A REFLEXIVITY THEOREM FOR WEAKLY CLOSED SUBSPACES OF OPERATORS

BY

HARI BERCOCICI

Abstract. It was proved in [4] that the ultraweakly closed algebras generated by certain contractions on Hilbert space have a remarkable property. This property, in conjunction with the fact that these algebras are isomorphic to $H^X$, was used in [3] to show that such ultraweakly closed algebras are reflexive. In the present paper we prove an analogous result that does not require isomorphism with $H^X$, and applies even to linear spaces of operators. Our result contains the reflexivity theorems of [3, 2 and 9] as particular cases.

Let $\mathcal{L}(\mathcal{H})$ denote the algebra of (linear, bounded) operators acting on the Hilbert space $\mathcal{H}$, and let $\mathcal{M}$ denote a linear subspace of $\mathcal{L}(\mathcal{H})$. Then $\mathcal{M}$ is endowed with the weak and ultraweak topologies that it inherits from $\mathcal{L}(\mathcal{H})$ (cf. [6, §15]). For two arbitrary vectors $x, y \in \mathcal{H}$ we can define the (ultra) weakly continuous functional $[x \otimes y]$ on $\mathcal{M}$ by

$$[x \otimes y](A) = \langle Ax, y \rangle, \quad A \in \mathcal{M},$$

where $\langle \cdot, \cdot \rangle$ stands for the scalar product in $\mathcal{H}$.

Definition 1. Let $n$ be a natural number, $n \geq 1$. The subspace $\mathcal{M}$ has property $(B_n)$ [respectively $(A_n)$] if for every positive number $\varepsilon$ there exists a positive number $\delta = \delta(\varepsilon, n)$ such that for every system $\{\phi_{ij}: 1 \leq i, j \leq n\}$ of weakly [respectively ultraweakly] continuous functionals on $\mathcal{M}$ and every system $\{x_i, y_j: 1 \leq i, j \leq n\}$ of vectors in $\mathcal{H}$ satisfying the inequalities $\|x_i - x_j\| < \delta$ there exist vectors $\{x'_i, y'_j: 1 \leq i, j \leq n\}$ in $\mathcal{H}$ such that

$$\phi_{ij} = [x'_i \otimes y'_j], \quad 1 \leq i, j \leq n,$$

and

$$\|x_i - x'_i\| < \varepsilon, \quad \|y_j - y'_j\| < \varepsilon, \quad 1 \leq i, j \leq n.$$

Since every weakly continuous functional on $\mathcal{M}$ is also ultraweakly continuous, property $(B_n)$ is weaker than $(A_n)$. (Added in proof. It was pointed out by C. Apostol that $(B_n)$ and $(A_n)$ are in fact equivalent. This fact is not used below.)

We recall now from [8] that a linear subspace $\mathcal{M}$ of $\mathcal{L}(\mathcal{H})$ is said to be reflexive if it contains every operator $T \in \mathcal{L}(\mathcal{H})$ with the property that $Tx \in (\mathcal{M}x)^\sim$ for every
x ∈ ℌ. Of course, reflexive subspaces are weakly closed. This definition coincides with the usual definition (ℳ = Alg Lat ℳ) if ℳ is a subalgebra of L(ℋ).

We state now the main result of this paper.

**Theorem 2.** Let ℳ be a weakly closed subspace of L(ℋ). If ℳ has property (Bₙ) for every natural number n, then ℳ is reflexive. Moreover, every weakly closed subspace of ℳ is also reflexive.

Before going into the proof, we relate this result with the reflexivity theorem from [3]. It was proved in [4] that, if T is a (BCP)-operator, the ultraweakly closed algebra Aᵀ generated by T has property (Aⁿ) for every n = 1, 2, .... The reflexivity of Aᵀ follows then from Theorem 2 and the following lemma.

**Lemma 3.** Let ℳ be a linear subspace of L(ℋ) having property (Aⁿ). Then the weak and ultraweak closures of ℳ coincide, and the weak and ultraweak topologies coincide on the weak closure of ℳ.

