ON CHEBYSHEV SUBSPACES IN THE SPACE
OF MULTIVARIATE DIFFERENTIABLE FUNCTIONS

BY

ANDRÁS KROÓ

ABSTRACT. In the present paper we give a characterization of Chebyshev sub-
spaces in the space of (real or complex) continuously-differentiable functions
of two variables. We also discuss various applications of the characterization
theorem.

Introduction. One of the central problems in approximation theory consists
in determining the best approximation; that is, in a normed linear space X with
a prescribed subspace U we seek for each f in X its best approximation p in U
satisfying $\|f - p\| = \text{dist}(f, U)$. If U is finite dimensional, then every element f in X
possesses a best approximant. This raises the very important and delicate question
of *unicity* of best approximation. The finite-dimensional subspace U is called a
Chebyshev subspace of X if for each f in X its best approximant in U is unique.
Chebyshev subspaces have been widely investigated in different functional spaces.
In case of approximation with respect to the supremum norm these investigations
were initiated by the classical works of Chebyshev and Haar.

Let $C(K)$ denote the space of real or complex continuous functions endowed
with the supremum norm on the compact Hausdorff space K. The celebrated
Haar-Kolmogorov theorem gives a characterization of finite-dimensional Cheby-
shev subspaces of $C(K)$: the n-dimensional subspace $U_n \subset C(K)$ is a Chebyshev
subspace of $C(K)$ if and only if it satisfies the so-called Haar property, i.e. each
nontrivial element of U_n has at most $n - 1$ distinct zeros at K. (This result was first
proved by Haar [6] in the real case, and then by Kolmogorov [7] in the complex
case.) Later Mairhuber [9] showed that in the real case $C(K)$ possesses a Cheby-
shev subspace of dimension $n > 1$ if and only if K is homeomorphic to a subset of
the circle, i.e. the study of finite-dimensional Chebyshev subspaces of $C(K)$ is, in
fact, restricted in the real case to functions of one variable.

It is natural to expect that requiring unicity only with respect to a smaller
subspace may lead to the extension of the family of Chebyshev subspaces. This ap-
proach is well known from the theory of L_1-approximation. In case of Chebyshev
approximation such investigations were initiated by Garkavi [5] who gave a char-
acterization of Chebyshev subspaces in the space of *real* continuously-differentiable
functions endowed with the supremum norm on $[a, b]$. In a series of papers [1–3]
the analog of Garkavi's result was given for the real rational families.

In a recent paper [8] we characterized the finite-dimensional Chebyshev sub-
spaces in the space $C^1[a, b]$ of real or *complex* continuously-differentiable functions
with supremum norm on $[a, b]$. In order to formulate this result we shall need

the following definition: if U_n is an n-dimensional subspace of $C(K)$, then the set of m distinct points $\{x_k\}_{k=1}^m \subseteq K$, where $1 \le m \le n+1$ in the real case and $1 \le m \le 2n+1$ in the complex case, is called an *extremal set* of U_n if there exist numbers $a_k \ne 0$, $1 \le k \le m$ (real in the real case), such that

$$\sum_{k=1}^m a_k g(x_k) = 0$$

for any $g \in U_n$. The numbers $\{a_k\}_{k=1}^m$ (determined in general nonuniquely) are called *coefficients* of the extremal set $\{x_k\}_{k=1}^m$. (This definition is close to the notion of extremal signatures, see e.g. [11].) It can easily be verified that $U_n \subseteq C(K)$ satisfies the Haar property if and only if no nontrivial element of U_n can vanish on an extremal set of U_n, which leads to another version of Haar-Kolmogorov theorem.

Then as it is shown in [8] in order that an n-dimensional subspace $U_n \subset C^1[a,b]$ be a Chebyshev subspace of $C^1[a,b]$ it is necessary and sufficient that there does not exist an extremal set $\{x_k\}_{k=1}^m$ of U_n with coefficients $\{a_k\}_{k=1}^m$ and $p \in U_n\setminus\{0\}$ such that $p(x_k) = 0$ for all $1 \le k \le m$ and $\mathrm{Re}\, a_k p'(x_k) = 0$ for all $x_k \in (a,b)$. In the real case coefficients of extremal sets do not appear in the characterization theorem since the relation $\mathrm{Re}\, a_k p'(x_k) = 0$ reduces to $p'(x_k) = 0$. This reflects an essential difference between real and complex cases.

Various examples given in [5 and 8] show that the family of Chebyshev subspaces of $C^1[a,b]$ is essentially wider than that of $C[a,b]$ both in the real and complex cases. On the other hand we cannot obtain further extension of the family of Chebyshev subspaces assuming higher differentiability; that is, replacing $C^1[a,b]$ by $C^r[a,b]$, where r is greater than 1.

The main goal of the present paper is extension of these considerations to multivariate functions. We shall consider the space of continuously-differentiable functions on the unit disc and give a characterization of its finite-dimensional Chebyshev subspaces. This problem for functions of two variables turned out to be more delicate and complicated because of extension of boundary from a discrete set in case of one variable to a continuum in case of two variables.

In the first part of the paper our main theorem is formulated and different corollaries are given. In §2 we give a proof of the main result. Finally in §§3 and 4 we consider different applications for complex and real polynomials.

1. Main result. Set $Y^2 = \{(x,y) \in \mathbf{R}^2 : x^2 + y^2 \le 1\}$ and let $C^1(Y^2)$ be the space of real or complex functions $f(x,y)$ such that the partial derivatives f_x and f_y exist in a neighborhood of each $(x_0,y_0) \in Y^2$ and are continuous at (x_0,y_0). As above this space of continuously-differentiable functions on Y^2 is endowed with the norm $\|f\|_C = \max_{(x,y)\in Y^2} |f(x,y)|$. The following theorem characterizing the finite-dimensional Chebyshev subspaces of $C^1(Y^2)$ is our principal result.

