A REFLEXIVITY THEOREM FOR WEAKLY CLOSED
SUBSPACES OF OPERATORS

BY

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Abstract. It was proved in [4] that the ultraweakly closed algebras generated by
certain contractions on Hilbert space have a remarkable property. This property, in
conjunction with the fact that these algebras are isomorphic to $H^\infty$, was used in [3]
to show that such ultraweakly closed algebras are reflexive. In the present paper we
prove an analogous result that does not require isomorphism with $H^\infty$, and applies
even to linear spaces of operators. Our result contains the reflexivity theorems of [3, 
2 and 9] as particular cases.

Let $L(\mathcal{H})$ denote the algebra of (linear, bounded) operators acting on the Hilbert
space $\mathcal{H}$, and let $\mathcal{M}$ denote a linear subspace of $L(\mathcal{H})$. Then $\mathcal{M}$ is endowed with the
weak and ultraweak topologies that it inherits from $L(\mathcal{H})$ (cf. [6, §15]). For two
arbitrary vectors $x, y \in \mathcal{H}$ we can define the (ultra) weakly continuous functional 
$x \otimes y$ on $\mathcal{M}$ by

$$ [x \otimes y](A) = \langle Ax, y \rangle, \quad A \in \mathcal{M}, $$

where $\langle \cdot, \cdot \rangle$ stands for the scalar product in $\mathcal{H}$.

Definition 1. Let $n$ be a natural number, $n \geq 1$. The subspace $\mathcal{M}$ has property 
$(B^n)$ [respectively $(A^n)$] if for every positive number $\varepsilon$ there exists a positive number 
$\delta = \delta(\varepsilon, n)$ such that for every system $\{\phi_{ij}; 1 \leq i, j \leq n\}$ of weakly [respectively 
ultraweakly] continuous functionals on $\mathcal{M}$ and every system $\{x_i, y_j; 1 \leq i, j \leq n\}$ of 
vectors in $\mathcal{H}$ satisfying the inequalities $\|\phi_{ij} - [x_i \otimes y_j]\| < \delta$ there exist vectors 
$\{x_i', y_j'; 1 \leq i, j \leq n\}$ in $\mathcal{H}$ such that

$$ \phi_{ij} = [x_i' \otimes y_j'], \quad 1 \leq i, j \leq n, $$

and

$$ \|x_i - x_i'\| < \varepsilon, \quad \|y_j - y_j'\| < \varepsilon, \quad 1 \leq i, j \leq n. $$

Since every weakly continuous functional on $\mathcal{M}$ is also ultraweakly continuous,
property $(B^n)$ is weaker than $(A^n)$ (ADDED IN PROOF. It was pointed out by C. 
Apostol that $(B^n)$ and $(A^n)$ are in fact equivalent. This fact is not used below.)

We recall now from [8] that a linear subspace $\mathcal{M}$ of $L(\mathcal{H})$ is said to be reflexive if
it contains every operator $T \in L(\mathcal{H})$ with the property that $Tx \in (\mathcal{M}x)^-$ for every

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x ∈ ℋ. Of course, reflexive subspaces are weakly closed. This definition coincides with the usual definition (ℳ = Alg Lat ℳ) if ℳ is a subalgebra of ℒ(ℋ).

We state now the main result of this paper.

**Theorem 2.** Let ℳ be a weakly closed subspace of ℒ(ℋ). If ℳ has property (Bn) for every natural number n, then ℳ is reflexive. Moreover, every weakly closed subspace of ℳ is also reflexive.

Before going into the proof, we relate this result with the reflexivity theorem from [3]. It was proved in [4] that, if T is a (BCP)-operator, the ultraweakly closed algebra AT generated by T has property (A") for every n = 1, 2,... The reflexivity of AT follows then from Theorem 2 and the following lemma.

**Lemma 3.** Let ℳ be a linear subspace of ℒ(ℋ) having property (A`). Then the weak and ultraweak closures of ℳ coincide, and the weak and ultraweak topologies coincide on the weak closure of ℳ.

