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CLOSED TIMELIKE GEODESICS

BY

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ABSTRACT. It is shown that every *stable* free t-homotopy class of closed timelike curves in a compact Lorentzian manifold contains a longest curve which must be a closed timelike geodesic. This result enables one to obtain a Lorentzian analogue of a classical theorem of Synge. A criterion for stability is presented, and a theorem of Tipler is derived as a special case of the result stated above.

1. Introduction and statement of the main result. Among the most venerable problems in Riemannian geometry is the problem of establishing the existence of closed geodesics in a Riemannian manifold. In spite of all the activity in global Lorentzian geometry during the last two decades, very little work has been done on the corresponding Lorentzian problem. Indeed, the author is aware of only one result (due to Tipler [**12**]) which establishes, under quite special circumstances, the existence of a closed timelike geodesic in a compact space-time. The oldest and most fundamental *Riemannian* result (due to Hadamard [**6**] in two dimensions and Cartan [**4**] in arbitrary dimension) asserts that within any (nontrivial) free homotopy class of a compact Riemannian manifold there exists a shortest curve, which must be a (nontrivial) closed geodesic. The purpose of this paper is to present a Lorentzian analogue of the Hadamard-Cartan result, and to use this result to obtain, à la Synge [**11**], some relationships between the curvature and global structure of compact Lorentzian manifolds.

A Lorentzian manifold (M, g) is a smooth manifold M of dimension $d \geqslant 2$, equipped with a smooth Lorentzian metric g having signature $(-+++\cdots+)$. Here and throughout the paper we shall employ the following convention concerning timelike curves. A timelike curve γ: $[0, b] \to M$ is a piecewise smooth curve such that (1) each smooth segment of γ is timelike and (2) the tangents at a common end point of two consecutive smooth segments of γ point into the same null half-cone. Let γ_1 and γ_2 be closed timelike curves in M. We say that γ_1 is freely t-homotopic to γ_2 if there is a homotopy which deforms γ_1 into γ_2 via closed *timelike* curves. The present goal is to establish conditions which ensure that a given free t-homotopy class $\mathscr{C}$ of closed timelike curves contains a *longest* curve. It follows in the same manner as the Riemannian case that such a curve, if it exists, must be a closed timelike geodesic. (Recall that timelike geodesics locally *maximize* arc length). An obvious necessary

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condition for the existence of such a curve is

$$\sup_{\gamma \in \mathscr{C}} L(\gamma) < \infty, \tag{1}$$

where $L(\gamma)$ is the length of γ. Although in the Riemannian case such a condition is absurd, it is not unusual for condition (1) to hold in the Lorentzian case. (See the example depicted in Figure 1.)

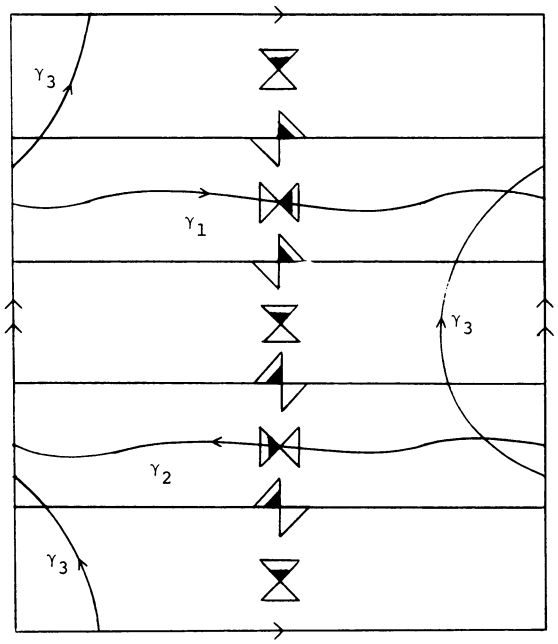

FIGURE 1

The diagram depicts a time orientable Lorentzian manifold which is topologically a torus. The Lorentzian metric is of the form $ds^2 = -(\cos\theta\, dy - \sin\theta\, dx)^2 + (\sin\theta\, dy + \cos\theta\, dx)^2$, where x, y are periodic coordinates, and θ is an appropriately chosen function of y. The t-homotopy classes determined by γ_1 and γ_2 are stable (as follows from the proposition in §3); the t-homotopy class determined by γ_3 is not.