**Proof.** Since every ultraweakly continuous functional on ℳ extends continuously to the ultraweak closure of ℳ, there is no loss of generality in assuming that ℳ is ultraweakly closed. Let δ = δ(1, 1) be as in Definition 1, and let ϕ be an arbitrary ultraweakly continuous functional on ℳ. Then ||ϕ/(2||ϕ||) − [0 ⊗ 0]| < δ so that we can find vectors x′ and y′ such that ||x'|| < 1, ||y'|| < 1 and δϕ/(2||ϕ||) = [x' ⊗ y'] or, equivalently, ϕ = [x ⊗ y] with x = (2||ϕ||/δ)¹/₂x', y = (2||ϕ||/δ)¹/₂y'. Thus we can write ϕ as [x ⊗ y] with ||x|| < (2/δ)¹/₂||ϕ||¹/₂, ||y|| < (2/δ)¹/₂||ϕ||¹/₂. We can now apply, e.g., the proof of [3, Theorem 1] to conclude that ℳ is weakly closed and the weak and ultraweak topologies coincide on ℳ.

We have therefore the following consequence of Theorem 2, which also implies the reflexivity results of [2 and 9].

**Corollary 4.** Let ℳ be an ultraweakly closed subspace of L(ℋ). If ℳ has property (Aₙ) for every natural number n, then ℳ is weakly closed and reflexive. Moreover, every weakly closed subspace of ℳ is also reflexive.

For the proof of Theorem 2, we need two lemmas. The first was proved in [3] for the case in which ℳ is a weakly closed algebra. The proof for linear subspaces of L(ℋ) is identical (and easy) so we content ourselves with the statement.

**Lemma 5.** Let ℳ be a linear subspace of L(ℋ). An operator T ∈ L(ℋ) is in the weak closure of ℳ if and only if for every natural n and every system {xᵢ, yᵢ; 1 ≤ i ≤ n} of vectors in ℋ such that ∑ᵢ=₁ⁿ[xᵢ ⊗ yᵢ] = 0, we have ∑ᵢ=₁ⁿ⟨Txᵢ, yᵢ⟩ = 0.

**Lemma 6.** Let ℳ be a linear subspace of L(ℋ). Assume that ℳ has property (Bₙ) for every natural number n. Then for every natural number n, every system {xᵢ, yᵢ; 1 ≤ i ≤ n} of vectors in ℋ and every ε > 0, there exist vectors {xᵢ, yᵢ; 1 ≤ i, j ≤ n} such that ∥[xᵢ ⊗ yᵢ] − δᵢⱼ[xᵢ ⊗ yⱼ]∥, 1 ≤ i, j ≤ n, and ∥xᵢ − xⱼ∥ < ε, ∥yⱼ − yᵢ∥ < ε, 1 ≤ i, j ≤ n. (Here δᵢⱼ denotes, as usual, Kronecker's symbol.)
Proof. Let $\delta = \delta(\varepsilon, n^2)$ be as in Definition 1. Set

$$\eta = \min \left\{ \frac{\delta}{2} \left\| [x_i \otimes y_k] \right\| : [x_i \otimes y_k] \neq 0 \right\}$$

and define $\phi_{i,j,k,l} = 0$ if $j \neq l$, $\phi_{i,1,k} = [x_i \otimes y_k]$, $\phi_{i,j,k} = \eta [x_i \otimes y_k]$, $j \geq 2$. The vectors $\{x_i^0, y_j^0 : 1 \leq i, j \leq n\}$ defined by $x_i^0 = \delta_{i1}x_i$, $y_j^0 = \delta_{j1}y_i$, $1 \leq i, j \leq n$, obviously satisfy the inequalities

$$\left\| \phi_{i,j,k,l} - \left[ x_i^0 \otimes y_j^0 \right] \right\| < \delta, \quad 0 \leq i, j, k, l \leq n,$$

and therefore, by property $(\text{B}_n^2)$, we can find vectors $\{x_i', y_j' : 1 \leq i, j \leq n\}$ in $\mathcal{H}$ such that

$$\phi_{i,j,k,l} = \left[ x_i' \otimes y_j' \right], \quad 0 \leq i, j, k, l \leq n,$$

and $\|x_i^0 - x_i'\| < \varepsilon$, $\|y_j^0 - y_j'\| < \varepsilon$, $0 \leq i, j \leq n$. Then the vectors $\{x_i, y_j : 1 \leq i, j \leq n\}$ defined by $x_i = x_i', x_{i1} = x_i, y_j = y_j'$, $1 \leq i \leq n, 2 \leq j \leq n$, satisfy the requirements of the lemma.