**Proof.** Since every ultraweakly continuous functional on ℳ extends continuously to the ultraweak closure of ℳ, there is no loss of generality in assuming that ℳ is ultraweakly closed. Let δ = δ(1, 1) be as in Definition 1, and let φ be an arbitrary ultraweakly continuous functional on ℳ. Then ||δφ/(2||φ||) - [0 ® 0|| < δ so that we can find vectors x' and y' such that ||x'|| < 1, ||y'|| < 1 and δφ/(2||φ||) = [x' ® y'] or, equivalently, φ = [x ® y] with x = (2||φ||/δ)1/2x', y = (2||φ||/δ)1/2y'. Thus we can write φ as [x ® y] with ||x|| < (2/δ)1/2||φ||1/2, ||y|| < (2/δ)1/2||φ||1/2. We can now apply, e.g., the proof of [3, Theorem 1] to conclude that ℳ is weakly closed and the weak and ultraweak topologies coincide on ℳ.

We have therefore the following consequence of Theorem 2, which also implies the reflexivity results of [2 and 9].

**Corollary 4.** Let ℳ be an ultraweakly closed subspace of ℒ(ℋ). If ℳ has property (B") for every natural number n, then ℳ is weakly closed and reflexive. Moreover, every weakly closed subspace of ℳ is also reflexive.

For the proof of Theorem 2, we need two lemmas. The first was proved in [3] for the case in which ℳ is a weakly closed algebra. The proof for linear subspaces of ℒ(ℋ) is identical (and easy) so we content ourselves with the statement.

**Lemma 5.** Let ℳ be a linear subspace of ℒ(ℋ). An operator T ∈ ℒ(ℋ) is in the weak closure of ℳ if and only if for every natural n and every system {xₙ, yₙ: 1 ≤ i ≤ n} of vectors in ℋ such that \( Σₙ=xₙ ® yₙ = 0 \), we have \( Σₓₙ=1(Txₙ, yₙ) = 0 \).

**Lemma 6.** Let ℳ be a linear subspace of ℒ(ℋ). Assume that ℳ has property (B") for every natural number n. Then for every natural number n, every system \{xₙ, yₙ: 1 ≤ i ≤ n\} of vectors in ℋ and every ε > 0, there exist vectors \{xᵢⱼ, yᵢⱼ: 1 ≤ i, j ≤ n\} such that \( [xᵢⱼ ® yᵢⱼ] = δᵢⱼ[xᵢ ® yᵢ], \) 1 ≤ i, j, k, l ≤ n, and \( ||xᵢ - xᵢⱼ|| < ε, ||yⱼ - yᵢⱼ|| < ε, \) 1 ≤ i, j ≤ n. (Here \( δᵢⱼ \) denotes, as usual, Kronecker's symbol.)
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PROOF. Let $\delta = \delta(\epsilon, n^2)$ be as in Definition 1. Set $$\eta = \min\{\delta/(2||x_i \otimes y_k||): [x_i \otimes y_k] \neq 0\}$$ and define $\phi_{i,j,k,l} = 0$ if $j \neq l$, $\phi_{i,k} = [x_i \otimes y_k]$, $\phi_{i,j} = \eta[x_i \otimes y_k]$, $j \geq 2$. The vectors \(\{x_{ij}^0, y_{ij}^0: 1 \leq i, j \leq n\}\) defined by $x_{ij}^0 = \delta_{ij}x_i$, $y_{ij}^0 = \delta_{ij}y_i$, $1 \leq i, j \leq n$, obviously satisfy the inequalities $$||\phi_{ij,k,l} - \frac{x_{ij}^0 \otimes y_{ij}^0}{\phi_{ij,k,l}}|| < \delta, \quad 0 \leq i, j, k, l \leq n,$$ and therefore, by property (B,\(n^2\)), we can find vectors \(\{x_{ij}^0, y_{ij}^0: 1 \leq i, j \leq n\}\) in \(\mathcal{H}\) such that $$\phi_{ij,k,l} = \frac{x_{ij}^0 \otimes y_{ij}^0}{\phi_{ij,k,l}}, \quad 0 \leq i, j, k, l \leq n,$$ and $||x_{ij}^0 - x_{ij}|| < \epsilon$, $||y_{ij}^0 - y_{ij}|| < \epsilon$, $0 \leq i, j \leq n$. Then the vectors \(\{x_{ij}, y_{ij}: 1 \leq i, j \leq n\}\) defined by $x_{ij} = x_{ij}^0$, $y_{ij} = y_{ij}^0$, $x_{ij} = \eta^{-1/2}x_{ij}^0$, $y_{ij} = \eta^{-1/2}y_{ij}^0$, $1 \leq i \leq n, 2 \leq j \leq n$, satisfy the requirements of the lemma.

