SOME ESTIMATES FOR NONDIVERGENCE STRUCTURE,
SECOND ORDER ELLIPTIC EQUATIONS

BY

LAWRENCE C. EVANS

Abstract. We obtain various formal estimates for solutions of nondivergence
structure, second order, uniformly elliptic PDE. These include interior lower bounds
and also gradient estimates in $L^p$, for some $p > 0$.

1. Introduction. This paper presents some new estimates for solutions of nondiver-
gence structure, second order, uniformly elliptic PDE of the form

\[
\begin{cases}
-a_{ij}u_{,x_i,x_j} = \lambda E & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

and, more generally,

\[
\begin{cases}
-a_{ij}u_{,x_i,x_j} = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Here $\Omega$ is a bounded, smooth, open subset of $\mathbb{R}^n$, $E \subset \Omega$ is measurable, $f$ is bounded
and measurable, and the symmetric coefficients $a_{ij}$ satisfy the uniform ellipticity
condition:

\[
\theta |\xi|^2 \leq a_{ij}(x) \xi_i \xi_j \leq \Theta |\xi|^2 \quad (\xi \in \mathbb{R}^n, x \in \Omega).
\]

Moreover we will implicitly assume throughout that the $a_{ij}$ are smooth on $\bar{\Omega}$, so that
solutions of (1.1), (1.2) exist and belong at least to the Sobolev spaces $W^{2,p}(\Omega)$
($1 \leq p < \infty$). However all the bounds we derive are independent of the moduli
of continuity of the $a_{ij}$ and so depend on these coefficients only through $\theta$ and $\Theta$.

Estimates for equation (1.2) which do not depend on the continuity of the
coefficients have long been studied, both for their own sake and for applications to
nonlinear PDE. The principal discoveries so far include the two dimensional
estimates of Bernstein and of Nirenberg [4, §12.2], the supremum-norm bounds of
Aleksandrov-Bakelman-Pucci [4, §9.1], and the recent, important Hölder and
Harnack estimates of Krylov-Safonov [6]. (See also [4, §9.8, 7, 9].) One special case
of these latter results states, roughly, that if $\Omega' \subset \Omega$ is a connected and compactly

---

Received by the editors February 29, 1984 and, in revised form, May 23, 1984.
1980 Mathematics Subject Classification. Primary 35J15.
Key words and phrases. Second order elliptic PDE, nondivergence structure, mean sojourn time.
Supported in part by NSF grant MCS-8301265.

@1985 American Mathematical Society
0002-9947/85 $1.00 + $.25 per page

701
contained subdomain, \( E \subset \Omega' \), and \( u \) solves

\[
\begin{align*}
-a_{ij}u_{x_i x_j} &= 0 \quad \text{in } \Omega \setminus E, \\
u &= 1 \quad \text{on } \partial E, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

then

\[
\inf_{\Omega'} u \geq C|E|^l
\]

for certain constants \( C, l > 0 \), depending only on \( \theta, \Theta, \Omega' \) and \( \Omega \). This says probabilistically that a sample path of the diffusion associated with the above elliptic operator will with probability at least \( C|E|^l \) hit the set \( E \) before it first exits \( \Omega \), provided it begins inside \( \Omega' \).

Our first theorem below provides a complementary estimate, this on the mean sojourn time of sample paths within the set \( E \). In PDE terms we prove

**Theorem 1.** Let \( \Omega' \subset \subset \Omega \) be a connected subdomain. There exist positive constants \( m = m(n, \theta, \Theta) \) and \( C(\Omega') = C(\Omega', \theta, \Theta) \) such that

\[
\inf_{\Omega'} u \geq C(\Omega')|E|^m,
\]

whenever \( E \) is a measurable subset of \( \Omega' \) and \( u \) solves (1.1).

Probabilistically, a sample path of the diffusion will, if it begins in \( \Omega' \), on the average thus spend at least \( C(\Omega')|E|^m \) units of time in \( E \) before it first exits \( \Omega \).

Notice this is not a consequence of the Krylov-Safonov result: that a sample path hits the set \( E \) is no guarantee it will spend a positive amount of time within \( E \). Nevertheless our proof in §§2 and 3 is strongly based on methods from [6], especially the idea of considering small cubes large fractions of whose interiors belong to \( E \).

