ON THE HIGHER WHITEHEAD GROUPS
OF A BIEBERBACH GROUP

BY

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ABSTRACT. Let Γ be a Bieberbach group, i.e. the fundamental group of a compact flat Riemannian manifold. In this paper we show that if $p > 2$ is a prime, then the p-torsion subgroup of $\mathrm{Wh}_i(\Gamma)$ vanishes for $0 \le i \le 2p - 2$, where $\mathrm{Wh}_i(\Gamma)$ is the ith higher Whitehead group of Γ. The proof involves Farrell and Hsiang's structure theorem for Bieberbach groups, parametrized surgery, pseudoisotopy, and Waldhausen's algebraic K-theory of spaces.

0. Introduction. A closed aspherical manifold is a closed manifold whose universal covering space is contractible. There is the following conjecture concerning the algebraic K-theory of such manifolds.

Conjecture. Let Γ be the fundamental group of a closed aspherical manifold. Then $\mathrm{Wh}_i(\Gamma) = 0$ for $i \ge 0$, where Wh_i is the ith higher Whitehead group of Γ.

In particular, if Γ is a Bieberbach group, i.e. the fundamental group of a compact flat Riemannian manifold, Farrell and Hsiang have shown $\mathrm{Wh}_i(\Gamma) = 0$, $i = 0, 1$ [5], and $\mathrm{Wh}_i(\Gamma) \otimes Q = 0$ for $i \ge 0$ [6]. In [12 and 13] it was shown that $\mathrm{Wh}_i(\Gamma) \otimes \mathbf{Z}[\frac{1}{2}] = 0$, $i = 2, 3$.

In this paper the techniques of [5, 12, 13] and Waldhausen's algebraic K-theory of topological spaces [14] are used to prove

MAIN THEOREM. *Let Γ be a Bieberbach group and $p > 2$ a prime. Then*

$$\mathrm{Wh}_j(\Gamma)_{(p)} = 0 \quad for\ 0 \le j \le 2p - 2.$$

The notation $A_{(p)}$ is used here to denote the p-torsion subgroup of the abelian group A. The main technical ingredient is Proposition 1.5 which asserts a certain vanishing property for the transfer map of $\pi_j(\mathrm{Wh}^{\mathrm{PL}}(M))$ for compact flat Riemannian M. Using a different approach, Frank Quinn has announced that he has developed new techniques which will prove $\mathrm{Wh}_i(\Gamma) = 0$, $i \ge 0$, for a Bieberbach group Γ.

1. A vanishing lemma for the transfer. Recall from [12 and 13] the following

DEFINITION 1.1. Let COV be the category whose objects are closed manifolds and whose morphisms $\tilde{M} \to M$ are covering projections. Suppose F: COV $\to$ Ab is a contravariant functor where Ab is the category of abelian groups. F is said to satisfy the vanishing condition for flat manifolds if for each compact flat Riemannian manifold M and $b \in F(M)$ there is a positive integer N_b (depending on b) such

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that if $\phi\colon M \to M$ is s-expansive (see [5] for the definition) and $s > N_b$, then $F(\phi)(b) = 0$.

Let M be a smooth manifold with (a possibly empty) boundary ∂M. Let $P(M, \partial)$ be the simplicial group of topological pseudoisotopies of M relative to ∂M and $P_{\mathrm{DIFF}}(M, \partial)$ be the simplicial group of smooth pseudoisotopies of M relative to ∂M. $P(M, \partial)$ and $P_{\mathrm{DIFF}}(M, \partial)$ have natural involutions; furthermore, there are "suspension" maps

$$\Sigma\colon P(M, \partial) \to P(M \times I, \partial) \quad \text{and} \quad \Sigma\colon P_{\mathrm{DIFF}}(M, \partial) \to P_{\mathrm{DIFF}}(M \times I, \partial)$$

