BEST APPROXIMATION AND QUASITRIANGULAR ALGEBRAS

BY

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ABSTRACT. If $\mathcal{P}$ is a linearly ordered set of projections on a Hilbert space and $\mathcal{X}$ is the ideal of compact operators, then $\text{Alg } \mathcal{P} + \mathcal{X}$ is the quasitriangular algebra associated with $\mathcal{P}$. We study the problem of finding best approximants in a given quasitriangular algebra to a given operator: given $T$ and $\mathcal{P}$, is there an $A$ in $\text{Alg } \mathcal{P} + \mathcal{X}$ such that $\|T - A\| = \inf\{\|T - S\| : S \in \text{Alg } \mathcal{P} + \mathcal{X}\}$? We prove that if $\mathcal{A}$ is an operator subalgebra which is closed in the weak operator topology and satisfies a certain condition $\Delta$, then every operator $T$ has a best approximant in $\mathcal{A} + \mathcal{X}$. We also show that if $\mathcal{P}$ is an increasing sequence of finite rank projections converging strongly to the identity then $\text{Alg } \mathcal{P}$ satisfies the condition $\Delta$. Also, we show that if $T$ is not in $\text{Alg } \mathcal{P} + \mathcal{X}$ then the best approximants in $\text{Alg } \mathcal{P} + \mathcal{X}$ to $T$ are never unique.

1. Introduction. The concept of quasitriangular operators on a Hilbert space was introduced by Halmos in [5], where an operator $T$ is said to be quasitriangular if there is a sequence $\{E_n\}$ of finite rank projections strongly converging to the identity such that $\|(1 - E_n)TE_n\| \to 0$.

For a fixed increasing sequence $\{P_n\}$ of finite rank projections strongly converging to the identity, Arveson [2] defined the quasitriangular algebra $QT(\{P_n\})$ to be the set of all operators $T$ for which $\|(1 - P_n)TP_n\| \to 0$. He proved a distance formula for $QT(\{P_n\})$ and showed that $QT(\{P_n\}) = \text{Alg } \{P_n\} + \mathcal{X}$, where $\text{Alg } \{P_n\} = \{T : (1 - P_n)TP_n = 0 \text{ for all } n\}$ is the triangular algebra associated with $\{P_n\}$ and $\mathcal{X}$ is the ideal of compact operators.

For any linearly ordered set $\mathcal{P}$ of projections which is closed in the strong operator topology and contains 0 and 1, Fall, Arveson, and Muhly [4] showed that the algebra $\text{Alg } \mathcal{P} + \mathcal{X}$ is norm closed, where $\text{Alg } \mathcal{P}$ is the triangular algebra associated with $\mathcal{P}$, namely $\text{Alg } \mathcal{P} = \{T : (1 - P)TP = 0, \text{ all } P \in \mathcal{P}\}$. They also gave a characterization of $\text{Alg } \mathcal{P} + \mathcal{X}$ as a generalized quasitriangular algebra.

In this paper we study the problem of finding best quasitriangular approximants to a given operator: given an operator $T$ does there exist an operator $A$ in $\text{Alg } \mathcal{P} + \mathcal{X}$ for which $\|T - A\| = \inf\{\|T - S\| : S \in \text{Alg } \mathcal{P} + \mathcal{X}\}$? We prove that if $\mathcal{A}$ is an operator subalgebra which is closed in the weak operator topology and satisfies a certain condition $\Delta(\mathcal{A})$, then every operator $T$ has a best approximant in $\mathcal{A} + \mathcal{X}$.
We also show that if \( \{P_n\} \) is an increasing sequence of finite rank projections strongly converging to 1, then \( \text{Alg}(\{P_n\}) \) satisfies the condition \( \Delta(\text{Alg}(\{P_n\})) \). Hence, best approximants in \( \text{Alg}(\{P_n\}) + \mathcal{K} \) exist for every operator \( T \). Moreover, we show that if \( T \not\in \text{Alg}(\{P_n\}) + \mathcal{K} \), then such best approximants are never unique.

Some of our results are reminiscent of those proved in [3] by Axler, Berg, Jewell, and Shields, where it is shown, for example, that every \( L^\infty \) function on the unit circle has a best approximant in the algebra \( H^\infty + C \). In fact, the general approach to proving our main result is inspired by that paper.

