{J}$$

and

$$\operatorname{div}_g(H \circ \mathscr{J}) = -\operatorname{div}_g(\mathscr{J} \circ H) = +\left(\mathscr{J}^i_e H^e_j\right)_{|k} g^{jk};$$

since $\nabla\mathscr{J} = 0$ (see remark preceding (4.9)), this is equal to

$$+\mathscr{J}^i_e H^i_{j|k} g^{jk} = -\mathscr{J}^i_e(\operatorname{div}_g H)^e = -\mathscr{J} \circ (\operatorname{div}_g H) = 0.$$

Thus, $D\Phi(g)^{TT}$ maps $S_2^{TT}(g)$ into $(T_{\mathscr{J}}\mathscr{A})^0$. Also, it must be surjective, since if $\dot{\mathscr{J}} \in (T_{\mathscr{J}}\mathscr{A})^0$, we may set $h = -g^{-1}(\mathscr{J} \circ \dot{\mathscr{J}})$. Then $\operatorname{tr}_g h = \operatorname{tr}(\mathscr{J} \circ \dot{\mathscr{J}}) = 0$, h is symmetric as above, and

$$\delta_g h = \operatorname{div}_g(\mathscr{J} \circ \dot{\mathscr{J}}) = \mathscr{J} \circ \operatorname{div}_g \dot{\mathscr{J}} = 0.$$

Thus, $h \in S_2^{TT}(g)$. This complete the proof of (4.14) and (4.12). □

THEOREM (4.15). *The differential of the map $\mathscr{J} \to \mu_{g(\mathscr{J})}$ vanishes on those $\dot{\mathscr{J}} \in T_{\mathscr{J}}\mathscr{A}$ which are horizontal.*

PROOF. By (4.12) it suffices to show that $g \mapsto \mu_g$ has a differential which vanishes on those $h \in S_2$ which are trace free. The differential of $g \mapsto \mu_g$ is easily computed to be $-\frac{1}{2}(\operatorname{tr}_g h) \cdot g$ and the result follows.

We are now ready to show that the Teichmüller space with the induced Hermitian metric $\langle\ ,\ \rangle$ is Kähler. From our previous results it only remains to show that

$d\Omega(\tilde{X}, \tilde{Y}, \tilde{Z}) = 0$ for horizontal fields $\tilde{X}, \tilde{Y}, \tilde{Z}$, where Ω is the Kähler form on $\mathscr{A}$ determined by the metric (,) and the almost complex structure Φ. Thus

$$\Omega(\tilde{Y}, \tilde{Z}) = \int_M \operatorname{tr}(\mathscr{J}\tilde{Y}\tilde{Z})\, d\mu_{g(\mathscr{J})}.$$

We again consider the terms in (4.3) for $d\Omega$.

LEMMA (4.16). *For horizontal fields* $\tilde{X}, \tilde{Y}, \tilde{Z}$,

$$\tilde{X}(\Omega(\tilde{Y}, \tilde{Z})) = \int_M \operatorname{tr}(\tilde{X}\tilde{Y}\tilde{Z})\, d\mu_{g(\mathscr{J})} + \int_M \operatorname{tr}(\mathscr{J}(D\tilde{Y}(\tilde{X}))\tilde{Z})\, d\mu_{g(\mathscr{J})}$$
$$+ \int_M \operatorname{tr}(\mathscr{J}\tilde{Y}(D\tilde{Z}(\tilde{X})))\, d\mu_{g(\mathscr{J})}.$$

PROOF. This follows at once, using the fact that $T \to \mu_{g(\mathscr{J})}$ has a differential which vanishes on horizontal fields.

LEMMA (4.17). *If* $\tilde{X}, \tilde{Y}, \tilde{Z} \in T_{\mathscr{J}}\mathscr{A}$ *then* $\operatorname{tr}(\tilde{X}\tilde{Y}\tilde{Z}) = 0$.

