RATE OF APPROACH TO MINIMA AND SINKS—
THE MORSE-SMALE CASE

BY

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ABSTRACT. The dynamical systems herein are Morse-Smale diffeomorphisms and flows on C^∞ compact manifolds. We show the asymptotic rate of approach of orbits to the sinks of the systems to be bounded by an expression of the form $K\exp(-DN)$, where D may be any number smaller than $C = \min_P\{1/m \log \operatorname{Jac} D_P f^m | W^u(P)\}$. Here the minimum is taken over all nonsink P in the nonwandering set of f, and m is the period of P. We extend our theorems to the entire manifold, so that there is no restriction on the location of the initial points of trajectories.

1. Introduction. The dynamical systems herein are Morse-Smale diffeomorphisms and flows on compact manifolds of finite dimension. Determining the asymptotic rate of approach of orbits to sinks amounts to comparing the Riemannian measure of the entire manifold to the measure of the set of points whose orbits remain outside a neighborhood of the sinks after N iterations for diffeomorphisms, or time T for flows.

Let P be a fixed point for f which is either a source or a saddle and let U be a neighborhood of P on M. If $x \in U$, then unless x is on the stable manifold of P, the orbit of P under f will leave U. This is simply the familiar fact from the stable manifold theorem that for U small enough $\bigcap_{n=0}^{\infty} f^{-n}(U)$ is the local stable manifold of P. The volume lemmas of Bowen and Ruelle [2] and Fried and Shub [5] add to this statement that the measure of $\bigcap_{n=0}^{\infty} f^{-n}(U)$ decays exponentially with the rate. By this we mean that the exponential constant is related to the logarithm of the Jacobian determinant of the unstable part of f at P. However, these results are local in nature; they only concern the rate at which orbits leave a neighborhood of P. We prove the related global results for all Morse-Smale diffeomorphisms and flows.

Throughout the paper M will be a C^∞ compact Riemannian manifold, and μ will be a measure derived from the Riemannian metric on M. For $r > 1$, let $\mathrm{Diff}(M)$ be the set of C^r diffeomorphisms of M. For the definitions of nonwandering point,

stable and unstable manifold, and Morse-Smale diffeomorphism we refer the reader to [7, 10]. Here we define filtrations.

DEFINITION. Let $f \in \text{Diff}(M)$. A filtration for f is a sequence of compact manifolds with boundary such that

$$M = M_k \supset M_{k-1} \supset \cdots \supset M_1 \supset M_0 = \varnothing$$

($\dim M_i = \dim M$, for $i = 1, \ldots, k$) and $f(M_k) \subset \text{int } M_k$. Given a filtration, $K_i = \bigcap_{n \in Z} f^n(M_i - \text{int } M_{i-1})$ is the maximal invariant set contained in $(M_i - M_{i-1})$. If $K_i = \Omega \cap (M_i - M_{i-1})$ for all i, we say that the filtration is a fine filtration for f. Finally, if we are given closed invariant sets which are disjoint $\Lambda_1, \ldots, \Lambda_k$, we say $M = M_k \supset M_{k-1} \supset \cdots \supset M_1 \supset M_0 = \varnothing$ is a filtration for $\Lambda_1, \ldots, \Lambda_k$ if $\Lambda_i = K_i$.

DEFINITION. $S(i, j, N) = M_i - f^{-N}(\text{int } M_j)$.

Given a filtration we may assume without loss of generality that M_1 contains all the sinks. We will denote $(M_i - \text{int } M_j)$ by $M[i, j]$.

DEFINITION. $\text{Jac } Df^u P_j = \text{Jac}(Df \colon E_{P_j}^u \to E_{P_j}^u)$.

The following theorem is our main result.

THEOREM A. *Let f be any Morse-Smale diffeomorphism on M. Let l be the least common multiple of the periods of the periodic orbits of f. Let $\delta > 0$ be given. Then for all N, and $i > j$ we have*

$$\mu(S(i, j, N)) \leqslant K(1 + \delta)^N \exp((-C/l)N)$$

with $K > 0$ independent of N, and

$$C/l = \text{Min}\left\{ 1/m \log \text{Jac } D_{p_j} f^m | W^u(p_j) \right\},$$

where the minimum is taken over all nonsink p_j in the nonwandering set for f, p_j not a sink and $f^m p_j = p_j$.

The proof of Theorem A consists of 3 parts and is presented in detail in §4. The following is a sketch.

(1) First let $g = f^l$. Then g is a Morse-Smale diffeomorphism with $\Omega(g) = \text{Fix}(g)$. Once the result is obtained for g, its extension to f is immediate.

(2) We know from [10] that g has a fine filtration: $M = \overline{M}_r \supset \overline{M}_{r-1} \supset \cdots \supset \overline{M}_1 \supset \overline{M}_0 = \varnothing$ such that the maximal invariant set contained in $\text{int}(\overline{M}_1 - \overline{M}_{i-1})$ is a fixed point which we label P_i. Also $P_1, P_2, \ldots, P_s$ for $1 \leqslant s < r$ are sinks. So for simplicity we define a new filtration by

$$M = M_{r-s+1} \subset M_{r-s} \subset \cdots \subset M_2 \subset M_1 \subset M_0 = \varnothing$$

where $M_1 = \overline{M}_s$, $M_2 = \overline{M}_{s+1}$, etc. Thus all the sinks of g are located in M_1.

We then make use of

THEOREM B. *Let f be a Morse-Smale diffeomorphism on the C^∞ compact manifold M with $\Omega(f) = \text{Fix}(f)$. Let μ be a measure derived from the Riemannian metric on M. Let $\delta > 0$ be given. Then for each $P_i \in \Omega(f)$, $i = s + 1, \ldots, r$, there is a compact neighborhood $U(P_i)$ such that with $U_N(P_i) = \{ x \in U(P_i) \colon f^k(x) \notin M_1 \text{ for } k \leqslant N \}$.*

Furthermore

$$\mu(U_N(P_i)) \leqslant K(1 + \delta)^N \exp(-CN),$$

where $K > 0$ *is independent of* N *and* $\exp(C) = \min_{j=s+1,...,i}\{\operatorname{Jac} Df^u P_j\}$.
  This means that $\mu(U_N(P_i))$ *decays exponentially with* N.

We note that Theorem B establishes Theorem A in the special case that the trajectories are constrained to begin in a neighborhood of a periodic point. Theorem B and the technical lemmas needed are proven in §3.