2. Preliminaries. In what follows, \( H \) will be a separable infinite-dimensional Hilbert space with \( \mathcal{L}(H) \) denoting the algebra of all bounded linear operators on \( H \) and \( \mathcal{K}(H) \), or simply \( \mathcal{K} \), denoting the ideal of compact operators in \( \mathcal{L}(H) \). All subspaces of \( H \) are assumed to be closed and all projections are selfadjoint. For a projection \( P \) let \( P^\perp = 1 - P \).

If \( \mathcal{P} \) is any subset of \( \mathcal{L}(H) \) and \( T \in \mathcal{L}(H) \), then the distance of \( T \) from \( \mathcal{P} \) is given by \( d(T, \mathcal{P}) = \inf\{\|T - S\|: S \in \mathcal{P}\} \). Also, Lat \( \mathcal{P} \) will denote the set of all projections \( P \) for which \( PSP = SP \) whenever \( S \in \mathcal{P} \). If \( \mathcal{P} \) is a set of projections in \( \mathcal{L}(H) \), then \( \text{Alg} \mathcal{P} \) denotes the set of all operators \( T \in \mathcal{L}(H) \) for which \( PTP = TP \) whenever \( P \in \mathcal{P} \). A subalgebra \( \mathcal{A} \subset \mathcal{L}(H) \) is said to be reflexive if \( \text{Alg} \text{Lat} \mathcal{A} = \mathcal{A} \).

A nest is a family of projections which is linearly ordered by range inclusion, contains 0 and 1, and is closed in the strong operator topology (SOT). A nest algebra is a subalgebra \( \mathcal{A} \) of \( \mathcal{L}(H) \) for which \( \mathcal{A} = \text{Alg} \mathcal{P} \) for some nest \( \mathcal{P} \). Equivalently, it is not hard to see that a nest algebra is a reflexive algebra \( \mathcal{A} \) such that \( \text{Lat} \mathcal{A} \) is linearly ordered (cf. [9]).

In [2] Arveson established the following distance formula for a nest algebra \( \mathcal{A} \).

\[
(2.1) \quad d(T, \mathcal{A}) = \sup\{\|P^\perp TP\|: P \in \text{Lat} \mathcal{A}\} \quad \text{for } T \in \mathcal{L}(H).
\]

For a nest \( \mathcal{P} \) define the quasitriangular algebra associated with \( \mathcal{P} \) by \( \text{QT}(\mathcal{P}) = \text{Alg} \mathcal{P} + \mathcal{K}(H) \). In [4] Fall, Arveson, and Muhly showed that \( \text{QT}(\mathcal{P}) \) is a norm closed algebra and that

\[
\text{QT}(\mathcal{P}) = \{T \in \mathcal{L}(H): (i) \quad P^\perp TP \in \mathcal{K}(H), \text{ for all } P \in \mathcal{P},
\text{the map } P \mapsto P^\perp TP \text{ is continuous}
(ii) \text{with respect to the SOT on } \mathcal{P} \text{ and the norm topology on } \mathcal{K}(H)\}.
\]

In the case when \( \mathcal{P} = \{P_n\} \) is an increasing sequence of finite rank projections converging strongly to 1, this yields the definition of \( \text{QT}(\{P_n\}) \) given by Arveson in [2]. For this special case Arveson has established the following distance formula.

\[
(2.2) \quad d(T, \text{QT}(\{P_n\})) = \lim_{n \to \infty} \|P_n^\perp TP_n\|, \quad n \to \infty, \quad \text{for } T \in \mathcal{L}(H).
\]

In this case (2.1) can be written as

\[
(2.1') \quad d(T, \text{Alg}(\{P_n\})) = \sup\{\|P_n^\perp TP_n\|: \text{all } n\} \quad \text{for } T \in \mathcal{L}(H).
\]

We also need the following known result.
Lemma 2.3. If $\mathcal{A} \subset \mathcal{L}(H)$ is closed in the weak operator topology (WOT), then every $T$ in $\mathcal{L}(H)$ has a best approximant in $\mathcal{A}$.

Proof. The proof is a standard argument using the compactness, in the weak operator topology, of the closed unit ball in $\mathcal{L}(H)$. □

Finally, we observe that if $\mathcal{P}$ is a nest then $\text{Alg} \mathcal{P}$ is closed in the WOT. Indeed, if $\{A_\lambda\} \subset \text{Alg} \mathcal{P}$ is a net of operators such that $A_\lambda \to A$ (WOT), then, for each $P \in \mathcal{P}$, $0 = P^\perp A_\lambda P \to P^\perp AP$ (WOT), which implies that $A \in \text{Alg} \mathcal{P}$.