PROOF. Since these fields anticommute with $\mathscr{J}$ we get

$$-\operatorname{tr}(\tilde{X}\tilde{Y}\tilde{Z}) = \operatorname{tr}(\mathscr{J}\tilde{X}\tilde{Y}\tilde{Z}\mathscr{J}) = -\operatorname{tr}(\mathscr{J}\tilde{X}\tilde{Y}\mathscr{J}\tilde{Z}) = \operatorname{tr}(\mathscr{J}\tilde{X}\mathscr{J}\tilde{Y}\tilde{Y}\tilde{Z})$$
$$= -\operatorname{tr}(\mathscr{J}^2\tilde{X}\tilde{Y}\tilde{Z}) = \operatorname{tr}(\tilde{X}\tilde{Y}\tilde{Z}). \quad \square$$

LEMMA (4.18). *On horizontal fields*

$$\tilde{X}(\Omega(\tilde{Y}, \tilde{Z})) = \int_M \operatorname{tr}(\mathscr{J}(D\tilde{Y}(\tilde{X}))\tilde{Z})\, d\mu_{g(\mathscr{J})} + \int_M \operatorname{tr}(\mathscr{J}\tilde{Y}(D\tilde{Z}(\tilde{X})))\, d\mu_{g(\mathscr{J})}.$$

PROOF. Immediate from (1) and (2).

Similarly, we see that

$$\tilde{Y}(\Omega(\tilde{Z}, \tilde{X})) = \int_M \operatorname{tr}(\mathscr{J}(D\tilde{Z}(\tilde{Y}))\tilde{X})\, d\mu_{g(\mathscr{J})} + \int_M \operatorname{tr}(\mathscr{J}\tilde{Z}(D\tilde{X}(\tilde{Y})))\, d\mu_{g(\mathscr{J})} \tag{4.19}$$

$$\tilde{Z}(\Omega(\tilde{X}, \tilde{Y})) = \int_M \operatorname{tr}(\mathscr{J}(D\tilde{X}(\tilde{Z}))\tilde{Y})\, d\mu_{g(\mathscr{J})} + \int_M \operatorname{tr}(\mathscr{J}\tilde{X}(D\tilde{Y}(\tilde{Z})))\, d\mu_{g(\mathscr{J})}. \tag{4.20}$$

Moreover,

$$\Omega([\tilde{X}, \tilde{Y}], \tilde{Z}) = \int_M \operatorname{tr}(\mathscr{J}\{D\tilde{Y}(\tilde{X}) - D\tilde{X}(\tilde{Y})\}\tilde{Z})\, d\mu_{g(\mathscr{J})}, \tag{4.21}$$

$$\Omega([\tilde{Y}, \tilde{Z}], X) = \int_M \operatorname{tr}(\mathscr{J}\{D\tilde{Z}(\tilde{Y}) - D\tilde{Y}(\tilde{Z})\}\tilde{X})\, d\mu_{g(\mathscr{J})}, \tag{4.22}$$

$$\Omega([\tilde{Z}, \tilde{X}], \tilde{Y}) = \int_M \operatorname{tr}(\mathscr{J}\{D\tilde{X}(\tilde{Z}) - D\tilde{Z}(\tilde{X})\}\tilde{Y})\, d\mu_{g(\mathscr{J})}. \tag{4.23}$$