Theorem 1 has an immediate application in deriving a (quite weak) gradient estimate for the solution of (1.2):

**Theorem 2.** Assume \( u \) solves (1.2). Then there exist positive constants \( C \) and \( p \), depending only on \( n, \Omega, \theta, \) and \( \Theta \), such that

\[
\|Du\|_{L^p} \leq C\|f\|_{L^p}.
\]

The proof, in §4, is based upon a duality argument of a type employed by Bensoussan, Lions and Papanicolaou in [2, §3.5.2]. As noted, this bound is rather poor; but in light of known pathologies for the operator \( L = -a_{ij}\partial^2/\partial x_i \partial x_j \) (cf. Miller [8], Bauman [1]), it is not at all clear what kind of gradient estimate could be expected.

A consequence of Theorem 2 is this

**Corollary.** Assume \( F: \mathbb{R}^{n \times n} \to \mathbb{R} \) is smooth and satisfies the uniform ellipticity condition

\[
\theta|\xi|^2 \leq \frac{\partial F}{\partial r_{ij}}(r)\xi_i \xi_j \leq \Theta|\xi|^2 \quad (r \in \mathbb{R}^{n \times n}, \xi \in \mathbb{R}^n)
\]
for some $0 < \theta < \Theta$. Let $u$ be a smooth solution of the fully nonlinear elliptic PDE

$$F(D^2 u) = 0 \text{ in } \Omega.$$  

Then for each $\Omega' \subset \subset \Omega$ there exist positive constants $p$ and $C$, depending only on $n$, $\Omega'$, $\Omega$, $\theta$ and $\Theta$ such that

$$\|D^2 u\|_{L^p(\Omega')} \leq C\|u\|_{W^{2,p}(\Omega)}.$$

This bound is fairly pitiful, but is to my knowledge the only a priori second derivative estimate for (1.6) available if $n \geq 3$ under just the natural hypothesis (1.5). Of course if $F$ satisfies additional structure assumptions, far stronger estimates can be had: see [4, §17.4] for $F$ concave, [3, §§2–3] for $F$ “linear at infinity”, and [5, p. 42] for $F$ “close to linear”. N. S. Trudinger has noted that a variant of the corollary provides formal estimates on the third derivatives in $L^p_{\text{loc}}$ (for some $p > 0$) for a solution of Bellman’s equation; this is explained in the concluding remark of §5.

Notation and terminology. $A(x) = ((a_{ij}(x)))$, $|E| = \text{Lebesgue measure of the (measurable) set } E$.

All cubes mentioned below are closed and have sides parallel to some fixed coordinate axes; if $Q$ is a cube with side $R$, $aQ$ denotes the cube with the same center and side $aR$.

We employ the summation convention throughout; the letter $C$ denotes various constants depending only on known quantities.

Acknowledgement. I wish to thank A. Bensoussan for bringing to my attention to duality argument from [2] and N. S. Trudinger for several useful comments, including the remark mentioned above. I also thank the referee for a very thorough reading of this paper.

Noted added in proof. A forthcoming paper of E. Fabes and D. Stroock presents a new approach to [6, 7, 9] and this paper, via the theory of $A_\infty$ weights.

2. Estimates concerning cubes. We temporarily assume, in addition to hypothesis (A), that

(B) $\Omega$ and $\Omega'$ are cubes, $\Omega = 4\Omega'$,

and

(C) $\{ \text{there exists a real number } \alpha \text{ such that } \text{tr} A(x) \geq \alpha > 2\Theta \text{ for } x \in \Omega \}.$

These restrictions will be removed in §3.

Lemma 2.1. There exist constants $0 < \beta$, $\gamma < 1$ such that if $Q \subset \Omega'$ is a cube, $E \subset \Omega'$ is measurable and

$$|Q \cap E| \geq \beta |Q|,$$

then

$$u \geq \gamma v \text{ in } \Omega,$$
where $u$ and $v$ solve the PDE

\[
\begin{align*}
-a_{ij} v_{x_i x_j} &= \chi_{\Omega} & \text{in } \Omega, \\
v &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

\[
\begin{align*}
-a_{ij} u_{x_i x_j} &= \chi_{E \cap Q} & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega.
\end{align*}
\]

The constants $\beta$ and $\gamma$ here depend only on $n, \theta, \Theta$ and $\alpha$, and not on the size of $Q$.

