A REFLEXIVITY THEOREM FOR WEAKLY CLOSED
SUBSPACES OF OPERATORS

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Abstract. It was proved in [4] that the ultraweakly closed algebras generated by
certain contractions on Hilbert space have a remarkable property. This property, in
conjunction with the fact that these algebras are isomorphic to $H^\infty$, was used in [3]
to show that such ultraweakly closed algebras are reflexive. In the present paper we
prove an analogous result that does not require isomorphism with $H^\infty$, and applies
even to linear spaces of operators. Our result contains the reflexivity theorems of [3,
2 and 9] as particular cases.

Let $\mathcal{L}(\mathcal{H})$ denote the algebra of (linear, bounded) operators acting on the Hilbert
space $\mathcal{H}$, and let $\mathcal{M}$ denote a linear subspace of $\mathcal{L}(\mathcal{H})$. Then $\mathcal{M}$ is endowed with the
weak and ultraweak topologies that it inherits from $\mathcal{L}(\mathcal{H})$ (cf. [6, §15]). For two
arbitrary vectors $x, y \in \mathcal{H}$ we can define the (ultra) weakly continuous functional
$[x \otimes y]$ on $\mathcal{M}$ by

$$[x \otimes y](A) = \langle Ax, y \rangle, \quad A \in \mathcal{M},$$

where $\langle \cdot, \cdot \rangle$ stands for the scalar product in $\mathcal{H}$.

Definition 1. Let $n$ be a natural number, $n \geq 1$. The subspace $\mathcal{M}$ has property
($B_n$) [respectively ($A_n$)] if for every positive number $\epsilon$ there exists a positive number
$\delta = \delta(\epsilon, n)$ such that for every system $\{\phi_{ij}: 1 \leq i, j \leq n\}$ of weakly [respectively
ultraweakly] continuous functionals on $\mathcal{M}$ and every system $\{x_i, y_j: 1 \leq i, j \leq n\}$ of
vectors in $\mathcal{H}$ satisfying the inequalities $\|\phi_{ij} - [x_i \otimes y_j]\| < \delta$ there exist vectors
$\{x'_i, y'_j: 1 \leq i, j \leq n\}$ in $\mathcal{H}$ such that

$$\phi_{ij} = [x'_i \otimes y'_j], \quad 1 \leq i, j \leq n,$$

and

$$\|x_i - x'_i\| < \epsilon, \quad \|y_j - y'_j\| < \epsilon, \quad 1 \leq i, j \leq n.$$

Since every weakly continuous functional on $\mathcal{M}$ is also ultraweakly continuous,
property ($B_n$) is weaker than ($A_n$). (Added in proof. It was pointed out by C.
Apostol that ($B_n$) and ($A_n$) are in fact equivalent. This fact is not used below.)

We recall now from [8] that a linear subspace $\mathcal{M}$ of $\mathcal{L}(\mathcal{H})$ is said to be reflexive if
it contains every operator $T \in \mathcal{L}(\mathcal{H})$ with the property that $Tx \in (\mathcal{M}x)^-$ for every

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140
HARI BERCOVICI

Of course, reflexive subspaces are weakly closed. This definition coincides with the usual definition \( \mathcal{M} = \text{Alg Lat } \mathcal{M} \) if \( \mathcal{M} \) is a subalgebra of \( \mathcal{L}(\mathscr{H}) \).

We state now the main result of this paper.

**Theorem 2.** Let \( \mathcal{M} \) be a weakly closed subspace of \( \mathcal{L}(\mathscr{H}) \). If \( \mathcal{M} \) has property \( (\mathcal{B}_n^*) \) for every natural number \( n \), then \( \mathcal{M} \) is reflexive. Moreover, every weakly closed subspace of \( \mathcal{M} \) is also reflexive.

Before going into the proof, we relate this result with the reflexivity theorem from [3]. It was proved in [4] that, if \( T \) is a \( (\text{BCP}) \)-operator, the ultraweakly closed algebra \( \mathcal{A}_T \) generated by \( T \) has property \( (\mathcal{A}_n^*) \) for every \( n = 1, 2, \ldots \). The reflexivity of \( \mathcal{A}_T \) follows then from Theorem 2 and the following lemma.

