SOBOLEV SPACE PROJECTIONS
IN STRICTLY PSEUDOCONVEX DOMAINS[1]

BY

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ABSTRACT. The orthogonal projection from a Sobolev space $W^s(\Omega)$ onto the subspace of holomorphic functions is studied. This analogue of the Bergman projection is shown to satisfy regularity estimates in higher Sobolev norms when Ω is a smooth bounded strictly pseudoconvex domain in $\mathbf{C}^n$.

The Bergman projection P_0: $L^2(\Omega) \to L^2(\Omega) \cap \{$holomorphic functions$\}$, where $\Omega \subset \mathbf{C}^n$ is a smooth bounded domain, has proved to be a key element in the study of boundary behavior of holomorphic mappings (see [4, 7, 13] and their references). In the important special case in which Ω is strictly pseudoconvex, a great deal is known about the projection P_0 and the Bergman kernel function $K_0(w, z)$ which represents it (see e.g. [14, 16, 19]). In particular the following two regularity theorems are well known consequences of Kohn's theory of the $\bar{\partial}$-Neumann problem [15, 17].

THEOREM A [17]. *Let $\Omega \subset \mathbf{C}^n$ be a smooth bounded strictly pseudoconvex domain. Then the Bergman projection P_0 admits both global and local regularity estimates in Sobolev norms*:

(i)
$$\|P_0 u\|_r \leqslant C_r \|u\|_r, \qquad r \geqslant 0,$$

and more generally, if $\zeta_1, \zeta_2 \in C_0^\infty(\mathbf{C}^n)$ are real-valued cut-off functions with $\zeta_2 = 1$ in a neighborhood of the support of ζ_1, then

(ii)
$$\|\zeta_1 P_0 u\|_r \leqslant C_r \big(\|\zeta_2 u\|_r + \|u\|_0\big), \qquad r \geqslant 0.$$

THEOREM B [16]. *Let $\Omega \subset \mathbf{C}^n$ be a smooth bounded strictly pseudoconvex domain. Then the Bergman kernel function $K_0(w, z)$ is smooth up to the boundary off the boundary diagonal, that is,*

$$K_0(w, z) \in C^\infty\big(\overline{\Omega} \times \overline{\Omega} \setminus \{z = w \in b\Omega\}\big).$$

The objects studied in this paper are the analogous projection P_s: $W^s(\Omega) \to W^s(\Omega) \cap \{$holomorphic functions$\}$, where Ω is a smooth bounded strictly pseudoconvex domain and $W^s(\Omega)$ is the Sobolev space of functions with s square-integrable

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derivatives, and the kernel function $K_s(w, z)$ that represents P_s. The main result is that the analogues of Theorems A and B hold.

THEOREM A $_s$. *Let $\Omega \subset \mathbf{C}^n$ be a smooth bounded strictly pseudoconvex domain. Then the projection P_s admits both global and local regularity estimates in Sobolev norms*:

(i) $\|P_s u\|_r \leqslant C_{r,s}\|u\|_r, \qquad r \geqslant s,$

and more generally, if $\zeta_1, \zeta_2 \in C_0^\infty(\mathbf{C}^n)$ are real-valued cut-off functions with $\zeta_2 = 1$ in a neighborhood of the support of ζ_1, then

(ii) $\|\zeta_1 P_s u\|_r \leqslant C_{r,s}\big(\|\zeta_2 u\|_r + \|u\|_s\big), \qquad r \geqslant s.$

THEOREM B $_s$. *Let $\Omega \subset \mathbf{C}^n$ be a smooth bounded strictly pseudoconvex domain. Then the kernel function $K_s(w, z)$ is smooth up to the boundary off the boundary diagonal, that is*

$$K_s(w, z) \in C^\infty\big(\overline{\Omega} \times \overline{\Omega} \setminus \{ z = w \in b\Omega \}\big).$$

The key to understanding the projection P_s is to prove regularity estimates for the operators U_s and T_s given by integration against a kernel in the "wrong" space:

$$U_s u(w) = \big\langle u(\cdot), K_s(\cdot, w)\big\rangle_0, \qquad T_s u(w) = \big\langle u(\cdot), K_0(\cdot, w)\big\rangle_s.$$

From these operators, first mentioned by Bell [5], one recaptures P_s via the formula $P_s = U_s T_s$.

The key estimates for U_s and T_s are proved in §5 using results from the $\bar{\partial}$-Neumann theory. The proofs of Theorems A $_s$ and B $_s$, based on these estimates, are given in §§2 and 3: the global part in §2 and the more delicate local part in §3.

I thank N. Kerzman, who suggested study of the K_s kernel as a thesis problem. My MIT doctoral dissertation, written under his direction, was a first step toward the results presented here. The papers of S. Bell, especially [5], have also provided inspiration in this work.

1. Preliminaries. Throughout, Ω is a smooth bounded strictly pseudoconvex domain in $\mathbf{C}^n$. This means that there is a bounded C^∞ real-valued defining function $\rho: \mathbf{C}^n \to \mathbf{R}$ such that $\Omega = \{ z \in \mathbf{C}^n: \rho(z) < 0 \}$, the boundary $b\Omega = \{ z \in \mathbf{C}^n: \rho(z) = 0 \}$, the gradient of ρ does not vanish on $b\Omega$, and the complex Hessian of ρ is strictly positive definite on the complex part of the tangent space. It will be assumed that ρ is normalized so that the gradient of ρ has length one on the boundary. In that case the globally defined vector field $\partial/\partial\rho$ given in the underlying real coordinates by

$$\frac{\partial}{\partial\rho} = \sum_{j=1}^{2n} \frac{\partial\rho}{\partial x_j} \frac{\partial}{\partial x_j}$$

agrees on the boundary with the unit outer normal.

The Hilbert space $L^2(\Omega)$ of square-integrable functions on Ω carries the usual norm $\| \ \|_0$ induced by the inner product $\langle u, v\rangle_0 = \int_\Omega u\bar{v}$. When s is a positive integer the inner product

$$\langle u, v\rangle_s = \sum_{|\alpha| \leqslant s} \big\langle D^\alpha u, D^\alpha v\big\rangle_0$$

induces the norm $\| \quad \|_s$ on the Sobolev space $W^s(\Omega)$. (Here $\alpha = (\alpha_1, \ldots, \alpha_{2n})$ is a multi-index and $D_j = \partial/\partial x_j$.) When s is a positive real number that is not an integer, $W^s(\Omega)$ can be defined by an interpolation procedure (see e.g. [18]). The closure in $W^s(\Omega)$ of the space $C_0^\infty(\Omega)$ of smooth functions with compact support in Ω is denoted $W_0^s(\Omega)$. The intersection of all the spaces $W^s(\Omega)$, taken with the usual inverse limit topology, is the space $C^\infty(\overline{\Omega})$ of functions smooth up to the boundary; it is a dense subspace of each $W^s(\Omega)$.