PROOF. Upon translation and scaling, we may suppose $Q$ to be the unit cube, with side 2 and center 0. Let $w$ be the solution of

\[
\begin{align*}
-a_{ij} w_{x_i x_j} &= \chi_Q & \text{in } \Omega, \\
w &= 0 & \text{on } \partial \Omega.
\end{align*}
\]

Estimate on $u$ and $w$ from above. Define

\[ r(x) = \frac{1}{|x|^p} \quad (x \neq 0), \]

for $p > 0$ to be selected in a moment. We compute

\[
-a_{ij} r_{x_i x_j} = \frac{p}{|x|^{p+2}} \left( \text{tr} A - (p + 2) a_{ij} \frac{x_i x_j}{|x|^2} \right)
\]

\[
\geq \frac{p}{|x|^{p+2}} \left( \text{tr} A - (p + 2) \Theta \right).
\]

From condition (C) it follows that $-a_{ij} r_{ij} \geq C_1/|x|^{p+2}$ for some $C_1 > 0$, provided $p > 0$ is taken sufficiently small. Thus (2.3) implies

\[
0 \leq v \leq C_2/|x|^p \quad (x \in \Omega, x \neq 0).
\]

This bound and the estimate of Aleksandrov-Bakelman [4, p. 220] give also

\[
0 \leq v \leq C_3 \quad (x \in \Omega).
\]

Since $0 \leq w \leq v$ in $\Omega$, we have

\[
0 \leq w \leq C_3/|x|^p, \quad 0 \leq w \leq C_3.
\]

Estimate on $w$ from below. Set

\[ s(x) = \epsilon \left( 1 - |x|^2 \right) \]

and fix $\epsilon > 0$ so that

\[-a_{ij} s_{x_i x_j} \leq 1 \quad \text{in } B(1).\]

Since $B(1) \subset Q$ we have $w \geq s$ in $B(1)$, whence

\[
w(x) \geq C_4 \quad \text{in } B(1/2)
\]

for some $C_4 > 0$. In view of the weak Harnack inequality for supersolutions [4, p. 246] it follows that

\[
w \geq C_5 \quad \text{in } 3Q
\]

for some $C_5 > 0$. 

Estimate on $w - u$. Extend $w - u \geq 0$ to be zero in $\mathbb{R}^n \setminus \Omega$; then

\begin{equation}
- a_{ij}(w - u)_{x_i x_j} \leq \chi_{Q \setminus \Omega} \quad \text{in } \mathbb{R}^n.
\end{equation}

For each $\mu > 0$ the Aleksandrov-Bakelman estimate gives

\begin{equation}
\max_{B(\mu)} |w - u| \leq C\mu \|\chi_{Q \setminus \Omega}\|_{L^n} + \max_{\partial B(\mu)} |w - u| \leq C\mu |Q \setminus \Omega|^{1/n} + \max_{\partial B(\mu)} |w| \leq C_6 \left[ \mu(1 - \beta)^{1/n} + \frac{1}{\mu^p} \right],
\end{equation}

by (2.1) and (2.8).

Completion of proof. In light of (2.10), (2.12) and (2.8) we have

\begin{equation}
u = w - (w - u) \geq C_5 - C_6 \left[ \mu(1 - \beta)^{1/n} + 1/\mu^p \right] > C_5/2 \geq C_7 w \quad \text{in } Q
\end{equation}

for some $C_7 > 0$, provided we first choose $\mu$ large enough and then $\beta$ sufficiently close to 1.

Furthermore

\begin{equation}
-a_{ij}(u - C_7 w)_{x_i x_j} = 0 \quad \text{in } \Omega \setminus Q,
\end{equation}

\begin{equation}
u = u - C_7 w = 0 \quad \text{on } \partial \Omega,
\end{equation}

and so

\begin{equation}
u \geq C_7 w \quad \text{in } \Omega.
\end{equation}

Next observe from (2.7) and (2.10) that for some $C_8 > 0$

\begin{equation}
u \geq C_8 v \quad \text{in } 3Q;
\end{equation}

whence follows (2.2). $\square$

Notation. If $Q$ is a cube, we denote by $\hat{Q}$ any one of the $2^n$ subcubes formed by bisecting the sides of $Q$.