PROOF OF THEOREM 2. Let $T \in \mathcal{L}(\mathcal{H})$ satisfy the property that $Tx \in (\mathcal{M}x)^{-}$ for every $x \in \mathcal{H}$. We first note that the equality $[x \otimes y] = 0$, $x, y \in \mathcal{H}$, means that $y$ is orthogonal to $(\mathcal{M}x)^{-}$, and hence it implies $\langle Tx, y \rangle = 0$.

In order to show that $T \in \mathcal{M}$, we must prove, according to Lemma 5, that the equality $\sum_{i=1}^{n}[x_i \otimes y_i] = 0$, $x_i, y_i \in \mathcal{H}$, $1 \leq i \leq n$, implies $\sum_{i=1}^{n}\langle Tx_i, y_i \rangle = 0$. By what has just been said, this property is satisfied for $n = 1$. Assume therefore that $n \geq 2$, $x_i, y_i \in \mathcal{H}$, $1 \leq i \leq n$, and $\sum_{i=1}^{n}[x_i \otimes y_i] = 0$. For every $\epsilon > 0$ we can find, using Lemma 6, vectors $x_{ij} = x_{ij}(\epsilon), y_{ij} = y_{ij}(\epsilon), 0 \leq i, j \leq n$, satisfying

1. $[x_{ij} \otimes y_{kl}] = \delta_{ij}[x_i \otimes y_k]$, $1 \leq i, j, k, l \leq n,$

and

2. $||x_i - x_{ii}|| = ||x_i - x_{ii}(\epsilon)|| < \epsilon$, $1 \leq i, j \leq n.$

We now remark that by (1)

$$\sum_{i=1}^{n}x_{ii} \otimes \sum_{i=1}^{n}y_{ii} = \sum_{i=1}^{n}[x_{ii} \otimes y_{ii}] + \sum_{i \neq j}[x_{ii} \otimes y_{jj}]$$

and therefore

$$\sum_{i=1}^{n}[x_{ii} \otimes y_{ii}] = 0$$

and therefore

$$\langle T\left(\sum_{i=1}^{n}x_{ii}\right) , \sum_{i=1}^{n}y_{ii} \rangle = 0.$$ 

Since $[x_{ii} \otimes y_{jj}] = 0$ for $i \neq j$, we also have $\langle Tx_{ii}, y_{jj} \rangle = 0$ for $i \neq j$ so that (3) can be rewritten as

