CHARACTERISTIC, MAXIMUM MODULUS
AND VALUE DISTRIBUTION

BY

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ABSTRACT. Let \( f \) be an entire function such that \( \log M(r, f) \sim T(r, f) \) on a set \( E \) of positive upper density. Then \( f \) has no finite deficient values. In fact, if we assume that \( E \) has density one and \( f \) has nonzero order, then the roots of all equations \( f(z) = a \) are equidistributed in angles. In view of a recent result of Murai [6] the conclusions hold in particular for entire functions with Fejér gaps.

1. Introduction. In a recent paper Murai [6] proved among other things that if \( f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n} \) is an entire function with Fejér gaps, i.e.

\[
\sum \lambda_n^{-1} < \infty,
\]

then \( f(z) \) can have no deficient values. In the course of his proof Murai showed that for such a function

\[
T(r, f) \sim \log M(r, f)
\]

as \( r \to \infty \) outside a set of finite logarithmic measure, where \( T(r, f) \) is the Nevanlinna characteristic and \( M(r, f) \) the maximum modulus of \( f \). In this paper we show that the condition (1.2) suffices in order that a transcendental entire function should have no deficient values and, subject to certain growth conditions, that the roots of all equations \( f(z) = a \) are equidistributed in angles. It is clear that some additional growth condition is necessary for this. In fact if \( f(z) \) is an entire function of genus zero, \( n(r) \) is the counting function of its zeros and

\[
N(r) = \int_0^r \frac{n(t)}{t} \, dt,
\]

then [4, (4.11), p. 133]

\[
n(r) = o\{N(r)\}
\]

implies (1.2), but (1.3) is unaffected by the arguments of the zeros. We shall see that a weaker gap condition than (1.1), namely Fabry gaps

\[
\lambda_n/n \to \infty,
\]

is sufficient or alternatively a growth condition, namely that \( f(z) \) has positive order and satisfies (1.2) on a set of density one.
2. Statement of results. We take for granted the usual notation of Nevanlinna theory. Let \( f(z) \) be a transcendental entire function of order \( \lambda \) and lower order \( \mu \), where \( 0 \leq \mu \leq \lambda \leq \infty \).

**Theorem 1.** Suppose that \( f(z) \) is an entire function such that

\[
(2.1) \quad \lim_{r \to \infty} \frac{T(r, f)}{\log M(r, f)} = 1
\]

as \( r \to \infty \) on a set \( E \) of positive upper density \( \delta \). Then there exists a set \( F \) of density zero, such that for every complex \( a \) we have

\[
(2.2) \quad N(r, a, f) \sim T(r, f)
\]

as \( r \to \infty \) in \( E \setminus F \). In particular, \( \delta(a, f) = 0 \) for every \( a \).

We write \( n(r, \theta_1, \theta_2, a) \) for the number of roots of the equation \( f(z) = a \) in the sector

\[
S(r, \theta_1, \theta_2) : 0 < |z| < r, \quad \theta_1 < \arg z < \theta_2,
\]

and

\[
(2.3) \quad N(r, \theta_1, \theta_2, a) = \int_0^r \frac{n(t, \theta_1, \theta_2, a)}{t} \, dt.
\]

Our next result is

**Theorem 2.** If \( \lambda > 0 \) and \( f(z) \) satisfies (2.1) as \( r \to \infty \) on a set \( E_1 \) of density one, then there exists a set \( E_2 \) of upper density one such that

\[
N(r, \theta_1, \theta_2, a) \sim \frac{\theta_2 - \theta_1}{2\pi} T(r, f)
\]

as \( r \to \infty \) on \( E_2 \) for every complex \( a \) and every pair \( \theta_1, \theta_2 \) such that \( \theta_1 < \theta_2 \leq \theta_1 + 2\pi \).

Our results have a natural extension to subharmonic functions when we consider the Riesz mass on set \( G \) of a subharmonic function \( u(z) \) to be the analogue of the number of zeros on \( G \) of the function \( f(z) - a \). We can then apply the subharmonic result to \( u(z) = \log|f(z) - a| \), provided that the set \( G \) is chosen independent of \( a \).

3. A growth result for real functions. In order to obtain Theorems 1 and 2 we prove an extension of a growth lemma of Edrei and Fuchs [1] to entire functions of arbitrary growth. Such an extension is possible if we work with the maximum modulus instead of the characteristic. However, in order to do this we need a sharpened version of an inequality for real functions of Hayman and Stewart [5]. We assume in this section that \( f(x) \) is a real function such that for sufficiently large positive \( x \), \( f^{(n-1)}(x) \) is convex. Thus for large \( x \), \( f^{(n)}(x) \) exists and is increasing outside a countable set. If, in addition, \( f^{(n)}(x) > 0 \) for large \( x \), we say that \( f(x) \in B(n) \) and define

\[
f_n(x) = \inf_{h > 0} \frac{f(x + h)}{h^n}.
\]
It was proved in [5] that for \( f(x) \in B(n) \) we have, given \( K > 1 \),
\[
(3.1) \quad f_n(x) < (eK/n)^n f^{(n)}(x)
\]
on a set \( E \) of positive lower density. In this paper we need to prove that the lower density is close to one if \( K \) is large. More precisely we have

**Theorem 3.** If \( E \) is the set of all \( x \) for which (3.1) is true when \( f(x) \in B(n) \), then if \( \delta(E) \) denotes the lower density of \( E \), we have \( \delta(E) \geq (K - 1)/(K - 1 + n) \).