(3) Next consider: $S(i, 1, N) = M_i - f^{-N}(\operatorname{int} M_1)$. We write

$$S(i, 1, N) = S(i - 1, 1, N) \cup S(i, i - 1, N) \cup A_N,$$

where A_N is the set of points in $M[i, i - 1]$ whose trajectories enter M_{i-1} but do not reach M_1 through N iterations. We proceed by induction so that the volume estimate for $S(i - 1, 1, N)$ is assumed known. By the filtration we show that the measure of $S(i, i - 1, N)$ is related by bounded factors to the measure of $U_N(P_i)$. Again by the filtration we show that A_N subdivides further into two parts—one whose orbits pass to M_{i-1} in a bounded number of iterations and one where orbits enter $U(P_i)$ again in bounded iterations. The measure of the first is thus related by bounded factors to $\mu(S(i - 1, 1, N))$ and the measure of the second is related by bounded factors to $\mu(U_N(P_i))$. Thus we obtain the result for $\mu(S(i, 1, N))$. Since $S(1, j, N) \subset S(i, 1, N)$ the proof is completed.

In §2 we begin the body of the paper with an example of a Morse-Smale diffeomorphism whose rate of volume decay is exactly that given in our theorems.

§5 concludes the paper with the proof that our results carry over to the case of Morse-Smale flows.

Without the assumption of transversality, we found the exponential constant to be more complicated and to yield a slower overall rate of decay. This is the case for general C^2 Axiom A systems with no cycles. For Axiom A systems we found the exponential constant relates to the topological pressure of f. We refer the interested reader to our paper [11].

2. An example with transversal intersection. Consider the torus $T^2 \subset R^3$ which is tilted back with respect to the horizontal plane. Let the gradient field on it be of the form

$$\dot{X} = -\operatorname{grad}(h),$$

where h is the height function of points in relation to the horizontal plane. So the flow is downward. Fix time t to τ, and consider the diffeomorphism x_τ. Now consider a diffeomorphism g described as follows: let $g = \operatorname{id}$ on the torus except in a

small band. In this band g moves points along their level curves. g rotates the level curves in the following manner: the uppermost curve rotates 0 radians, the lowermost rotates 2π radians, and the curves in between rotate from 0 to 2π going down the band. Set $h = X \circ g$. The effect of h is to push g down causing repetition of the pattern resulting in the transversal intersection of stable and unstable manifolds (Figure 1).

FIGURE 1
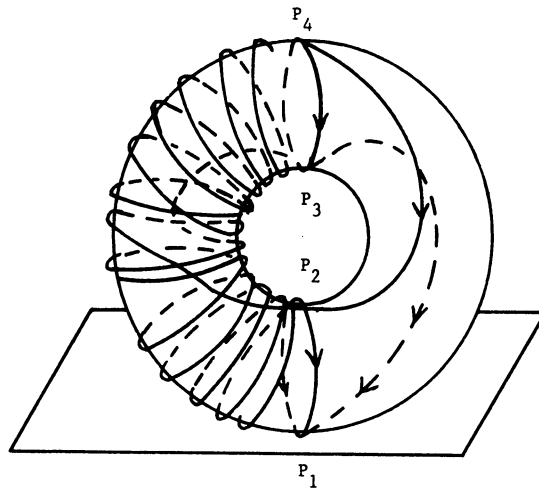
The intersection between P_3 and P_2 is illustrated in Figure 2. We note that in the figures, P_4 is the source, P_3 and P_2 are the saddles, and P_1 is the sink.

In Figure 2 there are an infinite number of such intersections tending forward to P_2. The λ Lemma of Palis [6] tells us that $W^u(P_3)$ in a neighborhood of P_2 becomes close to $W^u(P_2)$ in both distance and slope. In fact, it contains $W^u(P_2)$ in its closure.

Next in linearized neighborhoods of P_3 and P_2 the local diffeomorphisms can be given by

$$(**) \qquad \begin{aligned} X &= X_0 \exp(\gamma), \quad X = X_0 \exp(-\gamma), \\ Y &= Y_0 \exp(-\alpha), \quad Y = Y_0 \exp(\beta), \end{aligned}$$

respectively. The logarithms of the eigenvalues are γ, $-\alpha$, $-\gamma$, β. For sets moving from P_3 to P_2, the transversality property causes the two unstable directions to become aligned. This is shown in Figure 2, where we only consider one point of transversal intersection outside the nieghborhoods for clarity.

By the transversality property the height of the set which comes into the neighborhood of P_2 is independent of the number of iterations for which it stays in a neighborhood of P_3. So only $W^u(P_2)$ influences the set in the neighborhood.

Using the diffeomorphisms given in $(**)$ we show that the area of the set whose orbits remain in the neighborhood of P_3 for exactly n iterations and in the neighborhood of P_2 for at least m iterations equals $4\varepsilon^2 \exp(-\beta m) \exp(-\gamma n)$.

FIGURE 2

$$D = [\epsilon \exp(-\gamma), \epsilon] \times [-\epsilon, \epsilon]$$
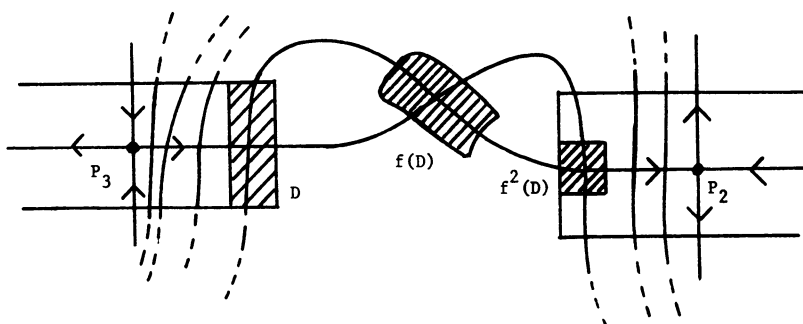
To see how this estimate is obtained consider the following illustrations. First consider the linearized neighborhood of P_3, in Figure 3.