Proof of Theorem 2. Let $T \in \mathcal{L}(\mathcal{H})$ satisfy the property that $Tx \in (\mathcal{M}x)^{-}$ for every $x \in \mathcal{H}$. We first note that the equality $[x \otimes y] = 0$, $x, y \in \mathcal{H}$, means that $y$ is orthogonal to $(\mathcal{M}x)^{-}$, and hence it implies $\langle Tx, y \rangle = 0$.

In order to show that $T \in \mathcal{M}$, we must prove, according to Lemma 5, that the equality $\sum_{i=1}^{n} [x_i \otimes y_i] = 0$, $x_i, y_i \in \mathcal{H}, 1 \leq i \leq n$, implies $\sum_{i=1}^{n} \langle Tx_i, y_i \rangle = 0$. By what has just been said, this property is satisfied for $n = 1$. Assume therefore that $n \geq 2$, $x_i, y_i \in \mathcal{H}, 1 \leq i \leq n$, and $\sum_{i=1}^{n} [x_i \otimes y_i] = 0$. For every $\varepsilon > 0$ we can find, using Lemma 6, vectors $x_i = x_i(\varepsilon), y_j = y_j(\varepsilon), 0 \leq i, j \leq n$, satisfying

(1) $[x_i \otimes y_j] = \delta_{ij} [x_i \otimes y_j], \quad 1 \leq i, j, k, l \leq n,$

and

(2) $\|x_i - x_i(\varepsilon)\| = \|x_i - x_i(\varepsilon)\| < \varepsilon,$

(3) $\|y_j - y_j(\varepsilon)\| = \|y_j - y_j(\varepsilon)\| < \varepsilon, \quad 1 \leq i, j \leq n.$

We now remark that by (1)

$$\sum_{i=1}^{n} x_i \otimes \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} [x_i \otimes y_i] + \sum_{i \neq j} [x_i \otimes y_j]$$

and therefore

(3) $\left\langle T \left( \sum_{i=1}^{n} x_i \right), \sum_{i=1}^{n} y_i \right\rangle = 0.$

Since $[x_i \otimes y_j] = 0$ for $i \neq j$, we also have $\left\langle Tx_i, y_j \right\rangle = 0$ for $i \neq j$ so that (3) can be rewritten as

(4) $\sum_{i=1}^{n} \left\langle Tx_i, y_i \right\rangle = 0.$
Assume now that $i \neq 1$. We have by (1)
\[
[(x_{ii} - x_{i1}) \otimes (y_{ii} + y_{i1})] = [x_{ii} \otimes y_{ii}] - [x_{i1} \otimes y_{i1}] + [x_{ii} \otimes y_{i1}] - [x_{i1} \otimes y_{i1}]
\]
\[= [x_i \otimes y_i] - [x_i \otimes y_i] = 0 \]

and therefore
\[
0 = \langle T(x_{ii} - x_{i1}), y_{ii} + y_{i1} \rangle
\]
\[= \langle Tx_{ii}, y_{ii} \rangle - \langle Tx_{i1}, y_{i1} \rangle + \langle Tx_{ii}, y_{i1} \rangle - \langle Tx_{i1}, y_{i1} \rangle. \]

The last two terms are zero because $[x_{ii} \otimes y_{i1}] = [x_{i1} \otimes y_{i1}] = 0$ and we conclude that $\langle Tx_{ii}, y_{i1} \rangle = \langle Tx_{i1}, y_{i1} \rangle$. Therefore (4) can now be written as $\sum_{i=1}^{n} \langle Tx_{ii}, y_{i1} \rangle = 0$. We now let $\varepsilon$ approach zero. We have $\lim_{\varepsilon \to 0} x_{ii}(\varepsilon) = x_i, \lim_{\varepsilon \to 0} y_{i1}(\varepsilon) = y_i$ so that
\[
\sum_{i=1}^{n} \langle Tx_{i1}(\varepsilon), y_{i1}(\varepsilon) \rangle = \lim_{\varepsilon \to 0} \sum_{i=1}^{n} \langle Tx_{i1}(\varepsilon), y_{i1}(\varepsilon) \rangle = 0
\]

and the reflexivity of $\mathcal{M}$ is proved by Lemma 5. The last statement of the theorem follows from [8, Theorem 2.3] (cf. also [7]).