The space $W^{-s}(\Omega)$, defined for each positive real number s, is the dual space of $W_0^s(\Omega)$. It is realized as a space of distributions containing $L^2(\Omega)$ as a dense subspace. If $u \in L^2(\Omega)$, then

$$\|u\|_{-s} = \sup\left\{ |\langle u, \varphi \rangle_0| : \varphi \in C_0^\infty(\Omega), \|\varphi\|_s = 1 \right\}.$$

The completion of $L^2(\Omega)$ in the stronger norm

$$\|u\|_{-s}^* = \sup\left\{ |\langle u, v \rangle_0| : v \in C^\infty(\overline{\Omega}), \|v\|_s = 1 \right\}$$

is the dual space $(W^s(\Omega))^*$ of $W^s(\Omega)$; it is not in general identified with a space of distributions. Always $\|u\|_{-s} \leqslant \|u\|_{-s}^*$, and if u is holomorphic it turns out that the two norms are equivalent (see Lemma 4.4).

The elements of $W^s(\Omega)$ represented by holomorphic functions comprise a closed subspace $H^s(\Omega)$. (N.B. This notation conflicts with common usage, in which H^s denotes the usual Sobolev space. It is convenient here to reserve the letter H to indicate a space of *holomorphic* functions.) The intersection of all the spaces $H^s(\Omega)$ with the topology inherited from $C^\infty(\overline{\Omega})$ is denoted $H^\infty(\overline{\Omega})$. The union of the spaces $H^{-s}(\Omega)$ with the usual inductive limit topology is denoted $H^{-\infty}(\Omega)$.

The objects of interest in this paper are, for each positive integer s, the orthogonal projection $P_s: W^s(\Omega) \to H^s(\Omega)$ and the kernel function $K_s(w, z)$ that represents it. When $u \in W^s(\Omega)$

$$P_s u(z) = \left\langle u(\cdot), K_s(\cdot, z) \right\rangle_s$$

and in particular for every holomorphic function h in $H^s(\Omega)$ the reproducing property $h(z) = \langle h(\cdot), K_s(\cdot, z) \rangle_s$ holds. For some elementary properties of K_s see [9], and for the general theory of reproducing kernels see [2].

The central idea is to relate the projection P_s to the usual Bergman projection P_0. A principal tool is the following integration by parts lemma, which holds in an arbitrary smooth bounded domain Ω (not necessarily pseudoconvex).

LEMMA 1.1. *For each positive integer s there is a linear differential operator L^{2s} of order $2s$ with coefficients in $C^\infty(\overline{\Omega})$ such that for every holomorphic function h in $H^s(\Omega)$ and every function u in $W^{2s}(\Omega)$*

$$(1.1) \qquad\qquad \left\langle h, u \right\rangle_s = \left\langle h, L^{2s}u \right\rangle_0.$$

If u is also holomorphic, then the top order term of $L^{2s}u$ reduces to $2^s(\partial/\partial\rho)^{2s}u$.

PROOF. Suppose at first that $h \in H^\infty(\overline{\Omega})$ and $u \in C^\infty(\overline{\Omega})$. Derivatives of h that are tangential near the boundary can be integrated by parts with no boundary terms appearing, and by the Cauchy-Riemann equations normal derivatives of the holomorphic function h can be rewritten as tangential derivatives. Hence all derivatives

on h can be moved to the other side of the inner product to make formula (1.1) appear.

To identify the top order term of L^{2s} observe that by Green's identity

$$\langle h, u \rangle_1 = \int_\Omega h\bar{u} + \int_\Omega \sum_{j=1}^{2n} (D_j h)(D_j \bar{u})$$

$$= \int_\Omega h\bar{u} + \int_{b\Omega} \frac{\partial h}{\partial \rho}\bar{u},$$

where Lebesgue measure is understood both on the boundary and on the interior. There is a real vector field $\partial/\partial\sigma$ on Ω, tangential at the boundary, with coefficients in $C^\infty(\overline{\Omega})$, such that $(\partial/\partial\rho) + i(\partial/\partial\sigma)$ is a vector field of type $(1,0)$. Then $\partial h/\partial\rho = i\partial h/\partial\sigma$, so integration by parts and another application of Green's identity give

$$\langle h, u \rangle_1 = \left\langle h, \left(1 + i(\Delta\rho)\frac{\partial}{\partial\sigma} - \frac{\partial^2}{\partial\sigma^2} + i\frac{\partial}{\partial\rho}\frac{\partial}{\partial\sigma}\right)u \right\rangle_0,$$

where Δ is the usual Laplace operator. It follows by induction that the leading term of L^{2s} is

$$\left(-\frac{\partial^2}{\partial\sigma^2} + i\frac{\partial}{\partial\rho}\frac{\partial}{\partial\sigma}\right)^s$$

which, when applied to a holomorphic function, reduces to $2^s(\partial/\partial\rho)^{2s}$ plus lower order terms.

Since $C^\infty(\overline{\Omega})$ is dense in $W^{2s}(\Omega)$, equation (1.1) continues to hold when u is only in $W^{2s}(\Omega)$. Passage to the limit over the interior approximating domains $\Omega_\delta = \{z \in \Omega : \rho(z) < -\delta\}$ shows that (1.1) persists also for h only in $H^s(\Omega)$.

2. Global regularity and duality. When $u \in L^2(\Omega)$, a holomorphic function $U_s u$ is defined by

$$U_s u(w) = \langle u(\cdot), K_s(\cdot, w) \rangle_0.$$

By Theorem B (Kerzman [16]) the Bergman kernel function $K_0(\cdot, w) \in C^\infty(\overline{\Omega})$ in every smooth bounded strictly pseudoconvex domain Ω, so when $u \in W^s(\Omega)$ a holomorphic function $T_s u$ is defined by

$$T_s u(w) = \langle u(\cdot), K_0(\cdot, w) \rangle_s.$$

The operators U_s and T_s, denoted L^s and Λ^s by Bell [5], are important because composing them recovers the projections P_0 and P_s.