THEOREM 1. *Let U_n be an n-dimensional subspace of $C^1(Y^2)$. Then U_n is a Chebyshev subspace of $C^1(Y^2)$ if and only if there does not exist an extremal set $\{(x_k, y_k)\}_{k=1}^m \subseteq Y^2$ of U_n with coefficients $\{a_k\}_{k=1}^m$ and $p \in U_n\setminus\{0\}$ such that for all $1 \le k \le m$*

(1) $p(x_k, y_k) = 0,$

(2) $\mathrm{Re}\, a_k p'_y(x_k, y_k) = \mathrm{Re}\, a_k p'_x(x_k, y_k) = 0 \quad \text{if } x_k^2 + y_k^2 < 1,$

(3) $$x_k \operatorname{Re} a_k p'_y(x_k, y_k) = y_k \operatorname{Re} a_k p'_x(x_k, y_k) \quad \text{if } x_k^2 + y_k^2 = 1.$$

Since in the real case, for each $1 \le k \le m$, $a_k \ne 0$ are real we can derive the following

COROLLARY 1. *In the real case in order that U_n be a Chebyshev subspace of $C^1(Y^2)$ it is necessary and sufficient that there does not exist an extremal set $\{(x_k, y_k)\}_{k=1}^m$ of U_n and $p \in U_n \setminus \{0\}$ such that for all $1 \le k \le m$*

(4) $$p(x_k, y_k) = 0,$$

(5) $$p'_y(x_k, y_k) = p'_x(x_k, y_k) = 0 \quad \text{if } x_k^2 + y_k^2 < 1,$$

(6) $$x_k p'_y(x_k, y_k) = y_k p'_x(x_k, y_k) \quad \text{if } x_k^2 + y_k^2 = 1.$$

Assume now that the functions in $C^1(Y^2)$ are complex and elements of U_n are analytic in the disc $\{|z| < 1 + \varepsilon\}$ for some $\varepsilon > 0$ $(z = x + iy)$. Then for any $p \in U_n$ we have $p'_y = ip'_x = ip'$ on $\{|z| < 1 + \varepsilon\}$. Hence in this case (2) is equivalent to $p'(z_k) = 0$ while (3) is equivalent to $\operatorname{Im} a_k z_k p'(z_k) = 0$ $(z_k = x_k + iy_k)$. Thus we obtain

COROLLARY 2. *Let the functions in $C^1(Y^2)$ be complex and assume that elements of U_n are analytic in $\{|z| < 1 + \varepsilon\}$ for some $\varepsilon > 0$. Then U_n is a Chebyshev subspace of $C^1(Y^2)$ if and only if there does not exist an extremal set $\{z_k = x_k + iy_k\}_{k=1}^m \subseteq Y^2$ of U_n with coefficients $\{a_k\}_{k=1}^m$ and $p \in U_n \setminus \{0\}$ such that for all $1 \le k \le m$*

(7) $$p(z_k) = 0,$$

(8) $$p'(z_k) = 0 \quad \text{if } |z_k| < 1,$$

(9) $$\operatorname{Im} a_k z_k p'(z_k) = 0 \quad \text{if } |z_k| = 1.$$

It can be seen that the essential difference between the cases of one and two variables consists in appearance in the multivariate case of certain boundary conditions (see (3), (6) and (9)).

2. Proof of Theorem 1. We shall need an auxiliary lemma which is equivalent to Theorem 1.3 of [**12**, p. 178].

LEMMA 1. *Let U_n be an n-dimensional subspace of $C(K)$ and consider an arbitrary $f \in C(K)$. Then in order that $p \in U_n$ be a best approximant of f it is necessary and sufficient that there exists an extremal set $\{x_k\}_{k=1}^m \subseteq K$ of U_n with coefficients $\{a_k\}_{k=1}^m$ such that*

$$f(x_k) - p(x_k) = \frac{\overline{a_k}}{|a_k|} \max_{x \in K} |f(x) - p(x)| \qquad (1 \le k \le m).$$

Furthermore, the following elementary proposition will be used in the proof.

PROPOSITION 1. *Assume that the real functions $\phi(x)$ and $\psi(x)$ $(x \in (\alpha, \beta))$ are differentiable at $\xi \in (\alpha, \beta)$. If $\phi(\xi) = \sup_{x \in (\alpha, \beta)} |\phi(x)|$, $\psi(\xi) = 0$ and*

(10) $$\sup_{x \in (\alpha, \beta)} |\phi(x) - \psi(x)| \le \sup_{x \in (\alpha, \beta)} |\phi(x)|,$$

then $\psi'(\xi) = 0$.

Indeed,

$$\phi(x) - \psi(x) = \phi(\xi) + \phi'(\xi)(x - \xi) + o(|x - \xi|) - \psi(\xi) - \psi'(\xi)(x - \xi) + o(|x - \xi|)$$
$$= \phi(\xi) - \psi'(\xi)(x - \xi) + o(|x - \xi|)$$

and this would contradict (10) if $\psi'(\xi) \neq 0$.

Let us now verify the sufficiency in Theorem 1.

Assume that U_n is not a Chebyshev subspace of $C^1(Y^2)$. Then there exist $f \in C^1(Y^2)$ and $p \in U_n \backslash \{0\}$ such that 0 and p are best approximants of f in U_n. By the above lemma there exists an extremal set $\{(x_k, y_k)\}_{k=1}^m \subseteq Y^2$ of U_n with coefficients $\{a_k\}_{k=1}^m$ such that

$$(11) \qquad\qquad f(x_k, y_k) = \frac{\overline{a}_k}{|a_k|} \|f\|_C \qquad (1 \leq k \leq m).$$

Let us verify that for the extremal set $\{(x_k, y_k)\}_{k=1}^m$, its coefficients $\{a_k\}_{k=1}^m$ and $p \in U_n \backslash \{0\}$ the relations (1)–(3) hold. Using that

$$(12) \qquad\qquad \|f - p\|_C = \|f\|_C$$

and relations (11) we have

$$(13) \qquad
\begin{aligned}
0 &\geq |f(x_k, y_k) - p(x_k, y_k)|^2 - \|f\|_C^2 \\
&= \left| \|f\|_C - \frac{a_k}{|a_k|} p(x_k, y_k) \right|^2 - \|f\|_C^2 \\
&= -\frac{2\|f\|_C}{|a_k|} \operatorname{Re} a_k p(x_k, y_k) + |p(x_k, y_k)|^2.
\end{aligned}$$

Hence $\operatorname{Re} a_k p(x_k, y_k) \geq 0$ for each $1 \leq k \leq m$. Since by definition of extremal sets $\sum_{k=1}^m a_k p(x_k, y_k) = 0$ it follows that $\operatorname{Re} a_k p(x_k, y_k) = 0$ $(1 \leq k \leq m)$. Finally this and (13) imply

$$(14) \qquad\qquad p(x_k, y_k) = 0 \qquad (1 \leq k \leq m).$$

Thus condition (1) is satisfied.