Using (4.3) and the above relationship we see that $3 \cdot d\Omega(\tilde{X}, \tilde{Y}, \tilde{Z})$ is given by (4.24)

$$3 \cdot d\Omega(\tilde{X}, \tilde{Y}, \tilde{Z}) = \int_M \operatorname{tr}\big(\mathscr{J}\tilde{Y}(D\tilde{Z}(\tilde{X}))\big)_{g(\mathscr{J})}\, d\mu_{g(\mathscr{J})} + \int_M \operatorname{tr}\big(\mathscr{J}\tilde{Z}(D\tilde{X}(\tilde{Y}))\big)\, d\mu_{g(\mathscr{J})}$$
$$+ \int_M \operatorname{tr}\big(\mathscr{J}\tilde{X}(D\tilde{Y}(\tilde{Z}))\big)\, d\mu_{g(\mathscr{J})} + \int_M \operatorname{tr}\big(\mathscr{J}\{D\tilde{Z}(\tilde{X})\}\tilde{Y}\big)\, d\mu_{g(\mathscr{J})}$$
$$+ \int_M \operatorname{tr}\big(\mathscr{J}\{D\tilde{X}(\tilde{Y})\}\tilde{Z}\big)\, d\mu_{g(\mathscr{J})} + \int_M \operatorname{tr}\big(\mathscr{J}\{D\tilde{Y}(\tilde{Z})\}\tilde{X}\big)\, d\mu_{g(\mathscr{J})}.$$

Since $\tilde{Z}\mathscr{J} = -\mathscr{J}\tilde{Z}$, we may differentiate this in the direction of $\tilde{X}$ to obtain

$$D\tilde{Z}(\tilde{X})\mathscr{J} + \tilde{Z}\tilde{X} = -\tilde{X}\tilde{Z} - \mathscr{J}\tilde{D}\tilde{Z}(\tilde{X}).$$

Therefore

$$\mathscr{J}D\tilde{Z}(\tilde{X}) = -\tilde{X}\tilde{Z} - \tilde{Z}\tilde{X} - D\tilde{Z}(\tilde{X})\mathscr{J}.$$

Since $\operatorname{tr}(\tilde{X}\tilde{Z}\tilde{Y}) = \operatorname{tr}(\tilde{Z}\tilde{X}\tilde{Y}) = 0$ (Lemma 2) we find that the fourth term in (4.24) is

$$\int_M \operatorname{tr}\big(\mathscr{J}\{D\tilde{Z}(\tilde{X})\}\tilde{Y}\big)\, d\mu_{g(\mathscr{J})} = -\int_M \operatorname{tr}\big(\{D\tilde{Z}(\tilde{X})\}\mathscr{J}\tilde{Y}\big)\, d\mu_{g(\mathscr{J})}$$
$$= -\int_M \operatorname{tr}\big(\mathscr{J}\tilde{Y}\{D\tilde{Z}(\tilde{X})\}\big)\, d\mu_{g(\mathscr{J})}.$$

Consequently, the sum of the first and fourth terms in (4.24) is zero. Similarly, the remaining terms cancel and $d\Omega(\tilde{X}, \tilde{Y}, \tilde{Z}) = 0$.

We summarize this in

THEOREM (4.25). *The Hermitian metric on the Teichmüller space of a two-dimensional surface $\mathscr{M}$ of genus greater than one, which is induced from the L_2-metric on the space of almost complex structures on $\mathscr{M}$, is Kähler.*

We conclude this paper by showing that this metric is indeed the Weil-Petersson metric. To accomplish this we need some results from [7].

THEOREM (4.26). *Let $\mathscr{M}$ be a compact orientable surface without boundary of genus p, $p > 1$. Then* $\dim S_2^{TT}(g) = 6p - 6$.

The space $\mathscr{D}$ of diffeomorphisms of $\mathscr{M}$ acts on the space of metrics $\mathfrak{M}$ of $\mathscr{M}$ via pull- back; i.e., for $u, v \in T_x\mathscr{M}$, $g \in \mathfrak{M}$, $f \in \mathscr{D}$,

$$(f^*g)(x)(u, v) = g(f(x))(df(x)u, df(x)v).$$

If $R\colon \mathfrak{M} \to C^\infty(\mathscr{M}, \mathbf{R})$ denotes the C^∞ scalar curvature map, then

$$R(f^*g) = f^*R(g) = R(g) \circ f.$$

Thus, if $R(g) = -1$, $R(f^*g) = -1$. Consequently, the action of $\mathscr{D}$ on $\mathfrak{M}$ induces an action of $\mathscr{D}$ on $\mathfrak{M}_{-1}$. We then have the following result from [7]:

THEOREM (4.27). *The triple $(\pi, \mathfrak{M}_{-1}, \mathfrak{M}_{-1}/\mathscr{D}_0)$ is a principal strong ILH fibre bundle with structure group $\mathscr{D}_0$ over the C^∞ $(6p - 6)$-dimensional manifold $\mathfrak{M}_{-1}/\mathscr{D}_0$. The map $\Xi\colon \mathfrak{M}_{-1} \to \mathscr{A}$; $g \to -g^{-1}\mu_g$ is a (strong ILH) bundle isomorphism. Moreover, $T_{[g]}(\mathfrak{M}_{-1}/\mathscr{D}_0)$ is canonically isomorphic to $S_2^{TT}(g)$.*

We would like to place a weak-Riemannian structure on the bundle $(\pi, \mathfrak{M}_{-1}, \mathfrak{M}_{-1}/\mathscr{D}_0)$ as we did with $(\pi, \mathscr{A}, \mathscr{A}/\mathscr{D}_0)$. Now there is a natural L_2-metric on the space $\mathfrak{M}$. For $g \in \mathfrak{M}$ we define, for $h, k \in S_2$,

$$(h, K)_g = \int_M h \cdot k \, d\mu_{(g)},$$

where $h \cdot k$ denotes the contraction of h and k via the metric g; in coordinates,

$$h \cdot k = g^{ab} g^{cd} h_{ac} k_{bd}.$$

It is not difficult to see that the map Ξ: $\mathfrak{M}_{-1} \to \mathscr{A}$, with $\mathscr{A}$ and $\mathfrak{M}_{-1}$ having their respective L_2-metrics, is not an isometry. The L_2-metric on $\mathfrak{M}_{-1}$ is, on the other hand, $\mathscr{D}$-invariant and so induces a metric on $\mathfrak{M}_{-1}/\mathscr{D}_0$. We then have the following result:

THEOREM (4.28). *If $\mathscr{A}/\mathscr{D}_0$ and $\mathfrak{M}_{-1}/\mathscr{D}_0$ are given the Riemannian metrics induced by the $\mathscr{D}$-invariant metrics on $\mathscr{A}$ and $\mathfrak{M}_{-1}$, then the induced map $\tilde{\Xi}$: $\mathfrak{M}_{-1}/\mathscr{D}_0 \to \mathscr{A}/\mathscr{D}_0$ is an isometry.*

PROOF. We know from (4.11) that every $\dot{\mathscr{J}} \in T_{\mathscr{J}}\mathscr{A}$ can be decomposed as $\dot{\mathscr{J}} = \dot{\mathscr{J}}^{TT} + L_X \mathscr{J}$, where $\dot{\mathscr{J}}^{TT}$ is divergence free and $L_X \mathscr{J}$ is the Lie derivative of $\mathscr{J}$ with respect to some (unique) $X \in \mathfrak{X}(\mathscr{M})$. This is an L_2-orthogonal decomposition. Similarly, as already pointed out, we know from [7] that every $h \in T_g \mathfrak{M}_{-1}$ has the L_2-orthogonal decomposition

$$h = h^{TT} + L_y g, \tag{4.29}$$

$L_Y g$ the Lie derivative of g with respect to a (unique) $Y \in \mathfrak{X}(\mathscr{M})$. We know from (4.12) that

$$D\Xi(g)\colon S_2^{TT}(g) \xrightarrow{\equiv} (T_{\mathscr{J}}\mathscr{A})^0$$

is an isomorphism. Again, let Φ: $\mathfrak{M} \to \mathscr{A}$ be the map $g \to -g^{-1}\mu_g$. Then $D\Phi(g)|\mathfrak{M}_{-1} = D\Xi(g)$. We must show that if $\dot{\mathscr{J}}_i = D\Phi(g)h_i$, $i = 1, 2$, then