LEMMA 2.1. (a) *The operator T_s maps $C^\infty(\overline{\Omega})$ continuously into itself and $U_s T_s = P_s$ on $C^\infty(\overline{\Omega})$.*

(b) *The operator U_s maps $L^2(\Omega)$ continuously into $H^s(\Omega)$ and $T_s U_s = P_0$ on $L^2(\Omega)$.*

PROOF. By Lemma 1.1 the equation $T_s u = P_0 L^{2s} u$ holds for u in $C^\infty(\overline{\Omega})$. Since the Bergman projection P_0 for a smooth bounded strictly pseudoconvex domain maps $C^\infty(\overline{\Omega})$ continuously into itself, it follows that T_s has the same property. In particular

$U_s T_s u$ is defined for u in $C^\infty(\overline{\Omega})$. Moreover

$$U_s T_s u(w) = U_s P_0 L^{2s} u(w) = \left\langle P_0 L^{2s} u(\cdot), K_s(\cdot, w) \right\rangle_0$$

$$= \left\langle L^{2s} u(\cdot), K_s(\cdot, w) \right\rangle_0 = \left\langle u(\cdot), K_s(\cdot, w) \right\rangle_s = P_s u(w).$$

This proves part (a).

Suppose $u \in C_0^\infty(\Omega)$. Since the norm of $K_s(z, \cdot)$ in $W^s(\Omega)$ is bounded by a constant depending on the distance from z to the boundary $b\Omega$, the holomorphic function $U_s u$ lies in $H^s(\Omega)$. Moreover if $h \in H^s(\Omega)$, Fubini's theorem implies that

$$(2.1) \qquad\qquad\qquad \left\langle U_s u, h \right\rangle_s = \left\langle u, h \right\rangle_0.$$

Therefore $\|U_s u\|_s \leqslant \|u\|_0$ for u in $C_0^\infty(\Omega)$. Since this space is dense in $L^2(\Omega)$ it follows that U_s is bounded from $L^2(\Omega)$ into $H^s(\Omega)$ and that equation (2.1) persists for u in $L^2(\Omega)$. Substituting $K_0(z, \cdot)$ for $h(z)$ in (2.1) shows that P_0 equals $T_s U_s$ on $L^2(\Omega)$, which proves part (b).

In view of Lemma 2.1 estimates for P_s will follow directly from estimates for U_s and T_s. The following global estimates hold for all h in $H^\infty(\overline{\Omega})$ and every real number r:

$$(2.2) \qquad\qquad C^{-1}\|h\|_{r+2s} \leqslant \|T_s h\|_r \leqslant C\|h\|_{r+2s},$$

$$(2.3) \qquad\qquad C^{-1}\|h\|_r \leqslant \|U_s h\|_{r+2s} \leqslant C\|h\|_r$$

with C independent of h. Thus T_s loses $2s$ derivatives and U_s gains $2s$ derivatives. These estimates will be proved at the end of §5 as a corollary of the key local estimates.

PROOF OF THE GLOBAL PART OF THEOREM A_s. If $u \in C^\infty(\overline{\Omega})$, then $P_s u = U_s T_s u = U_s P_0 L^{2s} u$ by Lemmas 1.1 and 2.1. It is well known that the Bergman projection is bounded on $W^r(\Omega)$ for every nonnegative r. (See e.g. [19] and further discussion of P_0 in §4.) Therefore by (2.3)

$$\|P_s u\|_r \leqslant C\|P_0 L^{2s} u\|_{r-2s} \leqslant C\|L^{2s} u\|_{r-2s} \leqslant C\|u\|_r$$

when $r \geqslant 2s$. Density of $C^\infty(\overline{\Omega})$ in $W^r(\Omega)$ implies that P_s is bounded on $W^r(\Omega)$ when $r \geqslant 2s$. Since by definition P_s is bounded on $W^s(\Omega)$, it follows by interpolation that P_s is bounded on $W^r(\Omega)$ when $r \geqslant s$. This proves the first part of Theorem A_s, granted estimate (2.3).

The estimates (2.2) and (2.3) imply in particular that T_s and U_s are continuous operators from $H^\infty(\overline{\Omega})$ into itself. In view of Lemma 2.1 the operators T_s and U_s are in fact mutually inverse isomorphisms of $H^\infty(\overline{\Omega})$ onto itself.

LEMMA 2.2 *The space* $H^\infty(\overline{\Omega})$ *is dense in* $H^r(\Omega)$ *for every real number* r.

This density lemma is true in arbitrary smooth bounded pseudoconvex domains (not necessarily strictly pseudoconvex): see [6] for a proof. Together with the a priori estimates (2.2) and (2.3) the lemma implies that T_s and U_s extend to mutually inverse isomorphisms of $H^{-\infty}(\Omega)$ onto itself; moreover the extensions give mutually inverse isomorphisms

$$T_s\colon H^{r+s}(\Omega) \to H^{r-s}(\Omega), \qquad U_s\colon H^{r-s}(\Omega) \to H^{r+s}(\Omega)$$

for every real number r. This generalizes a result of Bell [5], who established the isomorphism when $r = 0$.

Bell also showed that the H^0 pairing $\langle \ , \ \rangle_0 \colon H^\infty(\overline{\Omega}) \times H^\infty(\overline{\Omega}) \to \mathbf{C}$ extends uniquely to a continuous pairing of $H^{-\infty}(\Omega) \times H^\infty(\overline{\Omega})$ exhibiting the latter spaces as mutually dual. The same is true for the H^s pairing $\langle \ , \ \rangle_s$ in view of the equation $\langle g, h \rangle_s = \langle T_s g, h \rangle_0$, true for g and h in $H^\infty(\overline{\Omega})$.

3. Proof of the main theorems. In view of the relation $U_s T_s = P_s$ the local estimates for P_s will follow from local estimates for U_s and T_s.

THEOREM 3.1 (The key local estimates). *Let ζ_1 and ζ_2 be real-valued cut-off functions in $C_0^\infty(\mathbf{C}^n)$ such that $\zeta_2 = 1$ in a neighborhood of the support of ζ_1.*

(a) *For every real number r greater than or equal to s there is a constant C such that*

$$(3.1) \qquad \|\zeta_1 T_s u\|_{r-2s} \leqslant C\big(\|\zeta_2 u\|_r + \|u\|_s \big)$$

for all u in $C^\infty(\overline{\Omega})$. For holomorphic functions a stronger estimate holds: for every real number r (unrestricted) and every positive integer M there is a constant C such that

$$(3.2) \qquad \|\zeta_1 T_s h\|_{r-2s} \leqslant C\big(\|\zeta_2 h\|_r + \|h\|_{-M} \big)$$

for all h in $H^\infty(\overline{\Omega})$.