3. Main results.

Definition 3.1. A subalgebra $\mathcal{A}$ of $\mathcal{L}(H)$ satisfies condition $\Delta(\mathcal{A})$ provided that, for each $T \in \mathcal{L}(H)$, for each sequence of operators $\{A_n\} \subset \mathcal{L}(H)$ satisfying $A_n \to 0$ (SOT), and for each $\epsilon > 0$, there exists an $N$ such that

$$d(T + A_N, \mathcal{A}) \leq \epsilon + \max\{d(T, \mathcal{A}), d(T, \mathcal{A} + \mathcal{K}) + d(A_N, \mathcal{A})\}.$$ 

Two remarks are in order. First, if condition $\Delta(\mathcal{A})$ holds and $T$, $\{A_n\}$, and $\epsilon$ are chosen as indicated, then there exists an $N$ such that

$$d(T + \beta A_N, \mathcal{A}) \leq \epsilon + \max\{d(T, \mathcal{A}), d(T, \mathcal{A} + \mathcal{K}) + d(\beta A_N, \mathcal{A})\}$$

for all $\beta \in [0,1]$. Otherwise, for each $n$, take $\beta_n \in [0,1]$ such that the inequality fails for $\beta_n A_n$. The sequence $\{\beta_n A_n\}$ satisfies $\beta_n A_n \to 0$ (SOT), so the assumption that condition $\Delta(\mathcal{A})$ holds yields a contradiction. Secondly, for any fixed $M$, $N$ can be chosen so that $N > M$ by restricting attention to the sequence $\{A_n\}_{n \geq M}$.

The next result enables us to reduce the problem of finding best approximants in $\mathcal{A} + \mathcal{K}(H)$ to that of finding best approximants in $\mathcal{A}$.

Theorem 3.2. Let $\mathcal{A} \subset \mathcal{L}(H)$ be a subalgebra satisfying condition $\Delta(\mathcal{A})$. Choose $T \in \mathcal{L}(H) \setminus \mathcal{A} + \mathcal{K}$ and suppose the sequence $\{T_n\} \subset \mathcal{A} + \mathcal{K}$ satisfies $T_n \to T$ (SOT). Then there is a sequence $\{a_n\}$ of nonnegative real numbers satisfying $\sum a_n = 1$ and such that, if $K = \sum a_n T_n$, then $d(T - K, \mathcal{A}) = d(T, \mathcal{A} + \mathcal{K})$.

Proof. Let $A_n = T - T_n$ so that $A_n \to 0$ (SOT). For convenience let

$$r = d(T, \mathcal{A} + \mathcal{K}).$$

Claim. There exists an increasing sequence of positive integers $\{n(k)\}$ and a sequence $\{a_k\}$ of positive real numbers such that $\sum a_k = 1$ and such that, for all $N = 1, 2, \ldots,$

$$d\left(\sum_{k=1}^N a_k A_{n(k)}, \mathcal{A}\right) = r - \epsilon_N, \quad \text{where } \epsilon_N = r/3^N.$$

Proof of Claim. Choose $n(1) = 1$. Since $A_1 \notin \mathcal{A} + \mathcal{K}$, it follows that $d(A_1, \mathcal{A}) \neq 0$. Choose $a_1$ such that $a_1 \cdot d(A_1, \mathcal{A}) = r - \epsilon_1$. Since $a_1 \cdot d(A_1, \mathcal{A}) = d(a_1 A_1, \mathcal{A})$, it follows that $d(a_1 A_1, \mathcal{A}) = r - \epsilon_1$. The relations

$$d\left(A_1, \mathcal{A}\right) = d(T, \mathcal{A} + T_1) \geq d(T, \mathcal{A} + \mathcal{K})$$

imply $0 < a_1 < 1$.

Suppose $n(1), \ldots, n(N)$ and $a_1, \ldots, a_N$ have been chosen as required. Applying condition $\Delta(\mathcal{A})$ to the operator $\sum_{k=1}^N a_k A_{n(k)}$, the sequence $\{A_n\}$, and $\epsilon_{N+1}$, choose.
n(N + 1) > n(N) such that

\[ d \left( \sum_{k=1}^{N} \alpha_k A_n(k) + \beta A_{n(N+1)}, \mathcal{A} \right) \]

\[ \leq \varepsilon_{N+1} + \max \left( d \left( \sum_{k=1}^{N} \alpha_k A_n(k), \mathcal{A} \right), d \left( \sum_{k=1}^{N} \alpha_k A_n(k), \mathcal{A} + \mathcal{K} \right) \right) \]

\[ + d \left( \beta A_{n(N+1)}, \mathcal{A} \right) \]

for all \( \beta \in [0, 1] \).