$$(\dot{\mathscr{J}}_1, \dot{\mathscr{J}}_2) = (h_1, h_2)_{g(\mathscr{J})}.$$

Recall that

$$D\Phi(g)h = g^{-1}\left(\tfrac{1}{2}\operatorname{tr}_g h - h\right) d\mu_g.$$

On $S_2^{TT}(g)$,

$$D\Phi(g)h = -g^{-1} \cdot h \, d\mu_g,$$

i.e., in local coordinates,

$$D\Phi(g)h = -h^{ik}\mu_{kj}.$$

Now

$$(\dot{\mathscr{J}}_1, \dot{\mathscr{J}}_2)_{\mathscr{J}} = -\int_M \operatorname{tr}(\dot{\mathscr{J}}_1, \dot{\mathscr{J}}_2) \, d\mu_{g(\mathscr{J})},$$

But

$$\operatorname{tr}(\dot{\mathscr{J}}_1, \dot{\mathscr{J}}_2) = (D\Phi(g)h_1, D\Phi(g)h_2)_g.$$

Thus

$$(\dot{\mathscr{J}}_1, \dot{\mathscr{J}}_2)_{\mathscr{J}} = -\int_M \left(h_1^{ik}\mu_{kj}h_2^{al}\mu_{lb}g_{ia}g^{jb}\right) d\mu_g.$$

Using the facts that $\mu_i^k\mu_k^j = -\delta_k^j$ and $\mu_i^k\mu_{kj} = -g_{ij}$, we find that this is equal to

$$+\int_M \left(h_1^{ik}g_{kl}h_2^{al}g_{ia}\right) d\mu_{g(\mathscr{J})} = +\int_M \langle h_1, h_2\rangle_g \, d\mu_{g(\mathscr{J})} = (h_1, h_2)_{g(\mathscr{J})}. \quad \square$$

By this isometry result it suffices to consider the induced metric on $\mathfrak{M}_{-1}/\mathscr{D}_0$. The decomposition (4.29) says that the horizontal subspace of $(\pi, \mathfrak{M}_{-1}, \mathfrak{M}_{-1}/\mathscr{D}_0)$ at $g \in \mathfrak{M}_{-1}$ is precisely $S_2^{TT}(g)$.

THEOREM (4.31). *The induced L_2-metric on $\mathfrak{M}_{-1}/\mathscr{D}_0$ is the Weil-Petersson metric on Teichmüller space.*

So let $X_{[g]}$, $Y_{[g]} \in T_{[g]}(\mathfrak{M}_{-1}/\mathscr{D}_0)$. Then there are unique $\varphi, \psi \in S_2^{TT}(g)$ with $D\pi(g)\varphi = X_{[g]}$, $D\pi(g)\psi = Y_{[g]}$, and $\langle X_{[g]}, Y_{[g]}\rangle_{[g]} = (\varphi, \psi)_g$. Now let D be the unit disc in the plane. Then there is a unique complete metric of scalar curvature negative one pointwise conformally equivalent to the Euclidean metric. This metric, called the Poincaré metric, is given by

$$ds^2 = dx\,dy/(1 - |z|^2)^2.$$

It is well known (e.g. see [**14**]) that every orientable compact surface $\mathscr{M}$ without boundary of genus p, $p > 1$, can be represented isometrically as a quotient D/Γ, where Γ is a subgroup of the isometries of D with respect to this metric. Thus, if $\tilde{D}$ denotes a fundamental domain of this action,

$$(\varphi, \psi) = \int_{\tilde{D}} \varphi \cdot \psi \frac{dx\,dy}{(1 - |z|^2)^2} \tag{4.30}$$

(for a definition of $\varphi \cdot \psi$ see remarks following (4.27)).