Lemma 2.2. Let $E$ be a measurable subset of the cube $\Omega'$ satisfying $|E| < \beta|\Omega'|$. Then there exists a countable collection of cubes $\{Q_i\}$ in $\Omega'$ such that

(a) int $Q_k \cap$ int $Q_l = \emptyset$ ($k \neq l$),
(b) $|E \setminus \bigcup_i Q_i| = 0$,
(c) $\sum_i |Q_i| \geq |E|/\beta$, and
(d) for each cube $Q_i$ there exists at least one subcube $\hat{Q}_i$ satisfying $|E \cap \hat{Q}_i| \geq \beta|\hat{Q}_i|$.

This result is essentially due to Krylov and Safonov [6]; see also [4, pp. 226–227, 9, p. 853].

Lemma 2.3. Let $E \subset \Omega', |E| < \beta|\Omega'|$ and set $F = \bigcup_i Q_i$. Then

\begin{equation}
u \geq \gamma v \quad \text{in } \Omega,
\end{equation}

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
where \( u \) and \( v \) solve the PDE

\[
\begin{align*}
-\alpha_{ij} u_{x_i x_j} &= \chi_F \quad \text{in } \Omega, \\
v &= 0 \quad \text{on } \partial \Omega, \\
-\alpha_{ij} v_{x_i x_j} &= \chi_E \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

Here \( \beta, \gamma \) are the constants from Lemma 2.1 and \( \{ Q_i \} \) the cubes described in Lemma 2.2.

**Proof.** We have

\[
u = \sum_{i} \nu_i, \quad v = \sum_{i} v_i,
\]

where

\[
\begin{align*}
-\alpha_{ij} \nu_{i,j} &= \chi_{Q_i} \quad \text{in } \Omega, \\
\nu &= 0 \quad \text{on } \partial \Omega, \\
-\alpha_{ij} \nu_{i,j} &= \chi_{E \cup Q_i} \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

For each \( Q_i \) fix some subcube \( \hat{Q}_i \) satisfying

\[
|E \cap \hat{Q}_i| \geq \beta|\hat{Q}_i|.
\]

Introduce the functions \( \hat{u}_i \) and \( \hat{v}_i \) solving

\[
\begin{align*}
-\alpha_{ij} \hat{u}_{i,j} &= \chi_{\hat{Q}_i} \quad \text{in } \Omega, \\
\hat{u} &= 0 \quad \text{on } \partial \Omega, \\
-\alpha_{ij} \hat{v}_{i,j} &= \chi_{E \cap \hat{Q}_i} \quad \text{in } \Omega, \\
\hat{v} &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

According to Lemma 2.1 and (2.17) \( \hat{u}_i \geq \gamma \hat{v}_i \). But \( u_i \geq \hat{u}_i \) and \( \hat{v}_i \geq v_i \) (since \( Q_i \subset 3\hat{Q}_i \)). Thus \( u_i \geq \gamma v_i \) and so

\[
u = \sum_{i} \nu_i \geq \gamma \sum_{i} v_i = \gamma v. \quad \Box
\]

3. **Proof of Theorem 1.** Continue for the time being in assuming conditions (B) and (C).

We claim there exist constants \( m, C(\Omega') > 0 \) such that for \( k = 1, 2, \ldots \)

\[
\inf_{\Omega'} u \geq C(\Omega') |E|^m.
\]

The proof is by induction. First note that, upon our redefining \( \beta \) to be closer to 1 if necessary, the methods for §2 show \( \inf_{\Omega'} u \geq C(\Omega') \) for some constant \( C(\Omega') > 0 \).
provided \(|E| \geq \beta|\Omega'|\) and \(u\) solves (1.1). Now set

\[(3.2) \quad m = \log \gamma / \log \beta > 0\]

and observe then that (3.1) is valid for some possibly new constant \(C(\Omega') > 0\).

Now assume (3.1)_\(k\) and suppose \(u\) solves (1.1), where

\[\beta^k|\Omega'| > |E| \geq \beta^{k+1}|\Omega'|.\]

Define \(F = \bigcup_i Q_i\), the cubes \(\{Q_i\}\) taken from Lemma 2.2. We have

\[(3.3) \quad |F| = \sum |Q_i| \geq \frac{|E|}{\beta} > \beta^k|\Omega'|;\]

whence by the induction hypothesis

\[\inf_{\Omega'} v \geq C(\Omega')|F|^m,\]

\(v\) solving (2.15). Then Lemma 2.3 implies

\[\inf_{\Omega'} u \geq \gamma \inf_{\Omega'} v \geq C(\Omega') \gamma |F|^m\]

\[\geq C(\Omega') \frac{\gamma}{\beta^m} |E|^m \quad \text{by (3.3)}\]

\[= C(\Omega')|E|^m \quad \text{by (3.2)}.\]

This proves (3.1)_\(k+1\) and completes the induction.

