THE MACKEY TOPOLOGY AND COMPLEMENTED SUBSPACES
OF LORENTZ SEQUENCE SPACES $d(w, p)$ FOR $0 < p < 1$

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ABSTRACT. In this paper we continue the study of Lorentz sequence spaces $d(w, p)$, $0 < p < 1$, initiated by N. Popa [8]. First we show that the Mackey completion of $d(w, p)$ is equal to $d(v, 1)$ for some sequence $v$. Next, we prove that if $d(w, p) \nsubseteq l_1$, then it contains a complemented subspace isomorphic to $l_p$. Finally we show that if $\lim n^{-1}(\sum_{i=1}^{n} w_i)^{1/p} = \infty$, then every complemented subspace of $d(w, p)$ with symmetric bases is isomorphic to $d(w, p)$.

I. Introduction. A $p$-norm, $0 < p \leq 1$, on a vector space $X$ is a map $x \mapsto \|x\|$ such that:
1. $\|x\| > 0$ if $x \neq 0$.
2. $\|tx\| = |t| \|x\|$ for all $x \in X$ and all scalars $t$.
3. $\|x + y\|^p \leq \|x\|^p + \|y\|^p$ for all $x, y \in X$.

Let $B = \{x \in X : \|x\| \leq 1\}$; then the family $\{rB\}_{r>0}$ is a base of neighbourhoods of zero for a Hausdorff locally bounded vector topology on $X$ (see [9]). If $X$ is complete, we say that $X$ is a $p$-Banach space.

The Mackey topology $\mu$ of a locally bounded space $X$ with separating dual is the strongest locally convex topology on $X$ which is weaker than the original one (see [10]). It is easy to see that this normable topology is generated by neighbourhoods $\{r \overline{\text{conv}} B\}_{r>0}$. The Minkowski functional of the set $\overline{\text{conv}} B$ is called the Mackey norm on $X$. The completion of the space $(X, \mu)$ is called the Mackey completion of $X$ and denoted by $\hat{X}$. The extension of the Mackey norm to $\hat{X}$ is denoted by $\|\cdot\|$.

For every subset $E$ of $\omega (= \text{the space of all scalar sequences})$ we denote
$$E^+ = \{x = (x_i) \in E : x_i \geq 0 \text{ for } i = 1, 2, \ldots\}$$
and
$$E^{++} = \{x \in E^+ : x \text{ is nonincreasing} \}.$$

Let $0 < p < \infty$ and let $w = (w_i) \in l_1^+ \setminus l_1$. For $x = (x_i) \in \omega$ we define
$$\|x\|_{w,p} = \sup_\pi \left( \sum_{i=1}^{\infty} |x_{\pi(i)}|^p w_i \right)^{1/p},$$
where $\pi$ ranges over all permutations of the positive integers. The space $d(w, p) = \{x \in \omega : \|x\|_{w,p} < \infty\}$ equipped with the locally bounded vector topology induced by $\|\cdot\|_{w,p}$ is called the Lorentz sequence space.

It is well known that $d(w, p)$ is a $p$-Banach space for $0 < p < 1$ and a Banach space for $p \geq 1$. Moreover, the sequence of unit vectors $(e_i)$ is a symmetric basis of
$d(w,p)$. From the assumption $w \in l^{++}_1 \setminus l_1$ follows that $d(w,p) \subset c_0$. Therefore for every $x = (x_i) \in d(w,p)$ there exists a nonincreasing rearrangement $x^* = (x^*_i)$ of $x$ (i.e. a nonincreasing sequence obtained from $(|x_i|)$ by a suitable permutation of the integers) and $\|x\|_{w,p} = (\sum_{i=1}^{\infty} x^*_i p w_i)^{1/p}$.

Observe that $d(w,p) \approx l_p$ if and only if $w \notin c_0$ (cf. [6, p. 176]).

The first topic of the present paper is the Mackey topology of $d(w,p)$, $0 < p < 1$. Using a representation of the dual of $d(w,p)$, N. Popa [8] proved that the Mackey completion of $d(w,p)$ ($p = 1/k$, $k \in \mathbb{N}$, and $w$ satisfies some additional conditions) is isomorphic to $d(v,1)$ for a suitable sequence $v$. In §3 we show that the above theorem holds for any Lorentz sequence space $d(w,p)$, $0 < p < 1$. Our result is obtained without determining any dual space.

The last part of our paper is devoted to the study of complemented subspaces of $d(w,p)$, $0 < p < 1$.