(see [7]). K. Igusa has proved the stability theorem for smooth pseudoisotopy [8]. This result states that there is a function $\phi_1\colon \mathbf{Z}^+ \to \mathbf{Z}^+$ ($\mathbf{Z}^+$ is the set of nonnegative integers) satisfying the inequality $\phi_1(m) \geq (m - 10)/7$ such that, for any smooth manifold M^n with $n \geq 6$, $\Sigma\colon P_{\mathrm{DIFF}}(M, \partial) \to P_{\mathrm{DIFF}}(M \times I, \partial)$ is k-connected for $k \leq \phi_1(n)$. It follows from the results of [2] that there is a function $\phi_2\colon \mathbf{Z}^+ \to \mathbf{Z}^+$ with the property $\phi_2(m) \to \infty$ as $m \to \infty$ such that for any smoothable topological manifold M^n with $n \geq 6$ the map $\Sigma\colon P(M, \partial) \to P(M \times I, \partial)$ is k-connected for $k \leq \phi_2(n)$.

If N is a manifold and $p\colon \tilde{N} \to N$ a covering projection, then there is a transfer map $\mathrm{tr}(p)\colon P(N, \partial) \to P(\tilde{N}, p^{-1}(\partial N))$ defined using the homotopy lifting property (see [12]); furthermore, $\mathrm{tr}(p)$ commutes (up to homotopy) with the involution. The notation $\pi_k^+(P(N, \partial))$ and $\pi_k^-(P(N, \partial))$ will be used to denote respectively the positive and negative eigenspaces of the involution on $\pi_k(P(N, \partial))$. The following theorem was proved in [13].

PROPOSITION 1.2. $M \to \pi_k^-(P(M)) \otimes \mathbf{Z}[\frac{1}{2}]$ satisfies the vanishing condition for flat manifolds provided $\dim(M) \geq 6$ and $k < \phi_2(\dim(M))$.

There is an inclusion map $i\colon M \to M \times S^1$, $i(m) = (m, 1)$, where S^1 is the unit circle inside the complex numbers. Recall from [7] that i induces a map $i_*\colon P(M) \to P(M \times S^1)$ defined as follows: Let $J \subset S^1$ be a closed arc with $1 \in \mathrm{interior}(J)$. Then i_* is obtained from the composite

$$P(M) \overset{\Sigma}{\to} P(M \times J) \overset{h}{\to} P(M \times S^1),$$

where h is defined by extension. By [3, Chapter 6, Theorem 2], h induces a split injection in homotopy. Hence if $k < \phi_2(\dim(M))$, the map $i^*\colon \pi_k(P(M)) \to \pi_k(P(M \times S^1))$ is a split injection; furthermore, i^* anticommutes with the involution since Σ does [7, Appendix I]. In particular, i^* induces an injection $i^*\colon \pi_k^+(P(M)) \to \pi_k^-(P(M \times S^1))$.

Now suppose M^n is a compact flat Riemannian manifold with $n \geq 6$ and suppose $k < \phi_2(n)$. Let $b \in \pi_k^+(P(M)) \otimes \mathbf{Z}[\frac{1}{2}]$. Noting that $M \times S^1$ is also a compact flat Riemannian manifold, Proposition 1.2 implies that there is a positive integer N such that if $\tilde{\phi}\colon M \times S^1 \to M \times S^1$ is s-expansive and $s > N$, then $\mathrm{tr}(\tilde{\phi})(i_*(b)) = 0$ in $\pi_k^-(P(M \times S^1)) \otimes \mathbf{Z}[\frac{1}{2}]$. Suppose $\phi\colon M \to M$ is s-expansive and $s > N$. Let $\alpha_s\colon S^1 \to S^1$ be the map $\alpha_s(z) = z^s$. Note that $(\phi \times \alpha_s)i = i\phi$. There is a commutative diagram:

$$
\begin{array}{ccc}
\pi_k^+(P(M)) \otimes \mathbf{Z}[\frac{1}{2}] & \to & \pi_k^-(P(M \times S^1)) \otimes \mathbf{Z}[\frac{1}{2}] \\
\mathrm{tr}(\phi) \uparrow & & \uparrow \mathrm{tr}(\phi \times \alpha_s) \\
\pi_k^+(P(M)) \otimes \mathbf{Z}[\frac{1}{2}] & \to & \pi_k^-(P(M \times S^1)) \otimes \mathbf{Z}[\frac{1}{2}]
\end{array}
$$

Since $\phi \times \alpha_s$ is s-expansive and $\text{tr}(\phi \times \alpha_s)(i_*(b)) = 0$, the injectivity of i_* implies $\text{tr}(\phi)(b) = 0$. This proves

PROPOSITION 1.3. $M \to \pi_k^+(P(M)) \otimes \mathbf{Z}[\frac{1}{2}]$ *satisfies the vanishing condition or flat manifolds provided* $\dim(M) \geq 6$ *and* $k < \phi_2(\dim(M))$.