We have consequently established Theorem 1 under the additional assumptions (B) and (C), which we now remove.

First we retain hypothesis (C), but now drop (B). Thus assume \(\Omega' \subset \subset \Omega\) are open, bounded subsets of \(\mathbb{R}^n\), \(\Omega'\) connected. There exists a finite collection of cubes \(\{Q_k\}_{k=1}^N\) with equal sides and disjoint interiors such that

\[\Omega' \subset \bigcup_{k=1}^N Q_k \subset \bigcup_{k=1}^N 4Q_k \subset \Omega.\]

If \(E \subset \Omega'\), then \(|E \cap Q_k| \geq |E|/N\) for some \(k \in \{1, \ldots, N\}\). Let \(v\) solve

\[
\begin{cases}
-a_{ij}u_{x_i x_j} = \chi_{E \cap Q_k} \\ v = 0
\end{cases}
\quad \text{in } 4Q_k, \quad \text{on } \partial (4Q_k).
\]

Then the calculations before show

\[\inf_{Q_k} v \geq C(Q_k)|E \cap Q_k|^m \geq C_9|E|^m\]

for an appropriate constant \(C_9\). Since \(u \geq v\) and \(\Omega'\) is connected the estimates of Krylov and Safanov [6] give

\[\inf_{\Omega'} u \geq C(\Omega')|E|^m.\]

Finally we eliminate the restrictive hypothesis (C). Set

\[\tilde{\Omega} = \Omega \times (-1, 1) \times (-1, 1) \subset \mathbb{R}^{n+2},\]

\[\tilde{\Omega'} = \Omega' \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2}), \quad \tilde{E} = E \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2}),\]
and then define

$$
\tilde{A} = \begin{pmatrix}
    a_{11} & \cdots & a_{1n} & 0 & 0 \\
    \vdots & \ddots & \vdots & \vdots & \vdots \\
    a_{n1} & \cdots & a_{nn} & \cdots & \cdots \\
    0 & \cdots & \cdots & \cdots & \cdots \\
    0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \Theta \\
\end{pmatrix}_{(n+2) \times (n+2)}
$$

Since

$$
\operatorname{tr} \tilde{A} = \operatorname{tr} A + 2\Theta \geq n\theta + 2\Theta = \alpha > 2\Theta,
$$

condition (C) holds for \( \tilde{A} = ((\tilde{a}_{ij})) \). Thus if \( \tilde{u} \) is the solution of

$$
\begin{align*}
-\tilde{a}_{ij} \tilde{u}_{x_i x_j} &= \chi_{\tilde{F}} & &\text{in } \tilde{\Omega}, \\
\tilde{u} &= 0 & &\text{on } \partial \tilde{\Omega},
\end{align*}
$$

we have

$$(3.4) \quad \inf_{\tilde{\Omega}} \tilde{u} \geq C|\tilde{E}|^m$$

for certain constants \( C = C(\tilde{\Omega}', m) \). Now if \( u \) solves (1.1) and we regard \( u \) as really being a function of \( (x_1, \ldots, x_{n+2}) \), we have

$$
\begin{align*}
-\tilde{a}_{ij} u_{x_i x_j} &= \chi_{\tilde{F}} & &\text{in } \tilde{\Omega}, \\
u &\geq 0 & &\text{on } \partial \tilde{\Omega},
\end{align*}
$$

for

$$
\tilde{F} = E \times (-1, 1) \times (-1, 1) \supset \tilde{E}.
$$

Hence \( u \geq \tilde{u} \) in \( \tilde{\Omega} \); and so (3.4) implies \( \inf_{\Omega'} u \geq C|E|^m \). \( \square \)

4. Proof of Theorem 2. Choose some smooth, bounded, connected domain \( \Omega \subset \subset \Omega' \). According to the Fredholm alternative there exists a solution \( w \) of

$$(4.1) \quad \begin{cases}
- (a_{ij} w)_{x_i x_j} = 1 & \text{in } \Omega', \\
w = 0 & \text{on } \partial \Omega'.
\end{cases}$$