It is well known that every Lorentz sequence space $d(w,p)$, $p \geq 1$, has complemented subspace isomorphic to $l_p$ (see [6, Proposition 4.e.3]). N. Popa [8] showed that unlike the case $p \geq 1$ there are spaces $d(w,p)$, $0 < p < 1$, which contain no complemented subspaces isomorphic to $l_p$ and conjectured that it is true for each $d(w,p)$, $0 < p < 1$. In §4 we prove that if $\inf_n n^{-1} (\sum_{i=1}^{n} w_i)^{1/p} = 0$ (i.e. $d(w,p) \not\subset l_1$, see Proposition 1), then $d(w,p)$ has complemented subspace isomorphic to $l_p$. Moreover, if $\lim_{n \to \infty} n^{-1} (\sum_{i=1}^{n} w_i)^{1/p} = \infty$, then every complemented subspace of $d(w,p)$ with symmetric basis is isomorphic to $d(w,p)$.

Throughout the paper we denote by $B_{w,p}$ the closed unit ball in $d(w,p)$, $R^n = \text{span}\{e_i\}_{i=1}^{n}$, $B^n_{w,p} = B_{w,p} \cap R^n$, $n = 1,2,\ldots$. In addition we denote $S_n(x) = x_1 + \cdots + x_n$, $n = 1,2,\ldots$, $S_0(x) = 0$ for any sequence $x = (x_i) \in w$.

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II. Technical results. In this section we assume that $0 < p \leq 1$, $w = (w_i) \in l^{++}_1 \setminus l_1$, $\sigma_k = S_k(w)^{1/p}$, $f_k = \sigma_{k-1}^{-1} \sum_{i=1}^{k} e_i$ for $k = 1,2,\ldots$, and $f_0 = 0$.

**LEMMA 1.** Let $\| \cdot \|_n$ be the norm on $R^n$ defined by

$$\|x\|_n = \sum_{i=1}^{n} |x_i| (\sigma_i - \sigma_{i-1}) \quad \text{for} \quad x = (x_i) \in R^n,$$

and let

$$B^n = \{x = (x_i) \in R^n : \|x\|_n \leq 1\}, \quad n \in \mathbb{N}.$$

Then:

(a) $(B^n)^{++} = \text{conv}\{f_k : k = 0,1,\ldots,n\}$.

(b) $(B^n_{w,p})^{++} \subset (B^n)^{++}$.

(c) Let $0 < p < 1$ and $x = (x_i) \in (R^n)^{++}$. Then $\|x\|_n = \|x\|_{w,p} = 1$ if and only if $x = f_k$ for some $k = 1,2,\ldots,n$.

(d) If $p = 1$, then $(B^n_{w,p})^{++} = (B^n)^{++}$.

**PROOF.** (a) Every point $x \in R^n$ may be written in the form

$$x = \sum_{i=1}^{n-1} \sigma_i (x_i - x_{i+1}) f_i + \sigma_n x_n f_n.$$

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In addition, for each $x \in (\mathbb{R}^n)^+,$

$$\|x\|_n = \sum_{i=1}^{n-1} \sigma_i(x_i - x_{i+1}) + \sigma_n x_n.$$ 

Therefore every $x \in (\mathbb{R}^n)^+$ with $\|x\|_n = 1$ is a convex combination of the vector $f_1, \ldots, f_n.$ It implies $(B^n)^+ \subset \text{conv}\{f_0, f_1, \ldots, f_n\}.$

We observe that $\|f_k\|_{w,p} = \|f_k\|_n = 1$ for $k = 1, 2, \ldots, n.$ Both the sets $(\mathbb{R}^n)^+$ and $B^n$ are convex, so

$$\text{conv}\{f_0, \ldots, f_k\} \subset B^n \cap (\mathbb{R}^n)^+ = (B^n)^+.$$  

(b) By the proof of (a) and by concavity of the function $x \mapsto \|x\|_{w,p}$ on $(\mathbb{R}^n)^+,$

$$\|x\|_{w,p} \leq \sum_{i=1}^{n-1} \sigma_i(x_i - x_{i+1})\|f_i\|_{p,p} + \sigma_n x_n\|f_n\|_{p,p} = \|x\|_n$$

for $x \in (\mathbb{R}^n)^+, \|x\|_n = 1.$

Thus (b) follows from homogeneity of the functionals $\|\cdot\|_n$ and $\|\cdot\|_{w,p}.$

(c) Since the function $x \mapsto \|x\|_{w,p}$ is strictly concave on $(\mathbb{R}^n)^+ \setminus \{0\}, 0 < p < 1,$ the assertion (c) is clear.