Since
$$\pi_k(P(M)) \otimes \mathbf{Z}[\tfrac{1}{2}] = \pi_k^+(P(M)) \otimes \mathbf{Z}[\tfrac{1}{2}] \oplus \pi_k^-(P(M)) \otimes \mathbf{Z}[\tfrac{1}{2}],$$
Propositions 1.2 and 1.3 imply

PROPOSITION 1.4. $M \to \pi_k(P(M)) \otimes \mathbf{Z}[\frac{1}{2}]$ *satisfies the vanishing condition for flat manifolds provided* $\dim(M) \geq 6$ *and* $k < \phi_2(\dim(M))$.

The space $P^s(M)$ of stable pseudoisotopies of M is defined to be the direct limit of the sequence $P(M) \xrightarrow{\Sigma} P(M \times I, \partial) \xrightarrow{\Sigma} \cdots$. By the stability theorem for pseudo-isotopies the natural map $P(M) \to P^s(M)$ is k-connected for $k \leq \phi_2(\dim M)$.

Waldhausen [14, 16] has defined a functor $X \to \text{Wh}^{\text{PL}}(X)$ from the category of simplicial sets to the category of infinite loop spaces with the property that if X is a piecewise linear manifold, then $\text{Wh}^{\text{PL}}(X)$ is homotopy equivalent to a canonical 2-fold deloop of the space $P_{\text{PL}}^S(X)$ of stable piecewise linear pseudoisotopies of X. Note that by [1] $P_{\text{PL}}^S(X)$ is homotopy equivalent to $P^s(X)$ for triangulable X with $\dim(X) \geq 6$.

PROPOSITION 1.5. *For all* $k \geq 0$, $M \to \pi_k(\text{Wh}^{\text{PL}}(M)) \otimes \mathbf{Z}[\frac{1}{2}]$ *satisfies the vanishing condition for flat manifolds*.

PROOF. By the preceding discussion and Proposition 1.4 we have that $M \to \pi_k(\text{Wh}^{\text{PL}}(M)) \otimes \mathbf{Z}[\frac{1}{2}]$ satisfies the vanishing condition for flat manifolds provided $\dim(M) \geq 6$ and $k < \phi_2(\dim(M))$. Let M^n be any compact flat Riemannian manifold and let r be sufficiently large so that $n + r \geq 6$ and $k < \phi_2(r + n)$. Let $\alpha_s\colon S^1 \to S^1$ be the map $\alpha_s(z) = z^s$ and $i\colon M \to M \times (S^1)^r$ the inclusion $i(m) = (m, 1, \ldots, 1)$. Suppose $\phi\colon M \to M$ is s-expansive. Then $\phi \times (\alpha_s)^r$ is s-expansive and there is a commutative diagram:

$$
\begin{array}{ccc}
\pi_k(\text{Wh}^{\text{PL}}(M)) & \xrightarrow{i_*} & \pi_k(\text{Wh}^{\text{PL}}(M \times (S^1)^r)) \\
\text{tr}(\phi) \uparrow & & \uparrow \text{tr}(\phi \times \alpha_s^r) \\
\pi_k(\text{Wh}^{\text{PL}}(M)) & \xrightarrow{i_*} & \pi_k(\text{Wh}^{\text{PL}}(M \times (S^1)^r))
\end{array}
$$

Since i_* is a (split) injection the proposition follows.

2. Proof of the Main Theorem. Waldhausen has defined a homotopy functor $X \to A(X)$ from the category of simplicial sets to infinite loop spaces [14, 16]. There is a homotopy fibration

$$(2.1) \qquad\qquad h(X; A(*)) \to A(X) \to \text{Wh}^{\text{PL}}(X).$$

The space $h(X; A(*))$ is a homology theory with coefficients in $A(*)$ in the sense that $\pi_i h(X; A(*))$ is $h_i(X; A(*))$.