We follow the argument of [5] and define
\[
\beta(x) = \sup_{0 \leq v \leq n-1} \left( \frac{f^{(v)}(x)}{f^{(n)}(x)} \right)^{1/(n-v)}
\]
We need

**Lemma 1.** If \( f(x) \in B(n) \) and \( f, f', \ldots, f^{(n)} \) are all positive for \( x \geq x_0 \), then \( x - \beta(x) \) is increasing for \( x \geq x_0 \).

In [5, Lemma 3] it was shown that
\[
(3.2) \quad \beta\{x + \delta \beta(x)\} \leq e^{\delta} \beta(x).
\]
Suppose that there exist \( x_1 \) and \( x_2 \), such that \( x_0 \leq x_1 < x_2 \) and \( x_2 - \beta(x_2) < x_1 - \beta(x_1) \). Then there exists \( C > 1 \) such that \( Cx_2 - \beta(x_2) = Cx_1 - \beta(x_1) \). We write for a large positive integer \( N \)
\[
h = (x_2 - x_1)/N, \quad \xi_j = x_1 + jh, \quad j = 0, \ldots, N,
\]
and deduce that for at least one \( j, 0 \leq j \leq N - 1 \), we have
\[
C\xi_{j+1} - \beta(\xi_{j+1}) \leq C\xi_j - \beta(\xi_j),
\]
i.e.
\[
\beta(\xi_j + h) \geq \beta(\xi_j) + Ch = \beta(\xi_j)\left(1 + Ch/\beta(\xi_j)\right).
\]
Writing \( h = \delta \beta(\xi_j) \) we obtain
\[
\beta(\xi_j + h) \geq \beta(\xi_j)(1 + C\delta) > e^{\delta} \beta(\xi_j)
\]
if \( \delta \) is sufficiently small, i.e. \( N \) sufficiently large, since \( C > 1 \). This contradicts (3.2) and proves Lemma 1. We deduce

**Lemma 2.** Suppose that \( 0 < \theta < 1 \) and \( C > 0 \). Then for \( x \) on a set of lower density at least \((1 - \theta)/(1 - \theta + \theta C)\) we have
\[
(3.3) \quad \beta(x + h) > \theta \beta(x) \quad \text{for } 0 \leq h \leq C\theta \beta(x).
\]
We note that \( \beta(x) \) is continuous except on the countable set of jump increases of \( f^{(n)}(x) \), where \( \beta(x) \) has a jump decrease. At these points we define \( \beta(x) = \beta(x + 0) \), so that \( \beta(x) \) is continuous to the right. We suppose \( x_0 = x_0' \) to be as in Lemma 1, and if \( x_{j-1}' \) has already been defined, we define \( x_j \) to be the lower bound and so the least value of \( x \geq x_{j-1}' \) such that
\[
\beta(x + h) \leq \theta \beta(x) \quad \text{for some } h \leq C\theta \beta(x).
\]
We then choose the least such \( h \) and set \( x'_j = x_j + h \). Let \( E \) be the set of all \( x \) in the union of the intervals \((x'_j, x_{j+1})\). Then it is evident that (3.3) holds in \( E \). It remains to estimate the lower density of \( E \).

Suppose then that \( X > x_0 \), and assume first that \( X = x'_p \) for some \( p > 0 \). Since \( x - \beta(x) \) is nondecreasing we note that

\[
\sum_{j=0}^{p} \left\{ x'_j - x_j - \beta(x'_j) + \beta(x_j) \right\} \leq X - x_0 - \beta(X) + \beta(x_0) < X + O(1).
\]

Again by our construction

\[
\beta(x_j) - \beta(x'_j) \geq (1 - \theta) \beta(x_j) \geq \frac{1 - \theta}{\theta C} (x'_j - x_j).
\]

Thus

\[
\left\{ 1 + \frac{1 - \theta}{\theta C} \right\} \sum_{j=0}^{p} (x'_j - x_j) < X + O(1).
\]

So if \( E(X) = E \cap [x_0, X] \) and \( |E(X)| \) denotes the length of \( E(X) \) we see that

\[
|E(X)| \geq X \left( 1 - \frac{C \theta}{C \theta + (1 - \theta)} \right) + O(1) = \frac{1 - \theta}{C \theta + 1 - \theta} X + O(1).
\]

Next if \( x_p \leq X \leq x'_p \), \( X \) is smaller while \( |E(X)| \) is the same, so that (3.5) is still valid. Again if \( x'_p \leq X < x_{p+1} \), \( X \) is larger, so that (3.4) and (3.5) are still valid. Thus (3.5) holds in all cases and Lemma 2 is proved.

**Lemma 3.** Suppose that for some numbers \( x = x_0, \theta \) and \( C \) we have (3.3). Then

\[
f\{x_0 + C \theta \beta(x_0)\} \leq \left\{ \beta(x_0) \right\} e^{Cf^{(n)}(x_0)}.
\]

We write

\[
\beta = \theta \beta(x_0), \quad \alpha = (\beta/\theta)^n f^{(n)}(x_0), \quad \varphi(x) = \alpha \exp\{((x - x_0)/\beta)\},
\]

and suppose that (3.6) is false. From this we shall obtain a contradiction to (3.3).