The most elegant proofs of the Hadamard-Cartan result make use of certain global results in Riemannian geometry, for which there are no suitable Lorentzian analogues. In [**10**, Volume IV, p. 358f], Spivak presents a neat, elementary proof (based on a classical deformation technique) of the Hadamard-Cartan result which is more amenable to adaptation to the Lorentzian case than these other proofs, since the geometrical part of his proof uses only local methods. However, to successfully modify Spivak's proof we found it necessary to strengthen condition (1) a little.

Let g and g_0 be Lorentzian metrics on M. Then g is said to be wider than g_0, written $g > g_0$, if and only if for all nonzero vectors X, X is g-timelike ($g(X, X) < 0$) whenever X is g_0-nonspacelike ($g_0(X, X) \leqslant 0$). Intuitively the light cones in (M, g) open wider than the light cones in (M, g_0).

DEFINITION. Let $\mathscr{C}$ be a free t-homotopy class of closed timelike curves in a Lorentzian manifold (M, g_0). Then $\mathscr{C}$ is said to be stable if and only if there exists a Lorentzian metric $g > g_0$ such that

$$\sup_{\gamma \in \mathscr{C}} L_g(\gamma) < \infty, \tag{2}$$

where $L_g(\gamma)$ is the length of γ in the metric g.

In the Lorentzian manifold depicted in Figure 1, the t-homotopy class determined by γ_3 is not stable, but the t-homotopy classes determined by γ_1 and γ_2 are stable. There are also examples of compact *simply connected* Lorentzian manifolds with unstable t-homotopy classes (see the footnote on p. 200 of Penrose [**8**]). In §3 we present a simple criterion for the stability of free t-homotopy classes which is easily verifiable in many examples.

We are now ready to state our main result concerning the existence of closed timelike geodesics.

THEOREM 1. *Let (M, g_0) be a compact Lorentzian manifold. Then each stable free t-homotopy class in (M, g_0) contains a longest closed timelike curve. This curve is necessarily a closed timelike geodesic.*

The proof is presented in §2. In §3 we obtain a criterion for stability, and indicate how Tipler's theorem can be derived from Theorem 1. In §4 we present a Lorentzian version of Synge's theorem on connectivity.

2. Proof of Theorem 1. The proof is a bit long. So that the elements of the proof may stand out more clearly, we break the proof up into four parts (a)–(d). Although certain portions of the proof repeat arguments appearing in Spivak [**10**, Volume IV, p. 358f], for the sake of continuity and completeness we include all such arguments here.

Let $\mathscr{C}$ be a stable free t-homotopy class in (M, g_0). Let $g > g_0$ be a Lorentzian metric on M for which (2) is satisfied. Fix a Riemannian metric g_r on M.

(a) *Finite sup conditions.* For each $\gamma \in \mathscr{C}$, let $L_{g_r}(\gamma)$ and $L_{g_0}(\gamma)$ denote the g_r-length and the g_0-length of γ, respectively. We claim

$$\sup_{\gamma \in \mathscr{C}} L_{g_r}(\gamma) = L < \infty \tag{3}$$

and

$$\sup_{\gamma \in \mathscr{C}} L_{g_0}(\gamma) = L_0 < \infty. \tag{4}$$

To prove (3), consider the direction bundle DM of M. Represent each element of DM by an equivalence class of nonzero vectors $[X]$, where two nonzero vectors X and Y are equivalent if and only if there is a constant λ such that $Y = \lambda X$. DM is compact since M is. Let K be the compact subset of DM consisting of all g_0-nonspacelike directions on M. Introduce the norms $\|X\|_{g_r} = \sqrt{g_r(X, X)}$ and $\|X\|_g = \sqrt{-g(X, X)}$ for g_0-nonspacelike vectors X. The map $[X] \to \|X\|_{g_r}/\|X\|_g$ for

each $[X] \in K$ is well defined (by the homogeneity of the norms) and continuous (since $\|X\|_g \neq 0$ for all nonzero g_0-nonspacelike vectors X). Thus, since K is compact, there is a positive constant α such that $\|X\|_{g_r} \leqslant \alpha \|X\|_g$ for all g_0-nonspacelike vectors X. Hence,

$$L_{g_r}(\gamma) \leqslant \alpha L_g(\gamma) \quad \text{for each } \gamma \in \mathscr{C}. \tag{5}$$

(3) now follows from (2) and (5). The proof of (4) is similar.