According to [14, 15] there is a functor $\pi \to \text{Wh}(\pi)$ from the category of groups to the category of infinite loop spaces and a homotopy fibration

$$(2.2) \qquad\qquad h(B\pi; K(Z)) \to BGL(Z\pi)^+ \to \text{Wh}(\pi)$$

such that:

(i) $h(B\pi; K(Z))$ is the homology theory with coefficients in the Ω-spectrum $K(Z)$ with zeroth space $BGL(Z)^+$ and the induced map

$$h_i(B\pi; K(Z)) = \pi_i h(B\pi; K(Z)) \to \pi_i BGL(Z\pi)^+ = K_i(Z\pi)$$

is Loday's homomorphism [10] and $\pi_j \operatorname{Wh}(\pi) = \operatorname{Wh}_j(\pi)$.

(ii) There is a natural map of fibrations:

$$
\begin{array}{ccccc}
h(X; A(*)) & \to & A(X) & \to & \operatorname{Wh}^{\mathrm{PL}}(X) \\
\downarrow & & \downarrow & & \downarrow \\
h(B\pi_1 X; K(Z)) & \to & BGL(Z\pi)^+ & \to & \operatorname{Wh}(\pi_1 X)
\end{array}
$$

(2.3)

Let C_p be the Serre class of abelian groups consisting of those torsion abelian groups having no element with order a positive power of p. A map of spaces $f \colon X \to Y$ will be called k-connected mod C_p if $f_* \colon \pi_0 X \to \pi_0 Y$ is an isomorphism and, for any basepoint, $f_* \colon \pi_j X \to \pi_j Y$ is an isomorphism mod C_p for $j < k$ and $f_* \colon \pi_k X \to \pi_k Y$ is an epimorphism and mod C_p.

LEMMA 2.4. *Suppose p is a prime and X is an aspherical space; then the map*

$$\pi_j \operatorname{Wh}^{\mathrm{PL}}(X) \to \operatorname{Wh}_j(\pi_1 X)$$

is an isomorphism mod C_p *if* $0 < j < 2p - 2$ *and an epimorphism* mod C_p *if* $j = 2p - 2$.

PROOF. Let Q be the functor $\varinjlim \Omega^n \Sigma^n$. If X is a space let $\Omega^* X$ be the topological monoid of Moore loops on X. There is a natural map of A_∞ ring spaces (see [9] for the definition):

$$\theta \colon Q(\Omega^* X \cup *) \to Z(\Omega^* X),$$

where $\Omega^* X \cup *$ is the union with the disjoint base point $*$ and $Z(\Omega^* X)$ is the integral monoid ring of $\Omega^* X$. According to [14] θ is 1-connected. Now assume X is aspherical. Then $\Omega^* X$ has the homotopy type of a discrete set so that $\pi_j(Z\Omega^* X) = 0$ for $j > 0$ and any basepoint. Note that $\pi_j(Q(\Omega^* X \cup *))$ belongs to C_p for $0 < j < 2p - 3$ since the first p-torsion occurs in the stable homotopy of a point π_j^s when $j = 2p - 3$. Hence θ is $(2p - 3)$-connected mod C_p. By the mod C_p analog of [14, Proposition 1.1] for A_∞ ring spaces it follows that

$$\widehat{BGL}(Q(\Omega^* X \cup *))^+ \to \widehat{BGL}(Z\Omega^* X)^+$$

is $(2p - 2)$-connected mod C_p. ($\widehat{BGL}(Z\Omega^* X)^+$ is the K-theory of the topological ring $Z\Omega^* X$.) Since $\Omega^* X$ has the homotopy type of a discrete set we have for $j \geq 1$

$$\pi_j \widehat{BGL}(Z\Omega^* X)^+ = K_j(Z\pi_0 \Omega^* X) = K_j(Z\pi_1 X).$$

Note that $A(X)$ is (by one of its many definitions) $\widehat{BGL}(Q(\Omega^* X \cup *))^+$ (see [14]). In particular, $A(*) \to K(Z) \simeq BGL(Z)^+$ is $(2p-2)$-connected mod C_p. It follows that if Y is a finite complex, $h(Y; A(*)) \to h(Y; K(Z))$ is $(2p - 2)$-connected mod C_p. A standard diagram chase with the ladder of exact sequence obtained from the map of homotopy fibrations (2.3) yields the lemma.