We conclude with a condition implying property $(A_n)$ and which is sometimes easier to verify. For an arbitrary linear subspace $\mathcal{M}$ of $\mathcal{L}(\mathcal{H})$ we will denote by $\mathcal{M}_*$ the Banach space of all ultraweakly continuous functionals on $\mathcal{M}$. It is well known that the dual space of $\mathcal{M}_*$ coincides with the ultraweak closure of $\mathcal{M}_*$; we will not use this fact here. The following two definitions were given in [1] for ultraweakly closed algebras $\mathcal{M}$ (cf. [1, Definitions 1.4 and 1.5]).

**Definition 7.** Suppose $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ is a linear subspace and $0 \leq \theta < +\infty$. We denote by $X_{\theta}(\mathcal{M})$ the set of all $\phi$ in $\mathcal{M}_*$ such that there exist sequences $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ in $\mathcal{H}$ satisfying the following conditions:
\[
\|x_i\| \leq 1, \quad \|y_i\| \leq 1, \quad 1 \leq i \leq \infty,
\]
\[
\limsup_{i \to \infty} \|\phi - [x_i \otimes y_i]\| \leq \theta,
\]
and
\[
\lim_{i \to \infty} \left( \|x_i \otimes z\| + \|z \otimes x_i\| + \|y_i \otimes z\| + \|z \otimes y_i\| \right) = 0, \quad z \in \mathcal{H}.
\]

**Definition 8.** Suppose $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ is a linear subspace and $0 \leq \theta < \gamma < +\infty$. We say that $\mathcal{M}$ has property $X_{\theta, \gamma}$ if the closed absolutely convex hull of the set $X_{\theta}(\mathcal{M})$ contains the closed ball of radius $\gamma$ centered at the origin in $\mathcal{M}_*$:
\[
\overline{\text{aco} \ X_{\theta}(\mathcal{M})} \supset \{ \phi \in \mathcal{M}_* : \|\phi\| \leq \gamma \}.
\]

The following result coincides with [1, Theorem 1.9] if $\mathcal{M}$ is an ultraweakly closed algebra. However, neither the algebra structure, nor the ultraweak closedness of $\mathcal{M}$ has been used in the proof of that theorem, so that we refer to [1] for the proof.

**Theorem 9.** Suppose $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ is a linear subspace with property $X_{\theta, \gamma}$ for some $\gamma > \theta \geq 0$. Then for every $\phi \in \mathcal{M}_*$ there exist sequences $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ in $\mathcal{H}$ such that
\[
\phi = [x_i \otimes y_i], \quad 1 \leq i < \infty,
\]
\[
\limsup_{i \to \infty} \|x_i\| \leq (\gamma - \theta)^{-1/2} \|\phi\|^{1/2}, \quad \limsup_{i \to \infty} \|y_i\| \leq (\gamma - \theta)^{-1/2} \|\phi\|^{1/2},
\]
and

\[
\lim_{i \to \infty} \left( \| [x_i \otimes z] \| + \| [z \otimes x_i] \| + \| [y_i \otimes z] \| + \| [z \otimes y_i] \| \right) = 0, \quad z \in \mathcal{H}.
\]

It was seen in [1] that this theorem implies that \( \mathcal{M} \) has property \((A_n)\) for each \( n \); we recall that property \((A_n)\) requires the solvability for \( x_i \) and \( y_i \) of arbitrary systems of the form \( [x_i \otimes y_j] = \phi_{ij}, \phi_{ij} \in \mathcal{M}, \quad 1 \leq i, j \leq n \). In order to prove the stronger property \((A_n^*)\) we need the following lemma, whose proof is reminiscent of the techniques of Robel [9].

**Lemma 10.** Suppose \( \mathcal{M} \subset \mathcal{L}(\mathcal{H}) \) is a linear subspace with property \( X_\gamma, \gamma \) for some \( \gamma > 0 \). If \( n \) is a natural number, \( a > 0, \varepsilon > 0 \), and \( \phi_{ij} \in \mathcal{M}, \quad x_i, y_j \in \mathcal{H}, \quad 1 \leq i, j \leq n \), are such that

\[
\| \phi_{ij} - [x_i \otimes y_j] \| < a, \quad 1 \leq i, j \leq n,
\]

then there exist \( \{ x'_i, y'_j : 1 \leq i, j \leq n \} \) in \( \mathcal{H} \) such that

\[
\| \phi_{ij} - [x'_i \otimes y'_j] \| < \varepsilon, \quad 1 \leq i, j \leq n,
\]

and

\[
\| x_i - x'_i \| < n(\gamma - \theta)^{-1/2} a^{1/2}, \quad \| y_j - y'_j \| < n(\gamma - \theta)^{-1/2} a^{1/2}, \quad 1 \leq i, j \leq n.
\]

