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NILPOTENT AUTOMORPHISM GROUPS OF RIEMANN SURFACES

BY

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ABSTRACT. The action of nilpotent groups as automorphisms of compact Riemann surfaces is investigated. It is proved that the order of a nilpotent group of automorphisms of a surface of genus $g \geqslant 2$ cannot exceed $16(g-1)$. Exact conditions of equality are obtained. This bound corresponds to a specific Fuchsian group given by the signature $(0; 2, 4, 8)$.

0.0 Introduction. The study of automorphisms of Riemann surfaces has acquired a great importance from its relation with the problems of moduli and Teichmuller space. After Schwarz, who first showed that the group of automorphisms of a compact Riemann surface of genus $g \geqslant 2$ is finite in the late nineteenth century, fundamental results were obtained by Hurwitz [**8**], who obtained the best possible bound $84(g-1)$ for the order of such group. About the same time Wiman [**16**] made a thorough study of the cases $2 \leqslant g \leqslant 6$, as well as improved this bound for a cyclic group, by showing that an exact upper bound for the order of an automorphism is $2(2g+1)$. All this was done using classical algebraic geometry, without use of Fuchsian groups. There was not much movement in the subject between the early 1900s and 1961, when Macbeath [**10**], following up a remark of Siegel, proved that there are infinitely many values of g for which the Hurwitz bound is attained, as well as infinitely many g for which it is not attained. Macbeath used the theory of Fuchsian groups.

By then it was known that every finite group can be represented as a group of automorphisms of a compact Riemann surface of some genus $g \geqslant 2$ (see Hurwitz [**8**], Burnside [**1**] and Greenberg [**2**]).

The aim of the present paper is to make a fairly detailed study of nilpotent automorphism groups of a Riemann surface of genus $g \geqslant 2$. The groups involved are finite, by Schwarz' theorem, and since a finite nilpotent group is the product of its Sylow subgroups, the p-localization homomorphisms (which are analogous, in a way to the method of taking residues modulo p in number theory) provide a natural tool for the study of nilpotent automorphism groups.

The problem which I set out to solve is to find and prove the "nilpotent" analogue of Hurwitz' theorem. Not only does this paper present a complete solution to this

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problem, but the restriction to nilpotent groups enables me to obtain much more precise information than is available in the general case. Moreover, all nilpotent groups attaining the maximum order turn out to be 2-groups (i.e., their order is a power of 2). The results are as follows: *Suppose G is a nilpotent group of automorphisms of a Riemann surface X of genus $g \geqslant 2$. Then $|G| \leqslant 16(g-1)$. If $|G| = 16(g-1)$, then $g-1$ is a power of 2. Conversely, if $g-1$ is a power of 2, there is at least one surface X of genus g with an automorphism group of order $16(g-1)$, which must be nilpotent since its order is a power of 2. This bound corresponds to a specific Fuchsian group given by the signature* $(0; 2, 4, 8)$.

The necessary and sufficient condition "$g-1$ is a power of 2" gives much more precise and far-reaching information about maximal nilpotent automorphism groups than is available for Hurwitz groups. Specific Hurwitz groups known at the present time give the impression that their orders are distributed in a very chaotic fashion among the multiples of 84, and it does not seem realistic to expect precise information about them. Indeed, at the time of writing, no information is known about such basic questions as whether the values of g for which there is a Hurwitz group have or have not positive density among the integers. This relatively simple structure is clearly a result of the restriction that only nilpotent groups should be considered, and does not differentiate the covering group $(0; 2, 3, 7)$ (for the Hurwitz problem) from the covering group $(0; 2, 4, 8)$ for the "nilpotent" problem. Indeed, there are many nonnilpotent automorphism groups covered by $(0; 2, 4, 8)$ whose order is not a power of 2. For instance, it follows from the methods of Macbeath's paper [**12**] that $\mathrm{PSL}(2, 17)$ is a *smooth factor group* of $(0; 2, 4, 8)$ though it is certainly not nilpotent.

1.0 Bound for the order of the automorphism group. In this introductory section, I set out the basic methods by which the results of the last two theorems of this section on the best possible bound $16(g-1)$ are obtained.

The approach used here is based on the method of Fuchsian groups including Singerman's Theorem, as well as the standard group-theoretic algorithms of Todd and Coxeter, and Reidemeister and Schreier. It is essentially equivalent to the method of Wiman and Hurwitz.