Let $Z_{\langle p \rangle}$ be the subring of the rationals generated by fractions of the form a/b, where $b \neq 0$ and p does not divide b.

PROPOSITION 2.5. *Let Γ be a Bieberbach group and $p > 2$ a prime. Suppose $b \in \mathrm{Wh}_k(\Gamma) \otimes \mathbf{Z}_{\langle p \rangle}$, where $0 \le k \le 2p - 2$. Then there is an integer N_b such that if $\theta\colon \Gamma \to \Gamma$ is s-expansive and $s > N_b$, then $\mathrm{tr}(\theta)(b) = 0$.*

PROOF. In the cases $k = 0, 1$ this was proved in [5]. Suppose $\theta\colon \Gamma \to \Gamma$ is s-expansive (see [5] for the definition). By [4] there is a compact flat Riemannian manifold M with $\pi_1 M = \Gamma$ and an s-expansive map $H\colon M \to M$ such that $\theta = H_*\colon \pi_1 M \to \pi_1 M$. The result now follows from Proposition 1.5, Lemma 2.4 and the fact that $\mathrm{Wh}^{\mathrm{PL}}(M) \to \mathrm{Wh}(\pi_1 M)$ is natural with respect to the transfer.

The following weak form of Frobenius induction is valid for $\mathrm{Wh}_j(\Gamma)$.

LEMMA 2.6. *Let Γ be a group, $p\colon \Gamma \to G$ an epimorphism to a finite group, R a subring of the rationals and $j \ge 0$. Suppose $b \in \mathrm{Wh}_j(\Gamma) \otimes R$ and, for every hyperelementary subgroup E of G, $\mathrm{tr}(b) = 0$ where $\mathrm{tr}\colon \mathrm{Wh}_j(\Gamma) \otimes R \to \mathrm{Wh}_j(p^{-1}(E)) \otimes R$ is the transfer. Then $b = 0$.*

PROOF. For $H \subset G$ there is a long exact sequence from the fibration (2.2):

$$\to h_j(Bp^{-1}(H); K(Z)) \overset{l_j}{\to} K_j(Zp^{-1}(H))$$
$$\overset{q_j}{\to} \mathrm{Wh}_j(p^{-1}(H)) \to \cdots \to \mathrm{Wh}_0(p^{-1}H)) \to 0.$$

This sequence remains exact when tensored with R. Let

$$W_j(H) = \mathrm{Image}(q_j) \otimes R, \qquad V_j(H) = \ker(l_{j-1}) \otimes R \quad (\text{set } V_0(H) = 0).$$

Then for any $H \subset G$ there is a short exact sequence

$$(2.7) \qquad 0 \to W_j(H) \to \mathrm{Wh}_j(p^{-1}(H)) \otimes R \to V_j(H) \to 0.$$

According to [6], $K_j(Zp^{-1}(H))$ is a Frobenius module over Swan's Frobenius functor $G_0(H)$. This action induces an action of $G_0(H)$ on $W_j(H)$ and consequently W_j satisfies hyperelementary induction, in particular, if C is the collection of hyperelementary subgroups of G the product of the transfer ("restriction") maps is injective:

$$0 \to W_j(G) \to \prod_{H \in C} W_j(H).$$

Also the product of the transfer ("restriction") maps for V_j is injective (compare [11, Proposition 6.2.9]):

$$0 \to V_j(G) \to \prod_{H \in C} V_j(H).$$

It follows from (2.7) that the product over C of the transfer maps

$$\mathrm{Wh}_j(\Gamma) \otimes R \to \prod_{H \in C} \mathrm{Wh}_j(p^{-1}(H)) \otimes R$$

is injective, which yields the lemma.