Consider the quantity \( d(\sum_{k=1}^{N} \alpha_k A_n(k) + \alpha A_{n(N+1)}, \mathcal{A}) \) as a function of \( \alpha \). When \( \alpha = 0 \) this quantity equals \( r - \varepsilon_N \). Note that \( r - \varepsilon_N < r - \varepsilon_{N+1} \). As \( \alpha \to \infty \) this quantity also approaches \( \infty \). (Here we use the fact that \( A_k \) does not belong to \( \mathcal{A} \) for any \( k \).) Thus, there exists some value of \( \alpha \), call it \( \alpha_{N+1} \), for which

\[ d \left( \sum_{k=1}^{N} \alpha_k A_n(k) + \alpha_{N+1} \cdot A_{n(N+1)}, \mathcal{A} \right) = r - \varepsilon_{N+1}. \]

Note that

\[ r - \varepsilon_{N+1} = d \left( \sum_{k=1}^{N+1} \alpha_k A_n(k), \mathcal{A} \right) = d \left( \sum_{k=1}^{N+1} \alpha_k (T - T_n(k)), \mathcal{A} \right) \]

\[ \geq d \left( \sum_{k=1}^{N+1} \alpha_k T, \mathcal{A} + \mathcal{K} \right) \quad \text{since } T_n(k) \in \mathcal{A} + \mathcal{K} \]

\[ = \left( \sum_{k=1}^{N+1} \alpha_k \right) \cdot d(T, \mathcal{A} + \mathcal{K}) = \left( \sum_{k=1}^{N+1} \alpha_k \right) \cdot r \]

and, hence, \( \sum_{k=1}^{N+1} \alpha_k < 1 \). It remains to show that \( \sum_{k=1}^{N+1} \alpha_k = 1 \).

Referring to inequality (3.3), with \( \alpha_{N+1} \) in place of \( \beta \), suppose that

\[ d \left( \sum_{k=1}^{N+1} \alpha_k A_n(k), \mathcal{A} \right) \leq \varepsilon_{N+1} + d \left( \sum_{k=1}^{N} \alpha_k A_n(k), \mathcal{A} \right). \]

Then \( r - \varepsilon_{N+1} \leq \varepsilon_{N+1} + (r - \varepsilon_N) \), which implies that \( \varepsilon_N \leq 2\varepsilon_{N+1} \), a contradiction of the definition of \( \{ \varepsilon_n \} \). It follows that

\[ r - \varepsilon_{N+1} = d \left( \sum_{k=1}^{N+1} \alpha_k A_n(k), \mathcal{A} \right) \]

\[ \leq \varepsilon_{N+1} + d \left( \sum_{k=1}^{N} \alpha_k A_n(k), \mathcal{A} + \mathcal{K} \right) + d(\alpha_{N+1} A_{n(N+1)}, \mathcal{A}) \]

\[ = \varepsilon_{N+1} + \left( \sum_{k=1}^{N} \alpha_k \right) \cdot r + \alpha_{N+1} \cdot d(A_{n(N+1)}, \mathcal{A}). \]

If \( N \to \infty \) then \( \varepsilon_{N+1} \to 0 \) and, since \( \sum \alpha_k \leq 1 \), it follows that \( \alpha_{N+1} \to 0 \). Since \( A_n \to 0 \) (SOT) we see that \( \{ \| A_n \| \} \), and hence \( \{ d(A_n, \mathcal{A}) \} \), is a bounded set. Thus,
letting $N \to \infty$ in the above yields $r = d(\sum \alpha_k A_{n(k)}, \mathcal{A}) \leq (\sum \alpha_k) \cdot r$, which implies that $\sum \alpha_k \geq 1$. This completes the proof of the claim.

To complete the proof of the theorem, define the sequence $\{a_n\}$ by $a_{n(k)} = \alpha_k$ and $a_j = 0$ if $j$ is not of the form $n(k)$ for any $k$. Also, let $K = \sum a_n T_n = \sum \alpha_k T_{n(k)} = T - \sum \alpha_k A_{n(k)}$. This sum converges since $\sum \alpha_k = 1$ and since $\{|A_n|\}$ is a bounded set. It follows from the foregoing discussion that $d(T - K, \mathcal{A}_k) = d(\sum \alpha_k A_{n(k)}, \mathcal{A}) = r = d(T, \mathcal{A} + \mathcal{K})$, which completes the proof. □

Note that if $\mathcal{A} + \mathcal{K}(H)$ is norm closed then $K \in \mathcal{A} + \mathcal{K}(H)$. Also, if $\{T_n\}$ is taken to be a sequence of compact operators converging to $T$ (SOT), then $K \in \mathcal{K}(H)$, since $\mathcal{K}(H)$ is norm closed.