**Lemma 3.** Let \( \mathcal{M} \) be a linear subspace of \( \mathcal{L}(\mathscr{H}) \) having property \( (\mathcal{A}_n^+) \). Then the weak and ultraweak closures of \( \mathcal{M} \) coincide, and the weak and ultraweak topologies coincide on the weak closure of \( \mathcal{M} \).

**Proof.** Since every ultraweakly continuous functional on \( \mathcal{M} \) extends continuously to the ultraweak closure of \( \mathcal{M} \), there is no loss of generality in assuming that \( \mathcal{M} \) is ultraweakly closed. Let \( \delta = \delta(1, 1) \) be as in Definition 1, and let \( \phi \) be an arbitrary ultraweakly continuous functional on \( \mathcal{M} \). Then \( ||\delta\phi/(2\|\phi\|) - [0 \otimes 0]|| < \delta \) so that we can find vectors \( x' \) and \( y' \) such that \( ||x'|| < 1, ||y'|| < 1 \) and \( \delta\phi/(2\|\phi\|) = [x' \otimes y'] \) or, equivalently, \( \phi = [x \otimes y] \) with \( x = (2\|\phi\|/\delta)^{1/2}x', y = (2\|\phi\|/\delta)^{1/2}y' \). Thus we can write \( \phi \) as \( [x \otimes y] \) with \( ||x|| < (2/\delta)^{1/2}\|\phi\|^{1/2}, ||y|| < (2/\delta)^{1/2}\|\phi\|^{1/2} \). We can now apply, e.g., the proof of [3, Theorem 1] to conclude that \( \mathcal{M} \) is weakly closed and the weak and ultraweak topologies coincide on \( \mathcal{M} \).

We have therefore the following consequence of Theorem 2, which also implies the reflexivity results of [2 and 9].

**Corollary 4.** Let \( \mathcal{M} \) be an ultraweakly closed subspace of \( \mathcal{L}(\mathscr{H}) \). If \( \mathcal{M} \) has property \( (\mathcal{A}_n^+) \) for every natural number \( n \), then \( \mathcal{M} \) is weakly closed and reflexive. Moreover, every weakly closed subspace of \( \mathcal{M} \) is also reflexive.

For the proof of Theorem 2, we need two lemmas. The first was proved in [3] for the case in which \( \mathcal{M} \) is a weakly closed algebra. The proof for linear subspaces of \( \mathcal{L}(\mathscr{H}) \) is identical (and easy) so we content ourselves with the statement.

**Lemma 5.** Let \( \mathcal{M} \) be a linear subspace of \( \mathcal{L}(\mathscr{H}) \). An operator \( T \in \mathcal{L}(\mathscr{H}) \) is in the weak closure of \( \mathcal{M} \) if and only if for every natural \( n \) and every system \( \{x_i, y_i; 1 \leq i \leq n\} \) of vectors in \( \mathscr{H} \) such that \( \sum_{i=1}^{n}[x_i \otimes y_i] = 0 \), we have \( \sum_{i=1}^{n}\langle Tx_i, y_i \rangle = 0 \).

**Lemma 6.** Let \( \mathcal{M} \) be a linear subspace of \( \mathcal{L}(\mathscr{H}) \). Assume that \( \mathcal{M} \) has property \( (\mathcal{B}_n^+) \) for every natural number \( n \). Then for every natural number \( n \), every system \( \{x_i, y_i; 1 \leq i \leq n\} \) of vectors in \( \mathscr{H} \) and every \( \epsilon > 0 \), there exist vectors \( \{x_{ij}, y_{ij}; 1 \leq i, j \leq n\} \) such that \( [x_{ij} \otimes y_{ij}] = \delta_{ji}[x_i \otimes y_j], 1 \leq i, j \leq n \), and \( \|x_i - x_{ij}\| < \epsilon, \|y_j - y_{ij}\| < \epsilon, 1 \leq i, j \leq n \). (Here \( \delta_{ji} \) denotes, as usual, Kronecker's symbol.)
Proof. Let $\delta = \delta(\epsilon, n^2)$ be as in Definition 1. Set