**Proof.** Let \( \delta > 0 \) be such that \( (n^2 + 2n - 1)\delta < \varepsilon \). An application of Theorem 9 to \( \phi = \phi_{ij} - [x_i \otimes y_j] \) yields sequences \( \{ \xi_{ij}(k) \}_{k=1}^\infty, \{ \eta_{ij}(k) \}_{k=1}^\infty \) such that

\[
\phi_{ij} - [x_i \otimes y_j] = [\xi_{ij}(k) \otimes \eta_{ij}(k)], \quad 1 \leq k < \infty,
\]

\[
\| \xi_{ij}(k) \| < (\gamma - \theta)^{-1/2} a^{1/2}, \quad \| \eta_{ij}(k) \| < (\gamma - \theta)^{-1/2} a^{1/2}, \quad 1 \leq k < \infty,
\]

and

\[
\lim_{k \to \infty} \left( \| [\xi_{ij}(k) \otimes z] \| + \| [z \otimes \eta_{ij}(k)] \| \right) = 0, \quad z \in \mathcal{H}.
\]

An easy induction using (8) shows that we can find natural numbers \( k_{ij}, 1 \leq i, j \leq n \), such that the vectors \( \xi_{ij} = \xi_{ij}(k_{ij}) \) and \( \eta_{ij} = \eta_{ij}(k_{ij}) \) satisfy the inequalities

\[
\begin{align*}
\| [\xi_{ij} \otimes \eta_{kl}] \| &< \delta \quad \text{if} \quad (i, j) \neq (k, l), \\
\| [x_i \otimes \eta_{kl}] \| &< \delta, \quad 1 \leq i, k, l \leq n, \\
\| [\xi_{ij} \otimes y_k] \| &< \delta, \quad 1 \leq i, j, k \leq n.
\end{align*}
\]

We can now set

\[
x'_i = x_i + \sum_{k=1}^n \xi_{ik}, \quad y'_j = y_j + \sum_{l=1}^n \eta_{lj}
\]

and note that we obviously have from (7)

\[
\| x'_i - x_i \| \leq \sum_{k=1}^n \| \xi_{ik} \| < n(\gamma - \theta)^{-1/2} a^{1/2}, \quad 1 \leq i \leq n.
\]
and similarly
\[ \| y_j - y_j' \| < n(\gamma - \theta)^{-1/2}a^{1/2}, \quad 1 \leq j \leq n. \]

Finally, we observe that
\[
\phi_{ij} - [x_i' \otimes y_j'] = \phi_{ij} - [x_i \otimes y_j] - [\xi_{ij} \otimes \eta_{ij}] - \sum_{i=1}^{n} [x_i \otimes \eta_{ij}]
- \sum_{k=1}^{n} [\xi_{ik} \otimes y_j] - \sum_{(i,k) \neq (j,j)} [\xi_{ik} \otimes \eta_{ij}]
\]
and we obtain, using (6) and (9),
\[
\| \phi_{ij} - [x_i' \otimes y_j'] \| \leq n\delta + n\delta + (n^2 - 1)\delta < \varepsilon.
\]
The lemma follows.

A routine argument shows now that Lemma 10 is self-improving to yield the following result.

**Theorem 11.** Suppose \( \mathcal{M} \subset \mathcal{L}(\mathcal{H}) \) is a linear subspace with property \( X_{\gamma, \theta} \) for some \( \gamma > \theta > 0 \). If \( n \) is a natural number, \( a > 0 \) and \( \phi_{ij} \in \mathcal{M}, \ x_i, y_j \in \mathcal{H}, 1 \leq i, j \leq n, \) are such that
\[
\| x_i - x_i' \| < n(\gamma - \theta)^{-1/2}a^{1/2}, \quad \| y_j - y_j' \| < n(\gamma - \theta)^{-1/2}a^{1/2}, \quad 1 \leq i, j \leq n.
\]