$$\sum_{i=1}^{n}\langle Tx_{ii}, y_{ii} \rangle = 0.$$
Assume now that \( i \neq 1 \). We have by (1)
\[
\left( (x_{ii} - x_{i1}) \otimes (y_{ii} + y_{i1}) \right) = \left[ x_{ii} \otimes y_{ii} \right] - \left[ x_{i1} \otimes y_{i1} \right] + \left[ x_{ii} \otimes y_{i1} \right] - \left[ x_{i1} \otimes y_{ii} \right]
\]
and therefore
\[
0 = \left< T(x_{ii} - x_{i1}), y_{ii} + y_{i1} \right> = \left< Tx_{ii}, y_{ii} \right> - \left< Tx_{i1}, y_{i1} \right> + \left< Tx_{ii}, y_{i1} \right> - \left< Tx_{i1}, y_{ii} \right>.
\]
The last two terms are zero because \( [x_{ii} \otimes y_{i1}] = [x_{i1} \otimes y_{ii}] = 0 \) and we conclude that \( \left< Tx_{ii}, y_{i1} \right> = \left< Tx_{i1}, y_{ii} \right> \). Therefore (4) can now be written as \( \sum_{i=1}^{n} \left< Tx_{i1}, y_{ii} \right> = 0 \). We now let \( \epsilon \) approach zero. We have \( \lim_{\epsilon \to 0} x_{i1}(\epsilon) = x_{i1}, \lim_{\epsilon \to 0} y_{i1}(\epsilon) = y_{i1} \) so that
\[
\sum_{i=1}^{n} \left< Tx_{i1}(\epsilon), y_{ii}(\epsilon) \right> = \lim_{\epsilon \to 0} \sum_{i=1}^{n} \left< Tx_{i1}(\epsilon), y_{ii}(\epsilon) \right> = 0
\]
and the reflexivity of \( \mathcal{M} \) is proved by Lemma 5. The last statement of the theorem follows from [8, Theorem 2.3] (cf. also [7]).

We conclude with a condition implying property \((A_{\infty}')\) and which is sometimes easier to verify. For an arbitrary linear subspace \( \mathcal{M} \) of \( \mathcal{L}(\mathcal{H}) \) we will denote by \( \mathcal{M}_{*} \) the Banach space of all ultraweakly continuous functionals on \( \mathcal{M} \). It is well known that the dual space of \( \mathcal{M}_{*} \) coincides with the ultraweak closure of \( \mathcal{M} \); we will not use this fact here. The following two definitions were given in [1] for ultraweakly closed algebras \( \mathcal{M} \) (cf. [1, Definitions 1.4 and 1.5]).

**Definition 7.** Suppose \( \mathcal{M} \subset \mathcal{L}(\mathcal{H}) \) is a linear subspace and \( 0 \leq \theta < +\infty \). We denote by \( X_{\theta}(\mathcal{M}) \) the set of all \( \phi \in \mathcal{M}_{*} \) such that there exist sequences \( \{ x_{i} \}_{i=1}^{\infty} \) and \( \{ y_{i} \}_{i=1}^{\infty} \) in \( \mathcal{H} \) satisfying the following conditions:
\[
\|x_{i}\| \leq 1, \quad \|y_{i}\| \leq 1, \quad 1 \leq i \leq \infty,
\]
\[
\limsup_{i \to \infty} \|\phi - [x_{i} \otimes y_{i}]\| \leq \theta,
\]
and
\[
\lim_{i \to \infty} \left( \|x_{i} \otimes z\| + \|z \otimes x_{i}\| + \|y_{i} \otimes z\| + \|z \otimes y_{i}\| \right) = 0, \quad z \in \mathcal{H}.
\]

**Definition 8.** Suppose \( \mathcal{M} \subset \mathcal{L}(\mathcal{H}) \) is a linear subspace and \( 0 \leq \theta < \gamma < +\infty \). We say that \( \mathcal{M} \) has property \( X_{\theta,\gamma} \) if the closed absolutely convex hull of the set \( X_{\theta}(\mathcal{M}) \) contains the closed ball of radius \( \gamma \) centered at the origin in \( \mathcal{M}_{*} \):
\[
\overline{\text{aco}} \ X_{\theta}(\mathcal{M}) \supset \{ \phi \in \mathcal{M}_{*}: \|\phi\| \leq \gamma \}.
\]

The following result coincides with [1, Theorem 1.9] if \( \mathcal{M} \) is an ultraweakly closed algebra. However, neither the algebra structure, nor the ultraweak closedness of \( \mathcal{M} \) has been used in the proof of that theorem, so that we refer to [1] for the proof.