(b) *For every real number r and every positive integer M there is a constant C such that*

$$(3.3) \qquad \|\zeta_1 h\|_r \leqslant C\big(\|\zeta_2 T_s h\|_{r-2s} + \|h\|_{-M} \big)$$

for all h in $H^s(\Omega)$, and

$$(3.4) \qquad \|\zeta_1 U_s h\|_r \leqslant C\big(\|\zeta_2 h\|_{r-2s} + \|h\|_{-M} \big)$$

for all h in $H^0(\Omega)$.

REMARK. It is part of the theorem that finiteness of the right-hand side in (b) implies finiteness of the left-hand side. The upper bounds for T_s follow easily from standard estimates for the Bergman projection. The lower bound for T_s and corresponding upper bound for U_s require some work. The proof is postponed until §5.

THEOREM 3.2. *Let ζ_1 and ζ_2 be real-valued cut-off functions in $C_0^\infty(\mathbf{C}^n)$ such that $\zeta_2 = 1$ in a neighborhood of the support of ζ_1. If $u \in W^s(\Omega)$ and $\zeta_2 u \in W^r(\Omega)$ for some real number r greater than or equal to s, then $\zeta_1 P_s u \in W^r(\Omega)$ and*

$$(3.5) \qquad \|\zeta_1 P_s u\|_r \leqslant C\big(\|\zeta_2 u\|_r + \|u\|_s \big)$$

with C independent of u.

REMARK. This is a restatement of part (ii) of Theorem A$_s$. Note that the statement reduces to part (i) if ζ_1 is chosen to be identically 1 on Ω.

PROOF. Let η be a smooth real-valued cut-off function such that $\eta = 1$ in a neighborhood of the support of ζ_1 and $\zeta_2 = 1$ in a neighborhood of the support of η. Suppose at first that $u \in C^\infty(\overline{\Omega})$. The relation $U_s T_s = P_s$ of Lemma 2.1 together with the upper bound (3.4) for U_s implies

$$\|\zeta_1 P_s u\|_r \leqslant C\big(\|\eta T_s u\|_{r-2s} + \|T_s u\|_{-s} \big).$$

Now estimate the right-hand side by applying inequality (3.1) twice: with ζ_1 replaced by η it gives

$$\|\eta T_s u\|_{r-2s} \leq C\big(\|\zeta_2 u\|_r + \|u\|_s\big),$$

and with the cut-off functions set equal to 1 on Ω it gives $\|T_s u\|_{-s} \leq C\|u\|_s$. Combining the last three equations shows that (3.5) holds as an a priori estimate for u in $C^\infty(\overline{\Omega})$.

Now suppose only that $u \in W^s(\Omega)$ and $\zeta_2 u \in W^r(\Omega)$. There is a sequence of functions $u_1, u_2, \ldots$ in $C^\infty(\overline{\Omega})$ such that $u_j \to u$ in $W^s(\Omega)$ and $\eta u_j \to \eta u$ in $W^r(\Omega)$ as $j \to \infty$. (Such functions can be constructed by means of a partition of unity and the standard bounded extension operator that extends a $W^r(\Omega)$ function supported in a boundary chart to a $W_0^r(\mathbf{C}^n)$ function.) The a priori version of (3.5) implies that as $j \to \infty$ the $\zeta_1 P_s u_j$ converge in $W^r(\Omega)$ to some function v such that

$$\|v\|_r \leq C\big(\|\zeta_2 u\|_r + \|u\|_s\big).$$

Since P_s is by definition continuous in $W^s(\Omega)$, the $\zeta_1 P_s u_j$ converge to $\zeta_1 P_s u$ in $W^s(\Omega)$. By uniqueness of limits $v = \zeta_1 P_s u$, and so $\zeta_1 P_s u \in W^r(\Omega)$ and satisfies (3.5).

This completes the proof of Theorem A_s of the introduction. For compactly supported functions (which of course are not dense in $W^r(\Omega)$ when $r > 1/2$) the index of the global term in (3.5) can be made arbitrary. This improvement is required to prove Theorem B_s.

LEMMA 3.3. *Under the hypotheses of Theorem 3.2, if $\varphi \in C_0^\infty(\Omega)$ then for every positive integer M*

$$\|\zeta_1 P_s \varphi\|_r \leq C\big(\|\zeta_2 \varphi\|_r + \|\varphi\|_{-M}^*\big)$$

with C independent of φ.

REMARK. A stronger estimate holds: the index of the first term on the right-hand side can be reduced by $2s$. However, this gain of derivatives is irrelevant in the application.

PROOF. Since φ has compact support it is possible to integrate by parts to obtain

$$P_s \varphi(w) = \big\langle \varphi(\cdot), K_s(\cdot, w)\big\rangle_s = \big\langle \varphi(\cdot), K_s(\cdot, w)\big\rangle_0 = U_s P_0 \varphi(w).$$

If η is an intermediate cut-off function as in the previous proof then the upper bound (3.4) for U_s implies

$$\|\zeta_1 P_s \varphi\|_r \leq C\big(\|\eta P_0 \varphi\|_r + \|P_0 \varphi\|_{-M}\big) \leq C\big(\|\zeta_2 \varphi\|_r + \|\varphi\|_{-M}^*\big).$$

The last step follows from standard estimates for the Bergman projection (see Lemma 4.1).

COROLLARY 3.4 (Theorem B_s). *For each positive integer s the kernel function $K_s(w, z)$ is smooth up to the boundary off the boundary diagonal, that is*

$$K_s(w, z) \in C^\infty\big(\overline{\Omega} \times \overline{\Omega} \setminus \{(w, z) \in b\Omega \times b\Omega : w = z\}\big).$$

PROOF. The case of the usual Bergman kernel function ($s = 0$) was established by Kerzman [16]. Because of Lemma 3.3 essentially the same argument works in the new setting. The details are as follows.

Of course $K_s(w, z) \in C^\infty(\Omega \times \Omega)$ since it is holomorphic in w and conjugate holomorphic in z. What has to be checked is the behavior when w approaches the boundary and z stays away from w; possibly z also approaches the boundary. By Sobolev's lemma it is enough to show that if W is a small open set intersecting $b\Omega$ and Z is an open set whose closure is disjoint from the closure of W, then

$$(3.6) \qquad \sup_{w \in W \cap \Omega} \|D_w^\alpha K_s(w, z)\|_{W^r(Z \cap \Omega)} < \infty$$

for every multi-index α and every r greater than s.