FIGURE 3

A = Set of points which stay in the neighborhood of P_3 for exactly n iterations

$$= [\epsilon \exp(-\gamma(n + 1)), \epsilon \exp(-\gamma n)] \times [-\epsilon, \epsilon],$$

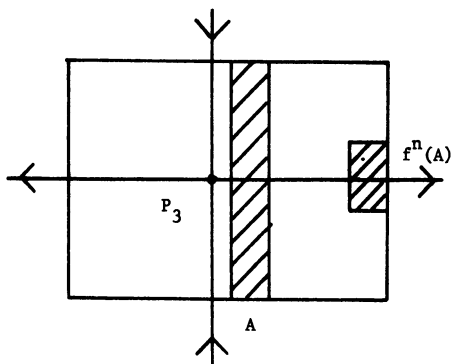$$f^n(A) = [\epsilon \exp(-\gamma), \epsilon] \times [-\epsilon \exp(-\alpha n), \exp(-\alpha, n)]$$

FIGURE 4

$B = f^{n+1}(A)$. The height of B = width A = $\epsilon(1 - \exp(-\gamma))$. Due to the transversality condition, the height is independent of the number of iterations the set was at P_3.

$$B = [-\epsilon, -\epsilon \exp(-\gamma)] \times [-(\epsilon/2)(1 - \exp(-\gamma), \epsilon/2)].$$
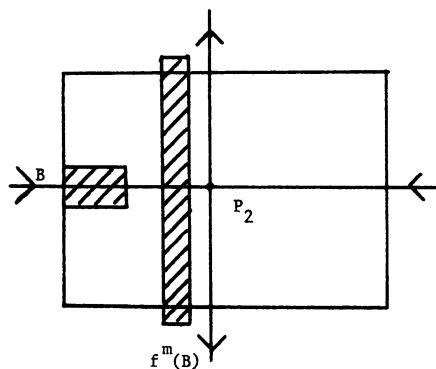
FIGURE 5

B' = Set of points which stay in the neighborhood of P_2 for m iterations. It is denoted by the shaded strip.
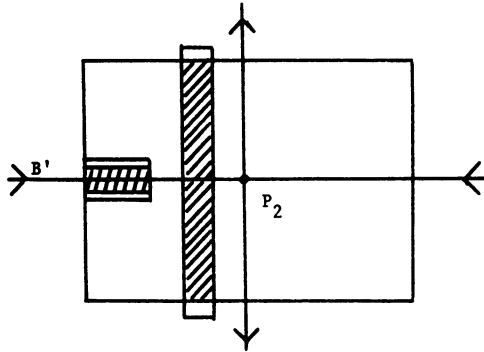
FIGURE 6

Translate the set B' back to P_3. "Go back" n iterations at P_3.
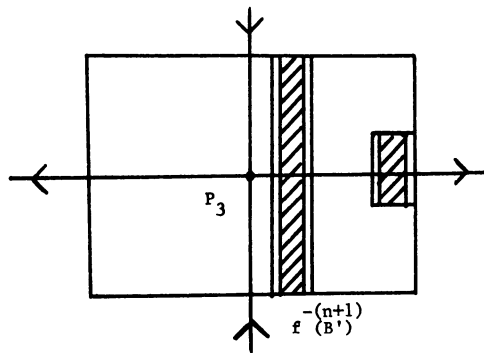
Let $E = \{f^{-(n+1)}\}(B')$. The width of $E = 2\varepsilon \exp(-\beta m) \exp(-\gamma n)$. The area of $E = 4\varepsilon^2 \exp(-\beta m) \exp(-\gamma n)$. Take $\beta = \gamma$ and call it C.

Furthermore, to obtain the area of the set in the neighborhood of P_3 whose orbits do not leave the neighborhoods of P_2 or P_3 through N iterations, sum over n and m in the relation $N = n + m + 1$. Considering the number of ways to add $n + m$ we have the area $= \sum_{n=0}^{N} 4\varepsilon^2 \exp(C) \exp(-CN)$, so we obtain that the area $= 4\varepsilon^2 (\exp C)(N + 1) \exp(-CN)$. We note that this example is a Morse-Smale diffeomorphism.

3. The Proof of Theorem B. We prove Theorem B in two stages. First we consider submanifolds of M in a neighborhood $U(P)$ of $P \in \Omega(f)$ which are of the same dimension as $W^u(P)$ and C^1 close to $W^u(P)$, showing that the volume of these submanifolds in $U_N(P)$ decays exponentially. Secondly, we cover $U_N(P)$ by a C^1 system of such manifolds and use Fubini's Theorem to arrive at our result for $U_N(P)$.

For $P \in \Omega(f)$ we know that $T_P M$ splits into the direct sum decomposition
$$T_P M = E^s \oplus E^u,$$

with stable (E^s) and unstable (E^u) subspaces. Let $(\,,\,)$ be a Riemannian metric on M adapted to f; that is for $v = (v_s, v_u) \in T_P M$,
$$\||D_P f(v_s)\|| \leqslant \lambda \||v_s\|| \quad \text{and} \quad \||D_P f(v_u)\|| \geqslant \lambda^{-1} \||v_u\||$$

when $0 < \lambda < 1$ is independent of P and $\||w\||^2 = (w, w)$.

We introduce a modified metric $\langle\,,\,\rangle$ on M by $\langle v, w \rangle = (v_s, w_s) + (v_u, w_u)$ and denote the stable and unstable seminorms $(v_s, v_s)^{1/2}$ and $(v_u, v_u)^{1/2}$ by $\|v\|_s$ and $\|v\|_u$. Then the norm defined by $\langle v, v \rangle^{1/2}$ is
$$\|v\| = \left(\|v\|_s^2 + \|v\|_u^2 \right)^{1/2}.$$

For a subspace $V \subset T_P M$, $v(\varepsilon) \equiv \{ v \in V : \|v\| \leqslant \varepsilon \}$. For small $\varepsilon > 0$ we can find a C^1 chart $\phi : T_P M(\varepsilon) \to M$ such that:

(1) $\phi(0) = P$ and $D_0 \phi : T_P M \to T_P M$ is the identity;

(2) $\phi(E^s(\varepsilon)) \subset W_P^s$; and

(3) $\phi(E^u(\varepsilon)) \subset W_P^u$

where W_P^s and W_P^u are the stable and unstable manifolds at P. The map $F = \phi^{-1} \circ f \circ \phi$ represents f at P in this coordinate system (Bowen and Ruelle [2] and Fried and Shub [5]). Such a chart can be chosen for each element of $\Omega(f)$ and we assume that ε suffices for all $P \in \Omega(f)$.

In Proposition 1 the slope of a smooth manifold N of $T_i M(\varepsilon)$ with $\dim(N) \leqslant \dim(E^u)$ at a point $n \in N$ refers to the supremum of the quotients $\|v^s\|/\|v^u\|$, taken over all tangent vectors $v = (v^s, v^u)$ to N at n. To say that N has slope less than or equal to ω relative to E_i^u means that for all $n \in N$, the slope of N at n is less than or equal to ω.

DEFINITION. $B_i(\varepsilon) = \{ x \in M : d(x, P_i) \leqslant \varepsilon \}$.