**Theorem 9.** Suppose \( \mathcal{M} \subset \mathcal{L}(\mathcal{H}) \) is a linear subspace with property \( X_{\theta,\gamma} \) for some \( \gamma > \theta \geq 0 \). Then for every \( \phi \in \mathcal{M}_{*} \) there exist sequences \( \{ x_{i} \}_{i=1}^{\infty} \) and \( \{ y_{i} \}_{i=1}^{\infty} \) in \( \mathcal{H} \) such that
\[
\phi = [x_{i} \otimes y_{i}], \quad 1 \leq i < \infty,
\]
\[
\limsup_{i \to \infty} \|x_{i}\| \leq (\gamma - \theta)^{-1/2}\|\phi\|^{1/2}, \quad \limsup_{i \to \infty} \|y_{i}\| \leq (\gamma - \theta)^{-1/2}\|\phi\|^{1/2},
\]
and
\[
\lim_{i \to \infty} \left( \left\| x_i \otimes z \right\| + \left\| z \otimes x_i \right\| + \left\| y_j \otimes z \right\| + \left\| z \otimes y_j \right\| \right) = 0, \quad z \in \mathcal{H}.
\]

It was seen in [1] that this theorem implies that \( \mathcal{M} \) has property \( (A_n) \) for each \( n \); we recall that property \( (A_n) \) requires the solvability for \( x_i \) and \( y_j \) of arbitrary systems of the form \( [x_i \otimes y_j] = \phi_{ij}, \phi_{ij} \in \mathcal{M}_*, 1 \leq i, j \leq n \). In order to prove the stronger property \( (A_n) \) we need the following lemma, whose proof is reminiscent of the techniques of Robel [9].

**Lemma 10.** Suppose \( \mathcal{M} \subset \mathcal{L}(\mathcal{H}) \) is a linear subspace with property \( X_{\theta, \gamma} \) for some \( \gamma > \theta > 0 \). If \( n \) is a natural number, \( a > 0, \epsilon > 0 \), and \( \phi_{ij} \in \mathcal{M}_*, x_i, y_j \in \mathcal{H}, 1 \leq i, j \leq n \), are such that
\[
\left\| \phi_{ij} - [x_i \otimes y_j] \right\| < a, \quad 1 \leq i, j \leq n,
\]
then there exist \( \{x'_i, y'_j; 1 \leq i, j \leq n\} \) in \( \mathcal{H} \) such that
\[
\left\| \phi_{ij} - [x'_i \otimes y'_j] \right\| < \epsilon, \quad 1 \leq i, j \leq n,
\]
and
\[
\left\| x_i - x'_i \right\| < n(\gamma - \theta)^{-1/2} a^{1/2}, \quad \left\| y_j - y'_j \right\| < n(\gamma - \theta)^{-1/2} a^{1/2}, \quad 1 \leq i, j \leq n.
\]