(b) *Special curves in* $\mathscr{C}$. Cover (M, g_0) by a finite number of geodesically convex neighborhoods $U_1, \ldots, U_s$. Without loss of generality we may assume that each U_α has compact closure and that $\overline{U}_\alpha$ is contained in a convex neighborhood W_α.

A timelike curve γ will be called *special* if there is a sequence of points p_0, $p_1, \ldots, p_N = p_0$ in M such that

(i) for each j, the points p_{j-1}, p_j both lie in some U_α and there is a timelike curve contained in U_α from p_{j-1} to p_j, and

(ii) γ is the union of the unique timelike geodesics γ_j from p_{j-1} to p_j.

For each γ: $[0, b] \to M$ in $\mathscr{C}$ there is a special curve $\bar{\gamma}$ in $\mathscr{C}$ such that $L_{g_0}(\bar{\gamma}) \geqslant L_{g_0}(\gamma)$. To prove this, consider the cover $\{\gamma^{-1}(U_\alpha)\}$ of $[0, b]$. By the Lebesgue covering lemma there is a sequence $0 = t_0 < \cdots < t_N = b$ such that $[t_{j-1}, t_j]$ is contained in some $\gamma^{-1}(U_\alpha)$, i.e. such that $\gamma_j = \gamma|_{[t_{j-1}, t_j]}$ is contained in U_α. Let $\bar{\gamma}_j$ be the unique geodesic segment in U_α from $p_{j-1} = \gamma(t_{j-1})$ to $p_j = \gamma(t_j)$; $\bar{\gamma}_j$ is necessarily timelike (see Lemma 2.16 in [**9**]) and has length greater than or equal to that of γ_j (see Proposition 7.2 in [**9**]). Since γ_j and $\bar{\gamma}_j$ both lie in some convex neighborhood U_α they are t-homotopic relative to their end points. (To ensure the existence of a homotopy via *timelike* curves Lemma 2.16 in [**9**] is needed.) Now, let $\bar{\gamma}$ be the union of the segments $\bar{\gamma}_j$. Clearly, $\bar{\gamma} \in \mathscr{C}$ and $L_{g_0}(\bar{\gamma}) \geqslant L_{g_0}(\gamma)$.

We now show that there is an upper bound to the number of points p_j needed in order to construct such special curves $\bar{\gamma}$ from curves $\gamma \in \mathscr{C}$.

Claim. There exists a positive integer N_0 such that for each $\gamma \in \mathscr{C}$ there is a special $\bar{\gamma} \in \mathscr{C}$, with $L_{g_0}(\bar{\gamma}) \geqslant L_{g_0}(\gamma)$, such that the number of points determining $\bar{\gamma}$ is no greater than N_0.

Proof of the Claim. M is a compact metric space with respect to the distance function ρ induced by the positive definite metric g_r. Thus, by the Lebesgue covering lemma there is an $\varepsilon > 0$ such that any set A with ρ-diameter $< \varepsilon$ lies entirely in some U_α. Given $\gamma \in \mathscr{C}$, let $\bar{\gamma}$ be a special curve, with $L_{g_0}(\bar{\gamma}) \geqslant L_{g_0}(\gamma)$, determined by the points $p_0, p_1, \ldots, p_N = p_0$, where N is as small as possible. For each i, $1 \leqslant i \leqslant [N/2]$, let $\bar{\eta}_i$ = segment of $\bar{\gamma}$ from $p_{2(i-1)}$ to p_{2i}. If, for some i, $\bar{\eta}_i$ had g_r-length less than ε, then $\bar{\eta}_i$ would lie entirely within some U_α, and so it could be replaced by a single maximal timelike geodesic in U_α, implying that N could be reduced. Thus,

$$L_{g_r}(\bar{\eta}_i) \geqslant \varepsilon, \qquad 1 \leqslant i \leqslant [N/2],$$

which, using (3), implies,

$$L \geqslant L_{g_r}(\bar{\gamma}) \geqslant [N/2]\varepsilon \geqslant (N/2 - 1)\varepsilon.$$

Hence, $N \leqslant 2(L/\varepsilon + 1)$. Setting N_0 equal to $[2(L/\varepsilon + 1)]$ completes the proof of the claim.

