THE AMALGAMATION PROPERTY FOR VARIETIES OF LATTICES

BY

ALAN DAY AND JAROSLAV JEŽEK

ABSTRACT. There are precisely three varieties of lattices that satisfy the amalgamation property: trivial lattices, distributive lattices, and all lattices.

1. Introduction. In [5], Jónsson showed that the variety of all lattices satisfied the (strong) amalgamation property and in [7] Pierce proved the similar (weak) version for distributive lattices. Grätzer, Jónsson and Lakser supplied the first negative results in [3] by showing that the only varieties of modular lattices that satisfied the amalgamation property were varieties of distributive lattices (there are two such varieties, $T = L(x = y)$ and $D$). This result is crucial in that it forces $N_5$, the pentagon, into any nondistributive $V$ satisfying (AP). Using this fact and the description of primitive lattices from Ježek and Slavík [4], Slavík, [8], showed that such a nondistributive variety, $V$, satisfying (AP) must contain all primitive lattices. In this paper we complete the process started by Slavík, though by slightly different methods, and show that $V = L$.

Slavík's approach involved ingenious arguments using his notion of $A$-decomposability. This notion defines when a lattice, $L$, has to be the amalgamation of two proper sublattices, $S_1$ and $S_2$, thus providing an inductive procedure to force larger lattices into $V$. Our first result is a complete characterization of this important idea. Slavík then used $A$-decomposability on the construction procedures for primitive lattices to produce his results. We apply it to $B$, the class of so-called bounded lattices introduced by McKenzie in [6]. By Day [1], we have that $\text{HSP}(B) = L$ and by Day [2], all members of $B$ are generated by the interval construction from the lattice, $I$. Our second result implies that for $L \in B$, if $L \in V$ then $L[I] \in V$, and this completes the proof that $V = L$.

The authors are indebted to V. Slavík and J. B. Nation for stimulating discussions on this problem.

2. $A$-decomposability. In this section we characterize Slavík's important notion of:

(2.1) DEFINITION [8]. Let $L$ be a finite lattice and let $S_1$ and $S_2$ be proper sublattices of $L$. $L$ is called $A$-decomposable by means of $S_1$ and $S_2$ [Notation: $L = A(S_1, S_2)$] if $L = S_1 \cup S_2$ and for any lattice, $Z$, and lattice monomorphisms $f_i : S_i \rightarrow Z$ with $f_1 \uparrow S_1 \cap S_2 = f_2 \uparrow S_1 \cap S_2$, $f = f_1 \cup f_2$ is a lattice monomorphism from $L$ into $Z$.

Since the variety of all lattices satisfies (AP) by [5], the free amalgamation of $S_1 \cap S_2 \rightarrow S_i$, $i = 1, 2$, always exists. If $L = A(S_1, S_2)$ then $L$ is this
free amalgamation in $L$. Moreover if $V$ satisfies (AP) and $S_1, S_2 \in V$, we obtain $L = A(S_1, S_2) \in V$.

Given lattices $S_1$ and $S_2$ with $S = S_1 \cap S_2$, there are three relatively simple ways to amalgamate the diagram

$$
S \leftarrow S_1 \\
\cap \\
S_2
$$

The most familiar method is, by [5], using the McNeil Completion of $(S_1 \cup S_2, \sqsubseteq)$ where $x \sqsubseteq y$ in $S_1 \cup S_2$ iff there exists a $z \in S$ with $x \leq z$ in $S_i$ and $z \leq y$ in $S_j$ for some $i, j \in \{0, 1\}$. Since $MC(S_1 \cup S_2, \sqsubseteq)$ preserves existing joins and meets, there are lattice monomorphisms $f_i : S_i \rightarrow MC(S_1 \cup S_2, \sqsubseteq)$ with $f_1 \uparrow S = f_2 \uparrow S$. This provides us with our first necessary condition for $L = A(S_1, S_2)$.

(2.2) LEMMA. If $L = A(S_1, S_2)$, then $S_1$ and $S_2$ satisfy $O(S_1, S_2)$: $x \leq y, x \in S_i$ and $y \in S_j$ imply there exists $z \in S$ with $x \leq z$ and $z \leq y$.

The McNeil Completion is not the only way in which $(S_1 \cup S_2, \sqsubseteq)$ can be completed. Let $IC(S_1, S_2)$ be the set of all $(S_1, S_2)$-ideals. That is: $I \in IC(S_1, S_2)$ iff

1. $x \in I$ and $y \sqsubseteq x$ imply $y \in I$ and
2. $x, y \in I \cap S_i$ imply $x \vee y \in I$.