**Proof.** Let \( \delta > 0 \) be such that \((n^2 + 2n - 1)\delta < \epsilon\). An application of Theorem 9 to \( \phi = \phi_{ij} - [x_i \otimes y_j] \) yields sequences \( \{\xi_{ij}(k)\}_{k=1}^{\infty}, \{\eta_{ij}(k)\}_{k=1}^{\infty} \) such that
\[
\phi_{ij} - [x_i \otimes y_j] = \left[ \xi_{ij}(k) \otimes \eta_{ij}(k) \right], \quad 1 \leq k < \infty,
\]
\[
\left\| \xi_{ij}(k) \right\| < (\gamma - \theta)^{-1/2} a^{1/2}, \quad \left\| \eta_{ij}(k) \right\| < (\gamma - \theta)^{-1/2} a^{1/2}, \quad 1 \leq k < \infty,
\]
and
\[
\lim_{k \to \infty} \left( \left\| \xi_{ij}(k) \otimes z \right\| + \left\| z \otimes \eta_{ij}(k) \right\| \right) = 0, \quad z \in \mathcal{H}.
\]
An easy induction using (8) shows that we can find natural numbers \( k_{ij}, 1 \leq i, j \leq n \), such that the vectors \( \xi_{ij} = \xi_{ij}(k_{ij}) \) and \( \eta_{ij} = \eta_{ij}(k_{ij}) \) satisfy the inequalities
\[
\left\| [\xi_{ij} \otimes \eta_{ij}] \right\| < \delta \quad \text{if } (i, j) \neq (k, l),
\]
\[
\left\| [x_i \otimes \eta_{ij}] \right\| < \delta, \quad 1 \leq i, k, l \leq n,
\]
\[
\left\| [\xi_{ij} \otimes y_k] \right\| < \delta, \quad 1 \leq i, j, k \leq n.
\]
We can now set
\[
x'_i = x_i + \sum_{k=1}^{n} \xi_{ik}, \quad y'_j = y_j + \sum_{l=1}^{n} \eta_{lj}
\]
and note that we obviously have from (7)
\[
\left\| x'_i - x_i \right\| \leq \sum_{k=1}^{n} \left\| \xi_{ik} \right\| < n(\gamma - \theta)^{-1/2} a^{1/2}, \quad 1 \leq i \leq n.
\]
and similarly
\[ \|y'_j - y_j\| < n(\gamma - \theta)^{-1/2}a^{1/2}, \quad 1 \leq j \leq n. \]

Finally, we observe that
\[
\begin{align*}
\phi_{ij} - [x'_i \otimes y'_j] = & \phi_{ij} - [x_i \otimes y_j] - [\xi_{ij} \otimes \eta_{ij}] - \sum_{l=1}^{n} [x_i \otimes \eta_{lj}] \\
& - \sum_{k=1}^{n} [\xi_{ik} \otimes y_j] - \sum_{(l, k) \neq (i, j)} [\xi_{ik} \otimes \eta_{lj}]
\end{align*}
\]
and we obtain, using (6) and (9),
\[ \|\phi_{ij} - [x'_i \otimes y'_j]\| \leq n\delta + n\delta + (n^2 - 1)\delta < \epsilon. \]

The lemma follows.

A routine argument shows now that Lemma 10 is self-improving to yield the following result.

**Theorem 11.** Suppose \(M \subset \mathcal{L}(H)\) is a linear subspace with property \(X_{\gamma, \theta}\) for some \(\gamma > \theta > 0\). If \(n\) is a natural number, \(a > 0\) and \(\phi_{ij} \in M, x_i, y_j \in H, 1 \leq i, j \leq n,\) are such that
\[ \|x_i - x'_i\| < n(\gamma - \theta)^{-1/2}a^{1/2}, \quad \|y_j - y'_j\| < n(\gamma - \theta)^{-1/2}a^{1/2}, \quad 1 \leq i, j \leq n. \]

**Proof.** Choose a positive number \(b\) such that
\[ \|\phi_{ij} - [x_i \otimes y_j]\| < b < a, \quad 1 \leq i, j \leq n, \]
and let \(\epsilon\) be a positive number to be specified later (\(\epsilon\) will only depend on \(a\) and \(b\)). By Lemma 10, we can find vectors \(\{x'_i, y'_j: 1 \leq i, j \leq n\}\) in \(H\) such that
\[ \|x_i - x'_i\| < n(\gamma - \theta)^{-1/2}a^{1/2}, \quad \|y_j - y'_j\| < n(\gamma - \theta)^{-1/2}a^{1/2}, \quad 1 \leq i, j \leq n. \]

and
\[ \|x'_i - x_i\| < n(\gamma - \theta)^{-1/2}b^{1/2}, \quad \|y'_j - y_j\| < n(\gamma - \theta)^{-1/2}b^{1/2}. \]