We define

\[
x_2 = \inf\{ x, x_0 \leq x \text{ and for some } \nu, 0 \leq \nu \leq n, f^{(\nu)}(x) > \varphi^{(\nu)}(x) \}.
\]

Since (3.6) is false we have

\[
\varphi(x_0 + C \beta) = \alpha e^C = (\beta/\theta)^n e^{C f^{(n)}(x_0)} < f(x_0 + C \beta).
\]

Again for \( \nu = 0, \ldots, n \) we have

\[
\varphi^{(\nu)}(x_0) = \alpha/\beta^\nu = \theta^{-\nu} \beta^{-\nu} f^{(n)}(x_0) \geq \beta(x_0)^{-\nu} f^{(n)}(x_0) \geq f^{(\nu)}(x_0).
\]

Thus \( x_2 \) exists and \( x_0 \leq x_2 < x_0 + C \beta \).

Suppose now that for some \( \nu < n \) we have

\[
\varphi^{(\nu)}(x_2) \leq f^{(\nu)}(x_2).
\]

Then we have by the definition of \( x_2 \)

\[
\varphi^{(\nu)}(x) \geq f^{(\nu)}(x), \quad 0 \leq x < x_2.
\]
Hence we deduce that
\[ \frac{d}{dx} \frac{q^{(r)}(x)}{f^{(r)}(x)} \leq 0 \]
at \( x = x_2 \), where differentiation denotes the left derivative. Thus
\[ 1 \leq \frac{f^{(r)}(x_2)}{q^{(r)}(x_2)} \leq \frac{f^{(r+1)}(x_2)}{q^{(r+1)}(x_2)}, \]
so that (3.7) holds with \( \nu + 1 \) instead of \( \nu \). Thus finally (3.7) must hold with \( \nu = n \) for the left derivative and so also the right derivative, while by the definition of \( x_2 \) we have (3.8) for \( \nu < n \) and \( x < x_2 \) and by continuity also for \( x = x_2 \). Thus
\[ \frac{f^{(r)}(x_2)}{f^{(n)}(x_2)} \leq \frac{q^{(r)}(x_2)}{q^{(n)}(x_2)} = \beta^{n-r}, \quad \nu = 0, \ldots, n - 1, \]
so that \( \beta(x_2) \leq \beta = \theta \beta(x_0) \). This contradicts (3.3) and so Lemma 3 is proved.

We can now complete the proof of Theorem 3. We set \( h = C \theta \beta(x_0) \) and deduce from (3.6) that if (3.3) holds with \( x = x_0 \) then
\[ f_n(x_0) \leq \frac{f(x_0 + h)}{h^n} \leq \frac{e^C}{(C \theta)^n} f^{(n)}(x_0). \]

By Lemma 2 we deduce that this inequality holds in a set of lower density at least
\[ \delta = \frac{1 - \theta}{1 - \theta + \theta C}. \]
Setting \( C = n, \theta = K^{-1} \) we deduce Theorem 3.

4. Proof of Theorem 1. In this section we suppose that \( u(z) \) is subharmonic and not constant in the plane and that \( u(0) = 0 \). We write
\[ (4.1) \quad B(r) = \sup_{|z|=r} u(z), \]
\[ (4.2) \quad b(r) = \int_0^r (r - t) B(t) \, dt, \quad b_2(r) = \inf_{h > 0} \frac{b(r + h)}{h^2}, \]
so that \( b''(r) = B(r) \). We also write \( n(z, h) \) for the Riesz mass of \( u \) in the disk \( |z' - z| < h \) and set
\[ (4.3) \quad N(z, h) = \int_0^h \frac{n(z, t)}{t} \, dt, \]
\[ (4.4) \quad u(z, h) = u(z) + N(z, h) = \frac{1}{2\pi} \int_0^{2\pi} u(z + he^{i\theta}) \, d\theta. \]
Suppose that \( f(z) \) is a transcendental entire function and that \( a \) is a complex constant. Then we have
\[ (4.5) \quad f(z) - a = c_\lambda z^\lambda + \cdots \]
and will apply our results to
\[ (4.6) \quad u_a(z) = \log \left| \frac{f(z) - a}{c_\lambda z^\lambda} \right| = \log |f_a(z)|. \]
We denote by \( A_1, A_2, A_3, \ldots \) positive absolute constants. We need
Lemma 4. If $0 < |z| = r < R$ and $h = A_1(R - r), \, 0 < A < 1$, we have

\begin{align}
(4.7) \quad u(z, \frac{1}{2} h) &> -\frac{A_2}{(R - r)^2} b(R) \\
\text{and} \\
(4.8) \quad n(z, h) &< \frac{A_3}{(R - r)^2} b(R).
\end{align}

Further if $0 < d < \frac{1}{2} h$ we have

\begin{align}
(4.9) \quad N\left(\zeta, \frac{1}{2} h\right) &< \frac{A_4}{(R - r)^2} \log \left(\frac{16 h}{d}\right) b(R)
\end{align}

for $|z - \zeta| < \frac{1}{2} h$ except possibly when $\zeta$ lies in a set of disks, the sum of whose radii is at most $d$.

The conclusions (4.7)-(4.9) are (14.1)-(14.3) of [2, p. 494]. The quantity $b(r)$ of the present paper is the $B_2(r)$ of [2].