(d) It is enough to observe that $\|x\|_{w,1} = \|x\|_n$ for every $x \in (\mathbb{R}^n)^+.$

**COROLLARY 1.** If $y = (y_i) \in (\mathbb{R}^n)^+$ and $S_k(y) \leq \sigma_k$ for $k = 1, \ldots, n,$ then

$$\sum_{i=1}^{n} x_i y_i \leq \left( \sum_{i=1}^{n} x_i^{p} w_i \right)^{1/p}$$

for every $x \in (x_i) \in (\mathbb{R}^n)^+.$

**PROOF.** Corollary 1 follows immediately from Lemma 1(b).

**COROLLARY 2.** For every $x = (x_i) \in (\mathbb{R}^n)^+,$

$$\left( \sum_{i=1}^{n-1} x_i^{p} w_i \right)^{1/p} + (\sigma_n - \sigma_{n-1})x_n \leq \left( \sum_{i=1}^{n} x_i^{p} w_i \right)^{1/p}.$$  

**PROOF.** It suffices to apply Corollary 1 with $\bar{w}_1 = S_{n-1}(w), \bar{w}_2 = w_n, \bar{x}_1 = \left( \sum_{i=1}^{n-1} x_i^{p} w_i \right)^{1/p} \sigma_{n-1}, \bar{x}_2 = x_n, y_1 = \sigma_n - \sigma_{n-1}, \text{ and } y_2 = \sigma_n - \sigma_{n-1}.$

**PROPOSITION 1.** Let $0 < p \leq 1, w = (w_i)$ and $v = (v_i)$ belong to $l_\infty^+ \setminus l_1.$ Then

$$d(w, p) \subset d(v, 1) \text{ if and only if } \inf_n \frac{S_n^{1/p}(w)}{S_n(v)} > 0.$$  

In particular

$$d(w, p) \subset l_1 \text{ if and only if } \inf_n n^{-1} S_n^{1/p}(w) > 0.$$  

**PROOF.** If $d(w, p) \subset d(v, 1),$ then, by the closed graph theorem, the inclusion map is continuous. Moreover, $\|f_n\|_{w,p} = 1$ for $n = 1, 2, \ldots.$ Thus

$$\sup_n \|f_n\|_{v,1} = \sup_n \frac{S_n(v)}{S_n^{1/p}(w)} < +\infty.$$  

If $\inf_n S_n^{1/p}(w)/S_n(v) > 0,$ then, by Corollary 1, $d(w, p) \subset d(v, 1).$
LEMMA 2. Let \( \inf_n \sigma_n/n = 0 \). Then there exist an increasing sequence of integers \((n_k)\) and a sequence of positive numbers \( q = (q_n) \in c_0 \) such that:
(a) \( S_n(q) \leq \sigma_n \) for \( n = 1, 2, \ldots \).
(b) \( S_{n_k}(q) = \sigma_{n_k} \) for \( k = 1, 2, \ldots \).
(c) The sequence \( (S_n(q)/n) \) is nonincreasing.

PROOF. We define \((n_k)\) by induction taking \( n_1 = 1 \) and
\[
n_{k+1} = \inf \left\{ n > n_k : \frac{\sigma_n}{n} < \frac{\sigma_{n_k}}{n_k} \right\}, \quad k = 1, 2, \ldots.
\]

Put \( Q_n = n \sigma_{n_k}/n_k \) for \( n_k \leq n < n_{k+1}, \ k = 1, 2, \ldots, \ q_n = Q_n - Q_{n-1} \) for \( n = 1, 2, \ldots, \) and \( Q_0 = 0 \). The assertions (a), (b) and (c) follow immediately from the construction.

LEMMA 3. If \( q = (q_n) \in c_0^+ \) and \( (S_n(q)/n) \in \omega^{++} \), then \( S_n(q) \leq S_n(q^*) \leq 2S_n(q) \) for \( n = 1, 2, \ldots \).

PROOF. Evidently \( S_n(q) \leq S_n(q^*) \). We define
\[ A = \{ i \in \{1, \ldots, n\} : q_i^* = q_j \text{ for some } j > n \}. \]

Since the sequence \( (S_n(q)/n) \) is nonincreasing, \( q_{n+1} \leq S_n(q)/n \) for \( n = 1, 2, \ldots \). Thus, if \( i \in A \) and \( q_i^* = q_j \) for \( j > n \), so \( q_i^* = q_j \leq S_{j-1}(q)/(j - 1) \leq S_n(q)/n \). Therefore
\[
S_n(q^*) = \sum_{i \in A} q_i^* + \sum_{i \leq n, i \not\in A} q_i \leq |A| \frac{S_n(q)}{n} + S_n(q) \leq 2S_n(q).
\]

LEMMA 4. Let \( \lim_{n \to \infty} \sigma_n/n = +\infty \) and let \( x_m = (x_{mi}) \) be a normalized sequence in \( d(w, p) \). Then \( \lim_{m \to \infty} \|x_m\|_{c_0} = 0 \) implies \( \lim_{m \to \infty} \|x_m\|_{l_1} = 0 \).