1.1 *Cocompact Fuchsian groups and signatures.* We consider Fuchsian groups acting on the upper half of the complex plane. A cocompact Fuchsian group Γ has presentation

$$(1.1.1) \qquad \left\langle x_j, a_k, b_k : x_j^{m_j}, x_1 \cdots x_r \prod_k [a_k, b_k],\ j = 1, \ldots, r,\ k = 1, \ldots, g \right\rangle$$

where $[a, b] = aba^{-1}b^{-1}$; g is the genus. We call the symbol

$$(1.1.2) \qquad S = (g; m_1, \ldots, m_r), \qquad r \geqslant 0, g \geqslant 0, m_i \geqslant 1,$$

the *signature* of Γ. If all $m_i \geqslant 2$, S is said to be *reduced*, otherwise *nonreduced*. If Γ has signature S, we write $\Gamma(S)$. Let $\bar{S}$ be obtained from S by dropping all $m_i = 1$. Thus $\Gamma(S) \cong \Gamma(\bar{S})$, but in what follows it is essential to consider S as well as $\bar{S}$. If there are no m_i (or if all $m_i = 1$), Γ is called a *surface group*.

Let $\Gamma = \Gamma(S)$ act on the complex upper half-plane H^2. Γ has a fundamental region F_Γ of hyperbolic area

$$\mu(F_\Gamma) = 2\pi\left[(2g - 2 + \sum_1^r \left(1 - \frac{1}{m_i}\right)\right]; \tag{1.1.3}$$

the rational number

$$\chi(S) = 2 - 2g + \sum_1^r \left(\frac{1}{m_i} - 1\right) \tag{1.1.4}$$

is its *Euler characteristic.*

It is known that if X is a compact Riemann surface of genus $g \geqslant 2$, then $X \simeq H^2/K$, where K is a Fuchsian surface group of genus g. Moreover, G is the automorphism group of X iff $G \simeq \Gamma(S)/K$, where $\Gamma(S)$ is Fuchsian and K is a surface group. Taking areas,

$$|G| = \frac{\mu(F_k)}{\mu(F_r)} = \frac{2 - 2g}{X(S)}; \qquad |G| = \text{order of } G, \tag{1.1.5}$$

this is the *Riemann-Hurwitz identity*. Note that $|G|$ is finite.

The signature S is called *degenerate* if

(a) $g = 0$ and $r = 1$, or

(b) $g = 0$ and $r = 2$, $m_1 \neq m_2$,

otherwise *nondegenerate*. If S is nondegenerate and Γ_1 is a subgroup of finite index in $\Gamma(S)$, then there exists a signature S_1 such that $\Gamma_1 = \Gamma(S_1)$ and

$$[\Gamma : \Gamma_1] = \chi(S_1)/\chi(S). \tag{1.1.6}$$

1.2 *More on degenerate signatures*. The degenerate signatures do, of course, define groups, but do so in such a way that the definition is in some sense uneconomical or redundant. For example, the signature $(0; m_1)$ gives an elaborate definition of the trivial group:

$$x_1^n = x_1^{-1} = 1. \tag{1.2.1}$$

The trivial group ought properly to belong to the signature

$$(0; \quad) \tag{1.2.2}$$

with empty set of periods and zero genus. With this signature the Euler characteristic of the trivial group is $+2$, which is consistent with the *index formula* (1.1.6). Therefore it is reasonable to regard (1.2.2) as a nondegenerate signature. The degenerate signatures are then characterized by the facts that:

(i) At least one of the relators can be replaced by an apparently stronger relator without affecting the group.

(ii) The index formula (1.1.6) is not valid if we use a degenerate signature to compute the *Euler characteristic*; that is why there is another family of degenerate signatures, namely,

$$g = 0, \qquad r = 2, \qquad m_1 \neq m_2. \tag{1.2.3}$$

Such a degenerate signature defines a cyclic group of order $d = \gcd(m_1, m_2)$; the proper signature for this group could be $(0; d, d)$, which is nondegenerate.

Certain signatures which yield positive χ are realized as finite groups acting on the 2-sphere, i.e., subgroups of the orthogonal group $O(3, R)$.