In the remainder of the paper, for $P_i \in \Omega(f)$, let $T_i M = T_{P_i} M$, $E_i^s = E_{P_i}^s$ and $E_i^u = E_{P_i}^u$.

DEFINITION. The graph transform Γ_i defined on C^1 functions g from $E_i^u(\varepsilon_i)$ to $E_i^s(\varepsilon_i)$ is defined by graph $\Gamma_i(g) = F_i$ (graph g) with $F_i = \phi_i^{-1} \circ f \circ \phi_i$.

We assume that ε is small enough so that we have C^1 charts as specified and graph transforms.

PROPOSITION 1. *For $\varepsilon, \delta > 0$ and $0 < \omega$ finite there exist constants $\varepsilon_i, \omega_i, i = 2, \ldots, r$, such that*

(1) $\varepsilon_i \leqslant \varepsilon$.

(2) $\omega_j \geqslant \omega_i/3$ *for* $i < j$.

(3) *If* $L \in \text{Hom}(E_i^u, E_i^s)$, $\|L\| \leqslant \omega_i$, *and* $v \in T_i M(\varepsilon_i)$, *then*

(*) $\qquad \text{Jac } D_v(F|v + \text{graph } L) \geqslant \text{Jac } D_0(F|E_i^u)/(1 + \delta)$.

(4) *For g a C^1 function from $E_i^u(\varepsilon_i)$ to $E_i^s(\varepsilon_i)$, slope $g \leqslant \omega_i$ implies that slope graph $\Gamma_i(g) \leqslant \omega$.*

(5) $T_i M(\varepsilon_i)$ *is contained in a neighborhood V_i of the origin in $T_i M$ such that for all* $P_k \in \Omega(f)$ *either* $W_k^u \cap W_i^s = \varnothing$ *or* $\phi_i^{-1}((W_k^u) \cap T_i M(\varepsilon_i))$ C^1-*fibers over the intersection of* $\phi_i^{-1}(W_k^u)$ *with* E_i^s *in* V_i, *each fiber having dimension equal to the dimension of* W_i^u *and slope no greater than* $\omega_i/3$.

(6) *For* $i > j$, *if* $W_i^u \cap W_j^s \neq \varnothing$, *then given any* C^1 *function h of* $E_i^u(\varepsilon_i)$ *to* $E_i^s(\varepsilon_i)$ *for which slope graph* $h \leqslant \omega_i$ *and* $(f^n \phi_i \, \text{graph } h) \cap B_j(\varepsilon_j) \neq \varnothing$, *then* $\phi_j^{-1} f^n \phi_i (\text{graph } h)$ C^1-*fibers over* $(\phi_j^{-1} W_i^u) \cap E_j^s$, *with the fibers having slope less than or equal to* ψ_j.

PROOF. Before beginning the proof we consider the following definition and lemma.

DEFINITION [6]. Let $f \in \text{Diff}(M)$ and P be a hyperbolic fixed point of f. We denote by $LS(P)$ and $LU(P)$ the local stable and unstable manifold of P, for $s = \text{dimension } LS(P)$. Let B^s be a cell neighborhood of P in $LS(P)$, such that $f(\partial B^s) \subset \text{int } B^s$. The existence of such a cell B^s follows from the fact that $f | LS(P)$ is a contraction. The embedded annulus in $LS(P)$ whose boundaries are B^s, $f(\partial B^s)$, is called a fundamental domain $G^s(P)$ of $W^s(P)$.

We note that $W^s(P) = \bigcup_{n \in \mathbf{Z}} f^n(G^s(P)) \cup P$. Any neighborhood $N^s(P)$ of $G^s(P)$ in M, disjoint from $LU(P)$, is called a fundamental neighborhood associated with $W^s(P)$.

Dually we can define $G^u(P)$ and $N^u(P)$.

LEMMA (1.11 OF [6]). *Fix a cell neighborhood B^r of P in $LU(P)$. There exists a neighborhood V or P, such that $W^u(P_i) \cap V = \varnothing$, or $W^u(P_i) \cap V$ is an r-cell fiberbundle over $W^u(P_i) \cap V \cap LS(P)$ with the fibers C^1 close to B^r. ($P_i \in \Omega(f)$.)*

Moreover, the fibering can be chosen to be C^1.

We now begin the proof of Proposition 1. For simplicity we assume that f has one sink. The proof is by induction using [5]. Let $i = 2$. Take $\bar{\varepsilon}_2 \leqslant \varepsilon$ and $0 < \omega_2 < \omega$ so that $(*)$ holds in $T_2 M(\bar{\varepsilon}_2)$. Furthermore take $\varepsilon_2' \leqslant \bar{\varepsilon}_2$ so that if $g: E_2^u(\varepsilon_2') \to E_2^s(\varepsilon_2')$ is a C^1 function with slope $\leqslant \omega_2$ then the graph transform of g by f, $\Gamma_2 g$, also has slope $\leqslant \omega$ (Fried and Shub [5]). This satisfies (1)–(4) of the proposition, for $i = 2$.

Next we consider (5). By the preceding results we can take $\varepsilon_2 \leqslant \varepsilon_2'$ so that $T_2 M(\varepsilon_2)$ is contained within the neighborhood V of P_2 guaranteed by Lemma 1.11 of [6] for $P_i \in \Omega(f)$, with $r = \dim LU(P)$. Thus, if $\phi_2^{-1}(W_k^u) \cap T_2 M(\varepsilon_2)$ is not empty we get that it C^1-fibers over its intersection with E_2^s, with all of its fibers having slope no greater than $\omega_2/3$.

By taking $T_2 M(\varepsilon_2)$ properly contained in V we can get the C^1 fiber structure. Thus (5) is satisfied, and this completes the proof for $i = 2$, since P_1 is the sink.

Assume that ε_j, ω_j have been chosen as to satisfy (1)–(6) for $j = 1, \ldots, i - 1 < r$. We show how to choose ε_i and ω_i. First take $\bar{\varepsilon}_i \leqslant \varepsilon$ and $0 < \bar{\omega}_i \leqslant \omega < 1$ so that the Jacobian inequality $(*)$ of Proposition 1 holds. Then take $\omega_i \leqslant \bar{\omega}_i$ and $\omega_i \leqslant \omega_{i-1}/3$ (this implies $\omega_i \leqslant \omega_j/3$ for $j < 1$). Next choose $\varepsilon_i' \leqslant \bar{\varepsilon}_i$ so that the graph transform Γ_i satisfies $\|Dh\| \leqslant \omega_i$. Then $\|D\Gamma_i(h)\| \leqslant \omega$ for $h \in C^1$, $h: E_i^u(\varepsilon_i') \to E_i^s(\varepsilon_i')$ (Fried and Shub [5]).