We are now in a position to prove one of our main results on the existence of best approximants.

**Theorem 3.4.** Let $\mathcal{A} \subset \mathcal{L}(H)$ be a subalgebra which is WOT-closed and satisfies condition $\Delta(\mathcal{A})$, and suppose $T \in \mathcal{L}(H)$. Then there exists $B \in \mathcal{A} + \mathcal{K}(H)$ such that $\|T - B\| = d(T, \mathcal{A} + \mathcal{K}(H))$.

**Proof.** Assume $T \not\in \mathcal{A} + \mathcal{K}(H)$, since otherwise the result is obvious. Let $\{e_j: j \geq 0\}$ be an orthonormal basis for $H$ and define $E_n$ to be the projection onto the subspace spanned by $\{e_j: j \leq n\}$. Each $E_n$ has finite rank and $E_n \to 1$ (SOT). Set $T_n = E_n T E_n$. Each $T_n$ is compact and $T_n \to T$ (SOT).

By Theorem 3.2 there is a sequence $\{a_n\}$ of nonnegative real numbers satisfying $\sum a_n = 1$ and such that, if $K = \sum a_n T_n$, $d(T - K, \mathcal{A}) = d(T, \mathcal{A} + \mathcal{K})$. Note that $K \in \mathcal{K}$. By Lemma 2.3 there exists $A \in \mathcal{A}$ such that $\|(T - K) - A\| = d(T - K, \mathcal{A})$. Therefore, the operator $B = A + K$ is in $\mathcal{A} + \mathcal{K}$ and satisfies

$$\|T - B\| = d(T, \mathcal{A} + \mathcal{K}).$$

In other words, $B$ is a best approximant to $T$ in $\mathcal{A} + \mathcal{K}$. □

We remarked earlier that every nest algebra is WOT-closed, so Theorem 3.4 applies, in particular, to any nest algebra $\mathcal{A}$ which satisfies condition $\Delta(\mathcal{A})$.

The following corollary shows that if $\mathcal{A} + \mathcal{K}$ is norm closed, then the operator $K$ in the conclusion of Theorem 3.2 is not unique.

**Corollary 3.5.** Let $\mathcal{A}$, $T$, and $\{T_n\}$ be as in the statement of Theorem 3.2, and also suppose that $\mathcal{A} + \mathcal{K}$ is norm closed. Then there exist two sequences $\{a_n\}$ and $\{b_n\}$ of nonnegative real numbers satisfying $\sum a_n = \sum b_n = 1$ and such that, if $K = \sum a_n T_n$ and $K_1 = \sum b_n T_n$, then $K \neq K_1$ and $d(T - K, \mathcal{A}) = d(T, \mathcal{A} + \mathcal{K}) = d(T - K_1, \mathcal{A}) = d(T, \mathcal{A} + \mathcal{K})$.

**Proof.** Let $\{a_n\}$ and $K = \sum a_n T_n$ be as in the conclusion of Theorem 3.2. Then $(T_n - K) \to (T - K)$ (SOT). Let $\mathcal{O}$ be a convex neighborhood of $T - K$ in the strong operator topology whose closure does not contain $\mathcal{O}$. Deleting a finite number of terms if necessary, assume that $T_n - K \in \mathcal{O}$ for all $n$.

Since $\mathcal{A} + \mathcal{K}$ is norm closed, we see that $K \in \mathcal{A} + \mathcal{K}$ and, hence, $(T_n - K) \in \mathcal{A} + \mathcal{K}$ for all $n$. Thus by Theorem 3.2 we can construct a sequence $\{b_n\}$ such that $\sum b_n = 1$ and such that if $K' = \sum b_n (T_n - K)$, then

$$d((T - K) - K', \mathcal{A}) = d(T - K, \mathcal{A} + \mathcal{K}) = d(T, \mathcal{A} + \mathcal{K}).$$
Thus, the operator \( K_1 = K + K' \) satisfies \( d(T - K_1, \mathcal{A}) = d(T, \mathcal{A} + \mathcal{N}) \). Since \( K' \) is a convex combination of elements of \( \emptyset \), it follows that \( K' \neq 0 \) and, hence, \( K_1 \neq K \). This proves the corollary. \( \square \)

We noted earlier that \( \mathcal{A} + \mathcal{N} \) is norm closed whenever \( \mathcal{A} \) is a nest algebra, so Corollary 3.5 applies, in particular, to any nest algebra satisfying condition \( \Delta (\mathcal{A}) \). Also note that if \( \{T_n\} \) is taken to be a sequence of compact operators, then \( K \) is compact as well and the requirement that \( \mathcal{A} + \mathcal{N} \) be norm closed is superfluous.