(c) *Construction of the curve* γ. By (4) and the Claim in (b), there is a sequence of special curves $\gamma^{(k)} \in \mathscr{C}$ such that $L_{g_0}(\gamma^{(k)}) \to L_0$ as $k \to \infty$. Furthermore, by adding points if necessary we can assume that each $\gamma^{(k)}$ is determined by exactly N_0 points, $p_0^{(k)}, p_1^{(k)}, \ldots, p_{N_0}^{(k)} = p_0^{(k)}$.

Since M is compact we may assume by taking subsequences that for each $j = 0, \ldots, N_0$, we have

$$\lim_{k \to \infty} p_j^{(k)} = p_j \in M. \tag{6}$$

Now consider a fixed j. For each k, the points $p_{j-1}^{(k)}, p_j^{(k)}$ both lie in some U_α. Since there are only finitely many U_α, one of them, $U_{\alpha(j)}$ say, must contain both $p_{j-1}^{(k)}$ and $p_j^{(k)}$ for infinitely many k. By taking a subsequence we can assume that all $p_{j-1}^{(k)}, p_j^{(k)}$ are in $U_{\alpha(j)}$. There are only finitely many j to consider, so by taking subsequences we may assume that

$$\text{all } p_{j-1}^{(k)}, p_j^{(k)} \text{ are in some } U_{\alpha(j)}, \qquad j = 1, \ldots, N_0, \tag{7}$$

and thus,

$$p_{j-1} \text{ and } p_j \text{ are in } \overline{U}_{\alpha(j)} \subset W_{\alpha(j)}, \qquad j = 1, \ldots, N_0. \tag{8}$$

Let γ_j be the unique geodesic segment in $W_{\alpha(j)}$ joining p_{j-1} to p_j. Note that since γ_j is the limit of timelike segments it is either null (and possibly trivial if $p_{j-1} = p_j$) or timelike. Define γ to be the union of the γ_j's.

We end this part of the proof by establishing the following Claim which will be needed in (d).

Claim. The sequence of curves $\gamma^{(k)}$ can be chosen so that (6) holds and $p_{j-1} \neq p_j$ for each j.

PROOF OF THE CLAIM. Let the sequence of curves $\gamma^{(k)}$ and the points p_j be constructed as above. If $p_{j-1} \neq p_j$ for all j then there is nothing to do. Consider the special case in which there is *exactly one* value $j < N_0$ such that $p_{j-1} = p_j$. It follows from (6) and (8) that for all k sufficiently large, $p_{j-1}^{(k)}, p_j^{(k)}, p_{j+1}^{(k)}$ are in $W_{\alpha(j+1)}$. Let $\bar{\gamma}_j^{(k)}$ be the unique timelike geodesic segment in $W_{\alpha(j+1)}$ from $p_{j-1}^{(k)}$ to $p_{j+1}^{(k)}$. Let $\bar{\gamma}^{(k)}$ be the curve obtained from $\gamma^{(k)}$ by replacing $\gamma_j^{(k)} \cup \gamma_{j+1}^{(k)}$ by $\bar{\gamma}_j^{(k)}$. Each $\bar{\gamma}^{(k)}$ is t-homotopic to $\gamma^{(k)}$ and hence is in $\mathscr{C}$. Furthermore, since $L_{g_0}(\bar{\gamma}^{(k)}) \geqslant L_{g_0}(\gamma^{(k)})$, we have $L_{g_0}(\bar{\gamma}^{(k)}) \to L_0$ as $k \to \infty$. Thus, the claim is established in this special case. However, by repeating the above procedure a finite number of times in an appropriately systematic manner, the claim is established in general.

It is assumed in the remainder of the proof that the sequence $\gamma^{(k)}$ has been chosen according to the Claim.

(d) *Demonstration that* $L_{g_0}(\gamma) = L_0$ *and* $\gamma \in \mathscr{C}$. Let d_j be the local Lorentzian distance function on $W_{\alpha(j)}$ (see [**3**, p. 103]). Using (6), (8) and the continuity of d_j we have,

$$L_{g_0}(\gamma) = \sum_{j=1}^{N_0} L_{g_0}(\gamma_j) = \sum_{j=1}^{N_0} d_j(p_{j-1}, p_j) = \lim_{k \to \infty} \sum_{j=1}^{N_0} d_j\left(p_{j-1}^{(k)}, p_j^{(k)}\right)$$

$$= \lim_{k \to \infty} \sum_{j=1}^{N_0} L_{g_0}\left(\gamma_j^{(k)}\right) = \lim_{k \to \infty} L_{g_0}\left(\gamma^{(k)}\right) = L_0.$$