PROOF. We can assume that \( x_m = x_m^* \) for every \( m \in \mathbb{N} \). Fix \( \varepsilon > 0 \). There is \( n_0 \in \mathbb{N} \) such that \( 2n/\varepsilon \leq \sigma_n \) for every \( n \geq n_0 \). Let
\[
y_i = \begin{cases} 
0 & \text{if } i < n_0, \\
2/\varepsilon & \text{if } i \geq n_0.
\end{cases}
\]

Then \( S_k(y) \leq \sigma_k \) for every \( k \in \mathbb{N} \). From Corollary 1 follows
\[
\sum_{i=1}^{n} x_{mi} y_i \leq \left( \sum_{i=1}^{n} x_{mi}^p w_i \right)^{1/p} \leq \|x_m\|_{w, p} = 1, \quad n, m = 1, 2, \ldots.
\]

Thus
\[
\frac{2}{\varepsilon} \sum_{i=n_0}^{\infty} x_{mi} \leq 1 \quad \text{for } m = 1, 2, \ldots.
\]

Finally
\[
\sum_{i=n_0}^{\infty} x_{mi} \leq \frac{\varepsilon}{2} \quad \text{for } m = 1, 2, \ldots.
\]
Lemma 5. Let $0 < p < 1$ and $x = (x_t) \in d(w, p)^{++}$. If $\|x\|_{u,p} = \|x\|_{w, p} = 1$, then $x = f_k$ for some $k = 1, 2, \ldots$.

Proof. Let $x^{(n)} = \sum_{i=1}^{n} x_i e_i$ and let $\|\cdot\|_n$ be as in Lemma 1. Every point $f_k$ is of the form $f_k = (\alpha, \alpha, \ldots, \alpha, 0, \ldots)$ for some $\alpha > 0$. Suppose that $x \neq f_k$ for $k = 1, 2, \ldots$. Then there is $t \in \mathbb{N}$ such that $x_{t-1} > x_t > 0$. Therefore by Lemma 1(b) $\|x^{(t)}\|_t \leq \|x^{(t)}\|_{w, p}$ and by Lemma 1(c) we see that the equality cannot hold. Thus for some $\varepsilon > 0$ we have

$$\|x^{(t)}\|_t \leq \|x^{(t)}\|_{w, p} - \varepsilon.$$ 

From this, using Corollary 2, we get by induction

$$\|x^{(n)}\|_{w, p} \leq \|x^{(n)}\|_n \leq \|x^{(n)}\|_{w, p} - \varepsilon \quad \text{for } n \geq 1.$$ 

Thus $\|x\|_{w, p} \leq \|x\|_{w, p} - \varepsilon$.

III. The Mackey topology of $d(w, p)$, $0 < p < 1$.

Theorem 1. Let $0 < p < 1$ and $w = (w_t) \in l_\infty^+ \setminus l_1$. Then there exists a sequence $v = (v_t) \in l_\infty^+ \setminus l_1$ such that $d(w, p) \subset d(v, 1)$ and the Mackey topology of $d(w, p)$ is induced from $d(v, 1)$.

The sequence $v \in c_0$ if and only if $\inf_n n^{-1} S_n^{1/p}(w) = 0$.

Proof. If $\inf_n n^{-1} S_n^{1/p}(w) > 0$, then by Proposition 1 $d(w, p) \subset l_1 = d(v, 1)$ for $v = (1, 1, \ldots)$. By [8, Proposition 3.4], the Mackey topology of $d(w, p)$ is induced from $l_1$.

Let $\inf_n n^{-1} S_n^{1/p}(w) = 0$. We choose sequences $(n_k) \subset \mathbb{N}$ and $(q_n)$ according to Lemma 2. Put $v_n = q_n^*, n = 1, 2, \ldots$.

We will show that

$$B_n \cap \text{conv } B_{w, p}^n \subset 2B_{v, 1}^n$$ 

Indeed, by Lemma 3, $S_k(v) = S_k(q^*) \leq 2S_k(q) \leq 2S_k^{1/p}(w)$, for $k = 1, 2, \ldots$. Thus, using Corollary 1 with $y_k = \frac{1}{k} v_k$, we obtain $(B_{w, p}^n)^{++} \subset 2(B_{v, 1}^n)^{++}$. Hence the right inclusion follows from the convexity of $B_{v, 1}$.

It is obvious that if $(B_{v, 1}^n)^{++} \subset \text{conv } B_{w, p}^n$, then the left inclusion holds. Since $(B_{v, 1}^n)^{++} = \text{conv } \{g_j : j = 0, 1, \ldots, n\}$, where $g_j = S_j^{-1}(v) \sum_{i=1}^{j} e_i$, $g_0 = 0$ (see Lemma 1(a) and (b)), it suffices to prove that $g_j \in \text{conv } B_{w, p}^n$ for $j = 1, \ldots, n$.