The Main Theorem is now easily proved by following the argument of [5] which was successfully copied in [12 and 13] with Proposition 2.5 providing the crucial vanishing result for the transfer and Lemma 2.6 showing Frobenius induction remains valid. We summarize this argument as follows:

If Γ is a Bieberbach group and $A \subset \Gamma$ its unique maximal normal abelian subgroup of finite index, then Γ/A is called the holonomy group of Γ and the rank

of Γ is defined to be the rank of A. A homomorphism $f\colon \Gamma \to \Gamma$ is said to be s-expansive, where s is an integer, if its restriction to A is multiplication by s and it induces the identity map on Γ/A. If s is a positive integer define $\Gamma_s = \Gamma/sA$. Hsiang and Farrell proved the following structure theorem.

THEOREM 2.8 [5]. *Let Γ be a Bieberbach group of rank n and holonomy group G. Then either:*

(1) $\Gamma = \Pi \times_\alpha T$; *i.e. Γ is the semidirect product of subgroups Π and T, where Π is a Bieberbach group of rank $n-1$; or*

(2) $\Gamma = B *_D C$; *i.e. is the amalgamated free product of subgroups of rank $n-1$, where D has index 2 in both B and C; or*

(3) *there is an infinite sequence of positive integers S_n with $S_n \equiv 1 \bmod |G|$ ($|G|$ =order of G) such that any hyperelementary subgroup of Γ_s which projects onto G via the canonical map $\Gamma \to G$ projects isomorphically to G.*

The proof that $\mathrm{Wh}_j(\Gamma)_{(p)} = 0$ for $0 \le j \le 2p-2$ proceeds by induction on j, on the rank of Γ and on the order of the holonomy group of Γ. The induction starts with the result $\mathrm{Wh}_0(\Gamma) = 0$ from [5]. Assuming the induction hypothesis, if we are in the first or second case of the above theorem then the result follows from Waldhausen's theory concerning the Whitehead groups of generalized free products [9] (the Nil terms vanish since the ring $Z\Gamma$ is left-regular and noetherian). In the third case let $b \in \mathrm{Wh}_j(\Gamma)_{(p)}$ and s be one of the integer S_n given by Theorem 2.8(3) subject to the constraint $s > N_b$, where N_b comes from Proposition 2.5. Let $p\colon \Gamma \to \Gamma_s$ be the canonical map. By Lemma 2.6 we will have $b = 0$ if b vanishes under the transfer maps associated to the hyperelementary subgroups of Γ_s. If E is a hyperelementary subgroup of Γ_s which projects via p to a proper subgroup of G, then the holonomy group of $p^{-1}(E)$ has order less than the order of G; thus, $\mathrm{Wh}_j(p^{-1}(E))_{(p)} = 0$ by the induction hypothesis. Otherwise E projects isomorphically to G by Theorem 2.8(3). Since all such subgroups are conjugate [5] it suffices to consider one of them. Let $E = pf(\Gamma)$, where $f\colon \Gamma \to \Gamma$ is s-expansive. However, $f^*(b) = 0$ by Proposition 2.5; hence, b vanishes under all appropriate transfer maps, completing the proof of the Main Theorem.

REMARKS. (1) As a corollary of the proof of Proposition 2.5 and the Main Theorem we have

$$\pi_j P^s(M)_{(p)} = \pi_{j+2}\mathrm{Wh}^{\mathrm{PL}}(M)_{(p)}$$
$$= \mathrm{Wh}_{j+2}(\pi_1 M)_{(p)} \quad \text{for } 0 \le j \le 2p-5 \text{ and } p > 2$$
$$= 0,$$

where M is a compact flat Riemannian manifold. As a consequence, for $p > 2$, the p-torsion of $\pi_j \mathrm{TOP}(M)$ vanishes in the range $0 \le j \le 2p-5$ and $j < \phi_2(\dim(M))$, where $\mathrm{TOP}(M)$ is the group of homeomorphisms of M (compare [13, §2]).

(2) If $\pi_k(\mathrm{Wh}^{\mathrm{PL}}(M)) \otimes \mathbf{Z}[\frac{1}{2}] \to \mathrm{Wh}_k(\pi_1 M) \otimes \mathbf{Z}[\frac{1}{2}]$ is onto for $k \ge 0$ and for compact flat Riemannian M, then the techniques used in this paper show $\mathrm{Wh}_k(\Gamma) \otimes \mathbf{Z}[\frac{1}{2}] = 0$, $k \ge 0$, for any Bieberbach group.

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