Clearly the intersection of $(S_1, S_2)$-ideals is again such and therefore $IC(S_1, S_2)$ is indeed a lattice. It is easy to check that $f_i : S_i \rightarrow IC(S_1, S_2)$ by $f_i(x) = \downarrow x = \{y \in S_1 \cup S_2 : y \sqsubseteq x\}, i = 1, 2$, are lattice monomorphisms with $f_1 \uparrow S = f_2 \uparrow S$. Dually we can define the $(S_1, S_2)$-filter completion, $FC(S_1, S_2)$.

We need to describe certain joins and meets of $IC(S_1, S_2)$ in the special case where $L = S_1 \cup S_2$ for sublattices $S_1$ and $S_2$ satisfying $O(S_1, S_2)$. Clearly $\downarrow x \cap \downarrow y = \downarrow x \vee y$ for all $x, y \in L$. In order to calculate $\downarrow x \vee \downarrow y$ we need a technical lemma.

(2.3) LEMMA. For $x \in S_1 \setminus S_2$ and $y \in S_2 \setminus S_1$, there exists $x' \in S_1$ and $y' \in S_2$ such that $\downarrow x \vee \downarrow y = \downarrow x' \vee \downarrow y'$ and there exists $z \in S$ with $z \leq x' \wedge y'$.

PROOF. Take $x \in S_1 \setminus S_2$ and $y \in S_2 \setminus S_1$. Since $x \wedge y \in L$ we have $x \wedge y \in S_1$ or $x \wedge y \in S_2$. Without loss of generality assume $x \wedge y \in S_1$. By $O(S_1, S_2)$, there exists $u \in S$ with $x \wedge y \leq u \leq y$ and by definition $x' = x \vee u \in \downarrow x \vee \downarrow y$. Now $x' \in S_1$ and by defining $y' = y$ we get $\downarrow x \vee \downarrow y = \downarrow x' \vee \downarrow y'$.

(2.4) DEFINITION. For $i \in \{1, 2\}$ and $S'_i = S_i \cup \{0, 1\}$ define $\alpha_i : L \rightarrow S'_i$ by $\alpha_i(x) = \{u \in S'_i : u \leq x\}$.

Note that for $x \in S_1 \setminus S_2$ and $y \in S_2 \setminus S_1$ we have:

1. $x \wedge y \leq \alpha_2(x) < x$ or $x \wedge y \leq \alpha_1(y) < y$,
2. $\alpha_2(x) < z \leq x$ implies $z \in S_1 \setminus S_2$,
3. $\alpha_2(x) \in \{0, 1\} \cup S$ (by 2.2).

(2.5) LEMMA. For $x \in S_1 \setminus S_2$ and $y \in S_2 \setminus S_1$ define $x_0 = x$, $y_0 = y$ and $x_{n+1} = x_n \vee \alpha_1(y_n), y_{n+1} = y_n \vee \alpha_2(x_n)$. Then $\downarrow x \vee \downarrow y = \bigcup \{\downarrow x_n \cup \downarrow y_n : n \geq 0\}$.

PROOF. Easy induction gives $x_n \in S_1$ and $y_n \in S_2$ for each $n$ as well as $x_n, y_n \in \downarrow x \vee \downarrow y$. However the union is clearly an $(S_1, S_2)$-ideal.

If $a$ is covered by $b$ in $L$ we call $a$ (resp. $b$) a lower (resp. upper) neighbour of $b$ (resp. $a$). We let $LN(a)$ (resp. $UN(a)$) be the set of all lower (resp. upper) neighbours of $a$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
(2.6) THEOREM. Let \( L \) be a finite lattice with \( L = S_1 \cup S_2 \) for proper sublattices \( S_1 \) and \( S_2 \) and define \( S = S_1 \cap S_2 \). \( L \) is \( A \)-decomposable by means of \( S_1 \) and \( S_2 \) if and only if \( S_1 \) and \( S_2 \) also satisfy:

- \( O(S_1, S_2) \): \( x \in S_1 \), \( y \in S_2 \) and \( x \leq y \) imply the existence of \( z \in S \) with \( x \leq z \) and \( z \leq y \).
- \( LN(S_1, S_2) \): For all \( x \in S \), \( LN(x) \subseteq S_1 \) or \( LN(x) \subseteq S_2 \).
- \( UN(S_1, S_2) \): For all \( x \in S \), \( UN(x) \subseteq S_1 \) or \( UN(x) \subseteq S_2 \).

PROOF. Assume firstly that \( L = A(S_1, S_2) \). By Lemma (2.3), \( O(S_1, S_2) \) holds and by definition the map \( x \mapsto \downarrow x \) of \( L \) into \( IC(S_1, S_2) \) must be a lattice monomorphism. If there existed an element \( u \in S \) with \( x \in LN(u) \setminus S_2 \) and \( y \in LN(u) \setminus S_1 \), \( \downarrow x \cup \downarrow y \) would be an \( (S_1, S_2) \)-ideal (observe that \( S_2 \cap (\downarrow x \cup \downarrow y) \subseteq \downarrow y \)) and therefore \( \downarrow x \lor \downarrow y \neq \downarrow (x \lor y) \). Therefore \( LN(S_1, S_2) \) and dually \( UN(S_1, S_2) \) hold.