Let φ be a smooth, nonnegative, radially symmetric function supported in the unit ball of $\mathbf{C}^n$ such that the integral of φ over $\mathbf{C}^n$ equals 1. Define

$$\varphi_w(t) = \mathrm{dist}(w, b\Omega)^{-2n} \varphi((t - w)/\mathrm{dist}(w, b\Omega)).$$

For each w in Ω the function φ_w is smooth with compact support in Ω, and the integral of φ_w over $\mathbf{C}^n$ equals 1. Integration by parts and the mean-value property of holomorphic functions imply

$$(-1)^{|\alpha|} D_w^\alpha K_s(w, z) = \langle K_s(\cdot, z), D^\alpha \varphi_w(\cdot) \rangle_0 = \overline{P_s D^\alpha \varphi_w(z)} .$$

Let ζ be a smooth real-valued cut-off function that is identically zero in a neighborhood of the closure of W and identically one in a neighborhood of the closure of Z. It is no loss of generality to assume that for every w in W the supports of φ_w and ζ are disjoint. When $r \geqslant s$ and $w \in W$ it follows from the above formula and Lemma 3.3 that

$$(3.7) \qquad \|D_w^\alpha K_s(w, z)\|_{W^r(Z \cap \Omega)} \leqslant C\left(\|\zeta D^\alpha \varphi_w\|_r + \|D^\alpha \varphi_w\|_{-M}^* \right) \leqslant C \|D^\alpha \varphi_w\|_{-M}^*$$

with C independent of w. If $u \in C^\infty(\overline{\Omega})$, then

$$\left| \langle D^\alpha \varphi_w, u \rangle_0 \right| = \left| \langle \varphi_w, D^\alpha u \rangle_0 \right| \leqslant \left(\int_\Omega \varphi_w \right) \sup_\Omega |D^\alpha u| \leqslant C \|u\|_M$$

as soon as $M > n + |\alpha|$. Hence $\|D^\alpha \varphi_w\|_{-M}^* \leqslant C$ for such M, with C independent of w in W. Thus (3.7) implies the required estimate (3.6). This completes the proof of the second main result, Theorem B_s.

4. Estimates for the $\bar{\partial}$-Neumann problem and for holomorphic functions. This section summarizes firstly some standard estimates for the $\bar{\partial}$-Neumann problem and secondly some special Sobolev estimates for holomorphic functions. A good reference for the $\bar{\partial}$-Neumann problem is [15]; details about Sobolev norms of holomorphic functions will appear in a forthcoming article [10].

The lemmas in this section should be understood to carry the following assumptions: Ω is a smooth bounded strictly pseudoconvex domain in $\mathbf{C}^n$; the functions ζ_1 and ζ_2 are real-valued cut-off functions in $C_0^\infty(\mathbf{C}^n)$ with ζ_2 identically 1 in a neighborhood of the support of ζ_1; and M is an arbitrary positive integer.

Consider $\bar{\partial}$ as a closed densely-defined operator from the Hilbert space of (p, q)-forms with square-integrable coefficients to the Hilbert space of $(p, q + 1)$-forms with square-integrable coefficients. The Hilbert space adjoint of $\bar{\partial}$ is denoted $\bar{\partial}^*$. The Neumann operator N on $(0, 1)$-forms is the inverse of $\bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*$. It is well known that N admits strong estimates in Sobolev norms (Kohn [17, 15]). Roughly speaking N gains one derivative, the combination $\bar{\partial}^*N$ gains one-half derivative, and the combination $\bar{\partial}^*N\bar{\partial}$ preserves the number of derivatives. The formula $P_0 = \text{Id} - \bar{\partial}^*N\bar{\partial}$ for the Bergman projection, together with the estimates for N, leads to pseudolocal estimates for P_0.

LEMMA 4.1. *For every nonnegative real number r there is a constant C such that*

$$\|\zeta_1 P_0 u\|_r \leqslant C\left(\|\zeta_2 u\|_r + \|u\|_{-M}^*\right),$$

$$\|\zeta_1 P_0 u\|_{-r}^* \leqslant C\left(\|\zeta_2 u\|_{-r}^* + \|u\|_{-M}^*\right)$$

for all u in $C^\infty(\bar{\Omega})$. In particular

$$\|P_0 u\|_r \leqslant C\|u\|_r, \qquad \|P_0 u\|_{-r}^* \leqslant C\|u\|_{-r}^*.$$

The first inequality is well known. It is commonly written with ambient term $\|u\|_0$, but the stronger form given here follows from the usual proof. The second inequality, with negative indices, follows by duality.

An immediate corollary is the following local density statement for holomorphic functions.

LEMMA 4.2. *Let r and s be positive real numbers such that $r \geqslant s$. If h is a holomorphic function such that $h \in W^s(\Omega)$ and $\zeta_2 h \in W^r(\Omega)$, then there is a sequence $h_1, h_2, \ldots$ of holomorphic functions in $C^\infty(\bar{\Omega})$ such that*

(a) $\|h - h_k\|_s \to 0$, *and*

(b) $\|\zeta_1(h - h_k)\|_r \to 0$

as $k \to \infty$.

PROOF. Let η be a real-valued cut-off function in $C_0^\infty(\mathbf{C}^n)$ such that $\eta = 1$ in a neighborhood of the support of ζ_1 and $\zeta_2 = 1$ in a neighborhood of the support of η. Take a sequence $u_1, u_2, \ldots$ of functions in $C^\infty(\bar{\Omega})$ such that $u_k \to h$ in $W^s(\Omega)$ and $\eta u_k \to \eta h$ in $W^r(\Omega)$, and set h_k equal to $P_0 u_k$. It follows from Lemma 4.1 that the holomorphic functions h_k have the required properties.

To state estimates for $\bar{\partial}^*N$ in norms with negative indices it is necessary to introduce tangential Sobolev norms. Each boundary point of Ω has a neighborhood in which it is possible to choose smooth real coordinates $t_1, \ldots, t_{2n-1}, \rho$, where ρ is a given defining function for Ω. A function u supported in such a boundary chart has tangential Fourier transform

$$\hat{u}(\tau, \rho) = \int_{\mathbf{R}^{2n-1}} e^{-i\langle t, \tau\rangle} u(t, \rho) \, dt$$

and tangential Sobolev norm

$$\|u\|_r^2 = \int_{-\infty}^0 \int_{\mathbf{R}^{2n-1}} \left(1 + |\tau|^2\right)^r |\hat{u}(\tau, \rho)|^2 \, d\tau \, d\rho.$$

Of course norms of forms are defined componentwise. When $r > 0$ the tangential norm is dominated by the usual Sobolev norm, but when $r < 0$ this relation is reversed (by duality).