Assume now that $x_k^2 + y_k^2 < 1$ for some $1 \leq k \leq m$. Then setting $\phi(x) = \operatorname{Re} a_k f(x, y_k)$, $\psi(x) = \operatorname{Re} a_k p(x, y_k)$ $(|x| \leq \sqrt{1 - y_k^2})$ and using (11), (12) and (14) we can conclude that, for $\phi(x), \psi(x)$ and $\xi = x_k \in (-\sqrt{1 - y_k^2}, \sqrt{1 - y_k^2})$, the conditions of Proposition 1 are fulfilled. Hence $\psi'(x_k) = 0$, i.e. $\operatorname{Re} a_k p'_x(x_k, y_k) = 0$. Analogously we can prove that $\operatorname{Re} a_k p'_y(x_k, y_k) = 0$. Thus (2) also holds.

Finally in the case $x_k^2 + y_k^2 = 1$ we set $x_k = \cos t_k$, $y_k = \sin t_k$, $\phi(t) = \operatorname{Re} a_k f(\cos t, \sin t)$ and $\psi(t) = \operatorname{Re} a_k p(\cos t, \sin t)$. Applying again Proposition 1 we arrive at $\psi'(t_k) = 0$ which is evidently equivalent to (3).

This completes the proof of sufficiency in Theorem 1.

Let us now verify the necessity of conditions imposed in Theorem 1.

Assume that there exist an extremal set $\{(x_k, y_k)\}_{k=1}^m \subseteq Y^2$ of U_n with coefficients $\{a_k\}_{k=1}^m$ and $p \in U_n \backslash \{0\}$ such that (1)–(3) hold. This will enable us to construct a function $f \in C^1(Y^2)$ having nonunique best approximation in U_n.

Consider the function

$$(15) \qquad\qquad \omega^*(t) = \omega(p'_x, t) + \omega(p'_y, t) + t \qquad (t > 0),$$

where $\omega(\tilde{g}, t) = \max\{|\tilde{g}(x_1, y_1) - \tilde{g}(x_2, y_2)| \colon (x_1, y_1), (x_2, y_2) \in Y^2, (x_1 - x_2)^2 + (y_1 - y_2)^2 \le t^2\}$ $(t > 0)$ denotes the modulus of continuity of $\tilde{g} \in C(Y^2)$. Furthermore, set

$$
(16) \qquad g(\phi) = \begin{cases} \displaystyle\int_0^\phi \omega^*(t)\,dt, & 0 \le \phi \le \pi/2, \\[2mm] \displaystyle c_1 - \int_0^{\pi-\phi} \omega^*(t)\,dt, & \pi/2 \le \phi \le \pi, \end{cases}
$$

where $c_1 = 2\int_0^{\pi/2} \omega^*(t)\,dt$. (Here and in what follows $c_1, c_2, \ldots$ denote positive constants depending only on p, the extremal set $\{(x_k, y_k)\}_{k=1}^m$ and its coefficients $\{a_k\}_{k=1}^m$.) Obviously, g is a real positive continuously-differentiable function on $[0, \pi]$ and

$$
(17) \qquad g'(0) = g'(\pi) = 0.
$$

Consider the extremal set $\{(x_k, y_k)\}_{k=1}^m \subseteq Y^2$. Without loss of generality we may assume that $x_k^2 + y_k^2 < 1$ for $1 \le k \le s$ and $x_k^2 + y_k^2 = 1$ for $s+1 \le k \le m$ $(0 \le s \le m)$. Let us introduce the functions

$$
(18) \qquad F_k(x, y) = \begin{cases} \displaystyle\int_0^{\sqrt{(x-x_k)^2 + (y-y_k)^2}} \omega^*(t)\,dt, & 1 \le k \le s, \\[3mm] \displaystyle c_1(1 - x^2 - y^2) + (x^2 + y^2)g\left(\arccos\frac{xx_k + yy_k}{\sqrt{x^2 + y^2}}\right), \\[3mm] \hfill s+1 \le k \le m, \end{cases}
$$

where $(x, y) \in \mathbf{R}^2$. It can easily be shown that

$$
(19) \quad F_k(x_k, y_k) = 0 \text{ and } F_k(x, y) > 0 \text{ if } (x, y) \in Y^2 \backslash \{(x_k, y_k)\} \qquad (1 \le k \le m).
$$

Furthermore functions F_k are real and we claim that $F_k \in C^1(Y^2)$ $(1 \le k \le m)$. If $1 \le k \le s$, then using that $\omega^*(t) \to 0$ as $t \to +0$ we can derive that $F_k(x, y)$ is continuously differentiable at each point $(x, y) \in \mathbf{R}^2$. Now let $s+1 \le k \le m$. Then, we can easily obtain from (18) that

$$
(20) \qquad \begin{aligned} (F_k(x, y))'_x &= -2c_1 x + 2xg\left(\arccos\frac{xx_k + yy_k}{\sqrt{x^2 + y^2}}\right) \\[2mm] &\quad + g'\left(\arccos\frac{xx_k + yy_k}{\sqrt{x^2 + y^2}}\right) y\,\mathrm{sign}(xy_k - yx_k). \end{aligned}
$$

The continuity of this function should be checked only at the line $xy_k = yx_k$. Since

$$
|(F_k(x, y))'_x| \le c_2|x| + c_3|y|
$$

it follows that $(F_k(x, y))'_x$ is continuous at the origin. Assume now that $\tilde{x}y_k = \tilde{y}x_k$ and $\tilde{x}^2 + \tilde{y}^2 > 0$. Then taking into account that $x_k^2 + y_k^2 = 1$ we obtain $|\tilde{x}x_k + \tilde{y}y_k|/\sqrt{\tilde{x}^2 + \tilde{y}^2} = 1$. Therefore (17) yields that function (20) is continuous at $(\tilde{x}, \tilde{y})$. Analogously it can be shown that $(F_k(x, y))'_y$ is continuous on $\mathbf{R}^2$. Thus $F_k \in C^1(Y^2)$, $1 \le k \le m$.