For the next step observe that if any orbit of f leaves the neighborhood $\phi_i(T_i M(\varepsilon_i'))$ of P_i on M then there is a bounded number of iterations N_i by which (1) the orbit is

in a neighborhood of the sink P_1 or (2) the orbit has entered (and possibly left) one of the neighborhoods $\phi_j(T_j M(\varepsilon_j))$. Thus we suppose that N is a C^1 submanifold of $T_i M(\varepsilon_i')$ and that $f^k \phi_i N \cap B_j(\varepsilon) \neq \varnothing$, for some $k \leqslant N_i$. By taking N in a tubular neighborhood of $E_i^u(\varepsilon_i')$ we get that $\phi_j^{-1} f^k \phi_i N \cap T_j M(\varepsilon_j)$ inherits a C^1 fiber structure over its intersection with E_j^s in V_j from $\phi_j^{-1} W_i^u$.

If we choose the width of the tubular neighborhood of $E_i^u(\varepsilon_i')$ small enough, say ε_i^*, we can insure that since $k \leqslant N_i$, that the slope of the fibers of $\phi_j^{-1} f^k \phi_i N - \phi_j^{-1}(W_i^u)$ relative to E_j^u is no more than ω_i larger than the slope of N relative to E_i^u. Because we are assuming via the induction hypothesis that the slope of $\phi_i^{-1}(W_i^u)$ relative to E_j^u is not larger than $\omega_j/3$, the slopes of the fibers of $\phi_j^{-1} f^k \phi_i N$, relative to E_i^u is bounded by $\omega_j/3 + \omega_i +$ (slope N relative E_i^u). Beginning with an N with slope ω_i, relative to E_i^u, gives that the slope of the fibers of $\phi_j^{-1} f^k \phi_i N$, relative to E_j^u, is less than or equal to $\omega_j/3 + 2\omega_i \leqslant \omega_j$. Using ε_i^* for the neighborhood of P_i in $T_i M$ satisfies (6). So we only need to prove (5). But here we argue as for $i = 2$ to get $\varepsilon_i \leqslant \varepsilon_i^*$ so that (5) is satisfied.

Thus the ε_i, ω_i have been chosen as to satisfy (1)–(6), and the proof of Proposition 1 is complete.

PROOF OF THEOREM B. For simplicity assume that there is exactly one sink. We note that the following $T_i M(\varepsilon)$ is defined in terms of the norm $\|v\| = \max\{\|v\|_s, \|v\|_u\}$.

Given $\delta > 0$, let $\delta' < \delta$. Then we can find for all $i = 2, \ldots, r$ a pair $(\varepsilon_i, \omega_i)$ which satisfy (1)–(6) of Proposition 1 with δ' in place of δ.

Take $\varepsilon > 0$ such that $\varepsilon < \min \varepsilon_i$ and consider the compact neighborhood of 0 in $T_i M$, $\operatorname{cl} T_i M(\varepsilon) = \operatorname{cl}(E_i^u(\varepsilon) + E_i^s(\varepsilon))$. Then take $U(P_i) = \phi_i(\operatorname{cl} T_i M(\varepsilon))$. These are our neighborhoods of the fixed points. Fix i and let $S_N = \phi_i^{-1}(U_N(P_i))$; $S_N \subset \operatorname{cl} T_i M(\varepsilon)$.

We prove that $\nu(S_N) \leqslant K(1 + \delta)^N \exp(-CN)$, where ν is the ordinary Lebesgue measure in $T_i M$. Passing back to the measure μ on M involves only multiplication by constants bounded away from 0 and ∞.

For $v \in \operatorname{cl} E_i^s(\varepsilon)$ consider the linear variety $H_v = \{(u, v) \in T_i M : u \in E_i^u\}$. Consider the subgraphs $S_N \cap H_v$. These are compact sets which cover S_N in the sense that $S_N = \bigcup_{v \in E_i^s(\varepsilon)}[S_N \cap H_v]$.

We appeal to Lemma 2 in order to get an area estimate for the subgraphs $(S_N \cap H_v)$.

LEMMA 2. *For $i \geqslant 2$, if h: $\operatorname{cl} E^u(\varepsilon_i) \to E^s(\varepsilon_i)$ is a C^1 function with $\|Dh\| \leqslant \omega_i$, then*

$$\operatorname{area}(\operatorname{graph} h \cap S_N) \leqslant K(1 + \delta)^N \exp(-CN),$$

where $K > 0$, $\exp(C) = \min_{2 \leqslant j \leqslant i}\{\operatorname{Jac} Df^u(P_j)\}$, and area *refers to the induced r-dimensional measure along* graph h, *where $r = \dimension(E^u)$.*

PROOF. The area measures are induced by the Riemannian metric induced by the inner product $\langle v, w \rangle = (v^u, w^u) + (v^s, w^s)$. The proof is by induction on i. If $i = 2$, we argue as in [5]. If $x \in S_N$, then $F^k x \in \operatorname{cl} T_2 M(\varepsilon)$ for $k = 0, \ldots, N - l$, where l is a fixed integer representing the maximum number of iterations for an orbit leaving

$U(P_2)$ to enter M_1, the neighborhood of the sink(s) on M (recall $F = \phi_i^{-1} \circ f \circ \phi_2$). The same statement thus applies to $F^k(\text{graph } h \cap S_N)$. Moreover we know that successive iterates of (graph $h \cap S_N$) are themselves graphs of C^1 functions of cl $E_2^u(\varepsilon)$ to cl $E_2^s(\varepsilon)$ with slope $\leqslant \omega_2$. This is because ε was chosen to insure that the graph transform of h, $\Gamma_2 h$, is a C^1 function with slope $\leqslant \omega_2$.

Thus $F^{N-l}(\text{graph } h \cap S_N)$ is indeed the graph of a C^1 function of cl $E^u(\varepsilon_2)$ into cl $E^s(\varepsilon_2)$ with slope $\leqslant \omega_i$ so that

(I) area($F^{N-l}(\text{graph } h \cap S_N)$) $\leqslant K$, where K is a constant depending only on ε and ω_2. Moreover,

$$K \geqslant \int_{\text{graph } h \cap S_N} \text{Jac } D_v\big(F^{N-l}|v + E_2^u\big)\, d\mu_r.$$

But the Jacobian inequality (∗) holds in cl $T_2 M(\varepsilon)$. That is if $L \in \text{Hom}(E_2^u, E_2^s)$ with $\|DL\| \leqslant \omega_2$ and $v \in \text{cl } T_2 M(\varepsilon)$ we have

$$\text{Jac } D_v\big(F|v + \text{graph } L\big) \geqslant (1 + \delta')^{-1} \text{Jac } D_0\big(F|E_2^u\big).$$

Here $0 < \delta' < \delta$ is the constant in the proof of Theorem B.