It remains to show that γ belongs to $\mathscr{C}$. We first show that γ is timelike, in the sense described in the introduction. Each segment γ_j of γ is a nontrivial null or timelike geodesic. For each j and for each k sufficient large, γ_j and $\gamma_j^{(k)}$ will have the same direction (past or future) relative to a time orientation of $W_{\alpha(j)}$. This observation and the fact that each $\gamma^{(k)}$ is timelike implies that γ satisfies the second requirement in the definition of a timelike curve given in the introduction. Thus, γ is timelike provided each γ_j is timelike. We now show that

$$\text{each } \gamma_j \text{ is timelike.} \tag{9}$$

Since $L_{g_0}(\gamma) > 0$, some γ_j is timelike. If some γ_j is null then there must be two consecutive segments γ_{j-1}, γ_j, one of which is timelike and the other of which is null. Hence γ_{j-1}, γ_j form a corner at p_{j-1} where they meet. The corner at p_{j-1} can be trimmed by a timelike geodesic segment so as to obtain a curve $\hat{\gamma}$ whose length is greater than L_0. By the manner in which $\gamma^{(k)}$ approaches γ, it is not hard to see that for k sufficiently large the corner at $p_{j-1}^{(k)}$ which must form can be trimmed by a timelike geodesic segment to obtain a timelike curve $\hat{\gamma}^{(k)}$ which is t-homotopic to $\gamma^{(k)}$ and has length greater than L_0. But this contradicts the choice of the sequence $\gamma^{(k)}$, and hence (9) is established.

(9) can now be used to show that $\gamma \in \mathscr{C}$. There exist neighborhoods V_0, $V_1, \ldots, V_{N_0} = V_0$ of $p_0, p_1, \ldots, p_{N_0} = p_0$, respectively, such that for each j, any point in V_{j-1} can be joined to any point in V_j by a timelike curve. For each j, let $\beta_{j,k}$ be the unique geodesic in $W_{\alpha(j)}$ from p_j to $p_j^{(k)}$. Since there are only finitely many values of j to consider, (6) and (8) imply that for any k sufficiently large, $\beta_{j,k}$ is contained in V_j for all j. By joining $\beta_{j-1,k}$ to $\beta_{j,k}$ by a suitable family of timelike geodesics in $W_{\alpha(j)}$, one sees that γ_j and $\gamma_j^{(k)}$ are t-homotopic (relative to $\beta_{j-1,k}, \beta_{j,k}$) for each $j = 1, \ldots, N_0$. These t-homotopies can be glued together to obtain a free t-homotopy deforming γ into $\gamma^{(k)}$. Thus, $\gamma \in \mathscr{C}$.

Finally, γ must be a closed geodesic, for, if not, there would be a curve freely t-homotopic to γ with length greater than L_0 (by the local maximizing property of timelike geodesics).

REMARKS. 1. We do not have an example which shows that Theorem 1 is false if (2) is replaced by the weaker assumption (4). However, one can conceive of the following sort of difficulty. (4) may hold because the curves in $\mathscr{C}$ are "turning null" in some sense. Thus a sequence $\gamma^{(k)}$ of curves in $\mathscr{C}$ whose lengths approach L_0 may be approaching a closed null geodesic.

2. The basic method of proof used here can also be used to give a simple, elementary proof of the fundamental result that two causally related points in a globally hyperbolic space-time can be joined by a maximal nonspacelike geodesic.

3. The assumption in Theorem 1 that M be compact can be weakened. The proof shows that it suffices to assume that there exists an open set $U \subset M$ with compact closure, such that each curve $\gamma \in \mathscr{C}$ is contained in U.

3. A criterion for stability and a derivation of Tipler's theorem. The following proposition establishes a criterion for the stability of free t-homotopy classes, which

in many examples (e.g. the space-time depicted in Figure 1) is easy to verify. A closed timelike curve γ: $S^1 \to M$ is said to *meet the open set U k times* if and only if the number of connected components of $\gamma^{-1}(U)$ in S^1 is k.

PROPOSITION. *Let (M, g_0) be a compact Lorentzian manifold and let $\mathscr{C}$ be a free t-homotopy class of closed timelike curves. Suppose there exists a positive integer k such that each $p \in M$ is contained in a sufficiently small neighborhood U_p with the property that each curve $\gamma \in \mathscr{C}$ meets U_p at most k times. Then $\mathscr{C}$ is a stable free t-homotopy class.*

By *sufficiently small* we mean that U_p is contained in some convex neighborhood.