Conversely assume the three conditions hold and take lattice monomorphisms \( f_i: S_i \to Z \) with \( f_1 \uparrow S = f_2 \uparrow S \). We must show \( g = f_1 \cup f_2 \) is a monomorphism.

Claim 1. \( x < y \) implies \( g(x) < g(y) \).

Assume \( x \in S_1 \setminus S_2 \) and \( y \in S_2 \setminus S_1 \). By \( O(S_1, S_2) \) there is a \( z \in S \) with \( x < z < y \). Moreover one of these inequalities must be strict. Therefore \( g(x) \leq g(z) \leq g(y) \) with one of these inequalities strict, hence \( g(x) < g(y) \).

Claim 2. \( g(x \lor y) = g(x) \lor g(y) \).

We need only consider the case where \( x \in S_1 \setminus S_2 \) and \( y \in S_2 \setminus S_1 \). By easy induction we obtain \( g(x) \lor g(y) = g(x_n) \lor g(y_n) \) for all \( n \).

Now if \( x_n, y_n < x \lor y \) for all \( n \), there exists \( k \) with \( x_{k+1} = x_k \in S_1 \setminus S_2 \) and \( y_{k+1} = y_k \in S_2 \setminus S_1 \). Therefore \( \alpha_1(y_k) \leq x_k \) and \( \alpha_2(x_k) \leq y_k \). By 2.4(1) we get \( x_k \land y_k \in \{\alpha_1(y_k), \alpha_2(x_k)\} \). Hence \( x_k \land y_k \in S \) by 2.4(3). Now 2.4(2) supplies the contradiction to \( UN(S_1, S_2) \). Therefore for some \( n \), \( x_n = x \lor y \geq y_n \) and \( g(x) \lor g(y) = g(x \lor y) \).

Claim 3. \( g(x \land y) = g(x) \land g(y) \).

By duality.

Therefore \( g: L \to Z \) is indeed a lattice monomorphism and \( L = A(S_1, S_2) \).

The above characterization makes it trivial to obtain certain properties of \( A \)-decomposable lattices.

(2.7) COROLLARY. If \( L = A(S_1, S_2) \) and \( S_i \leq T_i < L \), \( i = 1, 2 \), then \( L = A(T_1, T_2) \).

(2.8) COROLLARY. If there exists \( 0 < a \leq b < 1 \) in \( L \) with \( L = \downarrow a \cup \downarrow b \), then \( L = A(\downarrow a, \downarrow b) \).

3. \( \mathcal{V} = \mathcal{L} \). Let \( \mathcal{V} \) be a nondistributive variety of lattices satisfying the amalgamation property. By [3] we have \( N_5 \in \mathcal{V} \). We wish to show \( \mathcal{V} = \mathcal{L} \).

(3.1) LEMMA. Let \( L \) be a finite lattice with \( I = [u, v] \leq L \). If there exists \( \theta \in \text{Con}(L) \) with \( I = [I] \theta \), then \( L[I] \) is a sublattice of a product of \( L \) and \( L/\theta[I/\theta] \).

PROOF Let \( \psi = \text{Ker} f \) for the canonical \( f: L[I] \to L \) and define \( \bar{\theta} \in \text{Con}(L[I]) \) by \( \bar{x} \bar{y} \) if \( x, y \in L \setminus I \) and \( x \bar{y} \) or \( x, y \in I \times \{i\} \) and \( f(x) \bar{f}(y) \) for \( i \in 2 \). Easy calculations show that \( \bar{\theta} \in \text{Con}(L[I]), L[I]/\bar{\theta} \leq (L/\theta)[u/\theta, v/\theta] \) and \( \psi \land \bar{\theta} = \Delta_{L[I]} \).
(3.2) Corollary. If \( L \) is semidistributive, \( 0 < u < v < 1 \), and \( I = [u, v] \), then \( L[I] \) is a sublattice of a product of \( L \) and \( N_5 \).

Proof. Let \( \kappa(u) = \bigvee \{ x \in L : x \land u = 0 \} \) and \( \lambda(v) = \bigwedge \{ y \in L : y \lor v = 1 \} \). Then we have a homomorphism \( f : L \to 2^2 \) with congruence classes \( [u, v] \), \( [\lambda(v), \kappa(u)] \), \( [0, v \land \kappa(u)] \) and \( [u \lor \lambda(v), 1] \).

We would have liked a direct proof that \( L \in \mathcal{B} \cap \mathcal{V} \) implies \( L[I] \in \mathcal{B} \cap \mathcal{V} \) but this seems impossible. The following variation, however, does do the job.