$$\eta = \min \{ \delta/(2||[x_i \otimes y_k])|| : [x_i \otimes y_k] \neq 0 \}$$

and define $\phi_{i, j, k, l} = 0$ if $j \neq l$, $\phi_{1, k, l} = [x_i \otimes y_k]$, $\phi_{j, k, j} = \eta[x_i \otimes y_k]$, $j \geq 2$. The vectors $\{x_i^0, y_j^0 : 1 \leq i, j \leq n\}$ defined by $x_i^0 = \delta_{ij}x_i$, $y_j^0 = \delta_{ij}y_i$, $1 \leq i, j \leq n$, obviously satisfy the inequalities

$$||\phi_{i, j, k, l} - [x_i^0 \otimes y_j^0]|| < \delta, \quad 0 \leq i, j, k, l \leq n,$$

and therefore, by property $(B_n^2)$, we can find vectors $\{x_i', y_j' : 1 \leq i, j \leq n\}$ in $\mathcal{H}$ such that

$$\phi_{i, j, k, l} = [x_i' \otimes y_j'], \quad 0 \leq i, j, k, l \leq n,$$

and $||x_i^0 - x_i'|| < \epsilon$, $||y_j^0 - y_j'|| < \epsilon$, $0 \leq i, j \leq n$. Then the vectors $\{x_{i, j}, y_{i, j} : 1 \leq i, j \leq n\}$ defined by $x_{i, j} = x_{i, i} = x_i$, $y_{i, j} = y_{i, i} = y_i$, $1 \leq i \leq n$, $2 \leq j \leq n$, satisfy the requirements of the lemma.

Proof of Theorem 2. Let $T \in \mathcal{L}(\mathcal{H})$ satisfy the property that $Tx \in (\mathcal{M}x)^- \forall x \in \mathcal{H}$. We first note that the equality $[x \otimes y] = 0$, $x, y \in \mathcal{H}$, means that $y$ is orthogonal to $(\mathcal{M}x)^-$, and hence it implies $\langle Tx, y \rangle = 0$.

In order to show that $T \in \mathcal{M}$, we must prove, according to Lemma 5, that the equality $\sum_{i=1}^{n}[x_i \otimes y_i] = 0$, $x_i, y_i \in \mathcal{H}$, $1 \leq i \leq n$, implies $\sum_{i=1}^{n}\langle Tx_i, y_i \rangle = 0$. By what has just been said, this property is satisfied for $n = 1$. Assume therefore that $n \geq 2$, $x_i, y_i \in \mathcal{H}$, $1 \leq i \leq n$, and $\sum_{i=1}^{n}[x_i \otimes y_i] = 0$. For every $\epsilon > 0$ we can find, using Lemma 6, vectors $x_{i, j} = x_{i, j}(\epsilon), y_{i, j} = y_{i, j}(\epsilon), 0 \leq i, j \leq n$, satisfying,

1. $[x_{i, j} \otimes y_{i, j}] = \delta_{ij}[x_i \otimes y_i], \quad 1 \leq i, j, k, l \leq n,$

and

2. $||x_{i, i} - x_{i, i}(\epsilon)|| = ||x_{i, i} - x_{i, i}(\epsilon)|| < \epsilon,$

3. $||y_{j, j} - y_{j, j}(\epsilon)|| = ||y_{j, j} - y_{j, j}(\epsilon)|| < \epsilon, \quad 1 \leq i, j \leq n.$

We now remark that by (1)

$$\left[\sum_{i=1}^{n} x_{i, i} \otimes \sum_{i=1}^{n} y_{i, i}\right] = \sum_{i=1}^{n} [x_{i, i} \otimes y_{i, i}] + \sum_{i \neq j} [x_{i, i} \otimes y_{j, j}]$$

$$= \sum_{i=1}^{n} [x_i \otimes y_i] = 0$$

and therefore

$$\langle T\left(\sum_{i=1}^{n} x_{i, i}\right), \sum_{i=1}^{n} y_{i, i}\rangle = 0.$$ 