Fix $j \in \{1, \ldots, n\}$. We find $n_k$ such that $n_k \leq j < n_{k+1}$. Let $C$ be the family of all subsets of cardinality $n_k$ in the set $\{1, \ldots, j\}$. We define

$$x_C = S_{n_k}^{-1/p}(w) \sum_{i \in C} e_i \quad \text{for some } C \in C.$$
We have \( \|x_C\|_{w,p} = 1 \) and
\[
\frac{1}{|C|} \sum_{C \in C} x_C = \left( \binom{j}{n_k} \right)^{-1} \sum_{i \in C} S_{n_k}^{-1}(w) \sum_{i \in C} e_i
\]
\[
= \left( \binom{j}{n_k} \right)^{-1} \left( \frac{j}{n_k} - 1 \right) \sum_{i=1}^{n_k} S_{n_k}^{-1/p}(w) e_i
\]
\[
= \frac{1}{j} \sum_{i=1}^{n_k} S_{n_k}^{-1/p}(w) e_i = \frac{1}{j} S_{n_k}^{-1}(q) \sum_{i=1}^{j} e_i
\]
\[
= \frac{S_j(q^*)}{S_j(q)} g_j.
\]

Thus \( (S_j(q^*)/S_j(q)) g_j \in \text{conv} B_{w,p}^n \). Since \( S_j(q) \leq S_j(q^*) \) and the set \( \text{conv} B_{w,p}^n \) is balanced, \( g_j \in \text{conv} B_{w,p}^n \). Therefore the assertion \( (*) \) holds. Thus the Mackey topology of \( d(w,p) \) and the \( d(v,1) \)-topology coincide on the subspace of all finitely supported sequences. Since this subspace is dense in \( d(w,p) \), these two topologies coincide on \( d(w,p) \).

If \( \inf_n n^{-1} S_n^{1/p}(w) = 0 \), then \( v \in C_0 \) by Lemma 2.

As a simple application of Theorem 1 we obtain the representation of the dual \( d(w,p)' \) of \( d(w,p) \), \( 0 < p < 1 \).

**Corollary 3.** Let \( 0 < p < 1, w = (w_i) \in l_{1+}^{+} \setminus l_1 \). Then

\( d(w,p)' = l_\infty \) if \( \inf_n \frac{S_n^{1/p}(w)}{n} > 0 \);

\( d(w,p)' = \left\{ y \in C_0 : \sup_n \frac{S_n(y^*)}{S_n^{1/p}(w)} < +\infty \right\} =: E(w,p) \) if \( \inf_n \frac{S_n^{1/p}(w)}{n} = 0 \).

**Proof.** If \( \inf_n S_n^{1/p}(w)/n > 0 \), then by Theorem 1 \( d(w,p) = l_1 \), so \( d(w,p)' = l_\infty \). Let \( \inf_n S_n^{1/p}(w)/n = 0 \). Then by Theorem 1 there exists \( v = (v_i) \in C_0^{+} \setminus l_1 \) such that \( d(w,p) = d(v,1) \). Therefore by Proposition 1 \( \sup_n S_n(v)/S_n^{1/p}(w) < +\infty \). By [4, Theorem 11], \( d(v,1) = \left\{ y \in C_0 : \sup_n S_n(y^*)/S_n(v) < +\infty \right\} \). Hence \( d(w,p)' = d(v,1)' \subset E(w,p) \).

The inclusion \( E(w,p) \subset d(w,p)' \) follows directly from Corollary 1.

**Remark 1.** Theorem 1 and Corollary 3 are respectively extensions of Theorem 6.3 and Proposition 6.1 in [8].

**IV. Complemented subspaces of** \( d(w,p), 0 < p < 1 \).

**Theorem 2.** Let \( 0 < p < 1 \) and let \( w = (w_i) \in C_0^{+} \setminus l_1 \). If \( \inf_n S_n^{1/p}(w)/n = 0 \), then there is a positive continuous projection from \( d(w,p) \) onto a sublattice order isomorphic to \( l_p \).