(3.3) Lemma. For \( L \in \mathcal{B} \cap \mathcal{V} \) and \( I = [u, v] \subseteq L \), \( i \in 2 \), then \( (L \times 2)[(u, i), (v, i)] \) \( \in \mathcal{V} \) (and \( \mathcal{B} \)).

Proof. By induction on \( |L| \). Assume \( i = 1 \). If \( v = 1 \), then \( (L \times 2)[(u, 1), (v, 1)] \leq L \times 3 \in \mathcal{V} \). If \( v < 1 \), there is a co-atom \( m \), \( v \leq m < 1 \), and for \( p = \lambda(m) \), \( L = [m \cup p] \). Therefore \( L \times 2 \) can be pictured as in Figure (i). Since \( J = [(0, 1), (m, 1)] \) is a congruence class of the homomorphism \( f : L \times 2 \to 2^2 \), we can double \( J \) to produce a lattice that is a subdirect product of \( L \times 2 \) and \( N_5 \), hence a lattice in \( \mathcal{V} \). The congruence classes modulo the homomorphism \( g : (L \times 2)[J] \to N_5 \) produce the diagram in Figure (ii). Again since \( B_0 \) is a congruence class of this lattice, we can double this interval to produce a lattice \( M \in \mathcal{V} \) as in Figure (iii). Now let \( J \) be the interval \( I \) considered as lying in the congruence class labelled \( B \) in Figure (iii), and consider the lattice \( M[J] = A \cup B_0 \cup B_1 \cup C \cup D \cup B[J] \). By defining \( S_1 = A \cup B_0 \cup B_1 \cup C \cup D \) and \( S_2 = B_0 \cup B_1 \cup B[J] \), we obtain \( M[J] = A(S_1, S_2) \). Since \( S_1 \) is the lattice of Figure (ii), \( S_1 \in \mathcal{V} \). Since \( S_2 = A(B_0 \cup B[J], B_1 \cup B[J]) \), \( S_2 \in \mathcal{V} \) if and only if these two lattices belong to \( \mathcal{V} \). But \( B = [0, m] \) with \( |B| < |L| \), and these two lattices are \( B \times 2 \) with the interval \( I \times \{i\} \) split upstairs and downstairs respectively. By induction then \( S_2 \in \mathcal{V} \) and hence \( M[J] \in \mathcal{V} \). Since \( (L \times 2)[(u, 1), (v, 1)] \cong A \cup B[J] \cup C \cup D \leq M[J] \), this lattice is in \( \mathcal{V} \).

![Figure (i)](https://www.ams.org/journal-terms-of-use)
The proof for \( i = 0 \) follows by symmetry.

(3.4) **Theorem.** The only varieties of lattices that satisfy the amalgamation property are \( \mathcal{T}, \mathcal{D}, \) and \( \mathcal{L} \).

**Proof.** If \( \mathcal{V} \) is a nondistributive variety satisfying satisfying (AP), then by [3], \( N_5 \in \mathcal{V} \). Lemma (3.3) implies that for every \( L \in \mathcal{B} \), if \( L \in \mathcal{V} \) then \( L[I] \in \mathcal{V} \) since \( L[I] \leq (L \times 2)[I \times \{1\}] \). By [2], \( \mathcal{B} \subseteq \mathcal{V} \) and by [1], \( \mathcal{L} = \text{HSP}(\mathcal{B}) \subseteq \mathcal{V} \).

Since our proof requires only that \( N_5 \in \mathcal{V} \) it would be of interest to have an elementary proof (as opposed to [3]) that if \( \mathcal{V} \) satisfies (AP) and \( M_3 \in \mathcal{V} \) then \( N_5 \in \mathcal{V} \). Such a proof is not known to the authors.

(3.5) **Corollary.** \( \mathcal{L} \) is the only variety of lattices satisfying the strong amalgamation property.

**Proof.** \( \mathcal{D} \) does not satisfy (SAP). Whether or not \( \mathcal{T} \) has the strong amalgamation property depends directly on whether or not the empty lattice, \( \phi \), is allowed. If \( \phi \in \mathcal{T} \), then

\[
\begin{align*}
\phi & \rightarrow \{x\} \\
\cap & \{y\}
\end{align*}
\]

has no strong amalgamation in \( \mathcal{T} \).
REFERENCES

2. ______, Characterizations of finite lattices that are bounded-homomorphic images or sublattices of free lattices, Canad. J. Math. 31 (1979), 69–78.

DEPARTMENT OF MATHEMATICS, LAKEHEAD UNIVERSITY, THUNDER BAY, ONTARIO, CANADA P7B 5E1

DEPARTMENT OF MATHEMATICS, CHARLES UNIVERSITY, MFF, SOKOLOVSKÁ 83, 18600 PRAHA 8, CZECHOSLOVAKIA