We apply this inequality repeatedly to get from (I) the following:

$$K \geqslant (1 + \delta')^{-N+l} \text{Jac } D_0\big(F^{N-l}|E_2^u\big) \text{area}(\text{graph } h \cap S_N)$$

so that

$$\text{area}(\text{graph } h \cap S_N) \leqslant K(1 + \delta')^{N-l}\big(\text{Jac } D_0\big(F^{N-l}|E_2^u\big)\big)^{-1}.$$

Now since f is assumed to have only fixed points,

$$\text{Jac } D_0\big(F^{N-l}|E_2^u\big) = \text{Jac } D_0\big(F|E_2^u\big)^{N-l}.$$

Then by redefining K,

$$\text{area}(\text{graph } h \cap S_N) \leqslant K(1 + \delta')^N\big(\text{Jac } D_0\big(F|E_2^u\big)\big)^{-N}.$$

By observing that the properties of ϕ_2 allow us to show that $\text{Jac } Df^u(P_2) = \text{Jac } D_0(F|E_2^u)$, we are done with the $i = 2$ case.

Now assume that the result has been proven for all $i < k$, where $k > 2$ and prove it holds for graphs in cl $T_k M(\varepsilon)$. Here the situation for (graph $h \cap S_N$) is different because $x \in S_N$ does not imply that $F^{N-l}(x) \in \text{cl } T_k M(\varepsilon)$, where l is some fixed integer. In fact on M an orbit beginning in $U(P_k)$ may leave $U(P_k)$ and enter some $U(P_j)$ before reaching V. Therefore we must consider $N + 1$ separate subgraphs of (graph $h \cap S_N$). We let B_m be that subgraph of (graph $h \cap S_N$) for which $F^{m+1}B_m \cap \text{cl } T_k M(\varepsilon) = \varnothing$, and $F^m B_m \subset \text{cl } T_k M(\varepsilon)$ for $0 \leqslant k \leqslant m$ (i.e. B_m leaves $T_k M(\varepsilon)$ at exactly the $(m + 1)$ iteration). Thus (graph $h \cap S_N$) $- E_k^s = \bigcup_{m=0}^{\infty} B_m$ where the union is disjoint. We shall modify B_m slightly, replacing B_m by cl B_m. Then graph($h \cap S_N$) $\subset \bigcup_{m=0}^{\infty} B_m$ with the union no longer disjoint but consisting of compact sets. Furthermore we only need to consider a finite union

$$(\text{graph } h \cap S_N) \subset A \cup B_0 \cup B_1 \cup \cdots B_{N-1}$$

where A is the closure of that portion of graph($h \cap S_N$) which remains in cl $T_k M(\varepsilon)$ for at least N iterations. Since all these sets are disjoint except on their boundaries

we have

$$\text{area}(\text{graph } h \cap S_N) = \text{area}(A) + \sum_{m=0}^{N-1} \text{area}(B_m).$$

Furthermore we know by the arguments used in the $i = 2$ case that

$$\text{area}(A) \leqslant K(1 + \delta')^N \exp(-C_k N),$$

and also that

$$\text{area}(B_m) \leqslant (1 + \delta')^m \text{area}(F^m B_m) \exp(-C_k m),$$

where $C_k = \text{Jac } Df^u(P_k)$. Consider $\text{area}(F^m B_m)$. The compact set $\phi_k F^m B_m \subset U(P_k)$ passes out of $U(P_k)$ on the next iteration, that is $f(\phi_k F^m B_m) \cap \text{int } U(P_k) = \varnothing$. Subgraphs of $\phi_k F^m B_m$ may pass under further iterations of f to any $U(P_j)$ for $j < k$ or to M_1. However, since $\phi_k B_m \subset U_N(P_k)$ any subgraph passing to M_1 must be in a B_m with $m \leqslant N - l$ for a fixed integer l. To define the subgraph of $\phi_k F^m B_k$ we perform the following procedure. $\phi_k F^m B_m$ is contained in the closed fundamental neighborhood (associated to W_k^u) given by $D_K = U(P_k) - \text{int } f^{-1} U(P_k)$. Consider $D_k \cap \bigcup_{n=1}^{\infty} f^{-n}(U(P_{k-1}))$; this is in fact equal to $D_k \cap \bigcup_{n=1}^{l_{k-1}} f^{-n}(U(P_{k-1}))$ for some fixed l_{k-1} representing the maximum number of iterations for an orbit of D_k to enter $U(P_{k-1})$ for the first time. Hence the intersection of D_k with inverse images of $U(P_{k-1})$ is compact. We also know that the subgraph of $\phi_k F^m B_m$ in this intersection is compact.

Now the iterates of $\phi_k F^m B_m \cap \bigcup_{n=1}^{l_{k-1}} f^{-n}(U(P_{k-1}))$ under f may not enter $U(P_k)$ at one iteration precisely, but for simplicity of notations we will assume this is the case. In any event we are faced with at most another finite decomposition of $\phi_k F^m B_m$ into subgraphs identified by which iteration they first enter $U(P_{k-1})$.

Hence we assume that $f^l \phi_k F^m B_m \subset U(P_{k-1})$. Since $F^m B_m$ is the graph of a C^1 function with slope $\leqslant \omega_k$, in cl $T_{P_k} M(\varepsilon)$ we know from Proposition 1 that

$$\left(\phi_{k-1}^{-1} f^l \phi_k F^m B_m \right) \cap \left(\text{cl } T_{P_{k-1}} M(\varepsilon) \right)$$

is a C^1 fiber bundle over $\phi_{k-1}^{-1} f^l \phi_k F^m B_m$ intersected with E_{k-1}^s in the fundamental neighborhood associated with W^s given by $B_{k-1}(\varepsilon) - f(B_{k-1}(\varepsilon))$. Furthermore the fibers are the graphs of C^1 functions from a closed subset of cl $E_{k-1}^u(\varepsilon)$ into cl $E_{k-1}^s(\varepsilon)$ and all of the slopes $\leqslant \omega_{k-1}$. By the induction hypothesis the area (dim E_{k-1}^u) of any fiber is bounded by $K(1 + \delta'')^{N-l-m} \exp(-C(N - l - m))$ where δ'' is between δ' and δ and C is the appropriate minimum Jacobian. From this we deduce the area(dim E_k^u) of $\phi_{k-1}^{-1} f^l \phi_k F^m B_m$ by using Fubini's theorem, which applies due to the C^1 fiber structure. Thus

$$\text{area}\left(\phi_{k-1}^{-1} f^l \phi_k F^m B_m \right) \leqslant K(1 + \delta'')^{N'} \exp(-CN')$$

for $N' = (N - l - m)$, and where we have redefined K. We now translate this back to cl $T_k M(\varepsilon)$ and thus find that for a subgraph of $F^m B_m$ its length is bounded by

$$(**) \qquad\qquad K(1 + \delta'')^{N'} \exp(-CN)$$

where again K is redefined to account for bounded factors in the translation, and $N' = (N - l - m)$.