Since $[x_{i, i} \otimes y_{j, j}] = 0$ for $i \neq j$, we also have $\langle Tx_{i, i}, y_{j, j}\rangle = 0$ for $i \neq j$ so that (3) can be rewritten as

$$\sum_{i=1}^{n} \langle Tx_{i, i}, y_{i, i}\rangle = 0.$$
Assume now that \( i \neq 1 \). We have by (1)
\[
[(x_{ii} - x_{i1}) \otimes (y_{ii} + y_{i1})] = [x_{ii} \otimes y_{ii}] - [x_{i1} \otimes y_{i1}] + [x_{ii} \otimes y_{i1}] - [x_{i1} \otimes y_{ii}]
\]
and therefore
\[
0 = \langle T(x_{ii} - x_{i1}), y_{ii} + y_{i1} \rangle = \langle Tx_{ii}, y_{i1} \rangle + \langle Tx_{i1}, y_{ii} \rangle - \langle Tx_{ii}, y_{ii} \rangle - \langle Tx_{i1}, y_{i1} \rangle.
\]
The last two terms are zero because \([x_{ii} \otimes y_{ii}] = [x_{i1} \otimes y_{i1}] = 0\) and we conclude that \( \langle Tx_{ii}, y_{ii} \rangle = \langle Tx_{i1}, y_{i1} \rangle \). Therefore (4) can now be written as \( \sum_{i=1}^{n} \langle Tx_{ii}, y_{i1} \rangle = 0 \).
We now let \( \varepsilon \) approach zero. We have \( \lim_{\varepsilon \to 0} x_{ii}(\varepsilon) = x_{ii} \), \( \lim_{\varepsilon \to 0} y_{i1}(\varepsilon) = y_{i1} \) so that
\[
\sum_{i=1}^{n} \langle Tx_{i}, y_{i} \rangle = \lim_{\varepsilon \to 0} \sum_{i=1}^{n} \langle Tx_{i}(\varepsilon), y_{i}(\varepsilon) \rangle = 0
\]
and the reflexivity of \( \mathcal{M} \) is proved by Lemma 5. The last statement of the theorem follows from [8, Theorem 2.3] (cf. also [7]).

We conclude with a condition implying property \((\Lambda^n)\) and which is sometimes easier to verify. For an arbitrary linear subspace \( \mathcal{M} \) of \( \mathcal{L}(\mathcal{H}) \) we will denote by \( \mathcal{M}_{*} \) the Banach space of all ultraweakly continuous functionals on \( \mathcal{M} \). It is well known that the dual space of \( \mathcal{M}_{*} \) coincides with the ultraweak closure of \( \mathcal{M} \); we will not use this fact here. The following two definitions were given in [1] for ultraweakly closed algebras \( \mathcal{M} \) (cf. [1, Definitions 1.4 and 1.5]).

**Definition 7.** Suppose \( \mathcal{M} \subset \mathcal{L}(\mathcal{H}) \) is a linear subspace and \( 0 < \theta < +\infty \). We denote by \( X_{\theta}(\mathcal{M}) \) the set of all \( \phi \in \mathcal{M}_{*} \) such that there exist sequences \( \{x_{i}\}_{i=1}^{\infty} \) and \( \{y_{i}\}_{i=1}^{\infty} \) in \( \mathcal{H} \) satisfying the following conditions:
\[
\|x_{i}\| \leq 1, \quad \|y_{i}\| \leq 1, \quad 1 \leq i \leq \infty,
\]
and
\[
\limsup_{i \to \infty} \|\phi - [x_{i} \otimes y_{i}]\| \leq \theta.
\]

**Definition 8.** Suppose \( \mathcal{M} \subset \mathcal{L}(\mathcal{H}) \) is a linear subspace and \( 0 \leq \theta < \gamma < +\infty \). We say that \( \mathcal{M} \) has property \( X_{\theta, \gamma} \) if the closed absolutely convex hull of the set \( X_{\theta}(\mathcal{M}) \) contains the closed ball of radius \( \gamma \) centered at the origin in \( \mathcal{M}_{*} \):
\[
\overline{a}X_{\theta}(\mathcal{M}) \supset \{ \phi \in \mathcal{M}_{*} : \|\phi\| \leq \gamma \}.
\]

The following result coincides with [1, Theorem 1.9] if \( \mathcal{M} \) is an ultraweakly closed algebra. However, neither the algebra structure, nor the ultraweak closedness of \( \mathcal{M} \) has been used in the proof of that theorem, so that we refer to [1] for the proof.