After these preparations we can construct our counterexample. Set

$$Q_k(h) = \{(x,y) \in Y^2 : (x - x_k)^2 + (y - y_k)^2 < h^2\} \qquad (0 < h < 1/2,\ 1 \leq k \leq m),$$

where h is chosen small enough so that the closures of these sets are disjoint. Furthermore there exists a function $\psi \in C^1(Y^2)$ such that $\|\psi\|_C = 1$ and

$$(21) \qquad \psi(x,y) = \bar{a}_k/|a_k| \quad \text{if } (x,y) \in Q_k(h), \qquad 1 \leq k \leq m.$$

(An explicit formula for ψ can easily be established; we omit the details.) Then the needed function can be given by

$$(22) \qquad f(x,y) = \psi(x,y)(1 - c_4 F_0(x,y)),$$

where $F_0(x,y) = \prod_{k=1}^m F_k(x,y)$ and $c_4 = 1/\|F_0\|_C$. Obviously, $f \in C^1(Y^2)$, $\|f\|_C = 1$, and by (19) and (21)

$$(23) \qquad f(x_k, y_k) = \psi(x_k, y_k) = \bar{a}_k/|a_k| \qquad (1 \leq k \leq m).$$

It follows by Lemma 1 and the above relations that 0 is a best approximant of f in U_n. We state that for ε small enough εp is best approximation, too. Thus we should verify that $\|f - \varepsilon p\|_C = 1$ if $\varepsilon \in \mathbf{R}$ is small enough.

Consider an arbitrary $(x_0, y_0) \in Y^2$. Assume at first that $(x_0, y_0) \in Y^2 \backslash Q(h)$, where $Q(h) = \bigcup_{k=1}^m Q_k(h)$. Since F_k vanishes only at (x_k, y_k) (see (19)) it follows that $|f| \leq \eta < 1$ on $Y^2 \backslash Q(h)$. Therefore setting $|\varepsilon| \leq (1 - \eta)/\|p\|_C$ we have

$$(24) \qquad |f(x_0, y_0) - \varepsilon p(x_0, y_0)| \leq \eta + 1 - \eta = 1.$$

Now let $(x_0, y_0) \in Q(h)$, i.e. $(x_0, y_0) \in Q_k(h)$ for some $1 \leq k \leq m$. Using again that F_k vanishes only at (x_k, y_k) we can easily derive that for any $(x,y) \in Q_k(h)$

$$(25) \qquad F_0(x,y) = \prod_{j=1}^m F_j(x,y) \geq c_5 F_k(x,y) \quad (1 \leq k \leq m).$$

Therefore by (22) and (21) for each $(x,y) \in Q_k(h)$

$$|f(x,y) - \varepsilon p(x,y)|^2 = \left| 1 - c_4 F_0(x,y) - \frac{\varepsilon}{|a_k|} a_k p(x,y) \right|^2$$

$$(26) \qquad = (1 - c_4 F_0(x,y))^2 - 2(1 - c_4 F_0(x,y))\frac{\varepsilon}{|a_k|}\operatorname{Re} a_k p(x,y) + \varepsilon^2 |p(x,y)|^2$$

$$\leq 1 - c_6 F_k(x,y) + \frac{2|\varepsilon|}{|a_k|}|\operatorname{Re} a_k p(x,y)| + \varepsilon^2 |p(x,y)|^2, \qquad 1 \leq k \leq m.$$

Now we shall give upper bounds for the two last terms in (26). Let us consider two cases.

$\quad$ *Case 1.* $(x_0, y_0) \in Q_k(h)$, where $1 \leq k \leq s$. Obviously for any $(x_1, y_1), (x_2, y_2) \in Y^2$ we have

$$(27) \qquad |p(x_1, y_1) - p(x_2, y_2)|^2 \leq c_7\{(x_2 - x_1)^2 + (y_2 - y_1)^2\}.$$

Furthermore, by definition of F_k for $1 \leq k \leq s$ (see (18))

$$(28) \qquad F_k(x,y) \geq \frac{\sqrt{(x - x_k)^2 + (y - y_k)^2}}{2} \omega^* \left(\frac{\sqrt{(x - x_k)^2 + (y - y_k)^2}}{2} \right).$$

Since $p(x_k, y_k) = 0$ we have by (27) and (28)

$$(29) \qquad |p(x_0, y_0)|^2 \le c_7\{(x_k - x_0)^2 + (y_k - y_0)^2\} \le 4c_7 F_k(x_0, y_0).$$

Now set $q(x, y) = (1/|a_k|)\mathrm{Re}\, a_k p(x, y)$. Then q is a real function belonging to $C^1(Y^2)$. Moreover, by (1) and (2)

$$q(x_k, y_k) = q'_x(x_k, y_k) = q'_y(x_k, y_k) = 0.$$

Therefore applying this and (28) we have

$$\frac{1}{|a_k|}|\mathrm{Re}\, a_k p(x_0, y_0)| = |q(x_0, y_0)| = |q(x_0, y_0) - q(x_k, y_k)|$$

$$\le \sqrt{(x_0 - x_k)^2 + (y_0 - y_k)^2}\left\{\omega\left(q'_y, \sqrt{(x_0 - x_k)^2 + (y_0 - y_k)^2}\right)\right.$$

$$\left. +\omega\left(q'_x, \sqrt{(x_0 - x_k)^2 + (y_0 - y_k)^2}\right)\right\}$$

$$\le \sqrt{(x_0 - x_k)^2 + (y_0 - y_k)^2}\left\{\omega\left(p'_y, \sqrt{(x_0 - x_k)^2 + (y_0 - y_k)^2}\right)\right.$$

$$\left. +\omega\left(p'_x, \sqrt{(x_0 - x_k)^2 + (y_0 - y_k)^2}\right)\right\}$$

$$\le \sqrt{(x_0 - x_k)^2 + (y_0 - y_k)^2}\,\omega^*\left(\sqrt{(x_0 - x_k)^2 + (y_0 - y_k)^2}\right)$$

$$\le 4F_k(x_0, y_0).$$

Applying this and (29) in (26) we arrive at

$$(30) \qquad \begin{aligned} &|f(x_0, y_0) - \varepsilon p(x_0, y_0)|^2 \\ &\le 1 - c_6 F_k(x_0, y_0) + 8|\varepsilon|F_k(x_0, y_0) + 4\varepsilon^2 c_7 F_k(x_0, y_0) \le 1 \end{aligned}$$

if $8|\varepsilon| + 4c_7\varepsilon^2 \le c_6$.