We now prove

**Theorem 4.** With the above notation there exists an absolute constant $A_5$, such that if $K > 0$, we have

\begin{align}
(4.10) \quad u(re^{i\theta}) &> -Kb_2(r) \\
\text{for } 0 \leq \theta \leq 2\pi, \text{ outside a set } e(r, K) \text{ of } \theta \text{ whose measure is at most } 4\pi \exp(-A_5K).
\end{align}

We start by finding $R$, such that $r < R \leq 2r$ and

\begin{align}
(4.11) \quad \frac{b(R)}{(R - r)^2} &\leq 4b_2(r).
\end{align}

If $R > 2r$, we deduce from the fact that $B(r)$ increases with $r$ that so does

\begin{align}
R^{-2}b(R) &= \int_0^1 (1 - t) B(Rt) \, dt.
\end{align}

Hence for $R > 2r$

\begin{align}
\frac{b(R)}{(R - r)^2} &\geq \frac{b(R)}{R^2} \geq \frac{b(2r)}{(2r)^2} = \frac{1}{4} \frac{b(2r)}{(2r - r)^2}.
\end{align}

Thus

\begin{align}
\inf_{R > 2r} \frac{b(R)}{(R - r)^2} &\geq \frac{1}{4} \frac{b(2r)}{(2r - r)^2}
\end{align}

and so

\begin{align}
b_2(r) &\geq \frac{1}{4} \min_{r < R \leq 2r} \frac{b(R)}{(R - r)^2}.
\end{align}

Thus $R$ exists satisfying (4.11). Having chosen $R$ to satisfy (4.11) we define $h$ as in Lemma 4 and apply that lemma. We define $p$ to be the smallest integer such that $p \geq 2$ and

\begin{align}
2 \sin(\pi/2p) &= |\exp(\pi i/p) - 1| < \frac{1}{2} h/r.
\end{align}
Then if $z_{r} = r \exp(2\pi iv/p)$, the disks $C_{r}$: $|z - z_{r}| < \frac{1}{2}h$, $v = 1, \ldots, p$, cover $|z| = r$. Also

$$
2\pi/p \geq \pi/(p - 1) \geq 2\sin(\pi/2(p - 1)) \geq \frac{1}{2}h/r,
$$

so that $p \leq 4\pi r/h$.

Again for $d < \frac{1}{2}h$ we have (4.9) in $C_{r}$ outside a set $E_{r}$ of disks the sum of whose radii is at most $d$. Since $d < \frac{1}{2}h < \frac{1}{2}r$ each exceptional disk $|z - z_{j}| < d_{j} \leq d$ meets $z = re^{i\theta}$ in an arc of diameter at most $2\delta$, and so length at most $\pi\delta_{j}$. Thus the total length of those arcs on $C_{r} \cap (|z| = r)$, which lie in the exceptional disks is at most $\pi d$. Thus (4.9) holds on $[\xi] = r$, outside a set of arcs of total length at most $\pi pd$, i.e. (4.9) holds for $\xi = re^{i\theta}$, $0 \leq \theta \leq 2\pi$, except for a set $e(r)$ of $\theta$ having measure

$$
|e(r)| \leq \pi pd/r \leq (\pi d/r)(4\pi r/h) = 4\pi^{2}d/h.
$$

Further, for $\theta$ outside $e(r)$ we have from (4.4), (4.7) and (4.9)

$$
u(re^{i\theta}) = \nu(re^{i\theta}, \frac{1}{2}h) - N(re^{i\theta}, \frac{1}{2}h) > -b_2(r)
$$

by (4.11). Suppose now that $K > 4A_{2} + 4A_{4}\log(16h/d)$ and deduce from (4.12) and (4.13) that (4.10) holds outside a set $\theta$ of measure

$$
|e(r, K)| \leq 4\pi^{2}d/h = 64\pi^{2}\exp(A_{2}/A_{4} - K/4A_{4}) \leq \exp(-K/8A_{4})
$$

if $K \geq A_{6}$. This proves Theorem 4 for $K \geq A_{6}$. Also, if $K < A_{6}$, (4.10) is trivial if $\exp(A_{5}A_{6}) < 2$. Thus Theorem 4 holds in all cases with $A_{6} = \inf(1/8A_{4}, (\log 2)/A_{6})$.

We deduce the following consequence from Theorem 4, which may be considered as an analogue of the Edrei-Fuchs small arcs lemma [1, p. 322].

**Theorem 5.** If $E$ is a set of measure $\delta < 2\pi$ on the interval $[0, 2\pi]$ then we have

$$
\int_{E}u(re^{i\theta})\,d\theta > -A_{7}b_{2}(r)\delta \log\left(\frac{4\pi}{\delta}\right).
$$

We denote by $e(K)$ the set of $\theta$ such that $u(re^{i\theta}) < -Kb_{2}(r)$ and by $m(K)$ the measure of $e(K)$. Then Theorem 4 gives

$$
\int_{e(K)} u(re^{i\theta})\,d\theta = b_{2}(r)\int_{K}^{\infty} t\,dm(t) = -b_{2}(r)\{Km(K) + \int_{K}^{\infty} m(t)\,dt\}
$$

$$
> -4\pi b_{2}(r)\{K\exp(-A_{5}K) + \int_{K}^{\infty}\exp(-A_{5}t)\,dt\}
$$

$$
> -A_{4}b_{2}(r)\exp(-A_{5}K).
$$

Given $E$ as in Theorem 5 we choose $K > 0$, and define $E_{1}, E_{2}$ to be the subsets of $E$, where $u < -Kb_{2}(r)$, $u > -Kb_{2}(r)$, respectively. Then

$$
\int_{E} u(re^{i\theta})\,d\theta = \int_{E_{1}} + \int_{E_{2}} \geq \int_{E_{1}} u(re^{i\theta})\,d\theta + \int_{E_{2}} u(re^{i\theta})\,d\theta
$$

$$
> -b_{2}(r)\{A_{6}\exp(-A_{5}K) + K\delta\}.
$$
We choose $K$ so that $A_9 \exp(-A_9 K) = \delta$, i.e. $K = (A_9)^{-1} \log (A_9/\delta)$, and deduce that
\[
\int_{E_r} u(re^{i\theta}) \, d\theta > -b_2(r) \delta \left( 1 + \frac{1}{A_9} \log \frac{A_9}{\delta} \right)
\]
which gives Theorem 5.

