AN APPLICATION OF FLOWS TO TIME SHIFT
AND TIME REVERSAL IN STOCHASTIC PROCESSES\textsuperscript{1}

BY

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Abstract. A simple proposition (Theorem 1) on flows allows the investigation of random time shift and time reversal in Markov processes without assuming any regularity of paths. Theorem 5 is a generalization of Nagasawa's time reversal theorem and Theorem 4 generalizes a recent result of Getoor and Glover.

1. Flows.

1.1. We consider a flow in a measurable space \((\Omega, \mathcal{F})\) that is a family of transformations \(\theta_t, t \in \mathbb{R}\), such that \(\theta_{s+t} = \theta_s \theta_t\) for all \(s, t \in \mathbb{R}\) and \(\{(t, \omega): \theta_t \omega \in A\} \in \mathcal{B}_\mathbb{R} \times \mathcal{B}\) for all \(A \in \mathcal{F}(\mathcal{B})\) is the Borel \(\sigma\)-algebra in \(\mathbb{R}\).

We put \((\theta_t Y)(\omega) = Y(\theta_t \omega), \quad A \in \mathcal{F},
\)

\[\widetilde{P} = \int_{\mathbb{R}} P \theta_t \, dt,\]

for every measure \(P\). We denote by \(\mathcal{A}\) the collection of all sets \(A \in \mathcal{F}\) such that \(\theta_t^{-1} A = A\) for all \(t \in \mathbb{R}\). A function \(Y\) is measurable with respect to the \(\sigma\)-algebra \(\mathcal{A}\) if and only if it is \(\mathcal{F}\)-measurable and invariant under all transformations \(\theta_t\). Obviously \(\tilde{P}(A) = 0\) or \(+\infty\) for all \(A \in \mathcal{A}\). Nevertheless we prove

Theorem 1. If \(\tilde{P}\) is \(\sigma\)-finite (over \(\mathcal{F}\)), it determines uniquely the values \(P(A)\) for all \(A \in \mathcal{A}\).

1.2. As a tool we use stationary times. A stationary time \(\tau\) is a measurable mapping from \(\Omega\) to the extended real line \([-\infty, +\infty]\) such that \(\theta_t \tau = \tau - t\) for all \(t \in \mathbb{R}\).

Suppose that \(\mathcal{F}\) coincides with its completion with respect to the class of all probability measures. Then the first hitting time of a set \(A \in \mathcal{F}\),

\[\tau_A^+ = \inf\{t: \theta_t \omega \in A\},\]

and the last exit time from \(A\),

\[\tau_A^- = \sup\{t: \theta_t \omega \in A\}\]

are stationary times\textsuperscript{2} (the measurability follows from [5, Chapter 3, §1]).
In general, every stationary time $\tau$ is the first hitting time of $\{\tau < 0\}$ and it is the last exit time from $\{\tau > 0\}$.

We denote the indicator function for the set $\{\tau \in \mathbb{R}\}$ by $\kappa_\tau$. Let $\rho$ be a positive Borel function on $\mathbb{R}$ such that $\int \rho(t) \, dt = 1$. We put $\rho(-\infty) = \rho(+\infty) = 0$. We note that $\theta_\tau \rho(t) = \rho(t - \tau)$. Hence

$$\kappa_\tau = \int \rho(t - \tau) \, dt = \int \theta_\tau \rho(t) \, dt$$

and, for every $\mathcal{A}$-measurable $Y$,

$$(1.1) \quad P\kappa,Y = PY \int \theta_\tau \rho(t) \, dt = \int P\theta_\tau(Y \rho(t)) = \overline{P}\rho(\tau)Y.$$

1.3. Suppose that there exists a strictly positive function $F$ such that

$$(1.2) \quad F_s = \int_{-\infty}^{s} \theta_t F \, dt \quad \text{is } P\text{-integrable for some } s \in \mathbb{R}.$$ 

Put

$$\tau_a = \inf\{s : F_s \geq a\}, \quad a > 0.$$ 

Note that $P\{\tau_a = -\infty\} = 0$ and $\{\tau_a = +\infty\} = \{F_{\infty} \leq a\}$. Hence $\kappa_\tau = 1_{F_{\infty} > a} \uparrow 1$ $P$-a.s. as $a \downarrow 0$. By (1.1),

$$(1.3) \quad PY = \lim_{a \downarrow 0} \overline{P}Yp(\tau_a).$$

The restriction of measure $P$ to $\mathcal{A}$ can be recovered from $\overline{P}$ by formula (1.3). An analogous formula can be written if

$$(1.4) \quad F_s = \int_{s}^{+\infty} \theta_t F \, dt \quad \text{is } P\text{-integrable for some } s \in \mathbb{R}.$$ 

Both conditions (1.2) and (1.4) are satisfied if $\overline{P}$ is $\sigma$-finite.