Case 2. $(x_0, y_0) \in Q_k(h)$ for some $s + 1 \le k \le m$. Then $x_k^2 + y_k^2 = 1$, hence we may set $x_k = \cos\phi_k$, $y_k = \sin\phi_k$; $x_0 = r_0\cos(\phi_k + \phi_0)$, $y_0 = r_0\sin(\phi_k + \phi_0)$ $(0 \le r_0 \le 1)$. Since $(x_0, y_0) \in Q_k(h)$, where $0 < h < 1/2$, it follows that $r_0 > 1/2$ and $|\phi_0| \le \pi/2$. By definition of F_k in case $s + 1 \le k \le m$ (see (18) and (16)) we have

$$(31) \qquad \begin{aligned} F_k(x_0, y_0) &\ge c_1(1 - x_0^2 - y_0^2) + \tfrac{1}{4}g(\arccos\cos\phi_0) \\ &\ge c_8(1 - x_0^2 - y_0^2 + g(|\phi_0|)) \ge c_9(1 - x_0^2 - y_0^2 + |\phi_0|\omega^*(|\phi_0|)). \end{aligned}$$

Consider the function

$$t(\phi) = \frac{1}{|a_k|}\mathrm{Re}\, a_k p(\cos(\phi + \phi_k), \sin(\phi + \phi_k)) \qquad (|\phi| \le \pi).$$

It follows from (1) and (3) that $t(0) = t'(0) = 0$. This immediately implies that

$$|t(\phi_0)| \le |\phi_0| \max_{|\xi - \eta| \le |\phi_0|} |t'(\xi) - t'(\eta)|$$

$$\le c_{10}|\phi_0|(\omega(p'_y, |\phi_0|) + \omega(p'_x, |\phi_0|) + |\phi_0|)$$

$$= c_{10}|\phi_0|\omega^*(|\phi_0|).$$

Hence and by (27) and (31)

$$\frac{1}{|a_k|}|\operatorname{Re} a_k p(x_0, y_0)| \le \frac{1}{|a_k|}|\operatorname{Re} a_k p(\cos(\phi_k + \phi_0), \sin(\phi_k + \phi_0))|$$

(32)
$$+ |p(x_0, y_0) - p(\cos(\phi_k + \phi_0), \sin(\phi_k + \phi_0))|$$

$$\le |t(\phi_0)| + \sqrt{c_7}(1 - \sqrt{x_0^2 + y_0^2})$$

$$\le c_{11}(1 - x_0^2 - y_0^2 + |\phi_0|\omega^*(|\phi_0|)) \le c_{12}F_k(x_0, y_0).$$

Analogously using (27), (1) and (31) we have

$$|p(x_0, y_0)|^2 \le 2|p(x_0, y_0) - p(\cos(\phi_k + \phi_0), \sin(\phi_k + \phi_0))|^2$$

$$+ 2|p(\cos(\phi_k + \phi_0), \sin(\phi_k + \phi_0))|^2 \le 2c_7(1 - \sqrt{x_0^2 + y_0^2})^2$$

(33)
$$+ 2|p(\cos(\phi_k + \phi_0), \sin(\phi_k + \phi_0)) - p(\cos\phi_k, \sin\phi_k)|^2$$

$$\le 2c_7(1 - x_0^2 - y_0^2) + 2c_7\phi_0^2 \le 2c_7(1 - x_0^2 - y_0^2 + |\phi_0|\omega^*(|\phi_0|))$$

$$\le c_{13}F_k(x_0, y_0).$$

Applying estimations (32) and (33) in (26) we again obtain that

$$|f(x_0, y_0) - \varepsilon p(x_0, y_0)|^2$$

$$\le 1 - c_6 F_k(x_0, y_0) + 2c_{12}|\varepsilon|F_k(x_0, y_0) + c_{13}\varepsilon^2 F_k(x_0, y_0) \le 1$$

provided $2c_{12}|\varepsilon| + c_{13}\varepsilon^2 \le c_6$. Finally, (24), (30) and the above estimation imply that $\|f - \varepsilon p\|_C \le 1$ if $|\varepsilon| \le \mathbf{C}_{14}$. Thus $f \in C^1(Y^2)$ has nonunique best approximation, i.e. U_n is not a Chebyshev subspace of $C^1(Y^2)$.

The proof of Theorem 1 is completed.

3. Application for complex polynomials. In this section we shall discuss possible application of Theorem 1 and in particular, Corollary 2 in the complex case. We shall establish a wide family of polynomial Chebyshev subspaces of $C^1(Y^2)$ which do not satisfy the Haar condition, i.e. are not Chebyshev in $C(Y^2)$. The space of linear polynomials $E_1 = \{az : a \in \mathbf{C}\}$ is a simplest example of non-Haar space which is Chebyshev in $C^1(Y^2)$. (Throughout this section we consider functions of complex variable $z = x + iy$.) Indeed, any $p \in E_1\backslash\{0\}$ can vanish only at the origin while $p'(0) \ne 0$. Thus conditions (7) and (8) cannot hold and it follows by Corollary 2 that E_1 is Chebyshev in $C^1(Y^2)$.

Let $P_n = \{\sum_{j=0}^{n-1} \tilde{a}_j z^j : \tilde{a}_j \in \mathbf{C}\}$ be the space of algebraic polynomials of degree at most $n - 1$. Evidently P_n satisfies the Haar condition, hence it is Chebyshev in $C(Y^2)$. Let us delete from P_n some basis functions different from 1 and z^{n-1} and consider the resulting space of lacunary polynomials

$$\tilde{P}_s = \left\{ \sum_{j=0}^{s-1} b_j z^{q_j} : b_j \in \mathbf{C} \right\},$$

where $0 = q_0 < q_1 < \cdots < q_{s-1} = n - 1$ are fixed integers and $2 \le s < n$. Then $\dim \tilde{P}_s = s \le n - 1$ while $p^*(z) = z^{n-1} - 1 \in \tilde{P}_s$ has $n - 1$ distinct zeros at Y^2. Hence, for each choice of integers q_j, $\tilde{P}_s$ is not Chebyshev in $C(Y^2)$. On the other hand, our next result implies that under an additional assumption $\tilde{P}_s$ is a Chebyshev subspace of $C^1(Y^2)$.

THEOREM 2. *If $\tilde{P}_s \supseteq P_r$, where $r = [(3n-1)/4]$, then $\tilde{P}_s$ is a Chebyshev subspace of $C^1(Y^2)$ $(n \geq 4)$.*

That is, if we preserve the first $[(3n-1)/4]$ powers z^j $(0 \leq j \leq [(3n-1)/4]-1)$ and delete in an arbitrary way some of the remaining $n-[(3n-1)/4]$ basis functions of P_n, then the resulting space of lacunary polynomials will still be Chebyshev in $C^1(Y^2)$. This shows that in the complex case the family of Chebyshev subspaces of $C^1(Y^2)$ is essentially wider than that of $C(Y^2)$.

In order to apply Corollary 2 (namely condition (9)) we need some information on coefficients of the extremal sets. The needed information is provided by the following lemma of Vidensky (see [**13**, pp. 441, 442]).