**Theorem 9.** Suppose \( \mathcal{M} \subset \mathcal{L}(\mathcal{H}) \) is a linear subspace with property \( X_{\theta, \gamma} \) for some \( \gamma > \theta \geq 0 \). Then for every \( \phi \in \mathcal{M}_{*} \) there exist sequences \( \{x_{i}\}_{i=1}^{\infty} \) and \( \{y_{i}\}_{i=1}^{\infty} \) in \( \mathcal{H} \) such that
\[
\phi = [x_{i} \otimes y_{i}], \quad 1 \leq i < \infty,
\]
\[
\limsup_{i \to \infty} \|x_{i}\| \leq (\gamma - \theta)^{-1/2} \|\phi\|^{1/2}, \quad \limsup_{i \to \infty} \|y_{i}\| \leq (\gamma - \theta)^{-1/2} \|\phi\|^{1/2},
\]
and
\[
\lim_{i \to \infty} \left( \| [x_i \otimes z] \| + \| [z \otimes x_i] \| + \| [y_i \otimes z] \| + \| [z \otimes y_i] \| \right) = 0, \quad z \in H.
\]

It was seen in [1] that this theorem implies that $M$ has property $(A_n)$ for each $n$; we recall that property $(A_n)$ requires the solvability for $x_i$ and $y_i$ of arbitrary systems of the form $[x_i \otimes y_j] = \phi_{ij}, \phi_{ij} \in M_*, 1 \leq i, j \leq n$. In order to prove the stronger property $(A_n)$ we need the following lemma, whose proof is reminiscent of the techniques of Robel [9].

**Lemma 10.** Suppose $M \subset L(H)$ is a linear subspace with property $X_{\theta,\gamma}$ for some $\gamma > \theta > 0$. If $n$ is a natural number, $a > 0$, $\epsilon > 0$, and $\phi_{ij} \in M_*, x_i, y_j \in H$, $1 \leq i, j \leq n$, are such that
\[
\| \phi_{ij} - [x_i \otimes y_j] \| < a, \quad 1 \leq i, j \leq n,
\]
then there exist \{ $x'_i, y'_j$ : $1 \leq i, j \leq n$ \} in $H$ such that
\[
\| \phi_{ij} - [x'_i \otimes y'_j] \| < \epsilon, \quad 1 \leq i, j \leq n,
\]
and
\[
\| x_i - x'_i \| < n(\gamma - \theta)^{-1/2} a^{1/2}, \quad \| y_j - y'_j \| < n(\gamma - \theta)^{-1/2} a^{1/2}, \quad 1 \leq i, j \leq n.
\]