1.4. Let $\tau$ be a stationary time. For every function $Z(\omega)$ we denote by $\theta_\tau Z$ the function $Y(\omega)$ defined by the formula

$$Y(\omega) = \begin{cases} Z(\theta_\tau(\omega)) & \text{if } \tau(\omega) \in \mathbb{R}, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $Y$ is invariant and therefore formulas (1.1) and (1.3) are applicable to $Y$. We conclude

If $P_1$ and $P_2$ are two measures on $(\Omega, \mathcal{F})$ and if $\overline{P}_1 = \overline{P}_2$ is a $\sigma$-finite measure, then $P_1 \theta_\tau = P_2 \theta_\tau$ for all stationary times $\tau$.

1.5. The definition of the measure $\overline{P}$ makes sense if an arbitrary locally compact group $G$ acts on $(\Omega, \mathcal{F})$ (the Lebesgue measure on $\mathbb{R}$ should be replaced by a left- or right-invariant measure on $G$). Formula (1.1) and its proof remain valid if $\tau$ is a $G$-valued function defined on an invariant set $\Omega_\tau$ such that $\theta_g \tau(\omega) = \tau(\omega)g^{-1}$ for all $g \in G$. (We put $\kappa_\tau = 1_{\Omega_\tau}$ and we set $\rho(\tau) = 0$ if $\tau(\omega)$ is not defined.)

However the construction in subsection 1.3 is not applicable if $G \neq \mathbb{R}$. It remains an open problem for which groups $G$ Theorem 1 is true.
2. Random time shift in Markov processes.

2.1. Let \( p(s, x; t, B) \) be a Markov transition function in \((E, \mathcal{E})\). We put

\[
\mu T^s_t(B) = \int \mu(dx) p(s, x; t, B) \quad \text{and} \quad T^s_t f(x) = \int p(s, x; t, dy) f(y).
\]

Suppose that a \( \sigma \)-finite measure \( \nu_t \) is given for every \( t \in \mathbb{R} \). We say that \( \nu = \{ \nu_t \} \) is an entrance rule if, for all \( B \in \mathcal{E} \),

\[
\nu T^s_t(B) \leq \nu_t(B) \quad \text{and} \quad \nu T^s_t(B) \uparrow \nu_t(B) \quad \text{as} \ s \uparrow t.
\]

\( \nu \) is an entrance rule at time \( r \) if \( \nu_t = 0 \) for \( t \leq r \) and \( \nu T^s_t = \nu_t \) for \( r < s < t \). In an analogous way, we define an entrance rule at time \(-\infty\). Every entrance rule has a representation

\[
\nu = \int_{(-\infty, +\infty]} \nu' (dr)
\]

where \( \nu' \) is an entrance rule at time \( r \). We note that \( \sigma(-\infty) = 0 \) if and only if \( \nu T^s_t \downarrow 0 \) as \( t \uparrow \infty \) for some \( s \) (see [1, Formula (5.5)]).

A family \( h = \{ h^t, t \in \mathbb{R} \} \) of positive \( \mathcal{E} \)-measurable functions is called an exit rule if

\[
T^s_t h^t \leq h^s \quad \text{and} \quad T^s_t h^t \uparrow h^s \quad \text{as} \ t \downarrow s.
\]

All concepts related to entrance rules have natural analogs for exit rules.

The following result is due to Kuznetsov [4]. Suppose that \((E, \mathcal{E})\) is a standard Borel space and let an entrance rule \( \mu \) and an exit rule \( h \) satisfy the condition

\[
\mu_t(h^t = +\infty) = 0 \quad \text{for all} \ t.
\]

Then there exists a stochastic process \((X_t, P^h_\mu)\) on a random time interval \((\alpha, \beta)\) such that, for every \( t \in \mathbb{R} \),

\[
P^h_\mu(\alpha < t, X_t \in dy, t < \beta) = \mu_t(dy) h^t(y),
\]

and for all \( t_1 < \cdots < t_n \in \mathbb{R} \),

\[
P^h_\mu(\alpha < t_1, X_{t_1} \in dy_1, \ldots, X_{t_n} \in dy_n, t_n < \beta)
= \mu_{t_1}(dy_1) p(t_1, y_1; t_2, dy_2) \cdots p(t_{n-1}, y_{n-1}; t_n, dy_n) h^{t_n}(y_n).
\]

The measure \( P^h_\mu \) is \( \sigma \)-finite.