From [7] we have

PROOF. If in a fundamental domain $\tilde{D}$ we represent

$$\varphi = \begin{pmatrix} \varphi_1, & -\varphi_2 \\ -\varphi_2, & -\varphi_1 \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi_1 & -\psi_2 \\ -\psi_2 & -\psi_1 \end{pmatrix},$$

then

$$\varphi_* = (\varphi_1 + i\varphi)(dx + idy)^2 \quad \text{and} \quad \psi_* = (\psi_1 + i\psi_2)(dx + idy)^2$$

represent holomorphic quadratic differentials on $\mathscr{M} = \mathscr{D}/\Gamma$.

From (4.30) we can easily compute $\varphi \cdot \psi$ since the metric g on the fundamental domain $\tilde{D}$ must be given by $dx\,dy/(1 - |z|^2)^2$. We see that

$$\varphi \cdot \psi = 2(1 - |z|^2)^4(\varphi_1\psi_1 + \varphi_2\psi_2) = 2(1 - |z|^2)^4 \operatorname{Re}(\varphi_*\bar{\psi}_*).$$

Thus

$$(X_{[g]}, Y_{[g]})_{[g]} = (\varphi, \psi)_g = 2\,\mathrm{Re}\int_{\mathscr{D}_0} \varphi_* \bar{\psi}_* (1 - |z|^2)^2\, dx\, dy.$$

From Ahlfors' formula (4.2) in [**2**, p. 186], this is a scalar multiple of the Weil-Petersson metric after identifying, in another fashion, the tangent space of Teichmüller space as the space of holomorphic quadratic differentials on $\mathscr{M}$.

References

1. Lars. V. Alhfors, *On quasiconformal mappings*, J. Analyse Math. **4** (1954), 1–58.
2. _____, *Some remarks on Teichmüller's space of Riemann surfaces*, Ann. of Math. (2) **74** (1961), 171–191.
3. _____, *The complex analytic structure on the space of closed Riemann surfaces*, Analytic Functions, Princeton Univ. Press, Princeton, N. J., 1960.
4. _____, *Curvature properties of Teichmüller's space*, J. Analyse Math. **9** (1961), 161–176.
5. M. Berger and D. Ebin, *Some decompositions of the space of symmetric tensors on a Riemannian manifold*, J. Differential Geom. **3** (1969), 379–392.
6. A. Fischer and J. Marsden, *Manifolds of conformally equivalent metrics*, Canad. J. Math. **29** (1977), 193–209.
7. A. Fischer and A. Tromba, *On a purely Riemannian proof of the structure and dimension of the unramified moduli space of a compact Riemann surface*, SFB 72, preprint series, Bonn.; Math. Ann. (to appear).
8. _____, *Almost complex principal fibre bundles and the complex structure on Teichmüller space*, SFB 72, preprint series, Bonn.; Crelle's J. (to appear).
9. S. Kobayashi, *Intrinsic metrics on complex manifolds*, Bull. Amer. Math. Soc. **73** (1967), 347–349.
10. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vols. I, II, Interscience, New York, 1963.
11. H. Omori, *On the group of diffeomorphisms on a compact manifold*, Proc. Sympos. Pure Math., Vol. 15, Amer. Math. Soc., Providence, R. I., 1970, pp. 167–183.
12. H. L. Royden, *Remarks on the Kobayashi metric*, Several Complex Variables, II (Maryland, 1970), Springer, Berlin, 1971, pp. 125–137.
13. O. Teichmüller, *Extremale quasikonforme Abbildungen und quadratische Differentiale*, Abh. Preuss. Acad. Wiss. Math.-Nat. Kl., No. 22 (1939), 197 pp.
14. J. Wolf, *Spaces of constant curvature*, McGraw-Hill, New York, 1967.

Department of Mathematics, University of California, Santa Cruz, California 95064

Sonderforschungsbereich 72, Universität Bonn, Bonn, Federal Republic of Germany