LEMMA 4.3. *Let u be a $(0, 1)$-form with coefficients in $C^\infty(\overline{\Omega})$ such that $\overline{\partial}u = 0$. If ζ_1 and ζ_2 are supported in a boundary chart then for every real number r*

$$\left\|\zeta_1\overline{\partial}^*Nu\right\|_{r+1/2} \leqslant C\left(\left\|\zeta_2 u\right\|_r + \left\|u\right\|_{-M}^*\right) \quad if \ r \geqslant 0,$$

$$\left\|\zeta_1\overline{\partial}^*Nu\right\|_r \leqslant C\left(\left\|\left|\zeta_2 u\right|\right\|_{r-1/2} + \left\|\left|\zeta_2\overline{\partial}^*Nu\right|\right\|_{-M} + \left\|u\right\|_{-M}^*\right) \quad if \ r \leqslant 1,$$

with C independent of u.

The first case is well known. The second case comes not from duality but from the same techniques [**15**, p. 53] used to prove the first case. The proof is omitted.

The following lemmas concern Sobolev norms of holomorphic functions. The first one says that three different norms are equivalent.

LEMMA 4.4. *If h is a holomorphic function, then for every positive real number r*

$$\left\|\zeta_1 h\right\|_{-r} \leqslant \left\|\zeta_1 h\right\|_{-r}^* \leqslant C\left(\left\|\zeta_2 h\right\|_{-r} + \left\|h\right\|_{-M}\right),$$

and if ζ_1 and ζ_2 are supported in a boundary chart, then

$$C^{-1}\left\|\zeta_1 h\right\|_{-r} \leqslant \left\|\left|\zeta_1 h\right|\right\|_{-r} \leqslant C\left(\left\|\zeta_2 h\right\|_{-r} + \left\|h\right\|_{-M}\right)$$

with C independent of h.

To compute the Sobolev norm of a holomorphic function it turns out to be enough to consider derivatives in the direction normal to the boundary. Recall that if ρ is a normalized defining function for Ω, then $\partial/\partial\rho$ means $\Sigma(D_j\rho)D_j$.

LEMMA 4.5. *If h is a holomorphic function then for every real number r and every positive integer k*

$$\left\|\zeta_1 h\right\|_r \leqslant C\left(\left\|\zeta_2(\partial/\partial\rho)^k h\right\|_{r-k} + \left\|h\right\|_{-M}\right)$$

with C independent of h.

An inequality in the other direction also holds. In fact a differential operator of order k maps $W^r(\Omega)$ continuously into $W^{r-k}(\Omega)$ except when $r = 1/2, 3/2, \ldots, k - 1/2$, and restricting to holomorphic functions eliminates the exceptions:

LEMMA 4.6. *If h is a holomorphic function and L is a linear differential operator of order k with coefficients in $C^\infty(\overline{\Omega})$, then for every real number r*

$$\left\|\zeta_1 Lh\right\|_r \leqslant C\left(\left\|\zeta_2 h\right\|_{r+k} + \left\|h\right\|_{-M}\right)$$

with C independent of h.

The last three lemmas are actually true in a much more general setting. They hold for harmonic functions in arbitrary smooth bounded domains. The proofs use only standard elliptic theory (see [**10**] for details).

5. Proof of the key local estimates. This section is devoted to the proof of Theorem 3.1 and the estimates (2.2) and (2.3). To keep track of the local behavior it is convenient to fix a sequence $\eta_0, \eta_1, \ldots$ of smooth real-valued cut-off functions such that $\eta_0 = \zeta_1$, each η_{j+1} is identically equal to one in a neighborhood of the support of η_j, and ζ_2 is identically one on the support of η_j for every j. It is also useful to introduce the notation

$$\| \quad \|_q^{(*)} = \begin{cases} \| \quad \|_q & \text{if } q \geqslant 0, \\ \| \quad \|_q^* & \text{if } q < 0 \end{cases}$$

in order to treat the cases of positive and negative indices together.

To prove (3.1) observe that

$$\langle \zeta_1 T_s u, \varphi \rangle_0 = \langle u, P_0(\zeta_1 \varphi) \rangle_s = \sum_{|\alpha| \leqslant s} \langle D^\alpha u, D^\alpha P_0(\zeta_1 \varphi) \rangle_0$$

for u in $C^\infty(\overline{\Omega})$ and φ in $C_0^\infty(\Omega)$. When $|\alpha| \leqslant s$ and $r \geqslant s$

$$(5.1) \qquad |\langle D^\alpha u, D^\alpha P_0(\zeta_1 \varphi) \rangle_0| \leqslant \|\eta_1 D^\alpha u\|_{r-s} \|D^\alpha P_0(\zeta_1 \varphi)\|_{-(r-s)}^*$$
$$+ \|D^\alpha u\|_0 \|(1 - \eta_1) D^\alpha P_0(\zeta_1 \varphi)\|_0.$$

By Lemmas 4.4 and 4.6 and the estimates of Lemma 4.1 for the Bergman projection, it follows that the first term on the right-hand side is at most a constant times

$$\left(\|\zeta_2 u\|_r + \|u\|_{-M} \right) \|\varphi\|_{-r+2s}^{(*)},$$

while by the disjointness of the supports of $(1 - \eta_1)$ and ζ_1 the second term is dominated by $\|u\|_s \|\varphi\|_{-M}^*$ for every positive M. Therefore

$$|\langle \zeta_1 T_s u, \varphi \rangle_0| \leqslant C \left(\|\zeta_2 u\|_r + \|u\|_s \right) \|\varphi\|_{-r+2s}^{(*)}.$$

Since $C_0^\infty(\Omega)$ is dense in both $W_0^t(\Omega)$ and $(W^t(\Omega))^*$ when $t \geqslant 0$, it follows by taking the supremum over φ such that $\|\varphi\|_{-r+2s}^{(*)} = 1$ that

$$\|\zeta_1 T_s u\|_{r-2s} \leqslant C \left(\|\zeta_2 u\|_r + \|u\|_s \right).$$

This proves the first inequality in Theorem 3.1.