LEMMA 2. *Let $\{z_k\}_{k=1}^m \subseteq Y^2$ be an extremal set of P_r with coefficients $\{a_k\}_{k=1}^m$ $(r+1 \leq m \leq 2r+1)$. Then there exists $u \in P_{m-r} \backslash \{0\}$ such that*

$$\text{(34)} \qquad a_k = u(z_k)/\omega'(z_k) \qquad (1 \leq k \leq m),$$

where $\omega(z) = \prod_{j=1}^m (z - z_j)$.

PROOF OF THEOREM 2. Assume to the contrary that $\tilde{P}_s$ is not a Chebyshev subspace of $C^1(Y^2)$. Then by Corollary 2 there exist an extremal set $\{z_k\}_{k=1}^m \subseteq Y^2$ of $\tilde{P}_s$ with coefficients $\{a_k\}_{k=1}^m$ and $p \in \tilde{P}_s \backslash \{0\}$ such that relations (7)-(9) hold. Since $p(z_k) = 0$, $1 \leq k \leq m$, and $p \in \tilde{P}_s \subset P_n$ it follows that $m \leq n - 1 \leq 2r + 1$. On the other hand $\tilde{P}_s \supset P_r$, hence it is obvious that $\{z_k\}_{k=1}^m$ is an extremal set of P_r with coefficients $\{a_k\}_{k=1}^m$ and $r + 1 \leq m \leq 2r + 1$. Therefore by Lemma 2 there exists $u \in P_{m-r} \backslash \{0\}$ such that (34) holds. Further, assume that $|z_k| = 1$ for $1 \leq k \leq m_0$ and $|z_k| < 1$ if $m_0 + 1 \leq k \leq m$ $(0 \leq m_0 \leq m)$. By (7) and (8) $p(z_k) = 0$ $(1 \leq k \leq m)$ and $p'(z_k) = 0$ if $m_0 + 1 \leq k \leq m$. Hence

$$\text{(35)} \qquad m_0 \geq 2m - n + 1 \geq 2r - n + 3$$

and

$$p(z) = \omega(z) \prod_{k=m_0+1}^m (z - z_k)\tilde{p}(z),$$

where $\omega(z) = \prod_{j=1}^m (z-z_j)$ and $\tilde{p} \in P_{n-2m+m_0} \backslash \{0\}$. Using the above representation we have

$$\text{(36)} \qquad p'(z_j) = \omega'(z_j) \prod_{k=m_0+1}^m (z_j - z_k)\tilde{p}(z_j) \qquad (1 \leq j \leq m).$$

Furthermore (9) implies that $\operatorname{Im} a_j z_j p'(z_j) = 0$ for each $1 \leq j \leq m_0$. This together with (34) and (36) yield

$$\text{(37)} \qquad \begin{aligned} 0 &= \operatorname{Im} a_j z_j p'(z_j) = \operatorname{Im} z_j u(z_j) \prod_{k=m_0+1}^m (z_j - z_k)\tilde{p}(z_j) \\ &= \operatorname{Im} p^*(z_j) \qquad (1 \leq j \leq m_0), \end{aligned}$$

where $p^*(z) = zu(z)\prod_{k=m_0+1}^m (z - z_k)\tilde{p}(z) \in P_{n-r} \backslash \{0\}$. Moreover $\operatorname{Im} p^*(e^{i\phi})$ is a trigonometric polynomial of degree at most $n - r - 1$ while (37) and (35) imply that it has at least $2r - n + 3$ distinct zeros on $[0, 2\pi)$. Furthermore p^* is not a constant function, hence $\operatorname{Im} p^*(e^{i\phi})$ cannot be identically zero. This yields that

$2r - n + 3 \leq 2n - 2r - 2$, i.e. $r \leq [(3n - 1)/4] - 1$, a contradiction. The theorem is proved.

EXAMPLE 1. Set $n = 4$. Then $r = [(3n - 1)/4] = 2$. Thus applying Theorem 2 we may conclude that $\tilde{P}_3 = \text{span}\{1, z, z^3\}$ is a Chebyshev subspace of $C^1(Y^2)$. On the other hand if we delete one more basis function and consider $\tilde{P}_2 = \text{span}\{1, z^3\}$, then the condition of Theorem 2 does not hold, and we can show that $\tilde{P}_2$ is not Chebyshev in $C^1(Y^2)$. Indeed, $\{e^{2\pi i/3}, e^{-2\pi i/3}\}$ is an extremal set of $\tilde{P}_2$ with coefficients $a_1 = 1$, $a_2 = -1$. Moreover, for $p(z) = z^3 - 1 \in \tilde{P}_2$, the relations (7)–(9) evidently hold. Hence by Corollary 2, $\tilde{P}_2$ is not a Chebyshev subspace of $C^1(Y^2)$. This example shows that the condition of Theorem 2 is in a certain sense sharp.

The above result presents a significant area of application of Theorem 1 in the complex case. As it was already mentioned there are not spaces of *lacunary* complex polynomials having Chebyshev property in $C(Y^2)$. Nevertheless Theorem 2 establishes a sufficiently wide class of lacunary complex polynomials which are Chebyshev in $C^1(Y^2)$. This justifies the importance of investigation of Chebyshev subspaces in $C^1(Y^2)$. It would be especially interesting to give a direct characterization of those spaces of complex lacunary polynomials which are Chebyshev in $C^1(Y^2)$.

4. Application in the real case. Let us denote by $C_0^1(Y^2)$ the space of real-valued functions in $C^1(Y^2)$. In this section we shall discuss possible application of our main result in the study of Chebyshev subspaces of $C_0^1(Y^2)$.

Consider the space $\tilde{L}_3 = \{ax + by + c \colon a, b, c \in \mathbf{R}\}$ of linear polynomials of two variables. Corollary 1 immediately implies that $\tilde{L}_3$ is a Chebyshev subspace of $C_0^1(Y^2)$. Indeed, each extremal set of $\tilde{L}_3$ consists of at least 3 points, hence if (4)–(6) were true for some $p \in \tilde{L}_3 \backslash \{0\}$, then the differential of p would vanish at one of the points of extremal set. The Chebyshev property of $\tilde{L}_3$ in $C_0^1(Y^2)$ was discovered in an earlier paper of Collatz [4].