2.2. Suppose that a transition function \( p \) is stationary. Excessive measures and functions can be characterized as entrance and exit rules independent of \( t \). Let \( \mu \) be an entrance rule. If the measure

\[
\tilde{\mu} = \int \mu_t dt
\]

is \( \sigma \)-finite, it is excessive. To every exit rule \( h \) there corresponds an excessive function

\[
\tilde{h} = \int h^t dt.
\]

\(^3\) In the literature, entrance rules at time 0 are called entrance laws.
We assume that the process \( X_t(\omega) \) can be chosen in such a way that there exists a flow \( \theta_t, t \in \mathbb{R} \), with the properties: \( \theta_t \alpha = \alpha - t, \theta_t \beta = \beta - t \) and \( \theta_t X_u = X_{u+t} \).

A random time shift of the process \( X_t \) is defined on the space \( \Omega_\tau = \{ \omega: -\infty < \tau < +\infty \} \) by the formula

\[
(2.5) \quad \bar{X}_t(\omega) = X_{\tau(\omega) + t}(\omega).
\]

Suppose that an entrance rule \( \mu \) and an exit rule \( h \) satisfy the conditions

\[
(2.6) \quad \bar{\mu} \text{ is } \sigma \text{-finite and } h < \infty \quad \text{\( \bar{\mu} \) a.e.}
\]

Then the conditions of Kuznetsov's theorem are satisfied for three pairs \((\bar{\mu}, \bar{h})\), \((\mu, h)\) and \((\mu, \bar{h})\) and therefore, the measures \( P^h_\mu, P^h_\mu \) and \( P^\bar{h}_\mu \) are defined and \( \sigma \)-finite. We note that

\[ P^h_\mu = P^\bar{h}_\mu = P^\mu_\bar{h}. \]

By applying Theorem 1 and subsection 1.4, we get the following result:

**Theorem 2.** Measures \( P^h_\mu \) and \( P^\bar{h}_\mu \) coincide on all invariant sets. If \( \tau \) is a stationary time, then

\[
(2.7) \quad P^h_\mu \theta_\tau = P^\bar{h}_\mu \theta_\tau.
\]

The shift \( \bar{X}_t \) of \( X_t \) defined by formula (2.5) has identical laws under \( P^h_\mu \) and \( P^\bar{h}_\mu \).


3.1. Along with forward transition functions \( p(s, x; t, dy) \) we consider backward transition functions \( q(s, dx; t, y) \). We say that a forward transition function \( p \) and a backward transition function \( q \) are \( m \)-related if

\[
(3.1) \quad m_s(dx)p(s, x; t, dy) = q(s, dx; t, y)m_t(dy) \quad \text{for all } s < t.
\]

It follows from this relation that \( m \) is an entrance rule for both \( p \) and \( q \).

If \( g = \{ g_t \} \) is an exit rule for \( q \), then \( (g \circ m)_t(dx) = g_t(x)m_t(dx) \) is an entrance rule for \( p \), and all entrance rules for \( p \) which are absolutely continuous with respect to \( m \) can be represented in this form.

To every statement on forward transition functions there corresponds a statement on backward transition functions. The “backward” version of Kuznetsov’s theorem is as follows. Let \( \nu = \{ \nu_t \} \) be an entrance rule and \( g = \{ g_t \} \) be an exit rule for a backward transition function \( q \). If

\[
(3.2) \quad \nu'(g_t = +\infty) = 0 \quad \text{for all } t,
\]

then there exists a stochastic process \( (X_t, Q^*_g) \) on a random time interval \( (\alpha, \beta) \) such that, for every \( t \in \mathbb{R} \),

\[
(3.3) \quad Q^*_g(\alpha < t, X_t \in dy, t < \beta) = g_t(y)\nu'(dy)
\]

\(4\) (The \( \sigma \)-algebra \( \mathcal{F} \) in \( \Omega \) is generated by the sets \( \{ X_t \in B \}, t \in \mathbb{R}, B \in \mathcal{B} \).) Counterexamples show that Theorems 2 through 5 are not true without this assumption.
and, for all \( t_1 < \cdots < t_n \in \mathbb{R} \),

\[
Q^\alpha_g(\alpha < t_1, X_1 \in dy_1, \ldots, X_n \in dy_n, t_n < \beta) = g_{t_1}(y_1)q(t_1, dy_1; t_2, y_2) \cdots q(t_{n-1}, dy_{n-1}; t_n, y_n)v^\alpha(dy_n).
\]

The measure \( Q^\alpha_g \) is \( \sigma \)-finite.