Now if $\chi(S) > 0$, $\Gamma(S)$ is finite, and by the Riemann-Hurwitz identity it has order

$$|\Gamma(S)| = 2(1 - g)/\chi(S). \tag{1.2.4}$$

But this implies $1 - g > 0$ or $g < 1$ which gives $g = 0$ for g is a nonnegative integer. Thus $\Gamma(S)$ acts on the Riemann 2-sphere $\overline{X} = S_2$ and has order $2/\chi(S)$. The only reduced nondegenerate signatures with $\chi(S) > 0$ are:

TABLE 1.1

Signature	Order	Type of Group
$(0; n, n)$	n	cyclic Z_n
$(0; 2, 2, n)$	$2n$	dihedral D_{2n}
$(0; 2, 3, 3)$	12	tetrahedral A_4
$(0; 2, 3, 4)$	24	octahedral S_4
$(0; 2, 3, 5)$	60	icosahedral A_5

If $\chi(S) = 0$, then the group $\Gamma(S)$ is *infinite* and solvable (and acts on the complex plane $\mathbf{C}$). In addition, this yields groups of isometries of the Euclidean plane:

TABLE 1.2

Signature	Order	Type of Groups
$r = 0$ $(1; \quad)$	∞	Free abelian group of rank 2
$r = 3$ $(0; 2, 4, 4)$ $(0; 2, 3, 6)$ $(0; 3, 3, 3)$	∞ ∞ ∞	Containing a free abelian group of rank 2 as a normal subgroup of finite index with cyclic factor group
$r = 4$ $(0; 2, 2, 2, 2)$	∞	Extension of Z_2 of free abelian group of rank 2.

REMARK. When $r = 3, 4$, the groups are called the space groups of 2-dimensional crystallography.

(c) Finally if $\chi(S) < 0$, then $\mu(F_\Gamma) > 0$, thus $\Gamma(S)$ can be realized as a Fuchsian group; that is, a discrete subgroup of PSL(2, R), the group of all Möbius transformations of the complex upper-half plane H^2.

1.3 *Smooth homomorphisms.*

1.3.1. A fundamental notion in this context is a *smooth homomorphism*, which is a homomorphism Φ from a *Fuchsian group* $\Gamma(S)$ onto a finite group G which preserves the *periods* of Γ; i.e. for every generator x_i, of order m_i, order of $\Phi(x_i)$ is also equal to m_i. If Φ: $\Gamma(S) \to G$ is smooth, then ker Φ is a Fuchsian surface group. A finite group which has such a homomorphism onto it will be called a *smooth quotient group*. If p is a prime number, then Φ is called *p-smooth* if the order of $\Phi(x_i)$ is divisible by the highest power p^{α_i} of p which divides m_i.

1.3.2. Φ is smooth if and only if Φ *is p-smooth for every prime divisor p of the product* $\prod_{i=1}^{r} m_i$ *of periods.*

THEOREM 1.3.1 [**6**, **7**]. *If S is a nondegenerate signature, then every torsion element (i.e., an element of finite order) in* $\Gamma(S)$ *is conjugate to some power of some* x_i. *Moreover, the order of* x_i *is precisely* m_i. *If* $\chi(S) \leqslant 0$, *every finite subgroup of* $\Gamma(S)$ *is cyclic.*

COROLLARY 1.3.1. *The identity homomorphism* id: $\Gamma(S) \to \Gamma(S)$ *is smooth if and only if the signature S is nondegenerate.*

1.4 *Automorphisms of compact Riemann surfaces.* Let X be any compact Riemann surface, and suppose $\tilde{X}$ is the universal covering space of X. The complex structure on X can now be lifted to $\tilde{X}$ so that the projection p: $\tilde{X} \to X$ is analytic. Let now G be a finite group of automorphisms, i.e., biholomorphic self-mappings of X. Then there is a group $\tilde{G}$ of automorphisms of $\tilde{X}$ of X obtained by taking all the liftings of all elements of G. See [**2**, **9**, **10**, **13**].

The group $\tilde{G}$ covers the Riemann surface automorphism group G. Then there is a homomorphism Φ: $\tilde{G} \to G$ of the covering group $\tilde{G}$ onto G such that its kernel is $\pi_1(X)$, the fundamental group of the surface X, and such that if $\tilde{\mathscr{F}}$: $\tilde{G} \times \tilde{X} \to \tilde{X}$ and $\mathscr{F}$: $G \times X \to X$ denote the group actions, the following diagram commutes:

$$\begin{array}{ccc} \tilde{G} \times \tilde{X} & \xrightarrow{\tilde{\mathscr{F}}} & \tilde{X} \\ \Phi \downarrow \quad p \downarrow & & p \downarrow \\ G \times X & \xrightarrow{\mathscr{F}} & X \end{array} \tag{1.4.1}$$