4. More main results. Throughout this section let \( \mathcal{P} = \{P_n\} \) be a fixed increasing sequence of finite rank projections such that \( P_n \to 1 \) (SOT). Let

\[
\mathcal{A} = \text{Alg}\{P_n\} = \{ T \in \mathcal{L}(H) : P_n T P_n = 0 \text{ for all } n \}
\]

and let

\[
QT = QT(\{P_n\}) = \{ T \in \mathcal{L}(H) : \|P_n T P_n\| \to 0, n \to \infty \}.
\]

The following result establishes the validity of condition \( \Delta (\mathcal{A}) \) in this special case. It then follows from Theorem 3.4 that best approximants in \( QT \) exist for every operator in \( \mathcal{L}(H) \).

**Proposition 4.1.** The algebra \( \mathcal{A} = \text{Alg}\{P_n\} \) satisfies condition \( \Delta (\mathcal{A}) \).

**Proof.** Choose \( T \in \mathcal{L}(H) \) and let \( \{A_n\} \subseteq \mathcal{L}(H) \) satisfy \( A_n \to 0 \) (SOT). Fix \( \varepsilon > 0 \). If condition \( \Delta (\mathcal{A}) \) is not satisfied, then by the distance formulas (2.2) and (2.1') there is a sequence \( \{m_n\} \) of nonnegative integers such that

\[
\|P_{m_n} (T + A_n) P_{m_n}\| > \varepsilon + \alpha_n,
\]

where

\[
\alpha_n = \max \left( \sup_{j \geq 0} \|P_j T P_{m_n}\|, \limsup_k \|P_k^+ T P_k\| + \sup_{j \geq 0} \|P_j^+ A_n P_j\| \right).
\]

Consider two cases.

*Case 1.* Suppose no nonnegative integer appears infinitely often in the sequence \( \{m_n\} \). Passing to a subsequence if necessary, assume that \( \{m_n\} \) is an increasing sequence. By the definition of \( \limsup \), there exists some \( N \) such that for \( n \geq N \), we have

\[
\|P_{m_n} (T + A_n) P_{m_n}\| \leq \|P_{m_n}^+ T P_{m_n}\| + \|P_{m_n} A_n P_{m_n}\| \leq \limsup_k \|P_k^+ T P_k\| + \varepsilon / 2 + \sup_j \|P_j^+ A_n P_j\| \leq \alpha_N + \varepsilon / 2.
\]

This contradicts the definition of the sequence \( \{m_n\} \).

*Case 2.* Suppose some nonnegative integer, call it \( M \), appears infinitely often in the sequence \( \{m_n\} \). Passing to a subsequence if necessary, assume that

\[
\|P_{m_n} (T + A_n) P_{m_n}\| > \varepsilon + \alpha_n \text{ for all } n.
\]

Since \( P_M \) is compact and \( A_n \to 0 \) (SOT), it follows that \( \|A_n P_M\| \to 0 \). Choose \( N \) such that \( \|A_N P_M\| < \varepsilon / 2 \). We then have

\[
\|P_M^+ (T + A_n) P_M\| \leq \|P_M^+ T P_M\| + \|P_M^+ A_n P_M\| \leq \sup_j \|P_j^+ T P_j\| + \varepsilon / 2 \leq \alpha_N + \varepsilon / 2.
\]
This yields a contradiction to the definition of the sequence \( \{ m_n \} \) and completes the proof of the proposition. \( \square \)

We now show that best approximants in \( QT \) are never unique for operators not in \( QT \).

**Proposition 4.2.** For each \( T \in \mathcal{L}(H) \setminus QT \) there exist operators \( B \) and \( B_1 \) in \( QT \) such that \( B \neq B_1 \) and \( \| T - B \| = \| T - B_1 \| = d(T, QT) \).