Now let U_n be an arbitrary n-dimensional subspace of $C_0^1(Y^2)$ and assume that $p \in U_n \backslash \{0\}$ has n distinct zeros $d_1, \ldots, d_n \in Y^2$. Then the evaluation functionals $E_{d_i} \in U_n^*$ given by $E_{d_i}(g) = g(d_i)$ $(g \in U_n)$, $1 \leq i \leq n$, are linearly dependent on U_n, i.e. the set $\{d_i\}_{i=1}^n$ or a proper subset of it is an extremal set of U_n. If, in addition, the differential of p also vanishes at $d_1, \ldots, d_n$, then it follows from Corollary 1 that U_n is not a Chebyshev subspace of $C_0^1(Y^2)$. Thus elements of a Chebyshev subspace of $C_0^1(Y^2)$ may have at most $n - 1$ "double" zeros. This statement proved originally by Rivlin and Shapiro [10] implies that the set of quadratic polynomials $\tilde{L}_6 = \{\sum_{j+k \leq 2} a_{jk} x^j y^k \colon a_{jk} \in \mathbf{R}\}$ is not Chebyshev in $C_0^1(Y^2)$. Thus in contrast to the complex case it is very unlikely that there exists a wide class of Chebyshev subspaces of $C_0^1(Y^2)$. It is not even clear whether there can be found Chebyshev subspaces of $C_0^1(Y^2)$ of arbitrary finite dimension. Collatz's result on linear polynomials presents a 3-dimensional Chebyshev subspace of $C_0^1(Y^2)$. We shall now establish a 4-dimensional Chebyshev subspace of $C_0^1(Y^2)$. Set $\tilde{L}_4 = \{axy + bx + cy + d \colon a, b, c, d \in \mathbf{R}\}$.

PROPOSITION 2. *$\tilde{L}_4$ is a Chebyshev subspace of $C_0^1(Y^2)$.*

PROOF. Assume the contrary. Then by Corollary 1 there exist an extremal set $\{(x_k, y_k)\}_{k=1}^m \subseteq Y^2$ of $\tilde{L}_4$ and $p(x, y) = axy + bx + cy + d \in \tilde{L}_4 \backslash \{0\}$ such that (4)–(6) hold. Evidently, $m \geq 3$ and $a \neq 0$. Furthermore, the differential of p vanishes only at $(-c/a, -b/a)$.

Assume at first that $m = 3$. Then

$$(38) \qquad \sum_{k=1}^3 a_k = \sum_{k=1}^3 a_k x_k = \sum_{k=1}^3 a_k y_k = 0$$

and

$$(39) \qquad \sum_{k=1}^3 a_k x_k y_k = 0,$$

where $a_k \neq 0$, $1 \leq k \leq 3$, are the corresponding coefficients of the extremal set $\{(x_k, y_k)\}_{k=1}^3$. Relations (38) imply that for some b_i, $1 \leq i \leq 3$, we have

$$(40) \qquad b_1 + b_2 x_k + b_3 y_k = 0 \qquad (1 \leq k \leq 3, \ b_2^2 + b_3^2 > 0).$$

If $b_2 b_3 \neq 0$, then all $y_k = -b_1/b_3 - (b_2/b_3) x_k$, $1 \leq k \leq 3$, are distinct. Moreover, by (38) and (39)

$$0 = \sum_{k=1}^3 a_k x_k y_k = \sum_{k=1}^3 a_k \left(-\frac{b_1}{b_2} - \frac{b_3}{b_2} y_k \right) y_k = -\frac{b_3}{b_2} \sum_{k=1}^3 a_k y_k^2,$$

i.e. $\sum_{k=1}^3 a_k y_k^2 = 0$. This and (38) imply that $a_1 = a_2 = a_3 = 0$, a contradiction. Thus either b_2 or b_3 should be zero. Assume for example that $b_2 = 0$ and $b_3 \neq 0$. Then we can derive from (40) that $y_1 = y_2 = y_3 = -b_1/b_3$, where $|b_1/b_3| < 1$. Furthermore, it follows from (4) that $p(x_k, -b_1/b_3) = 0$, $1 \leq k \leq 3$, where all x_k, $1 \leq k \leq 3$, should be distinct. Therefore $p(x, y) = (ax + c)(y + b_1/b_3)$, i.e. $p_x'(x_k, -b_1/b_3) = 0$, $1 \leq k \leq 3$. Using that $x_k \neq 0$ if $x_k^2 + y_k^2 = 1$ we obtain by (5) and (6) that $p_y'(x_k, -b_1/b_3) = 0$, $1 \leq k \leq 3$. But this is a contradiction since the differential of p vanishes only at one point.

Consider now the case when $m \geq 4$. By (5) the differential of p should vanish at each interior point of the extremal set, hence it follows that at least 3 points of the extremal set $\{(x_k, y_k)\}_{k=1}^m$ are on the unit circle $x^2 + y^2 = 1$. This together with (4) and (6) yields that the trigonometric polynomial of degree 2 given by

$$t(\phi) = p(\cos\phi, \sin\phi) = \frac{a}{2} \sin 2\phi + b \cos\phi + c \sin\phi + d$$

has at least 3 distinct double zeros on $[0, 2\pi)$, a contradiction. The proposition is proved.

Now we are going to show that methods developed in this paper can be used in the study of Zolotarjov type extremal problems for polynomials of two variables.

Let P_n^* be the set of real algebraic polynomials of one variable of degree at most n, $P_{n,m}^* = \{\sum_{i=0}^n \sum_{j=0}^m b_{i,j} x^i y^j : b_{i,j} \in \mathbf{R}\}$ denotes the real polynomials in x and y of degree n and m, respectively. Consider an arbitrary polynomial $p_n \in P_n^*$ and let $\tilde{p}_k \in P_k^*$ be its best Chebyshev approximation on $S = [-1, 1]$ out of P_k^* ($k < n$). Analogously taking a polynomial $p_m \in P_m^*$ we denote by $\tilde{p}_s \in P_s^*$ its best Chebyshev approximation on S out of P_s^* ($s < m$). Set $P_{n,s}^* + P_{k,m}^* = \{p + q : p \in P_{n,s}^*, \ q \in P_{k,m}^*\}$, $m, n \in \mathbf{N}$; $0 \leq k < n$; $0 \leq s < m$. Then by a result of Shapiro [**11**,

p. 35] the polynomial $p_n(x)\tilde{p}_s(y) + \tilde{p}_k(x)p_m(y) - \tilde{p}_k(x)\tilde{p}_s(y)$ is a best Chebyshev approximation of $p_n(x)p_m(y)$ on $S^2 = [-1,1] \times [-1,1]$ out of $P^*_{n,s} + P^*_{k,m}$. Now we give a sufficient condition for unicity in the above approximation problem.