Estimate on \( w \) from above. Select any \( f \in L^\infty(\Omega') \) and let \( v \) solve

$$
\begin{align*}
- a_{ij} v_{x_i x_j} &= f & &\text{in } \Omega', \\
v &= 0 & &\text{on } \partial \Omega'.
\end{align*}
$$

Then

$$
\begin{align*}
\int_{\Omega'} w f \, dx &= \int_{\Omega'} w (-a_{ij} v_{x_i x_j}) \, dx \\
&= \int_{\Omega'} -(a_{ij} w)_{x_i x_j} v \, dx \\
&= \int_{\Omega'} v \, dx \leq C \| f \|_{L^\infty(\Omega')},
\end{align*}
$$
and so

(4.2) \[ \|w\|_{L^{\infty}(\Omega')} \leq C_{10}. \]

Estimate on \( w \) from below. Let \( E \subset \Omega \) be measurable and suppose \( v \) now solves

\[
\begin{align*}
-a_{ij} v_{x_i x_j} &= \chi_E \quad \text{in } \Omega', \\
v &= 0 \quad \text{on } \partial\Omega'.
\end{align*}
\]

Then

\[ \int_E w \, dx = \int_{\Omega'} w(-a_{ij}v_{x_i x_j}) \, dx = \int_{\Omega'} v \, dx \geq C_{11}|E|^m \]

for certain constants \( m, C_{11} > 0 \), according to Theorem 1. Thus, in particular, \( w > 0 \) a.e. in \( \Omega' \). Fix \( \lambda > 0 \) and take \( E = \{ x \in \Omega | 1/w(x) > \lambda \} \). Then

\[ |E|/\lambda \geq \int_E w \, dx \geq C_{11}|E|^m. \]

Consequently

\[ |E| \leq C_{12} \lambda^{-1/(m-1)}, \]

and this in turn implies

(4.3) \[ \int_\Omega \frac{dx}{w(x)^\alpha} \leq C_{13} \]

for \( C_{13} \) and \( 0 < \alpha < 1/(m-1) \).

Estimate on \( Du \). Now assume \( u \) solves (1.2) in \( \Omega \). We multiply both sides of the equation by \( wu \) and integrate by parts to obtain

(4.4) \[ \int_\Omega (a_{ij}wu)_{x_i} u_{x_j} \, dx = \int_\Omega fwu \, dx. \]

The term on the left equals

\[
\begin{align*}
\int_\Omega w a_{ij} u_{x_i} u_{x_j} + (a_{ij}w)_{x_j} u_{x_i}^2 \frac{u^2}{2} \, dx \\
= \int_\Omega w a_{ij} u_{x_i} u_{x_j} - (a_{ij}w)_{x_i} u_{x_j} \frac{u^2}{2} \, dx \\
\geq \int_\Omega \theta w |Du|^2 + \frac{u^2}{2} \, dx.
\end{align*}
\]

Since \( \|u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^r(\Omega)} \), the calculation above, (4.4) and (4.2) yield

(4.5) \[ \int_\Omega w |Du|^2 \, dx \leq C_{14} \|f\|_{L^r(\Omega)}^2. \]

Finally observe that

\[ \int_\Omega |Du|^p \, dx \leq \left( \int_\Omega |Du|^p \right)^{\frac{w^q}{w^q}} \left( \int_\Omega \frac{dx}{w^{qr}} \right)^{\frac{1}{s}} \]
for any \( p, q, r, s > 0, 1/r + 1/s = 1 \). Choose
\[
p = \frac{2\alpha}{\alpha + 1}, \quad q = \frac{\alpha}{\alpha + 1}, \quad r = \frac{\alpha + 1}{\alpha}, \quad s = \alpha + 1,
\]
where \( \alpha \) is the constant from (4.3). Then
\[
\int_{\Omega} |Du|^p \, dx \leq \left( \int_{\Omega} |Du|^2 w \, dx \right)^{\alpha/(\alpha + 1)} \left( \int_{\Omega} \frac{dx}{w^\alpha} \right)^{1/(\alpha + 1)}
\leq \left( C_{14} \|f\|_{L^m}^2 \right)^{\alpha/(\alpha + 1)} C_{13}^{1/\alpha} = C_{15} \|f\|_{L^p(\Omega)},
\]
owing to (4.3) and (4.5).  \( \square \)