**Proof.** Consider two cases.

**Case 1.** Suppose there is a subsequence \( \{ n_k \} \) such that \( (P_{n_k} - P_{n_k}) TP_{n_0} \neq 0 \) for all \( k \geq 0 \). Set \( E_k = P_{n_k} \) and let \( T_k = E_k TE_k \). Now, let \( K = \sum a_k T_k \) and \( K_1 = \sum b_k T_k \) be as in the conclusion of Corollary 3.5. Note that \( K \) and \( K_1 \) are compact. By Lemma 2.3 we can find operators \( A \) and \( A_1 \) in \( \mathcal{A} \) such that \( \| T - K - A \| = d(T - K, \mathcal{A}) \) and \( \| T - K_1 - A_1 \| = d(T - K_1, \mathcal{A}) \). Thus, \( B = A + K \) and \( B_1 = A_1 + K_1 \) are best approximants in \( QT \) to \( T \). To show that \( B \neq B_1 \), it suffices to show that \( K - K_1 \notin \mathcal{A} \).

Suppose, to the contrary, that \( K - K_1 \in \mathcal{A} \). Then it follows, in particular, that

\[
0 = E_0^* (K - K_1) E_0 = \sum_{k \geq 0} (a_k - b_k) E_0^* E_k TE_k E_0
\]

Letting \( C_k = \sum_{j > k} (a_j - b_j) \), a summation by parts shows that

\[
\sum_{k=1}^{N} C_k (E_k - E_{k-1}) TE_0 = \sum_{k=1}^{N-1} (a_k - b_k) E_k TE_0 + C_N E_N TE_0 - C_1 E_0 TE_0.
\]

As \( N \to \infty \), \( |C_N| \to 0 \), so \( ||C_N E_N TE_0|| \to 0 \). Thus,

\[
\sum_{k=1}^{\infty} C_k (E_k - E_{k-1}) TE_0 = \sum_{k=1}^{\infty} (a_k - b_k) E_k TE_0 - C_1 E_0 TE_0.
\]

We thus have that

\[
\sum_{k=1}^{\infty} \left( \sum_{j > k} (a_j - b_j) \right) (E_k - E_{k-1}) TE_0 = \sum_{k=1}^{\infty} (a_k - b_k) E_k TE_0 - \sum_{k=1}^{\infty} (a_k - b_k) E_0 TE_0 = \sum_{k=1}^{\infty} (a_k - b_k) (E_k - E_0) TE_0 = 0.
\]

Since the range of \( (E_l - E_{l-1}) \) is orthogonal to that of \( (E_j - E_{j-1}) \) whenever \( l \neq j \), and since, by assumption, \( (E_k - E_{k-1}) TE_0 \neq 0 \) for \( k \geq 1 \), it follows that

\[
\sum_{j > k} (a_j - b_j) = 0 \quad \text{for } k \geq 1.
\]

The fact that \( \sum a_n = \sum b_n = 1 \) implies that \( \sum_{j > 0} (a_j - b_j) = 0 \) as well. Hence, \( a_j = b_j \) for all \( j \geq 0 \), which contradicts the assumption that \( K \neq K_1 \). Thus, \( K - K_1 \notin \mathcal{A} \) and, consequently, \( B \neq B_1 \).
Case 2. Suppose there is no subsequence \( \{n_k\} \) for which \( (P_{n_k} - P_{n})TP_{n_k} \neq 0 \) for all \( k \). Then for each \( k \) there is a smallest integer \( m(k) \) such that \( P_{m(k)}TP_k = 0 \). We claim that \( m(k) \geq k + 1 \) for infinitely many \( k \). Indeed, were this not so then there would exist \( N \) such that \( m(k) \leq k \) for \( k \geq N \). Hence, \( P_k^\perp TP_k = 0 \) for \( k \geq N \), which implies that \( d(T, QT) = 0 \), contradicting the assumption that \( T \notin QT \).

We make the following remarks.

(a) If \( m(k) \geq k + 1 \), then \( (P_{m(k)} - P_k)TP_k \neq 0 \). This follows from the choice of \( m(k) \) as the smallest integer such that \( P_{m(k)}TP_k = 0 \).

(b) It is clear that if \( (P_{m(k)} - P_k)TP_k \neq 0 \), then \( (P_j - P_k)TP_k \neq 0 \) for \( j \geq m(k) \).

Now, choose \( k_0 \) such that \( m(k_0) \geq k_0 + 1 \) and \( TP_{k_0} \neq 0 \). For \( j \geq 1 \) inductively choose \( k_j \) such that \( m(k_j) \geq k_j + 1 \) and \( k_j > m(k_{j-1}) \). Set \( E_j = P_{k_j} \) and let \( T_j = E_jTE_j \). From this we get \( K = \sum a_nT_n \) and \( K_1 = \sum b_nT_n \), as in the conclusion of Corollary 3.5. To complete the proof it suffices, as in the previous case, to show that \( K - K_1 \notin \mathcal{A} \).