We can now complete the proof of Theorem 1. Suppose that we have on the set $E$ of values of $r$

\[(4.14) \quad T(r, f) > \left(1 - \epsilon(r)^2\right) \log M(r, f),\]

where

\[(4.15) \quad \epsilon(r) \to 0, \quad \text{but} \quad \epsilon(r)^2 \log M(r)/\log r \to \infty\]
as $r \to \infty$. We define $u(z) = \log |f(z)|$.

Let $F$ be the set of all $r$, such that

\[(4.16) \quad b_2(r) > -\frac{1}{\epsilon(r)} B(r) = -\frac{1}{\epsilon(r)} b''(r), \quad \text{where} \quad B(r) = \log M(r, f).\]

Then given $K > 1$, we have for all large $r$ in $F$

\[
b_2(r) > \frac{e^2 K^2}{4} b''(r),
\]

so that $F$ has upper density at most $2/(K + 1)$ by Theorem 3. Since $K$ is arbitrary, $F$ has density zero.

Suppose now that $a$ is any complex number and replace $u(z)$ by the function $u_a(z)$ defined by (4.6). Then

\[
u_a(z) = \log |f(z) - a| + O(\log |z|)
\]

so that for $|z| = r$

\[
u_a^+(z) = \max(u_a(z), 0) = \{\log^+ |f(z)| + O(\log r)\}.
\]

Thus, since $f(z)$ is transcendental we have $B(r, u_a(z)) = B(r) + O(\log r)$ as $r \to \infty$, and similarly $T(r, f_a(z)) = T(r, f) + O(\log r)$. Hence, also we have as $r \to \infty$

\[
b(r, u_a) \sim b(r), \quad b_2(r, u_a) \sim b_2(r).
\]

We deduce from (4.16) that for any complex $a$ we have for $r \in E \setminus F$ and $r > r_0(a)$

\[(4.17) \quad b_2(r, u_a) < \frac{2}{\epsilon(r)} B(r, u_a),\]

and from (4.14) and (4.15) that

\[(4.18) \quad T(r, f_a) > \left(1 - 2\epsilon(r)^2\right) B(r, u_a).\]

Suppose now that for such a value of $r$, $e(r, a)$ is the set of all $\theta$ for which $u_a < 0$ and let $e'(r, a)$ be the complementary set of $\theta$. Then

\[
2\pi T(r, f_a) = \int_0^{2\pi} u_a^+(re^{i\theta}) \, d\theta = \int_{e} + \int_{e'} \leq (2\pi - |e(r, a)|) B(r, u_a),
\]
where $|e|$ denotes the measure of $e$. Thus

$$T(r, f_a) \leq \left(1 - \frac{|e(r, a)|}{2\pi}\right) B(r, u_a),$$

so that by (4.18), $|e(r, a)| \leq 4\pi r^2$. Thus Theorem 5 and (4.17) yield for large $r$ in $E \setminus F$

$$m(r, a) + O(\log r) = -\frac{1}{2\pi} \int_{e(r, a)} u_a(re^{i\theta}) \, d\theta$$

$$< A_7 b_2(r, u_a)|e(r, a)|\log \frac{4\pi}{|e(r, a)|}$$

$$= O\left( B(r, u_a) \epsilon(r) \log \frac{1}{\epsilon(r)} \right) = o\{ T(r, f) \},$$

and this proves Theorem 1, for $E \setminus F$ has positive upper density and so is unbounded.

5. Another growth lemma. In order to prove Theorem 2 we need

**Lemma 5.** Suppose that $B(r)$ is a positive increasing function of positive order, that $b(r)$ and $b_2(r)$ are defined by (4.2) and that $\varphi(r)$ is a positive function of $r$, such that

(5.1) $\varphi(r) = O\{ b_2(r) \}$ as $r \to \infty$

and for some function $\epsilon(r)$, which decreases to zero as $r \to \infty$, we have

(5.2) $\varphi(r) = O\{ \epsilon(r) b_2(r) \}$ as $r \to \infty$

on a set $E_1$ of density one. Then there exists a set $E_2$ of upper density one, depending only on $E_1$ and the function $\epsilon(r)$, such that

(5.3) $\int_1^r \varphi(t) \log \left( \frac{r}{t} \right) \frac{dt}{t} = o\{ B(r) \}$ as $r \to \infty$

in $E_2$.

We note that $b(r)$ and $b_2(r)$ also increase with $r$, and have positive order. In fact, the increasing property is obvious from (4.2) and

$$h^{-2} b(r + h) \geq h^{-2} \int_r^{r+h} (r + h - t) B(t) \, dt \geq \frac{1}{2} B(r)$$

so that $b_2(r) \geq \frac{1}{2} B(r)$ and $b(2r) \geq r^2 B(r)/2$ for all $r$. Thus if $B(r)$ has positive order $\lambda$, $b(r)$ has order at least $\lambda + 2$ and $b_2(r)$ has order at least $\lambda$. We now choose $\mu$ such that $0 < \mu < \lambda$ and a sequence $R_n$, which tends to $\infty$ with $n$ and is such that

(5.4) $b_2(r) \leq (r/R_n)^\mu b_2(R_n)$ for $1 \leq r < R_n$.