Suppose that \( p \) and \( q \) are \( m \)-related and that \( h \) is an exit rule for \( p \) and \( g \) is an exit rule for \( q \). It follows from (3.3) and (3.4) that

\[
P_{g \ast m}^h = Q_{g \ast m}^h.
\]

3.2. If a backward transition function \( q \) is stationary, then we write

\[
q_t(dx, y) = q(t, dx; 0, y), \quad t < 0.
\]

If \( p \) and \( q \) are both stationary and if \( m_t = m \) for all \( t \), then the condition (3.1) can be rewritten as follows

\[
m(dx)p_t(x, dy) = q_t(dx, y)m(dy).
\]

Suppose that \( \mu \) is an entrance rule for \( p \), \( v \) is an entrance rule for \( q \) and let \( \bar{\mu} \) and \( \bar{v} \) be \( \sigma \)-finite and absolutely continuous with respect to \( m \). The densities \( u = d\bar{\mu}/dm \) and \( v = d\bar{v}/dm \) can be chosen to be \( q \)-excessive and \( p \)-excessive respectively. By (3.5),

\[
\overline{P^u} = P^v = P_{u \ast m}^v = Q_{u \ast m}^v = Q^v_u = \overline{Q^v_u},
\]

and Theorem 1 and subsection 1.4 imply

THEOREM 3. Measures \( P^v_u \) and \( Q^v_u \) coincide on all invariant sets. For every stationary time \( \tau \), \( P^v_u \theta_\tau = Q^v_u \theta_\tau \). The shifted process \( \tilde{X}_t = X_{t+\tau} \) has identical laws under \( P^v_u \) and \( Q^v_u \).

We write \( (X_t, P) \equiv (X_t', P') \) if stochastic processes \( (X_t, P) \) and \( (X_t', P') \) have identical laws (i.e., if they have the same finite-dimensional distributions).

3.3. Formula

\[
p_t(x, dy) = q_-(dy, x)
\]

determines a stationary forward transition function which is in weak duality with \( p_t(x, dy) \) in the sense of Getoor and Sharpe [3]. We note that \((\tilde{X}_{-\tau}, Q_{u \ast m}^{v \ast \tau}) \equiv (X_t, \hat{P}_m^u)\) where an entrance rule \( v \) for \( \hat{p} \) and an entrance rule \( v^\ast \) for \( q \) are related by the formula \((v^\ast)_t = v_{-t}\). Analogously, \((u^\ast)_t = u^{-t}\). We can assume without any substantial loss of generality that there exists a measurable transformation \( r \) of the space \((\Omega, \mathcal{F})\) such that \( X_t(r\omega) = X_{-t}(\omega) \). If \( \tau \) is a stationary time, then so is \( \tau^\ast(\omega) = -\tau(\omega) \). Theorem 3 implies

THEOREM 4. Let stationary transition functions \( p \) and \( \hat{p} \) be in weak duality relative to \( m \). Let \( \mu \) be an entrance rule for \( p \) and \( v \) be an entrance rule for \( \hat{p} \). Suppose that \( \bar{\mu} \) and \( \bar{v} \) are \( \sigma \)-finite and absolutely continuous with respect to \( m \). Let \( u \) and \( v \) be excessive functions (for \( p \) and \( \hat{p} \) respectively) such that \( \bar{\mu} = u \circ m, \bar{v} = v \circ m \). Then, for every stationary time \( \tau \), \((X_{\tau - t}, P^v_m) \equiv (X_{\tau - t}, \hat{P}_m^u)\).
This is a generalization of a theorem of Getoor and Glover (see [2, Theorem (6.5)]) who have considered the situation when \( \tau = \beta \),

\[
\begin{align*}
\mu_t &= \mu_0 P_t \quad \text{for } t > 0, \\
\nu_t &= \nu_0 \hat{P}_t \quad \text{for } t > 0, \\
\mu_\tau &= \nu_\tau = 0 \quad \text{for } t \leq 0.
\end{align*}
\]

In this situation \( \beta^* = \alpha = 0 \) \( \hat{\mu} \)-a.s. and \( (X_{\beta^-}, P_\mu) \equiv (X_t, \hat{P}_\nu) \). Moreover

\[
\begin{align*}
u &= d(\mu_0 G)/dm, \\
\nu_\tau &= d(\nu_0 \hat{G})/dm
\end{align*}
\]

where

\[
\hat{G} = \int \hat{T}_t dt, \quad \hat{\hat{G}} = \int \hat{T}_t dt.
\]

(Actually, in [2] the process \( X(\beta^-)_t \) rather than \( X_{\beta^-} \), is considered and therefore certain regularity conditions for paths are required.)