First observe that remarks (a) and (b) imply that \( (E_n - E_j)TE_j \neq 0 \) for \( n \geq l + 1 \). Also, by the construction of the sequence \( \{E_n\} \), it follows that \( (E_{j+1} - E_j)TE_j = 0 \) for \( j \geq l + 1 \). Putting these together we see that, for \( n \geq l + 1 \),

\[
(E_n - E_j)TE_j = \sum_{j=l}^{n-1} (E_{j+1} - E_j)TE_j = (E_{l+1} - E_j)TE_j \neq 0.
\]

To see that \( K - K_1 \notin \mathcal{A} \), suppose the contrary. Then, for \( l \geq 0 \), we must have

\[
0 = E_l^\perp (K - K_1)E_l = \sum_{n \geq l+1} (a_n - b_n)E_l^\perp E_nTE_nE_l
\]

\[
= \sum_{n \geq l+1} (a_n - b_n)(E_n - E_l)TE_l
\]

\[
= \sum_{n \geq l+1} (a_n - b_n)(E_{l+1} - E_l)TE_l
\]

\[
= \left[ \sum_{n \geq l+1} (a_n - b_n) \right] (E_{l+1} - E_l)TE_l.
\]

Since \( (E_{l+1} - E_l)TE_l \neq 0 \), it follows that \( \sum_{n \geq l+1} (a_n - b_n) = 0 \) for \( l \geq 0 \). Since \( \sum a_n = \sum b_n = 1 \), it follows that \( \sum_{n \geq l} (a_n - b_n) = 0 \) for all \( l \geq 0 \) and, hence, \( a_n = b_n \) for all \( n \), contradicting the assumption that \( K \neq K_1 \). Hence, \( K - K_1 \notin \mathcal{A} \) and the corollary is proved.

5. Remarks. The obvious question is to ask which subalgebras \( \mathcal{A} \) satisfy the condition \( \Delta(\mathcal{A}) \). Our proof of Proposition 4.1 and Arveson's proof of the distance formula (2.2) both use the finite dimensionality of the projections \( P_n \) in an important way. Some means of eliminating this dependence would apparently be needed to establish a broader validity of condition \( \Delta(\mathcal{A}) \). A generalization of Proposition 4.2 to the setting of §3 would also be useful.

A question related to Theorem 3.2 is the following. If the operators \( \{T_n\} \) are taken to be compact, then the resulting \( K \) is also compact. It is possible that this \( K \) is a best compact approximant to \( T \)?
In [3] Axler, Berg, Jewell, and Shields employ what they call the “Basic Inequality” for a Banach space $X$. This inequality is similar to condition $\Delta(\mathcal{A})$ for $\mathcal{A} = \{0\}$, the zero operator. They show that the Basic Inequality is satisfied for $X = l^p$, $1 < p < \infty$. They also prove that the closed unit ball of $L^\infty / H^\infty + C$ has no extreme points.

Two questions which arise are whether $\Delta(\mathcal{A})$ holds when $\mathcal{A}$ is the algebra of operators on $l^p$ ($1 < p < \infty$) with upper triangular matrix representations with respect to the standard basis, and whether the closed unit ball of $\mathcal{L}(H) / \mathcal{A} + \mathcal{K}(H)$ has any extreme points if $\mathcal{A}$ is a nest algebra satisfying condition $\Delta(\mathcal{A})$.

Another line of questioning is related to the theory of $M$-ideals, introduced in 1972 by Alfsen and Effros [1]. Luecking [8] showed that $H^\infty + C / H^\infty$ is an $M$-ideal in $L^\infty / H^\infty$, and it seems reasonable to ask if $\mathcal{A} + \mathcal{K}(H) / \mathcal{A}$ is an $M$-ideal in $\mathcal{L}(H) / \mathcal{A}$ for any nest algebra $\mathcal{A}$. An affirmative answer would imply, by a result of Holmes, Scranton, and Ward [7], that the collection $\mathcal{P}_T = \{A + \mathcal{A} \in \mathcal{A} + \mathcal{K}(H) / \mathcal{A} : d(T - A, \mathcal{A}) = d(T, \mathcal{A} + \mathcal{K}(H))\}$ would algebraically span $\mathcal{A} + \mathcal{K}(H) / \mathcal{A}$ for each $T \in \mathcal{L}(H) \setminus \mathcal{A} + \mathcal{K}(H)$.

References


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