Inequality (3.2) for holomorphic functions follows from the same argument if in (5.1) the function u is replaced by h and the second term on the right-hand side is replaced by

$$\|D^\alpha h\|_{-M-s}^* \|(1 - \eta_1) D^\alpha P_0(\zeta_1 \varphi)\|_{M+s}.$$

Next consider inequality (3.3) for h in $H^s(\Omega)$. If ζ_1 has compact support in Ω then $\|\zeta_1 h\|_r \leqslant C \|h\|_{-M}$, so after taking a partition of unity it may be assumed that ζ_1 and ζ_2 are supported in a boundary chart. Suppose for the moment it has been shown that $\eta_{j+4} h \in W^q(\Omega)$ for a certain index j and a certain real number q less than or equal to $r - 1/2$. (This assumption holds for every j when $q \leqslant s$.) In view of the local density statement of Lemma 4.2 there is a sequence $h_1, h_2, \ldots$ of holomorphic functions in $C^\infty(\overline{\Omega})$ such that $h_k \to h$ in $W^s(\Omega)$ and $\eta_{j+3} h_k \to \eta_{j+3} h$ in $W^q(\Omega)$ as $k \to \infty$.

If $\varphi \in C_0^\infty(\Omega)$, then by Fubini's theorem

$$(5.2) \qquad \left\langle \eta_{j+1} T_s h, \varphi \right\rangle_0 = \left\langle h, P_0(\eta_{j+1}\varphi) \right\rangle_s = \lim_{k \to \infty} \left\langle h_k, P_0(\eta_{j+1}\varphi) \right\rangle_s$$

$$= \lim_{k \to \infty} \left\langle \eta_{j+1} T_s h_k, \varphi \right\rangle_0.$$

Writing $T_s = P_0 L^{2s}$ and $P_0 = \mathrm{Id} - \bar\partial^* N \bar\partial$ gives

$$(5.3) \qquad \left\langle \eta_{j+1} T_s h_k, \varphi \right\rangle_0 = \left\langle \eta_{j+1} L^{2s} h_k, \varphi \right\rangle_0 - \left\langle \eta_{j+1} \bar\partial^* N \bar\partial L^{2s} h_k, \varphi \right\rangle_0.$$

The main point of the argument is that the second term on the right-hand side is lower order than the first term. Lemma 4.3 implies

$$\left| \left\langle \eta_{j+1} \bar\partial^* N \bar\partial L^{2s} h_k, \varphi \right\rangle_0 \right| \leqslant C \Big(\left\| \eta_{j+2} \bar\partial L^{2s} h_k \right\|_{q-2s} + \left\|\left| \eta_{j+2} \bar\partial L^{2s} h_k \right\|\right|_{q-2s}$$

$$+ \left\|\left| \zeta_2 \bar\partial^* N \bar\partial L^{2s} h_k \right\|\right|_{-M-2s} + \left\| \bar\partial L^{2s} h_k \right\|^*_{-M-2s} \Big) \|\varphi\|^{(*)}_{2s-q-1/2}.$$

(One or the other of the first two terms in parentheses on the right-hand side is irrelevant, depending on whether $q - 2s$ is positive or negative.) Since $\bar\partial h_k = 0$ and the commutator $[\bar\partial, L^{2s}]$ is an operator of order $2s$, it follows from Lemma 4.6 that the first term in parentheses is at most a constant times $\|\eta_{j+3} h_k\|_q + \|h_k\|_{-M}$. In view of the norm equivalence stated in Lemma 4.4, the second term in parentheses admits the same bound. The third term in parentheses is at most

$$\left\|\left| \zeta_2 P_0 L^{2s} h_k \right\|\right|_{-M-2s} + \left\|\left| \zeta_2 L^{2s} h_k \right\|\right|_{-M-2s} \leqslant C \big(\| T_s h_k \|_{-M-2s} + \| h_k \|_{-M} \big),$$

and by estimate (3.2) already proved this is dominated by $\| h_k \|_{-M}$, as is the fourth term in parentheses. Thus

$$(5.4) \qquad \left| \left\langle \eta_{j+1} \bar\partial^* N \bar\partial L^{2s} h_k, \varphi \right\rangle_0 \right| \leqslant C \big(\| \eta_{j+3} h_k \|_q + \| h_k \|_{-M} \big) \|\varphi\|^{(*)}_{2s-q-1/2}.$$

Since φ has compact support

$$\lim_{k \to \infty} \left\langle \eta_{j+1} L^{2s} h_k, \varphi \right\rangle_0 = \left\langle \eta_{j+1} L^{2s} h, \varphi \right\rangle_0.$$

In view of estimate (5.4) it follows by combining (5.2) with (5.3) that

$$\left| \left\langle \eta_{j+1} L^{2s} h, \varphi \right\rangle_0 \right| \leqslant \left| \left\langle \eta_{j+1} T_s h, \varphi \right\rangle_0 \right| + C \big(\| \eta_{j+3} h \|_q + \| h \|_{-M} \big) \|\varphi\|^{(*)}_{2s-q-1/2}.$$

Taking the supremum over compactly supported φ such that $\|\varphi\|^{(*)}_{2s-q-1/2} = 1$ gives

$$(5.5) \qquad \left\| \eta_{j+1} L^{2s} h \right\|_{q-2s+1/2} \leqslant C \big(\left\| \eta_{j+1} T_s h \right\|_{q-2s+1/2} + \| \eta_{j+3} h \|_q + \| h \|_{-M} \big).$$

Recall from Lemma 1.1 that the top order term of L^{2s} on holomorphic functions is $2^s (\partial/\partial\rho)^{2s}$. In view of Lemma 4.5

$$\left\| \eta_j h \right\|_{q+1/2} \leqslant C \Big(\left\| \eta_{j+1} (\partial/\partial\rho)^{2s} h \right\|_{q-2s+1/2} + \| h \|_{-M} \Big)$$

$$\leqslant C \Big(\left\| \eta_{j+1} L^{2s} h \right\|_{q-2s+1/2} + \left\| \eta_{j+2} h \right\|_{q-1/2} + \| h \|_{-M} \Big).$$

Together with (5.5) this implies

$$(5.6) \qquad \|\eta_j h\|_{q+1/2} \leqslant C\Big(\|\zeta_2 T_s h\|_{r-2s} + \|\eta_{j+3} h\|_q + \|h\|_{-M}\Big).$$

If the right-hand side of (3.3) is finite, then (5.6) implies that $\eta_j h \in W^{q+1/2}(\Omega)$. By induction on q (starting from the value $\min(s, r - 1/2)$) it follows that $\eta_j h \in W^r(\Omega)$ for every j; moreover (5.6) holds whenever $q \leqslant r - 1/2$. The middle term on the right-hand side of (5.6) has the same form as the left-hand side, but with the index lowered by $1/2$. Iterating (5.6) at most $2[q + M + 1]$ times lowers this index to $-M$, resulting in

$$\|\eta_j h\|_{q+1/2} \leqslant C\Big(\|\zeta_2 T_s h\|_{r-2s} + \|h\|_{-M}\Big).$$

Now set j equal to 0 and q equal to $r - 1/2$ to obtain the desired estimate (3.3).