5. **Proof of the corollary.** Now assume \( u \) to be a smooth solution of (1.6) in \( \Omega \) and choose \( \Omega' \subset \subset \Omega \). Let \( w \) solve
\[
\begin{cases}
-(a_{ij}w)_{x_i x_j} = 1 & \text{in } \Omega, \\
w = 0 & \text{on } \partial \Omega.
\end{cases}
\]
Select a smooth cutoff function \( \xi \) which equals 1 in \( \Omega' \), 0 near \( \partial \Omega \). Differentiate (1.6) with respect to \( x_k \):
\[
\frac{\partial F}{\partial r_{ij}} (D^2 u)_{x_k x_i x_j} = 0.
\]
We set
\[
v = u_{x_k}, \quad a_{ij} = \frac{\partial F}{\partial r_{ij}} (D^2 u)
\]
to obtain
\[
-a_{ij} v_{x_i x_j} = 0 \quad \text{in } \Omega.
\]
Multiply by \( \xi^2 wv \) and integrate by parts to find
\[
\int_{\Omega} \xi^2 w a_{ij} v_{x_i x_j} + (a_{ij}w)_{x_i} \xi^2 w_{x_i} \, dx = -\int_{\Omega} 2 a_{ij} w \xi x_{x_i} v_{x_j} \, dx.
\]
The second term on the left equals
\[
-\int_{\Omega} (a_{ij}w)_{x_i x_j} \xi^2 \frac{\partial^2 u}{\partial x_i \partial x_j} + (a_{ij}w)_{x_j} \xi^2 v_{x_i} \, dx
\]
\[
= \int_{\Omega} \frac{\xi^2 \partial^2 u}{2} + a_{ij} w \left((\xi x_{x_i})_{x_j} v^2 + 2 \xi x_{x_i} v_{x_j} \right) \, dx.
\]
Consequently
\[
\theta \int_{\Omega} \xi^2 w |Dv|^2 \, dx \leq -\int_{\Omega} 4 a_{ij} w \xi x_{x_i} v_{x_j} + a_{ij} w (\xi x_{x_i})_{x_j} v^2 \, dx
\]
\[
\leq \frac{\theta}{2} \int_{\Omega} \xi^2 w |Dv|^2 \, dx + C \int_{\Omega} w v^2 |D\xi|^2 + \xi |D^2 \xi| \, dx;
\]
whence
\[ \int_{\Omega} |Dv|^2 \, dx \leq \int_{\Omega} w^2 \, dx \leq C \|v\|_{L^p(\Omega)}^2. \]
Since \( v = u_{x_i} \) is any derivative of \( u \), the estimate above and the calculations in §4 show
\[ \int_{\Omega} |D^2u|^p \, dx \leq C_{16} \|u\|_{W^{2,p}(\Omega)}^p. \]

**Remark.** The following observation is due to N. S. Trudinger. Suppose in addition to (1.5) that \( F \) is concave. Differentiate (1.6) twice in some direction \( \eta \) and discard the term involving \( D^2F \) to find
\[ -\frac{\partial F}{\partial \eta_i} (D^2u)_{\eta\eta, x_j} \leq 0 \quad \text{in } \Omega. \]
Define \( a_{ij} \) as above and set \( v = u_{\eta\eta} \). This gives
\[ (5.1) \quad -a_{ij} u_{x_i x_j} \leq 0 \quad \text{in } \Omega. \]
Set \( K = \|D^2u\|_{L^p} \), take \( \xi \) and \( w \) as above, multiply (5.1) by \( \xi^2 w(v + K) \) and mimic then the foregoing calculations to find
\[ \int_{\Omega} |Dv|^p \, dx \leq C \quad (C, p > 0). \]
We then derive the bound
\[ (5.2) \quad \int_{\Omega} |D^3u|^p \, dx \leq C_{17}, \]
\( C_{17} \) depending on \( \|u\|_{W^{2,p}} \) and the other usual quantities.
A review of the proof shows that the various constants here depend on \( F \) only through \( \theta, \Theta \) and \( \|DF\|_{L^\infty} \). Estimate (5.2) thus formally applies to the Bellman equation
\[ \inf_{\alpha \in A} \{ L^\alpha u \} = 0 \quad \text{in } \Omega, \]
where each \( L^\alpha \) is a linear, constant coefficient elliptic operator with ellipticity bounds \( \theta, \Theta \).
It is not at all clear if any of this can be made rigorous. \( \Box \)

**References**