We can then use Lemma 10 to construct inductively sequences \(\{x^k_i\}_{k=2}^{\infty}, \{y^k_j\}_{k=2}^{\infty}, 1 \leq i, j \leq n,\) such that
\[ \|\phi_{ij} - [x^k_i \otimes y^k_j]\| < \epsilon^k, \quad 1 \leq i, j \leq n, 2 \leq k < \infty, \]
and
\[ \|x^k_{i+1} - x^k_i\| < n(\gamma - \theta)^{-1/2}\epsilon^{k/2}, \quad \|y^k_{j+1} - y^k_j\| < n(\gamma - \theta)^{-1/2}\epsilon^{k/2}, \quad 1 \leq i, j \leq n, 1 \leq k < \infty. \]
It is obvious that the sequences \( \{ x^k_i \}_{k=1}^{\infty} \) and \( \{ y^k_j \}_{k=1}^{\infty} \), \( 1 \leq i, j \leq n \), are Cauchy and 
\[ x^*_i = \lim_{k \to \infty} x^k_i, \quad y^*_j = \lim_{k \to \infty} y^k_j, \quad 1 \leq i, j \leq n. \]

Finally,
\[
\| x^*_i - x_i \| \leq \| x^*_i - x_j \| + \sum_{k=1}^{\infty} \| x^{k+1}_i - x^*_i \| < n(\gamma - \theta)^{-1/2} \left( b^{1/2} + \sum_{k=1}^{\infty} \epsilon_k^{1/2} \right)
\]
\[ = n(\gamma - \theta)^{-1/2} \left( b^{1/2} + \epsilon_1^{1/2} (1 - \epsilon_1^{1/2})^{-1} \right), \quad 1 \leq i \leq n, \]
and analogously
\[
\| y^*_j - y_j \| < n(\gamma - \theta)^{-1/2} \left( b^{1/2} + \epsilon_1^{1/2} (1 - \epsilon_1^{1/2})^{-1} \right), \quad 1 \leq j \leq n.
\]

It suffices therefore to choose \( \epsilon \) so small that \( b^{1/2} + \epsilon_1^{1/2} (1 - \epsilon_1^{1/2})^{-1} < a^{1/2} \). The theorem is proved.

We are now able to prove the promised criterion.

**Corollary 12.** Suppose \( \mathcal{M} \subset \mathcal{L}(\mathcal{H}) \) is a linear subspace with property \( x_{\theta, \gamma} \) for some \( \gamma > \theta > 0 \). Then \( \mathcal{M} \) has property (\( A_n^+ \)) for every natural number \( n \). In particular the ultraweak closure \( \mathcal{M}^- \) of \( \mathcal{M} \) is weakly closed and reflexive.

**Proof.** The last part of the statement follows from the first part, combined with Lemma 3 and Corollary 4. To prove the first part we only have to use Theorem 11. Observe that we can take \( \delta(\epsilon, n) = \epsilon^2 n^{-2}(\gamma - \theta) \).

We finally note that one could give a definition analogous to Definition 8, in which the space \( \mathcal{M}_* \) is replaced by the set \( \mathcal{M}_- \) of all weakly continuous functionals on \( \mathcal{M} \). The property thus defined would however be stronger than \( X_{\theta, \gamma} \) since \( \mathcal{M}_* \) coincides with the norm closure of \( \mathcal{M}_- \); this is why we restricted ourselves to the space \( \mathcal{M}_* \) and the properties (\( A_n^+ \)). We do not know whether the weaker properties (\( A_n^- \)) imply reflexivity. Property (\( A_1^- \)) alone does not imply reflexivity. Indeed, the algebra \( \mathcal{M} \) of 2 \( \times \) 2 matrices defined as
\[
\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{C} \right\}
\]
is not reflexive, but it has property (\( A_1^- \)) (and even (\( A_1^\infty \)), as can be seen by an easy computation).

**References**


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