**Proof.** First we construct by induction an increasing sequence of integers \( \{n_k\}_{k=0}^{\infty} \) and a sequence \( q = (q_i) \in \omega^+ \) such that the following conditions are
satisfied for all \( k \geq 0 \):  

(1) \[
\left( \sum_{i=n_k+1}^{j} w_i \right)^{1/p} \geq \sum_{i=n_k+1}^{j} q_i \quad \text{for } n_k < j \leq n_{k+1};
\]

(2) \[
k \leq \left( \sum_{i=n_k+1}^{n_{k+1}} w_i \right)^{1/p} = \sum_{i=n_k+1}^{n_{k+1}} q_i;
\]

(3) the sequence \[
\left( \sum_{i=n_k+1}^{j} \frac{q_i}{j-n_k} \right)_{j=n_k+1}^{n_{k+1}}
\]
is nonincreasing;

(4) \[
\left( \sum_{i=1}^{n-n_k} w_i \right)^{1/p} \leq 2 \left( \sum_{i=n_k+1}^{n} w_i \right)^{1/p}.
\]

We start with \( n_0 = 0, q_0 = 0 \). Suppose that \( n_k \) has been already defined for some \( k \geq 0 \). Since \( w \notin l_1 \), there is \( r \in \mathbb{N}, r \geq n_k \) such that for every \( n > r \)

\[
\left( \sum_{i=n_k+1}^{n-n_k} w_i \right)^{1/p} \leq 2 \left( \sum_{i=n_k+1}^{n} w_i \right)^{1/p}.
\]

Applying Lemma 2 to the sequence \((w_i)_{i=n_k+1}^{\infty}\) we can find \( n_{k+1} > r \) and \((q_i)_{i=n_k+1}^{n_{k+1}}\) such that (1), (2) and (3) hold. As \( n_{k+1} > r \), the same is true of (4).

Let

\[
f_k = \left( \sum_{i=n_k+1}^{n_{k+1}} w_i \right)^{-1/p} \sum_{i=n_k+1}^{n_{k+1}} e_i, \quad k = 0, 1, 2, \ldots
\]

It follows from (4) that \( \|f_k\|_{w,p} \leq 2 \).

Now we define the projection \( P: d(w, p) \to \text{span}\{f_k\}_{k=0}^{\infty} \) by

\[
P(x) = \sum_{k=0}^{\infty} \left( \sum_{i=n_k+1}^{n_{k+1}} x_i q_i \right) f_k, \quad \text{where } x = (x_i) \in d(w, p).
\]

Let \( x = (x_i) \in d(w, p) \) and let \((\hat{x}_i)_{i=n_k+1}^{n_{k+1}}\) and \((\hat{q}_i)_{i=n_k+1}^{n_{k+1}}\), \( k = 0, 1, \ldots \), be respectively nonincreasing rearrangements of the sequences \((|x_i|)_{i=n_k+1}^{n_{k+1}}\) and \((q_i)_{i=n_k+1}^{n_{k+1}}\). Using (3) and Lemma 3 we have

\[
\sum_{i=n_{k+1}+1}^{1} \hat{q}_i \leq 2 \sum_{i=n_k+1}^{1} q_i, \quad l = n_k + 1, \ldots, n_{k+1}.
\]

Thus by (1) and Corollary 1 we get

\[
\|Px\|_{w,p}^p \leq 2^p \sum_{k=0}^{\infty} \left( \sum_{i=n_k+1}^{n_{k+1}} x_i q_i \right)^p \leq 2^p \sum_{k=0}^{\infty} \left( \sum_{i=n_k+1}^{n_{k+1}} \hat{x}_i \hat{q}_i \right)^p
\]

\[
\leq 2^{p+1} \sum_{k=0}^{\infty} \left( \sum_{i=n_k+1}^{n_{k+1}} \hat{x}_i \hat{q}_i \right)^p \leq 2^{p+1} \sum_{i=1}^{\infty} x_i^p w_i = 2^{p+1} \|x\|_{w,p}^p.
\]


Thus $P$ is continuous. By (2) and [8, Lemma 3.1] there is a strictly increasing sequence $(j_k)$ such that $(f_{j_k})$ is equivalent to the canonical basis of $l_p$. Therefore the desired result follows from unconditionality of the basic sequence $(f_k)$.

REMARK 2. Theorem 2 solves Problems 3 and 3a in [8].

COROLLARY 4. If $\inf_n n^{-1} S_n^{1/p}(w) = 0$, then $d(w,p) \oplus l_p$ is isomorphic to $d(w,p)$, $0 < p < 1$.

PROOF. By Theorem 2, $d(w,p) = X \oplus l_p$ for some $F$-space $X$. Therefore

$$d(w,p) = X \oplus l_p = X \oplus l_p \oplus l_p = d(w,p) \oplus l_p.$$ 

COROLLARY 5. Let $0 < p < 1$, and $\inf_n n^{-1} S_n^{1/p}(w) = 0$. Then $d(w,p)$ has uncountably many mutually nonequivalent unconditional bases.

PROOF. It is enough to know that $d(w,p)$ has at least two mutually nonequivalent bases (cf. [6, p. 118]). Thus our result follows from Corollary 4.