Since $b_2(r)/r^\mu$ is continuous and unbounded we may for instance choose $R_1 = 1$ and if $R_{n-1}$ has been defined let $R_n$ be the smallest number such that $R_n \geq 2R_{n-1}$ and

$$b_2(R_n)/R_n^\mu \geq \sup_{1 \leq R < 2R_{n-1}} b(r)/r^\mu.$$
We proceed to show that if $K_n$ tends to $\infty$ sufficiently slowly with $n$ and $E_2$ consists of all those points $r$ in the intervals $[R_n, K_nR_n]$ for which

\begin{equation}
(5.5) \quad b_2(r) < K_n^{\mu/2}B(r),
\end{equation}

then the set $E_2$ has the required property.

We note first that $E_2$ has upper density one. In fact, it follows from Theorem 3 that given $K > 1$ we have

\begin{equation}
(5.6) \quad b_2(r) < \left(\frac{eK}{2}\right)^2B(r)
\end{equation}

for a set of $r$ in $[0, K_nR_n]$ having measure at least $(K - 1)K_nR_n/(K + 1) + O(1)$ when $R_n$ is large and so in a set in $[R_n, K_nR_n]$ having measure at least

\[
\left\{ K - 1 \right\} \frac{1}{K + 1} K_nR_n + O(1).
\]

Thus, since (5.6) implies (5.5) for large $n$, we see that $E_2$ has upper density at least $(K - 1)/(K + 1)$, and since $K$ can be as large as we please $E_2$ has upper density one.

We next choose the quantities $K_n$. Let $E_1'$ be the complement of $E_1$, let $E_1'[r]$ be the intersection of $E_1'$ with the interval $[0, r]$, and let $|E_1'[r]|$ be the measure of $E_1'[r]$. Then we assume that $K_n$ tends to infinity so slowly that

\begin{equation}
(5.7) \quad K_n^{2+\mu} < r/|E_1'(r)|, \quad r \geq R_n.
\end{equation}

This is possible since $E_1'$ has density zero and $R_n \to \infty$ with $n$. We also assume that

\begin{equation}
(5.8) \quad K_n^{\mu}e(R_n/K_n) < 1,
\end{equation}

which is possible since $e(r) \to 0$ as $r \to \infty$. The set $E_2$ defined as above is independent of $\varphi(r)$ and has upper density one. It remains to show that (5.3) holds in $E_2$.

Assume that $r \in E_2$, $R_n \leq r \leq K_nR_n$, and write

\begin{equation}
(5.9) \quad I(r) = \int_1^r \varphi(t) \log \frac{r}{t} \frac{dt}{t} = I_0(r) + I_1(r) + I_1'(r),
\end{equation}

where $I_0(r)$, $I_1(r)$ and $I_1'(r)$ are the integrals over the ranges $[1, R_n/K_n]$, $e_1 = [R_n/K_n, r] \cap E_1$ and $e_1' = [R_n/K_n, r] \cap E_1'$, respectively. Then by (5.1), (5.4) and (5.5) we have

\[
I_0(r) = \int_1^{R_n/K_n} \varphi(t) \log \frac{R_n}{t} \frac{dt}{t} \leq 2 \int_1^{R_n/K_n} \varphi(t) \log \frac{R_n}{t} \frac{dt}{t}
\]

\[
= O\left( \int_1^{R_n/K_n} b_2(t) \log \frac{R_n}{t} \frac{dt}{t} \right) = O\left( b_2(R_n) \int_1^{R_n/K_n} \left( \frac{t}{R_n} \right) \mu \log \frac{R_n}{t} \frac{dt}{t} \right)
\]

\[
= O\left( b_2(R_n) K_n^{-\mu} \log \frac{1}{K_n} \right) = O\left( b_2(r) K_n^{-\mu} \log \frac{1}{K_n} \right)
\]

\[
= O\left( B(r) K_n^{-\mu/2} \log \frac{1}{K_n} \right) = o\left( B(r) \right).
\]
Again by (5.2), (5.5) and (5.8)
\[
I_1(t) = O\left(\varepsilon \left(\frac{R_n}{K_n}\right) \int_{e_1^t} b_2(t) \log \frac{r}{t} \, dt\right)
\]
\[
= O\left(\int_{e_1^t} b_2(r) \varepsilon \left(\frac{R_n}{K_n}\right) \frac{r}{t} \log \frac{r}{t} \, dt\right)
\]
\[
= O\left(\int_{e_1^t} b_2(r) \varepsilon \left(\frac{R_n}{K_n}\right) \left(\log K_n\right)^2 \right)
\]
\[
= O\left(B(r) \varepsilon \left(\frac{R_n}{K_n}\right) \log K_n^2 \right) = o\{B(r)\}.
\]

Finally, by (5.1), (5.5) and (5.7)
\[
I_1(t) = O\left(\frac{K_2}{r} \left(\log K_n\right)^2 \right) = o\{B(r)\}.
\]

Now (5.3) follows from (5.9) and Lemma 5 is proved.