We repeat this procedure to determine the areas of the other subgraphs of $F^m B_m$ noting that l may change. Since there are finitely many fixed points then the entire area of $F^m B_m$ is of the form (∗∗) with K modified to account for the number of points and the different l. Then translating (∗∗) back to the initial iteration $n = 0$ we immediately get

$$\text{area}(B_m) \leqslant (1 + \delta')^m K (1 + \delta'') \exp(-C_k m - CN')$$

for $N' = (N - l - m)$, where C is now the minimum of the Jac $Df^u(P_j)$ for $1 \leqslant j \leqslant k$. We redefine K to get

$$\text{area}(B_m) \leqslant k(1 + \delta'')^N \exp(-CN)$$

and finally arrive at

$$\text{area}(\text{graph } h \cap S_N) \leqslant \text{area } A + \text{area } B_0 + \cdots + \text{area } B_{N-1}$$
$$\leqslant (N + 1)K(1 + \delta'')^N \exp(-CN).$$

Since the exponential decay dominates algebraic growth we amy redefine K to get

$$\text{area}(\text{graph } h \cap S_N) \leqslant K(1 + \delta)^N \exp(-CN)$$

as desired. This completes the proof of Lemma 2.

Applying Lemma 2 we continue the proof of Theorem B. Since $S_N \cap H_v$ is contained in a linear variety, the area (·), here denoted ν^u, is an r-dimensional Lebesgue measure. Thus we get

$$\nu^u(S_n \cap H_v) \leqslant K(1 + \delta)^N \exp(-CN).$$

Now by application of Fubini's theorem,

$$\nu(S_N) = \int_{v \in E_i^s(\varepsilon)} \left[\nu^u(S_N \cap H_v) \right] d\nu^s$$

or

$$\nu(S_N) \leqslant K(1 + \delta)^N \exp(-CN)\nu^s(\text{cl } E_i^s(\varepsilon)),$$

$$\nu(S_N) \leqslant K(2\varepsilon) \dim E^s (1 + \delta)^N \exp(-CN).$$

Then redefine K as $K(2\varepsilon) \dim E^s$; we then get $\nu(S_N) \leqslant K(1 + \delta)^N \exp(-CN)$. Hence,

$$\mu(U_N(P_i)) \leqslant K(1 + \delta)^N \exp(-CN),$$

where again K is redefined. This completes the proof of Theorem B.

4. The Proof of Theorem A. Let $g = f^l$. Then g is a Morse-Smale diffeomorphism with $\Omega(g) = \text{Fix}(g)$. We will prove Theorem A for g by induction on i in the quantity $\mu(S(i, i - 1, N))$.

From M. Shub [9, part (e), p. 496], we have that there exist n_i and m_i such that $[g^{n_i}(M_i) - g^{-m_i}(\text{int } M_{i-1})] \subset U(P_i)$. Let $L_i = n_i + m_i$. Then it is evident that $x \in M_i$ implies either

(1) $$g^{n_i}(x) \in U(P_i)$$

or

$$(2) \qquad\qquad g^{L_i}(x) \in M_{i-1}.$$

Thus from the definition of $S(i, i-1, N+L_i)$ we have

$$S(i, i-1, N+L_i) \subset g^{-n_i}\big(U_{N+m_i}(P_i)\big).$$

Let $i = 2$. We consider $S(2, 1, N)$; $S(2, 1, N) = S(2, 1, (N-L_2)+L_2)$ so

$$S(2, 1, (N-L_2)+L_2) \subset g^{-n_2}U_{N-n_2}(P_2).$$

Thus by Theorem B,

$$\mu\big(S(2, 1, N)\big) \leqslant K^*\mu\big(U_{N-n_2}(P_2)\big) \leqslant K^*K(1+\delta)^{N-n_2}\exp\big(-C/l(N-n_2)\big)$$

for $N \geqslant L_2$.

By redefining K we get

$$\mu\big(S(2, 1, N)\big) \leqslant K(1+\delta)^N\exp\big(-C/lN\big).$$

Assuming that we have the estimate

$$\mu\big(S(i-1, 1, N)\big) \leqslant K(1+\delta)^N\exp\big(-C/lN\big) \quad \text{for } i < r$$

consider $S(i, 1, N)$. Then let A_N be defined by

$$S(i, 1, N) = S(i, i-1, N) \cup S(i-1, 1, N) \cup A_N.$$

Note. If $x \in A_N$ then $x \in M[i, i-1]$ and $f^N(x) \in \text{int } M_{i-1}$.

Now $\mu(S(i-1, 1, N))$ is known by hypothesis and $\mu(S(i, i-1, N))$ is gotten by a trivial modification of the $S(2, 1, N)$ case. So we consider A_N. Next write $A_N = D \cup E$, where

$$D = \big\{x \in A_N: g^{L_i}(x) \in M[i, i-1]\big\},$$
$$E = \big\{x \in A_N: g^{L_i}(x) \in \text{int } M_{i-1}\big\}.$$

Then $g^{L_i}(E) \subset S(i-1, 1, N-L_i)$ and $E \subset g^{-L_i}(S(i-1, 1, N-L_i))$ so that $\mu(E) \leqslant K'\mu(S(i-1, 1, N-L_i))$, where $K' = \max\{\text{Jac } g^{-L_i}\}$ over $M[i, i-1]$. Thus $\mu(E) \leqslant K(1+\delta)^N\exp(-CN)$, by modifying K.