In the proof of the next theorem we use the same ideas as in [7, Theorem 2.3].

THEOREM 3. Let $0 < p < 1$, $w = (w_i) \in l^{1+} \setminus l_1$. If $\lim_{n \to \infty} S_n^{1/p}(w)/n = \infty$, then each infinite-dimensional complemented subspace of $d(w,p)$ contains a subspace $Y$ which is isomorphic to $d(w,p)$ and complemented in $d(w,p)$.

PROOF. Let $P$ be a continuous projection from $d(w,p)$ onto an infinite-dimensional subspace $X$ of $d(w,p)$. Since $\lim_{n \to \infty} S_n^{1/p}(w)/n = \infty$, by Theorem 1 $d(w,p) = l_1$. Because $X$ is complemented in $d(w,p)$, so its Mackey topology is also induced from $l_1$. Since the $l_1$-closure of $\{P(e_i) : i \in \mathbb{N}\}$ is a neighbourhood of zero in $X$, the set $\{P(e_i) : i \in \mathbb{N}\}$ is not precompact in $l_1$. Therefore, using the standard gliding hump method, we can construct a strictly increasing sequence of the integers $(n_k)$ and sequences of vectors $(y_k)$ and $(z_k)$ such that:

1. $y_k = P(e_{n_{2k+1}} - e_{n_{2k}});$ 
2. $z_k = \sum_{i \in A_k} t_i e_i$ is a block basic sequence; 
3. $\sum_{k=1}^{\infty} \|y_k - z_k\|_{l^{1/p}} < 1;$ 
4. $0 < C_1 \leq \|z_k\|_{l_1} \leq \|z_k\|_{l^{w,p}} \leq C_2$ for $k \in \mathbb{N}$, where $C_1, C_2$ are some constants.

By Lemma 4 we have $\inf_{k \in A_k} |t_i| > 0$. Since $(e_k)$ is symmetric and $P$ is continuous, the sequence $(z_k)$ is equivalent to $(e_k)$. Thus, as in [3], we may define a continuous projection $Q$ by

$$Q(x) = \sum_{n=1}^{\infty} \frac{x_{i_n}}{t_{i_n}} z_n \quad \text{if} \quad x = (x_i) \in d(w,p),$$

where $i_n \in A_n$ and $|t_{i_n}| = \max\{|t_i| : i \in A_n\}$, $n = 1, 2, \ldots$. Using a stability theorem (cf. [6, Proposition 1.9] and [7, Proposition 1.2]) we conclude that $\text{span}\{P(e_{n_{2k+1}}) - P(e_{n_{2k}})\}_{k \geq k_0}$ is isomorphic to $d(w,p)$ and complemented in $d(w,p)$.

Our next result is an easy consequence of Theorem 3 and Pelczyński's decomposition method.

COROLLARY 6. Let $0 < p < 1$ and $w = (w_i) \in l^{1+} \setminus l_1$. If $\lim_{n \to \infty} S_n^{1/p}(w)/n = \infty$, then every infinite-dimensional complemented subspace of $d(w,p)$ with symmetric basis is isomorphic to $d(w,p)$. 

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COROLLARY 7. Let $0 < p < 1$, $w = (w_i) \in c_0^{++} \setminus l_1$ and $\lim_{n \to \infty} S_{n}^{1/p}(w)/n = \infty$. Then $d(w, p)$ contains a closed subspace $X$ nonisomorphic to $l_p$ and $d(w, p)$ such that $X \approx l_1$.

PROOF. It follows from Corollary 6 that $d(w, p) \oplus l_p \not\cong d(w, p)$. Moreover $d(w, p) \oplus l_p$ is isomorphic to some subspace $Z$ of $d(w, p) \oplus d(w, p) \approx d(w, p)$. Since $l_p \oplus d(w, p) = l_1 \oplus l_1 \approx l_1$ we get $Z \approx l_1$.

REMARK 3. Corollary 7 solves partially Problem 2 in [8].

PROPOSITION 2. Let $0 < p < 1$, $w = (w_i) \in l_1^{++} \setminus l_1$ and $w_1 < S_{n}^{1/p}(w)/n$ for $n > 1$. If $P : d(w, p) \to Y \subset d(w, p)$ is a constructive projection, then $Y = \text{span}\{e_i : i \in A\}$ for some set $A \subset N$.

PROOF. We can assume that $w_1 = 1$. Since $1 < n^{-1}S_{n}^{1/p}(w)$, by Theorem 1 and Corollary 1, we have $d(w, p) = l_1$ and $(B_{w,p}^n)^{++} \subset B_{l_1}$, $n = 1, 2, \ldots$. Thus $B_{w,p} \subset B_{l_1}$ and

$$
\hat{B} = \text{conv}^{l_1}B_{w,p} \subset B_{l_1} = \text{conv}^{l_1}\{e_i : i = 1, 2, \ldots\} \subset \text{conv}^{l_1}B_{w,p} = \hat{B},
$$

where $\hat{B} = \{x \in l_1 : \|x\|_{w,p} \leq 1\}$.