6. Proof of Theorem 2. In order to prove Theorem 2 we need a formalism used elsewhere. We suppose that $f(z)$ is a transcendental entire function such that $f(0) = 1$ and denote by $n(r, \theta_1, \theta_2)$ the number of zeros of $f(z)$ in $0 < |z| < r$, $	heta_1 < \arg z < \theta_2$ each counted with due multiplicity. We also write
\[
N(r, \theta_1, \theta_2) = \int_0^r n(t, \theta_1, \theta_2) \frac{dt}{t}.
\]

Next, if $f(z) \neq 0$ on the segment $z = t e^{i \theta}$, $0 < t < r$, we define $v(t, \theta)$ to be the continuous value of $\arg f(z)$ on this segment such that $v(0, \theta) = 0$, and we write,
\[
V(r, \theta) = \frac{1}{2\pi} \int_0^r v(t, \theta) \frac{dt}{t}.
\]

With this notation we have [3, Theorem 1]

**Lemma 6.** If $f(z) \neq 0$ on the segments $z = t e^{i \theta}$, $0 < t < r$, $\theta = \theta_1$ or $\theta_2$, then
\[
N(r, \theta_1, \theta_2) = \frac{1}{2\pi} \int_{\theta_1}^{\theta_2} \log |f(r e^{i \theta})| d\theta + V(r, \theta_1) - V(r, \theta_2).
\]

We need to transform the quantity $V(r, \theta)$ a little and note that
\[
v(r, \theta) = \int_0^r \frac{\partial v(t, \theta)}{\partial t} \frac{dt}{t} = \int_0^r - \frac{1}{t} \frac{\partial}{\partial \theta} \log |f(t e^{i \theta})| \frac{dt}{t}.
\]

Thus, for $\alpha < \beta < \alpha + 2\pi$ we have
\[
\int_{\alpha}^{\beta} V(r, \theta) \, d\theta = -\frac{1}{2\pi} \int_0^r \frac{ds}{s} \int_0^t \frac{dt}{t} \int_{\alpha}^{\beta} \log |f(t e^{i \theta})| \frac{dt}{t} = \frac{1}{2\pi} \int_0^r \left(\log |f(t e^{i \alpha})| - \log |f(t e^{i \beta})|\right) \frac{dt}{t}.
\]
We write
\begin{equation}
M(t) = \sup_{0 < \theta < 2\pi} |f(te^{i\theta})|, \quad B(t) = \log M(t)
\end{equation}
and
\[ \varphi(t, \alpha) = \frac{1}{2\pi} \log \frac{M(t)}{|f(te^{i\alpha})|}. \]

Thus
\begin{equation}
\int_{\alpha}^{\beta} V(r, \theta) \, d\theta = \int_{0}^{r} \left( \varphi(t, \beta) - \varphi(t, \alpha) \right) \log \frac{r}{t} \, dt.
\end{equation}

Our aim is to show that the positive function \( \varphi(t, \alpha) \) is on the average not too large. We also define \( \varphi_a(t, \alpha) \) to be the function \( \varphi(t, \alpha) \) defined as above w.r.t. the functions \( f_a(z) \) introduced in (4.6).

**Lemma 7.** If
\[ \varphi_a(r) = \frac{1}{2\pi} \int_{0}^{2\pi} \varphi_a(t, \alpha) \, d\alpha, \]

then under the hypotheses of Theorem 2 there exists a set \( E_2 \) of upper density one such that we have
\[ \int_{1}^{r} \varphi_a(t) \log \frac{r}{t} \, dt = o_B(r) \]
as \( r \to \infty \) in \( E_2 \) for each complex \( a \).

We define the set \( E_2 \) as in Lemma 5, where \( B(r) \) is given by (6.1) (i.e. for the function \( f_0(z) = f(z) \)). Then \( B(r) \) has positive order by hypothesis, so that Lemma 5 is applicable. We deduce from Theorem 1 that under the hypotheses of Theorem 1, we have for each complex \( a \) as \( r \to \infty \) in \( E_1 \)
\begin{equation}
\frac{1}{2\pi} \int_{0}^{2\pi} \log^+ |f_a(re^{i\theta})| \, d\theta = \frac{1}{2\pi} \int_{0}^{2\pi} \log^+ |f(re^{i\theta})| + O(\log r)
\end{equation}
\[ > B(r) - \varepsilon(r) B(r) + O(\log r), \]
where \( \varepsilon(r) \to 0 \) with \( r \) and \( \varepsilon(r) \) is independent of \( a \). Thus \( |f_a(re^{i\theta})| > 1 \) outside a set of \( \theta \) of measure at most \( O(\varepsilon(r)) \), provided that \( \varepsilon(r) \to 0 \) so slowly that \( \log r = o\{ \varepsilon(r) B(r) \} \) as \( r \to \infty \). Now Theorem 5 shows that as \( r \to \infty \) in \( E_1 \) we have
\begin{equation}
\int_{0}^{2\pi} \log^+ \left| \frac{1}{f_a(re^{i\theta})} \right| \, d\theta = O\left( \varepsilon(r) \log \frac{1}{\varepsilon(r)} b_2(r) \right)
\end{equation}
\[ = O\{ \varepsilon_1(r) b_2(r) \}, \]
where \( \varepsilon_1(r) \) is independent of \( a \). We note that if \( a \neq 1, f_a(z) = (f(z) - a)/(1 - a) \) while \( f_1(z) = (f(z) - 1)/z^\lambda \). Thus for any fixed \( a \) and large \( r \) we have
\[ M_a(r) = \sup_{|z| = r} |f_a(z)| < C_a M(r), \quad |z| = r, \]
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where the constant $C_a$ depends only on $a$. Also (6.3) and (6.4) yield