3.4. We say that a random time \( \tau \) is a renewal time for a Markov process \( (X_t, P) \) if \( (\hat{X}_t, P) \) where \( \hat{X}_t \) is defined by formula (2.5) has the same transition function as \( (X_t, P) \). If \( (X_t, P) \) is a strong Markov process with a stationary transition function, then every optional time is a renewal time.

Markov process \( (X_t, P_\nu) \) corresponding by Kuznetsov’s theorem to a stationary transition function \( \mu \), an entrance rule \( \nu \) and an excessive function \( u \) has the stationary transition function

\[
\begin{align*}
p_\nu(x, dy) &= u(x)^{-1} p_t(x, dy) u(y) \quad \text{for } 0 < u(x) < \infty, \\
p_\nu(x, B) &= 1_B(x) \quad \text{for } u(x) = 0 \text{ or } \infty.
\end{align*}
\]

We call \( p_\nu \) the \( u \)-transform of \( p \).

The following result is an immediate implication of Theorem 4:

**Theorem 5.** Let \( p, \hat{p}, \mu, \nu, u, v \) have the same meaning as in Theorem 4. If a stationary time \( \tau \) is a renewal time for \( (X_t, \hat{P}_\nu) \) then \( (X_{\tau^-}, P_\mu) \) is a Markov process with a stationary transition function \( \hat{p}_\nu \) (the \( v \)-transform of \( \hat{p} \)).

Theorem 5 is a generalization of Nagasawa’s theorem [6].

4. A property of invariant measures.

4.1. Let \( P \) be an invariant measure for a flow \( \theta_t \). For every positive measurable \( Y \) we put

\[
\bar{Y} = \int_{\mathbb{R}} \theta_t Y dt.
\]

We note that if \( \tau \) is a stationary time, then

\[
P_{\tau, Y} = P Y \int \theta_t \rho(\tau) dt = \int P \theta_t (\theta_{-\tau} Y \rho(\tau)) dt
\]

\[
= P \int \theta_{-\tau} Y \rho(\tau) dt = P \bar{Y} \rho(\tau)
\]

(cf. (1.1)).
4.2. An invariant measure \( P \) is called dissipative if \( \bar{Y} < \infty \) \( P \)-a.e. for every positive \( P \)-integrable \( Y \), and it is called conservative if \( \bar{Y} = 0 \) or \( \infty \) \( P \)-a.e. for every positive measurable \( Y \). It is well known (see, e.g., [7, §V.5]) that every \( \sigma \)-finite invariant measure \( P \) can be represented as the sum of a dissipative measure \( P_d \) and a conservative measure \( P_c \) which are mutually singular.

**Theorem 6.** Let \( P \) be a dissipative invariant measure. If \( \bar{Y}_1 = \bar{Y}_2 \) \( P \)-a.e., then \( PY_1 = PY_2 \).

Indeed, let \( F \) be a strictly positive \( P \)-integrable function. Since \( \bar{F} < \infty \) \( P \)-a.e., the condition (1.2) is satisfied and there exist stationary times \( \tau_a \) such that \( \kappa_{\tau_a} \uparrow 1 \) as \( a \downarrow 0 \). By (4.1),

\[
PY_i = \lim_{a \downarrow 0} P \bar{Y}_i \rho(\tau_a), \quad i = 1, 2.
\]

4.3. Let \( P \) be an arbitrary \( \sigma \)-finite invariant measure. There exists a partition of \( \Omega \) into disjoint sets \( \Omega_c \) and \( \Omega_d \) such that \( P_c(\Omega_d) = P_d(\Omega_c) = 0 \). Let \( F \) be a \( P \)-integrable function which is strictly positive on \( \Omega_c \) and equal to 0 on \( \Omega_d \). We have \( \bar{F} = 0 \) or \( \infty \) \( P_c \)-a.e., \( \bar{F} = 0 \) \( P_d \)-a.e. and therefore \( \bar{F} = 2 \bar{F} P \)-a.e. On the other hand, \( P(F) \neq P(2F) \) if \( P_c \neq 0 \). Therefore the statement of Theorem 6 is true only if \( P \) is dissipative.

**Remark.** It follows from subsections 4.2 and 4.3 that a \( \sigma \)-finite invariant measure \( P \) is dissipative if and only if it satisfies condition (1.2) (or (1.4)).

4.4. A measure \( P^1_m \) corresponding, by Kuznetsov’s theorem, to a stationary transition function \( p \), an excessive measure \( m \) and the exit rule \( h = 1 \) is invariant with respect to the flow \( \theta_t \). It is dissipative if and only if \( Gf < \infty, m \)-a.e. where \( G \) is defined in subsection 3.3.

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