Therefore $\|x\|_{w,p} = \|x\|_{l_1}$.

Hence a continuous extension $\hat{P}$ of $P$ is a contractive projection in $l_1 = d(w, p)$. By [5, Chapter 6, §17, Theorem 3] (see also [6, Theorem 2.a.4]),

$$
\hat{P}(x) = \sum_{j=1}^{m} h_j(x)u_j,
$$

where $\{u_j\}_{j=1}^{m}$ are vectors of norm 1 in $l_1$ ($m = \dim Y$ is either an integer or $\infty$), $u_j = \sum_{i \in A_j} t_i e_i$, with $A_j \cap A_k = \emptyset$ for $j \neq k$ and $\{h_j\}_{j=1}^{m} \subset l_1$ satisfy $\|h_j\|_{\infty} = h_j(u_j) = 1$, $j = 1, 2, \ldots$.

Since for every $x \in d(w, p)$ and $j = 1, 2, \ldots$,

$$
\|x\|_{w,p} \geq \|Px\|_{w,p} = \|\hat{P}x\|_{w,p} \geq \|h_j(x)u_j\|_{w,p},
$$

so $u_j \in d(w, p)$ and $Q_j(x) := h_j(x)u_j$ is a contractive projection from $d(w, p)$ onto a one-dimensional subspace $\text{span}\{u_j\}$.

Therefore $\|u_j\|_{w,p} = \|u_j\|_{w,p} = 1$. By Lemma 5, $u_j^* = f_k$ for some $k = 1, 2, \ldots$.

Since $1 < S_{n}^{1/p}(w)/n$ for $n > 1$, $\|f_k\|_{w,p} < \|f_k\|_{w,p}$ if $k > 1$. Thus $u_j^* = e_i$, $j = 1, 2, \ldots$.

COROLLARY 8. Let $0 < p < 1$, $w = (w_i) \in c_0^{++} \setminus l_1$ and $w_1 < S_{n}^{1/p}(w)/n$ for $n > 1$. Then $l_p$ is not isomorphic to the range of a contractive projection in $d(w, p)$.

REMARK 4. Corollary 8 is an extension of Theorem 5.5 in [8].

V. Open problems and remarks. If $\lim_{n \to \infty} S_{n}^{1/p}(w)/n = 0$, then by Theorem 2 there exists a continuous projection $P$ from $d(w, p)$ onto a subspace isomorphic to $l_p$. Moreover, if $\lim_{n \to \infty} S_{n}^{1/p}(w)/n = \infty$, then by Theorem 3 no subspace isomorphic to $l_p$ is complemented in $d(w, p)$.

PROBLEM 1. Let $0 < p < 1$ and $0 < \lim_{n \to \infty} S_{n}^{1/p}(w)/n < \infty$. Is there a continuous projection from $d(w, p)$ onto a subspace isomorphic to $l_p$?
PROBLEM 2. Let $0 < p < 1$ and $\lim_{n \to \infty} S_n^{1/p}(w)/n = 0$. Is there a contractive projection from $d(w, p)$ onto a subspace isomorphic to $l_p$?

The next result is an extension of Theorem 3.8 in [8].

PROPOSITION 3. Each symmetric basis $(y_k)$ of $d(w, p)$ ($0 < p < 1$) is equivalent to the canonical basis $(e_k)$ of $d(w, p)$.

PROOF. Using the standard gliding hump method we can find a strictly increasing sequence of natural numbers $(n_k)$ such that the sequence $x_k = y_{n_{2k}} - y_{n_{2k+1}}$ is equivalent to a block basic sequence $z_k = \sum_{i \in A_k} b_i e_i$. Since $x_k$ is symmetric and equivalent to $(y_k)$, by [8, Lemma 3.1] $\inf_i \max_{i \in A_k} |b_i| > 0$. Hence $(y_k)$ dominates $(e_k)$. If we interchange the roles of $(e_k)$ and $(y_k)$ we deduce the equivalence of these bases.

If $\lim_{n \to \infty} S_n^{1/p}(w)/n = 0$, then $d(w, p)$ has uncountable many mutually non-equivalent unconditional bases. However the above proposition and Corollary 6 suggest the following

PROBLEM 3. Let $0 < p < 1$ and $\lim_{n \to \infty} S_n^{1/p}(w)/n = \infty$. Are every two unconditional bases in $d(w, p)$ equivalent?

REFERENCES


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