**Proof.** Choose a positive number \( b \) such that
\[
\| \phi_{ij} - [x_i \otimes y_j] \| < b < a, \quad 1 \leq i, j \leq n,
\]
and let \( \varepsilon \) be a positive number to be specified later (\( \varepsilon \) will only depend on \( a \) and \( b \)). By Lemma 10, we can find vectors \( \{ x_i', y_j' \} : 1 \leq i, j \leq n \) in \( \mathcal{H} \) such that
\[
\| \phi_{ij} - [x_i' \otimes y_j'] \| < \varepsilon, \quad 1 \leq i, j \leq n,
\]
and
\[
\| x_i - x_i' \| < n(\gamma - \theta)^{-1/2}b^{1/2}, \quad \| y_j - y_j' \| < n(\gamma - \theta)^{-1/2}b^{1/2}.
\]
We can then use Lemma 10 to construct inductively sequences \( \{ x_i^k \}_{k=2}^{\infty}, \{ y_j^k \}_{k=2}^{\infty}, \)
\[
1 \leq i, j \leq n, \quad 1 \leq k < \infty,
\]
and
\[
\| x_i^{k+1} - x_i^k \| < n(\gamma - \theta)^{-1/2}b^{k/2}, \quad \| y_j^{k+1} - y_j^k \| < n(\gamma - \theta)^{-1/2}b^{k/2},
\]
\[
1 \leq i, j \leq n, \quad 1 \leq k < \infty.
\]
It is obvious that the sequences \( \{ x_i^k \}_{k=1}^\infty \) and \( \{ y_j^k \}_{k=1}^\infty \), \( 1 \leq i, j \leq n \), are Cauchy and \( \phi_{i,j} = [x_i^j \otimes y_j^j] \), \( 1 \leq i, j \leq n \), if
\[
x_i^j = \lim_{k \to \infty} x_i^k, \quad y_j^j = \lim_{k \to \infty} y_j^k, \quad 1 \leq i, j \leq n.
\]

Finally,
\[
\| x_i^j - x_i \| \leq \| x_i^j - x_i \| + \sum_{k=1}^\infty \| x_i^{k+1} - x_i^k \|
\]
\[
< n(\gamma - \theta)^{-1/2}(b^{1/2} + \sum_{k=1}^\infty e^{k/2})
\]
\[
= n(\gamma - \theta)^{-1/2}(b^{1/2} + e^{1/2}(1 - e^{1/2})^{-1}), \quad 1 \leq i \leq n,
\]
and analogously
\[
\| y_j^j - y_j \| < n(\gamma - \theta)^{-1/2}(b^{1/2} + e^{1/2}(1 - e^{1/2})^{-1}), \quad 1 \leq j \leq n.
\]

It suffices therefore to choose \( \varepsilon \) so small that \( b^{1/2} + e^{1/2}(1 - e^{1/2})^{-1} < a^{1/2} \). The theorem is proved.

We are now able to prove the promised criterion.

**Corollary 12.** Suppose \( M \subset \mathcal{L}(H) \) is a linear subspace with property \( X_{\theta,\gamma} \) for some \( \gamma > \theta > 0 \). Then \( M \) has property \( (A_n^-) \) for every natural number \( n \). In particular
the ultraweak closure \( M^- \) of \( M \) is weakly closed and reflexive.

**Proof.** The last part of the statement follows from the first part, combined with Lemma 3 and Corollary 4. To prove the first part we only have to use Theorem 11. Observe that we can take \( \delta(\varepsilon, n) = \varepsilon^2 n^{-2}(\gamma - \theta) \).

We finally note that one could give a definition analogous to Definition 8, in which the space \( M_* \) is replaced by the set \( M_- \) of all weakly continuous functionals on \( M \). The property thus defined would however be stronger than \( X_{\theta,\gamma} \) since \( M_* \) coincides with the norm closure of \( M_- \); this is why we restricted ourselves to the space \( M_* \) and the properties \( (A_n^-) \). We do not know whether the weaker properties \( (A_n^-) \) imply reflexivity. Property \( (A_1^-) \) alone does not imply reflexivity. Indeed, the algebra \( M \) of \( 2 \times 2 \) matrices defined as
\[
M = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{C} \right\}
\]
is not reflexive, but it has property \( (A_1^-) \) (and even \( (A_1^-) \), as can be seen by an easy computation).

**References**


Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Current address: Mathematical Sciences Research Center, 2223 Fulton Street, Berkeley, California 94720