THEOREM 3. *Let $n, m \in \mathbf{N}$; $0 \le k < n$; $0 \le s < m$, and assume that*

$$(41) \qquad\qquad \min\{2k + 2 - n, 2s + 2 - m\} > 0.$$

Furthermore, consider arbitrary polynomials $p_n \in P^*_n$, $p_m \in P^*_m$ and let $\tilde{p}_k \in P^*_k$ and $\tilde{p}_s \in P^*_s$ be their best Chebyshev approximants on S out of P^*_k and P^*_s, respectively. Then $p_n(x)\tilde{p}_s(y) + \tilde{p}_k(x)p_m(y) - \tilde{p}_k(x)\tilde{p}_s(y) \in P^*_{n,s} + P^*_{k,m}$ is the unique best Chebyshev approximation of $p_n(x)p_m(y)$ on S^2 out of $P^*_{n,s} + P^*_{k,m}$.

This theorem immediately gives explicit unique solutions to certain Zolotarjov type extremal problems. For example, if we set $k = n - 1$ and $s = m - 1$, then it follows that the polynomial $T_n(x)T_m(y)$, where $T_n(x) = 2^{-n+1}\cos n \arccos x$ denotes the Chebyshev polynomial of degree n, is the unique polynomial of the form

$$(42) \qquad\qquad p(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{m} b_{i,j}x^i y^j$$

with $b_{n,m} = 1$ having minimal Chebyshev norm on S^2. This result was originally proved by Zeller and Ehrlich [**14**]. Analogously setting $k = n - 1$, $s = m - 2$ $(n \ge 1, m \ge 3)$ and denoting by $Z_{m,c}$ the Zolotarjov polynomial of degree m we obtain that $T_n(x)Z_{m,c}(y)$ is the unique polynomial having the least Chebyshev deviation on S^2 among all polynomials of the form (42) with $b_{n,m} = 1$ and $b_{n,m-1} = -c$. This proposition seems to be new. (Recall that $Z_{m,c}(y)$ has the minimal Chebyshev norm on S among polynomials $y^m - cy^{m-1} + \sum_{i=0}^{m-2} b_i y^i$, where $c \in \mathbf{R}$ is fixed.)

PROOF OF THEOREM 3. We shall only outline the proof because it goes essentially along the same lines as that of Theorem 1.

Let $\{x_i\}_{i=1}^{k+2} \subset S$ and $\{y_j\}_{j=1}^{s+2} \subset S$ be points of Chebyshev alternation of the functions $p_n - \tilde{p}_k$ and $p_m - \tilde{p}_s$, respectively. That is

$$(43) \qquad p_n(x_i) - \tilde{p}_k(x_i) = \eta(-1)^i d \qquad (1 \le i \le k + 2; \eta = \pm 1)$$

and

$$(44) \qquad p_m(y_j) - \tilde{p}_s(y_j) = \xi(-1)^j h \qquad (1 \le j \le s + 2; \xi = \pm 1),$$

where d and h are the Chebyshev norms of $p_n - \tilde{p}_k$ and $p_m - \tilde{p}_s$ on S, respectively. Moreover, it can easily be verified that there exist $a_i > 0$ and $b_j > 0$ $(1 \le i \le k+2, 1 \le j \le s + 2)$ such that

$$\sum_{i=1}^{k+2}(-1)^i a_i x_i^r = 0 \qquad (0 \le r \le k),$$

$$\sum_{j=1}^{s+2}(-1)^j b_j y_j^l = 0 \qquad (0 \le l \le s).$$

Then for any $t \in P_{n,s}^* + P_{k,m}^*$

$$(45) \qquad \sum_{j=1}^{s+2} \sum_{i=1}^{k+2} (-1)^{i+j} a_i b_j t(x_i, y_j) = 0.$$

Consider now the function $g(x, y) = (p_n(x) - \tilde{p}_k(x))(p_m(y) - \tilde{p}_s(y))$. It follows by (43) and (44) that

$$(46) \qquad g(x_i, y_j) = \eta \xi (-1)^{i+j} \, dh, \qquad 1 \le i \le k+2, \ 1 \le j \le s+2,$$

where dh is equal to the Chebyshev norm of g on S^2. Now, in order to prove the needed statement we have to show that 0 is the unique best approximant of g on S^2 out of $P_{n,s}^* + P_{k,m}^*$. If we assume that, for some $\tilde{t} \in P_{n,s}^* + P_{k,m}^* \backslash \{0\}$, $\max_{(x,y) \in S^2} |g(x, y) - \tilde{t}(x, y)| \le dh$, then using (45), (46) and the same arguments as in the proof of Theorem 1 it can easily be shown that

$$(47) \qquad \tilde{t}(x_i, y_j) = 0 \qquad (1 \le i \le k+2, \ 1 \le j \le s+2),$$

$$(48) \qquad \tilde{t}'_x(x_i, y_j) = 0 \qquad (2 \le i \le k+1, \ 1 \le j \le s+2),$$

$$(49) \qquad \tilde{t}'_y(x_i, y_j) = 0 \qquad (1 \le i \le k+2, \ 2 \le j \le s+1).$$

Furthermore $t(x, y_j) \in P_n^*$ $(1 \le j \le s+2)$, while (47), (48) and (41) imply that $t(x, y_j)$ has at least $2k + 2 > n$ zeros counting with multiplicities. Thus $t(x, y_j)$, $1 \le j \le s+2$, are identically zero. Analogously it can be shown that $t(x_i, y)$, $1 \le i \le k+2$, are zero polynomials. Therefore the polynomial $\prod_{i=1}^{k+2} (x - x_i) \prod_{j=1}^{s+2} (y - y_j)$ should be a divisor of $\tilde{t}$. But this contradicts our assumption that $\tilde{t} \in P_{n,s}^* + P_{k,m}^* \backslash \{0\}$. The theorem is proved.

Now we give an example showing that condition (41) cannot be dropped in general.

EXAMPLE 2. Set $n = 2$, $m = 1$, $k = s = 0$, i.e. (41) is false. Consider the polynomials $x^2 + 4x - 1$ and y $(x, y \in S)$. Then 0 is the best approximant of $x^2 + 4x - 1$ and y by constant functions. On the other hand, we can verify by easy calculations that for any $0 \le |\alpha| < 1$, $\alpha(x^2 - 1) \in P_{2,0}^* + P_{0,1}^*$ is a best approximation of $(x^2 + 4x - 1)y$ on S^2 out of $P_{2,0}^* + P_{0,1}^*$.

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MATHEMATICAL INSTITUTE OF THE HUNGARIAN ACADEMY OF SCIENCES, BUDAPEST, REALTANODA U.13-15, H-1053, HUNGARY