**Proof.** Let $\delta > 0$ be such that $(n^2 + 2n - 1) \delta < \epsilon$. An application of Theorem 9 to $\phi = \phi_{ij} - [x_i \otimes y_j]$ yields sequences \{ $\xi_{ij}(k) \}_{k=1}^\infty$, \{ $\eta_{ij}(k) \}_{k=1}^\infty$ \} such that
\[
\phi_{ij} - [x_i \otimes y_j] = [\xi_{ij}(k) \otimes \eta_{ij}(k)], \quad 1 \leq k < \infty,
\]
\[
\| \xi_{ij}(k) \| < (\gamma - \theta)^{-1/2} a^{1/2}, \quad \| \eta_{ij}(k) \| < (\gamma - \theta)^{-1/2} a^{1/2}, \quad 1 \leq k < \infty,
\]
and
\[
\lim_{k \to \infty} \left( \| [\xi_{ij}(k) \otimes z] \| + \| [z \otimes \eta_{ij}(k)] \| \right) = 0, \quad z \in H.
\]
An easy induction using (8) shows that we can find natural numbers $k_{ij}, 1 \leq i, j \leq n$, such that the vectors $\xi_{ij} = \xi_{ij}(k_{ij})$ and $\eta_{ij} = \eta_{ij}(k_{ij})$ satisfy the inequalities
\[
\| [\xi_{ij} \otimes \eta_{kl}] \| < \delta \quad \text{if} \quad (i, j) \neq (k, l),
\]
\[
\| [x_i \otimes \eta_{kl}] \| < \delta, \quad 1 \leq i, k, l \leq n,
\]
\[
\| [\xi_{ij} \otimes y_k] \| < \delta, \quad 1 \leq i, j, k \leq n.
\]
We can now set
\[
x'_i = x_i + \sum_{k=1}^n \xi_{ik}, \quad y'_j = y_j + \sum_{l=1}^n \eta_{lj}
\]
and note that we obviously have from (7)
\[
\| x'_i - x_i \| \leq \sum_{k=1}^n \| \xi_{ik} \| < n(\gamma - \theta)^{-1/2} a^{1/2}, \quad 1 \leq i \leq n.
\]
and similarly
\[ \|y'_j - y_j\| < n(\gamma - \theta)^{-1/2} a^{1/2}, \quad 1 \leq j \leq n. \]
Finally, we observe that
\[
\phi_{ij} - [x'_i \otimes y'_j] = \phi_{ij} - [x_i \otimes y_j] - [\xi_{ij} \otimes \eta_{ij}] - \sum_{i=1}^n [x_i \otimes \eta_{ij}]
- \sum_{k=1}^n [\xi_{ik} \otimes y_j] - \sum_{(i,k) \neq (i,j)} [\xi_{ik} \otimes \eta_{ij}]
\]
and we obtain, using (6) and (9),
\[
\|\phi_{ij} - [x'_i \otimes y'_j]\| \leq n\delta + n\delta + (n^2 - 1)\delta < \epsilon.
\]
The lemma follows.

A routine argument shows now that Lemma 10 is self-improving to yield the following result.

**Theorem 11.** Suppose \( \mathcal{M} \subset \mathcal{L}(\mathcal{H}) \) is a linear subspace with property \( X_{\gamma, \theta} \) for some \( \gamma > \theta > 0 \). If \( n \) is a natural number, \( a > 0 \) and \( \phi_{ij} \in \mathcal{M} \), \( x_i, y_j \in \mathcal{H}, 1 \leq i, j \leq n \), are such that
\[
\|x'_i - x_i\| < n(\gamma - \theta)^{-1/2} a^{1/2}, \quad \|y'_j - y_j\| < n(\gamma - \theta)^{-1/2} a^{1/2}, \quad 1 \leq i, j \leq n.
\]
then there exist \( \{x'_i, y'_j: 1 \leq i, j \leq n\} \) in \( \mathcal{H} \) such that
\[
\phi_{ij} = [x'_i \otimes y'_j], \quad 1 \leq i, j \leq n,
\]
and
\[
\|x_i - x'_i\| < n(\gamma - \theta)^{-1/2} a^{1/2}, \quad \|y_j - y'_j\| < n(\gamma - \theta)^{-1/2} a^{1/2}, \quad 1 \leq i, j \leq n.
\]