PROOF. Let g be any Lorentzian metric on M wider than g_0. By assumption, for each $p \in M$ there exists a neighborhood U_p of p which is contained in some convex neighborhood N_p, such that each $\gamma \in \mathscr{C}$ meets U_p at most k times. Let $V_p \subset U_p$ be a neigborhood of p such that space-time $(V_p, g|V_p)$ has finite timelike diameter. By Propositions 4.8 and 4.9 in [**9**], there exists an open connected neighborhood $W_p \subset V_p$ of p with the property that any timelike curve lying entirely within N_p meets W_p at most once. It follows that each $\gamma \in \mathscr{C}$ meets W_p at most k times.

Consider a finite subcover $\{W_1, \ldots, W_n\}$ of the cover $\{W_p\colon p \in M\}$ of M. Let ρ_i = the timelike diameter of $(W_i, g|W_i)$, $i = 1, \ldots, n$. Since each W_i is contained in some V_p, ρ_i is finite for each i. Let $\mu = \max\{\rho_1, \ldots, \rho_n\}$. Since there are only n W_i's and $\gamma \in \mathscr{C}$ meets each W_i at most k times, it follows that $L_g(\gamma) \leqslant nk\mu$. Thus (2) holds, and $\mathscr{C}$ is stable.

As mentioned in the introduction, Tipler has established the existence of closed timelike geodesics in compact Lorentzian manifolds of a very special type. Here we want to briefly indicate how Tipler's result can be obtained from Theorem 1.

THEOREM (TIPLER [**12**]). *Let (M, g) be a compact Lorentzian manifold with a covering space which admits a compact Cauchy suface. The (M, g) contains a closed timelike geodesic.*

As John Beem pointed out to me, there is a small gap in Tipler's proof which can be filled by assuming that the covering space with compact Caucy surface be regular. In the proof of Tipler's theorem presented below it is assumed that the covering space is regular.

PROOF. Since M is compact it contains a closed timelike curve γ. (This result is usually stated for time-orientable Lorentzian manifolds, but, as a standard covering argument shows, it holds for non-time-orientable Lorentzian manifolds, as well.) Let $\tilde{M}$ be the given covering manifold with covering map π: $\tilde{M} \to M$. Let $\tilde{\gamma}$ be a lift of γ from $q_1 \in \tilde{M}$ to $q_2 \in \tilde{M}$. Without loss of generality it can be assumed q_2 is in the future of q_1. Let ψ: $\tilde{M} \to \tilde{M}$ be a deck transformation taking q_1 to q_2. Let $\tilde{S}_1$ be a compact Cauchy surface in $\tilde{M}$ passing through q_1, and let $\tilde{S}_2 = \psi(\tilde{S}_1)$.

Consider the quotient space $\hat{M} = \tilde{M}/G$, where G is the cyclic subgroup generated by ψ of the group of deck transformations. Then $\tilde{M}$ is a covering manifold of $\hat{M}$, where the natural projection map p: $\tilde{M} \to \hat{M}$ is the covering map (see e.g. Massey [**7**]). Since ψ is an isometry, $\hat{M}$ carries a Lorentzian metric such that p is a local

isometry. $\hat{M}$ has a simple geometric description. Let $K = J^+(\tilde{S}_1) \cap J^-(\tilde{S}_2)$. Since $\tilde{M}$ is globally hyperbolic, K is compact and $\partial K = \tilde{S}_1 \cup \tilde{S}_2$. $\hat{M}$ is obtained from K by identifying the points q and $\psi(q)$ for each $q \in \tilde{S}_1$. In particular, $\hat{M}$ is compact. Observe, also, that $\hat{S} = p(\tilde{S}_1)$ is an embedded compact spacelike hypersurface without boundary in $\hat{M}$.