Finally for D we have $g^{n_i}D \subset U(P_i)$ but since any point in $g^{N_i}D$ stays out of M_i for $N-n_i$ iterations we have $g^{n_i}D \subset U_{N-n_i}(P_i)$. Hence $\mu(D) \leqslant K^*\mu(U_{N-n_i}(P_i))$ or

$$\mu(D) \leqslant K^*K(1+\delta)^{N-n_i}\exp\big(-C/l(N-n_i)\big) \quad \text{for } N \geqslant L_i.$$

By redefining K, we get

$$\mu\big(S(i, l, N)\big) \leqslant \mu\big(S(i, i-1, N)\big) + \mu\big(S(i-1, j, N)\big) + \mu(D) + \mu(E)$$
$$\leqslant K(1+\delta)^N\exp\big(-C/lN\big)$$

as desired. Since $S(i, j, N) \subset S(i, 1, N)$ this completes the proof for g. To extend the result to f we note the following: if $\hat{U}_m(P_i) = \{x \in U(P_i): f^kx \notin M_1, k = 0, 1, 2, \ldots, m\}$, then

$$\hat{U}_{Nl}(P_i) \subset U_N(P_i) = \big\{x \in U(P_i): g^kx \notin M_1, k = 0, 1, \ldots, N\big\}.$$

Hence $\mu(\hat{U}_N(P_i)) \leqslant K(1 + \delta)^N \exp(-CN)$ or

$$\mu(U_N(P_i)) \leqslant \left[K(1 + \delta)^{1/l} \right] (1 + \delta)^N \exp(-C/l)N.$$

The proof is finished by noting that C/l is the minimum of the quantities $\{1/m \log \operatorname{Jac} D_{p_j} f^m | W^u(p_j)\}$ where the minimum is taken over all p_j in the non-wandering set for f which are not sinks and $f^m p_j = p_j$.

5. Morse-Smale flows. The goal of this section is to prove a corresponding theorem for Morse-Smale flows.

For x a fixed point of a flow ϕ_t we define

$$\phi^u(x) = \log \operatorname{Jac} D\phi_1 | E_x^u \to E_x^u.$$

For γ a closed orbit of ϕ_t with period τ we define

$$\phi^u(\gamma) = 1/\tau \left(\min_{y \in \gamma} \left\{ \log \operatorname{Jac} D\phi_\tau | E_y^u \to E_y^u \right\} \right).$$

A fixed point x or a closed orbit δ is an attractor of the flow ϕ_t if there is an open set U containing it satisfying $\phi_t(U) \subset U$ for all $t \geqslant 0$, and

$$\bigcap_{t \geqslant 0} \phi_t(U) = \begin{cases} \{\gamma\}, \\ \{x\}. \end{cases}$$

DEFINITION. Let X be a vector field on M, and denote by ϕ_t its induced flow. We say ϕ_t is a Morse-Smale flow if X satisfies the following conditions [6]:

(a) $\Omega(X)$ is the union of a finite number of fixed points $x_1, x_2, \ldots, x_m$ and a finite number of closed orbits $\gamma_1, \gamma_2, \ldots, \gamma_n$ of X.

(b) The x_i, γ_j, are all hyperbolic.

(c) The stable and unstable manifolds of the x_i, γ_j have transversal intersection.

THEOREM C. *Let ϕ_t be a Morse-Smale flow on M. There is a filtration $\mathcal{M} = \{M_i\}_{i=1}^r$ such that*:

(a) *All attractors for ϕ_t are in M_1.*

(b) *For all $\delta > 0$,*

$$\mu(S(i, 1, T)) \leqslant K(1 + \delta)^T \exp(-CT)$$

for $i > 2$, $K > 0$ independent of T, and $C = \min_{x_j, \gamma_j} \{\phi^u(x_j), \phi^u(\gamma_j)\}$ where x_j, γ_j are, respectively, the fixed points and closed orbits of ϕ_t which are not sinks or attractors.

PROOF. We show how to modify our previous arguments to this case.

If x is a fixed point of ϕ_t, then our previous work suffices to describe the behavior of trajectories in the neighborhood of x by considering the time one diffeomorphism $g = \phi_1$.

If γ is a closed orbit of ϕ_t which is not an attractor, then $T_\gamma M$ splits continuously into $E + E^s + E^u$ where E is the one-dimensional bundle tangent to the flow and dimension $E^u \geqslant 1$. For $g = \phi_\tau$ the time τ diffeomorphism where τ is the period of γ, γ is a closed, infinite, nonhyperbolic set in $\Omega(g)$. However, since $Dg(y)$ expands E_y^u more rapidly that E_y for any $y \in \gamma$, then given $\varepsilon > 0$ we can find δ, $0 < \delta < \varepsilon$, so that if $g^k(x) \in B_y(\varepsilon)$ for $0 \leqslant k \leqslant n$ but $g^{n+1}(x) \notin B_y(\varepsilon)$, then $d(g^{N+1}(x), \gamma) > \delta$. By compactness we can cover $B_\gamma(\delta)$ by a finite number of $\{B_y(\varepsilon)\}_{y \in \gamma}$.

The effect of this construction is to permit use of the previous established volume and area estimates in the neighborhood of γ by considering only a finite number of neighborhoods covering γ. If N is a C^1 submanifold of M in cl $B\gamma(\delta)$, then N decomposes into a finite number of submanifolds determined by its intersections with $B_y(\varepsilon)$. Let N_y be a typical one. Then $\phi_y^{-1}N_y$ is a C^1 submanifold of $T_yM(\varepsilon)$. If, in addition, it is the graph of a C^1 function of a closed subset of $E_y^u(\varepsilon)$ into $E_y^{cs}(\varepsilon) = E_y^s(\varepsilon) + E_y(\varepsilon)$ with prescribed slope, then under iteration by $G = \phi_y^{-1} \circ g \circ \phi_y$, it remains the graph of a C^1 function of the unstable space E_y^u into the center stable space E_y^{cs}. That is the graph transform properties from before apply here [2]. Locally γ behaves like a finite number of nonwandering points as far as the area lemma (Lemma 2) is concerned.

The only other issue is that of transversal intersection. The stable and unstable manifold of any two closed orbits, including fixed points, have transversal intersection. In the case of a closed orbit there is a third direction—the flow direction. We actually need each unstable manifold to intersect each center stable manifold transversally. This occurs since the stable and center stable manifolds intersect along the flow. Hence the analogue of Proposition 1 is true for Morse-Smale flows. Also by our previous remark the area lemma, Lemma 2, has an analogue. We thus prove our volume estimates locally using fixed time diffeomorphisms $g = \phi_\tau$ and finish the proof by normalizing the estimates by dividing the exponential constants by τ when necessary.

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