**Proof.** Choose a positive number \( b \) such that
\[
\|\phi_{ij} - [x_i \otimes y_j]\| < b < a, \quad 1 \leq i, j \leq n,
\]
and let \( \epsilon \) be a positive number to be specified later (\( \epsilon \) will only depend on \( a \) and \( b \)). By Lemma 10, we can find vectors \( \{x'_i, y'_j: 1 \leq i, j \leq n\} \) such that
\[
\|\phi_{ij} - [x'_i \otimes y'_i]\| < \epsilon, \quad 1 \leq i, j \leq n,
\]
and
\[
\|x'_i - x_i\| < n(\gamma - \theta)^{-1/2} b^{1/2}, \quad \|y'_j - y_j\| < n(\gamma - \theta)^{-1/2} b^{1/2}.
\]
We can then use Lemma 10 to construct inductively sequences \( \{x^k_i\}_{k=2}^\infty, \{y^k_j\}_{k=2}^\infty \),
\[ 1 \leq i, j \leq n, \text{ such that} \]
\[
\|\phi_{ij} - [x^k_i \otimes y^k_j]\| < \epsilon^k, \quad 1 \leq i, j \leq n, 2 \leq k < \infty,
\]
and
\[
\|x^k_i - x_i\| < n(\gamma - \theta)^{-1/2} \epsilon^{k/2}, \quad \|y^k_j - y_j\| < n(\gamma - \theta)^{-1/2} \epsilon^{k/2}, \quad 1 \leq i, j \leq n, 1 \leq k < \infty.
\]
It is obvious that the sequences \( \{x^k_i\}_{k=1}^{\infty} \) and \( \{y^k_j\}_{k=1}^{\infty} \), \( 1 \leq i, j \leq n \), are Cauchy and \( \phi_{i,j} = [x^i_i \otimes y^j_j] \), \( 1 \leq i, j \leq n \), if

\[
x^i_i = \lim_{k \to \infty} x^k_i, \quad y^j_j = \lim_{k \to \infty} y^k_j, \quad 1 \leq i, j \leq n.
\]

Finally,

\[
\|x^i_i - x_j\| \leq \|x^i_i - x^i_j\| + \sum_{k=1}^{\infty} \|x^{k+1}_i - x^k_i\| < n(\gamma - \theta)^{-1/2}\left(b^{1/2} + \sum_{k=1}^{\infty} e^{k/2}\right)
\]

\[
= n(\gamma - \theta)^{-1/2}\left(b^{1/2} + e^{1/2}(1 - e^{1/2})^{-1}\right), \quad 1 \leq i \leq n,
\]

and analogously

\[
\|y^j_j - y^i_i\| < n(\gamma - \theta)^{-1/2}\left(b^{1/2} + e^{1/2}(1 - e^{1/2})^{-1}\right), \quad 1 \leq j \leq n.
\]

It suffices therefore to choose \( \epsilon \) so small that \( b^{1/2} + e^{1/2}(1 - e^{1/2})^{-1} < a^{1/2} \). The theorem is proved.

We are now able to prove the promised criterion.

**Corollary 12.** Suppose \( \mathcal{M} \subseteq \mathcal{L}(\mathcal{H}) \) is a linear subspace with property \( x_{\theta, \gamma} \) for some \( \gamma > \theta \geq 0 \). Then \( \mathcal{M} \) has property \( (A_n^\gamma) \) for every natural number \( n \). In particular the ultraweak closure \( \mathcal{M}^- \) of \( \mathcal{M} \) is weakly closed and reflexive.

**Proof.** The last part of the statement follows from the first part, combined with Lemma 3 and Corollary 4. To prove the first part we only have to use Theorem 11. Observe that we can take \( \delta(\epsilon, n) = \epsilon^2 n^{-2}(\gamma - \theta) \).

We finally note that one could give a definition analogous to Definition 8, in which the space \( \mathcal{M}_* \) is replaced by the set \( \mathcal{M}_- \) of all weakly continuous functionals on \( \mathcal{M} \). The property thus defined would however be stronger than \( X_{\theta, \gamma} \) since \( \mathcal{M}_* \) coincides with the norm closure of \( \mathcal{M}_- \); this is why we restricted ourselves to the space \( \mathcal{M}_* \) and the properties \( (A_n^\gamma) \). We do not know whether the weaker properties \( (A_n^\gamma) \) imply reflexivity. Property \( (A_1) \) alone does not imply reflexivity. Indeed, the algebra \( \mathcal{M} \) of \( 2 \times 2 \) matrices defined as

\[
\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{C} \right\}
\]

is not reflexive, but it has property \( (A_1) \) (and even \( (A_1^\gamma) \), as can be seen by an easy computation).

**References**


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