The curve $\hat{\gamma} = p \circ \tilde{\gamma}$ is a closed timelike curve in $\hat{M}$ which intersects $\hat{S}$ exactly once. By intersection number theory, each curve in $\mathscr{C}(\hat{\gamma})$ (the t-homotopy class determined by $\hat{\gamma}$) must meet $\hat{S}$ exactly once, and hence has a unique lift in $\tilde{M}$ from $\tilde{S}_1$ to $\tilde{S}_2$. Using this fact, the fact that $\tilde{M}$ is strongly causal, and basic properties of coverings one easily shows that each point of $\hat{M}$ has a sufficiently small neighborhood which is met at most *once* by each curve in $\mathscr{C}(\hat{\gamma})$. Thus, by the Proposition, $\mathscr{C}(\hat{\gamma})$ is stable, and hence contains a closed timelike geodesic $\hat{\sigma}$. Let $\tilde{\sigma}$ be the lift of $\hat{\sigma}$ into $\tilde{M}$ from $\tilde{S}_1$ to $\tilde{S}_2$. Then $\sigma = \pi \circ \tilde{\sigma}$ is a closed timelike geodesic in M.

4. A Lorentzian version of Synge's theorem. A classical result relating the topology and curvature of a Riemannian manifold is Synge's theorem on connectivity which states that a compact even-dimensional orientable Riemannian manifold with positive sectional curvature is simply connected. In this section we present a Lorentzian analogue of this result.

Let $K(\Pi)$ denote the sectional curvature of any nondegenerate plane Π. Because of our signature convention (and other reasons) the assumption $K(\Pi) < 0$ for timelike planes Π in Lorentzian geometry corresponds to the assumption of positive sectional curvature in Riemannian geometry.

Observe that for two-dimensional Lorentzian manifolds the following conditions are incompatible.

(a) M is compact and orientable.

(b) $K(\Pi) < 0$ for all timelike planes Π in M.

Indeed, since M is compact and carries a Lorentzian metric, it must have vanishing Euler characteristic, $\chi(M) = 0$. However, by the Gauss-Bonnet theorem for indefinite metrics [**1**, **5**], $K(\Pi) < 0$ implies $\chi(M) \neq 0$.

We want to consider to what extent this incompatibility persists in higher dimensions. One need not look far to find space-times which satisfy both (a) and (b). Indeed, Avez [**2**, p. 130] has constructed an example of a Lorentzian manifold having topology S^3 in which (as a computation shows) (b) is satisfied. However, if we restrict attention to *even*-dimensional Lorentzian manifolds then something can be said.

THEOREM 2. *Let (M, g) be a Lorentzian manifold. Then the following conditions are incompatible.*

(i) *M is compact, even-dimensional and orientable.*

(ii) *$K(\Pi) < 0$ for all timelike planes Π.*

(iii) *M has a stable t-homotopy class.*

Whether or not Theorem 2, without condition (iii), is still valid is, as far as we know, an open question. When comparing with Synge's theorem, condition (iii) plays

a role analogous to the statement: M is nonsimply connected. It would naturally be of interest to have a better understanding of the connection between the causal and topological structure of a Lorentzian manifold and the existence of stable t-homotopy classes.

Theorem 2 follows from Theorem 1 and the following lemma.

SYNGE'S LEMMA FOR LORENTZIAN MANIFOLDS. *Let M be an orientable, even-dimensional Lorentzian manifold with $K(\Pi) < 0$ for all timelike planes Π. Let γ be a closed timelike geodesic in M. Then there is a variation of γ by timelike curves such that each varied curve has length greater than γ.*

PROOF OF SYNGE'S LEMMA. We shall only present a sketch of the proof here since it parallels so closely the Riemannian proof. (See e.g. Spivak [**10**, Volume IV] for complete details in the Riemannian case.)

Assume γ: $[0, b] \to M$ is parameterized proportional to arc length. Using the orientability and even-dimensionality of M, one can argue just as in the Riemannian case that there exists a unit vector field X parallely displaced along γ and perpendicular to γ such that $X(0) = X(b)$.

Consider the variation of γ defined by

$$\alpha(u, t) = \exp_{\gamma(t)} uX(t),$$

i.e. $\alpha(u, t)$ is the point reached by traveling along the geodesic through $\gamma(t)$ with initial tangent $X(t)$ a distance u. The varied curves $t \to \alpha(u, t)$ will be timelike for u sufficiently small, and will be closed since $X(0) = X(b)$. Let $l(u)$ denote the length of $t \to \alpha(u, t)$. Since γ is a geodesic the first variation of arc length formula (see Proposition 11.23 in [**3**]) implies $l'(0) = 0$. The curvature assumption implies that $l''(0) = I(X, X) > 0$, where I is the Lorentzian index form (see Definition 9.4 and Equation 9.8 in [**3**]). Hence, for u sufficiently small, the varied curve $t \to \alpha(u, t)$ is longer than γ.

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