$$\frac{1}{2\pi} \int_0^{2\pi} \log \frac{M_a(r)}{|f_a(re^{i\theta})|} d\theta = O\{\varepsilon(r) B(r) + \varepsilon_1(r) b_2(r) + \log r\}$$

$$= O\{\varepsilon_1(r) b_2(r)\}$$

as $r \to \infty$ in $E_1$. Thus, if $\varphi_a(t)$ is defined with $f_a(z)$ instead of $f(z)$, we see that for each complex $a$

$$\varphi_a(t) = \frac{1}{2\pi} \int_0^{2\pi} \varphi_a(t, \alpha) d\theta = O\{\varepsilon_1(t) b_2(t)\}$$

as $t \to \infty$ in $E_1$, while we have in any case

$$\varphi_a(t) = B(t) - \frac{1}{2\pi} \int_0^{2\pi} \log |f_a(te^{i\theta})| d\theta + O(1) \leq B(t) + O(1) = O\{b_2(t)\}.$$ 

Thus $\varphi_a(t)$ satisfies the hypotheses of Lemma 5 and we deduce that there exists a set $E_2$ of upper density one such that for each complex $a$ we have

$$(6.5) \quad \int_{1}^{r} \varphi_a(t) \log \frac{r dt}{t} = o\{B(r)\} \quad \text{as} \quad r \to \infty \text{ on } E_2.$$

This proves Lemma 7.

We now suppose that we are given $\varepsilon > 0$ and $\theta_1, \theta_2$, such that $\theta_1 \leq \theta_2 < \theta_1 + 2\pi$. We also take a fixed complex $a$ and assume that $r \to \infty$ on $E_2$. Then there exist $\alpha, \beta$ such that $\theta_2 < \alpha < \theta_2 + \varepsilon/3, \theta_2 + 2\varepsilon/3 < \beta < \theta_2 + \varepsilon$ and

$$0 \leq \int_{1}^{r} \varphi_a(t, \alpha) \log \frac{r dt}{t} \leq \frac{3}{\varepsilon} \int_{1}^{r} \log \frac{r dt}{t} \int_{\theta_2}^{\theta_2 + \varepsilon/3} \varphi_a(t, \theta) d\theta$$

$$\leq \frac{6\pi}{\varepsilon} \int_{1}^{r} \log \frac{r dt}{t} \varphi_a(t) \frac{dt}{t},$$

$$0 \leq \int_{1}^{r} \varphi_a(t, \beta) \log \frac{r dt}{t} \leq \frac{6\pi}{\varepsilon} \int_{1}^{r} \log \varphi_a(t) \frac{dt}{t}.$$ 

Using (6.2) and (6.5) we deduce that $|\int_{\alpha}^{\beta} V(r, \theta) d\theta| = o\{B(r)\}$. Hence there exists $\varphi_2 = \varphi_2(r)$ such that $\alpha < \varphi_2 < \beta$ and so $\theta_2 < \varphi_2 < \theta_2 + \varepsilon$ and $V(r, \varphi_2) = o\{B(r)\}$. Similarly, there exists $\varphi_1$, such that $\theta_1 - \varepsilon < \varphi_1 < \theta_1$, and $V(r, \varphi_1) = o\{B(r)\}$. Thus Lemma 6 shows that

$$N(r, \theta_1, \theta_2) \leq N(r, \varphi_1, \varphi_2) \leq \frac{\varphi_2 - \varphi_1}{2\pi} \{B(r) + O(1)\} + o\{B(r)\}$$

$$\leq \frac{\theta_2 - \theta_1 + 2\varepsilon + o(1)}{2\pi} B(r).$$

This gives

$$\lim_{r \to \infty} \frac{N(r, \theta_1, \theta_2)}{B(r)} \leq \frac{\theta_2 - \theta_1}{2\pi}.$$

Also, we may assume that $E_2$ is disjoint from the set $F$ of Theorem 1, since this does not affect the density. Then as $r \to \infty$ in $E_2$

$$(6.7) \quad N(r, \theta_2, \theta_1 + 2\pi) + N(r, \theta_1, \theta_2) = N(r) = (1 + o(1)) B(r)$$
by Theorem 1. We apply (6.6) with $\theta_2, \theta_1 + 2\pi$ instead of $\theta_1, \theta_2$ and obtain
\[
\lim \frac{N(r, \theta_2, \theta_1 + 2\pi)}{B(r)} \leq \frac{2\pi + \theta_1 - \theta_2}{2\pi}.
\]
Now (6.7) gives
\[
\lim \frac{N(r, \theta_1, \theta_2)}{B(r)} \geq \frac{\theta_2 - \theta_1}{2\pi}.
\]
Combining this with (6.6) we obtain
\[
\lim \frac{N(r, \theta_1, \theta_2)}{B(r)} = \frac{\theta_2 - \theta_1}{2\pi}
\]
as $r \to \infty$ in $E_2$, and this proves Theorem 2.

In conclusion we note that, by Theorem 2, (1.2) implies angular equidistribution of all $a$-values unless $f(z)$ has order zero. However, for functions of order zero it follows from Theorem 3 of [3] that $f(z)$ satisfies the conclusion of Theorem 2 if (1.4) holds and à fortiori if (1.1) holds. Thus (1.1) always implies equidistribution of the $a$-values.

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