Inequality (3.4) is an easy consequence of (3.3) and the relation $T_s U_s = P_0$ of Lemma 2.1. If $h \in H^0(\Omega)$, then by (3.3)

$$(5.7) \qquad \|\zeta_1 U_s h\|_r \leqslant C\Big(\|\zeta_2 T_s U_s h\|_{r-2s} + \|U_s h\|_{-M}\Big)$$

$$= C\Big(\|\zeta_2 h\|_{r-2s} + \|U_s h\|_{-M}\Big),$$

so it remains only to estimate the global term in (5.7). Since U_s is bounded from $H^0(\Omega)$ to $H^s(\Omega)$, inequality (5.7) implies in particular the global estimate

$$(5.8) \qquad \|U_s h\|_{M+2s} \leqslant C\|h\|_M$$

when $M \geqslant 0$. If $\varphi \in C_0^\infty(\Omega)$, then Fubini's theorem, equation (5.8), and the estimates for the Bergman projection yield

$$|\langle U_s h, \varphi \rangle_0| = |\langle h, U_s \varphi \rangle_0| = |\langle h, U_s P_0 \varphi \rangle_0|$$

$$\leqslant C\|h\|_{-M-2s}\|U_s P_0 \varphi\|_{M+2s} \leqslant C\|h\|_{-M-2s}\|\varphi\|_M.$$

Therefore

$$(5.9) \qquad \|U_s h\|_{-M} \leqslant C\|h\|_{-M-2s}$$

when $M \geqslant 0$, and so (5.7) implies (3.4).

This completes the proof of Theorem 3.1. The global a priori estimates (2.2) and (2.3) are an immediate corollary. First note that (3.2) and (3.4) imply that T_s and U_s map $H^\infty(\overline{\Omega})$ into itself. Setting ζ_1 equal to 1 on Ω in (3.2) and (3.3) gives

$$\|T_s h\|_{r-2s} \leqslant C\|h\|_r, \qquad \|h\|_r \leqslant C\Big(\|T_s h\|_{r-2s} + \|h\|_{-M}\Big).$$

To get (2.2) it remains only to observe by (5.9) that for h in $H^\infty(\overline{\Omega})$

$$\|h\|_{-M} = \|U_s T_s h\|_{-M} \leqslant C\|T_s h\|_{-M-2s}.$$

Finally (2.3) follows from (2.2) by replacing h with $U_s h$ and using that $T_s U_s h$ equals h for h in $H^\infty(\overline{\Omega})$.

6. Further remarks. (1) The full force of the *strict* pseudoconvexity of Ω is not used in the proofs. The key element is the knowledge that $\overline{\partial}^* N$ gains a fractional derivative, but the size of the gain is unimportant. Accordingly Theorems A_s and B_s hold more generally when Ω is a smooth bounded pseudoconvex domain such that

its Neumann operator admits subelliptic estimates. David Catlin [11] has recently characterized such domains as being the domains of finite type, in the sense that the maximum order of contact of complex varieties with the boundary is finite. Examples of such domains are strictly pseudoconvex domains with C^∞ boundary and weakly pseudoconvex domains with real analytic boundary. See [12] for further discussion of finite type conditions.

(2) In this paper I have considered the projection P_s only for integral values of s, but it can equally well be defined for fractional s. This has some interest because s equal to $1/2$ corresponds to the Szegö projection. Methods similar to the ones used here show that the Szegö projection admits regularity estimates in domains of finite type. This result will be proved in a forthcoming paper [10].

BIBLIOGRAPHY

1. R. A. Adams, *Sobolev spaces*, Academic Press, New York, 1975.

2. N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337–404.

3. D. Barrett, *Regularity of the Bergman projection on domains with transverse symmetries*, Math. Ann. **258** (1982), 441–446.

4. S. R. Bell, *Biholomorphic mappings and the $\bar\partial$-problem*, Ann. of Math. **114** (1981), 103–113.

5. _____, *A representation theorem in strictly pseudoconvex domains*, Illinois J. Math. **26** (1982), 19–26.

6. S. R. Bell and H. P. Boas, *Regularity of the Bergman projection and duality of holomorphic function spaces*, Math. Ann. **267** (1984), 473–478.

7. S. R. Bell and D. Catlin, *Boundary regularity of proper holomorphic mappings*, Duke Math. J. **49** (1982), 385–396.

8. S. Bergman, *The kernel function and conformal mapping*, 2nd ed., Math. Surveys, no. 5, Amer. Math. Soc., Providence, R. I., 1970.

9. H. P. Boas, *Holomorphic reproducing kernels in Reinhardt domains*, Pacific J. Math. **112** (1984), 273–292.

10. _____, *Regularity of the Szegö projection in domains of finite type*, in preparation.

11. D. Catlin, *Boundary invariants of pseudoconvex domains*, Ann. of Math. (in press).

12. J. P. D'Angelo, *Real hypersurfaces, orders of contact, and applications*, Ann. of Math. **115** (1982), 615–637.

13. K. Diederich and J. E. Fornaess, *Boundary regularity of proper holomorphic mappings*, Invent. Math. **67** (1982), 363–384.

14. C. Fefferman, *The Bergman kernel and biholomorphic mappings of pseudoconvex domains*, Invent. Math. **26** (1974), 1–65.

15. G. B. Folland and J. J. Kohn, *The Neumann problem for the Cauchy-Riemann complex*, Ann. of Math. Studies, no. 75, Princeton Univ. Press, Princeton, N. J., 1972.

16. N. Kerzman, *The Bergman kernel. Differentiability at the boundary*, Math. Ann. **195** (1972), 149–158.

17. J. J. Kohn, *Harmonic integrals on strongly pseudoconvex manifolds. I, II*, Ann. of Math. **78** (1963), 112–148; **79** (1964), 450–472.

18. J. L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications*, Vol. I, Springer-Verlag, New York, 1972.

19. D. H. Phong and E. M. Stein, *Estimates for the Bergman and Szegö projections on strongly pseudo-convex domains*, Duke Math. J. **44** (1977), 695–704.

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