In this case if g denotes the orbit genus of X, then $\tilde{X}$ will be one of the three simply-connected Riemann surfaces $\mathbf{C} = \mathbf{C}U\{\infty\}$, $\mathbf{C}$ or Δ, and $\tilde{G}$ will be a group of a signture S. The $\ker(\Phi) = \pi_1(X)$ will be the group of the signature $(g;\quad)$, and by (1.1.5)

$$|G| = (2 - 2g)/\chi(S). \tag{1.4.2}$$

Thus G is a Fuchsian group if and only if $\chi(S) < 0$ or $2 - 2g < 0$, i.e. if and only if $g \geqslant 2$, for if $g = 0$ then $\chi(S) > 0$, and if $g = 1$ then $\chi(S) = 0$. And since $\pi_1(X)$ is torsion-free the homomorphism Φ is smooth. Conversely if G is any finite group, any smooth homomorphism Φ: $\Gamma(S) \to G$ induces a group action of G as a group of automorphisms of the Riemann surface $\tilde{X}/\ker \Phi$. Therefore we have the following result.

1.4.3. *We can obtain all Riemann surface automorphism groups* (G, X) *with G finite and X compact by finding all the smooth homomorphisms* Φ *of the Fuchsian groups* $\Gamma(S)$ *onto finite groups G.*

1.5 *The localization of the signatures.*

1.5.1. Let p be a prime number, and as before let $S = (g; m_1, \ldots, m_r)$ be a *signature* and $\Gamma(S)$ the group defined by this signature. For each $i = 1, \ldots, r$, let p^{α_i} be the highest power of the prime p which divides m_i. Then we call the signature

$S_p = (g;\, p^{\alpha_1},\ldots,p^{\alpha_r})$ the *p-localization* of S. If every period of S is already some power of one fixed prime p, then we call the *signature* $S = S_p$ a *p-local signature*, and the group defined by S_p, i.e., $\Gamma(S_p)$, the *p-localized Fuchsian group*. This group has the following presentation:

$$\Gamma(S_p) = \left\langle x_1',\ldots,x_r',\, a_1',\, b_1',\ldots,a_g',\, b_g' \middle| (x_1')^{p^{\alpha_1}},\ldots,(x_r')^{p^{\alpha_r}},\ \prod_{i=1}^{r} x_i' \prod_{j=1}^{g} [a_j',\, b_j'] \right\rangle. \tag{1.5.1}$$

Using the hypothesis that $p^{\alpha_i}|m_i$, we have $(x_i')^{m_i} = 1$. And so the mapping defined on the generating set by

$$x_i \to x_i',\ a_j \to a_j',\ b_k \to b_k' \qquad \begin{matrix}(i = 1,\ldots,r)\\ (j,\, k = 1,\ldots,g)\end{matrix}$$

can be extended to a homomorphism

$$l_p\colon \Gamma(S) \to \Gamma(S_p)$$

which we shall call a *p-localization homomorphism*. We require the following theorems by A. M. Macbeath [**11**].

THEOREM 1.5.1. *If G_p is a finite p-group and $\phi\colon \Gamma(S) \to G_p$ is a homomorphism, then there is a unique homomorphism $\phi_p\colon \Gamma(S_p) \to G_p$ such that $\phi = \phi_p \circ l_p$.*

THEOREM 1.5.2. *Let G be a finite nilpotent group and, for each prime p, let G_p be its p-Sylow subgroup. For formal simplicity let $G_p = \{1\}$ if $p \nmid |G|$. Let $\phi\colon \Gamma(S) \to G$ be a homomorphism and let $\lambda_p\colon G \to G_p$ be the projection of G (as product of its Sylow subgroups) onto G_p. Then ϕ is smooth if and only if $(\lambda_p \circ \phi)_p\colon \Gamma(S_p) \to G_p$ is smooth for each prime divisor p of $\prod_{i=1}^{r} m_i$.*

Theorem 1.5.2 shows that one can study *nilpotent Riemann surface automorphism* groups by studying the *smooth homomorphisms* of *p-local groups* onto finite p-groups. We can observe this idea in detail in the following.

Let $\pi(S) = \{\, p : p | \prod_{i=1}^{r} m_i \}$, p = prime. Now if $p \notin \pi(S)$, S_p is free of periods and $\Gamma(S_p)$ is a surface group, and thus every homomorphism from $\Gamma(S_p)$ to a finite group is smooth. Let $p_1,\ldots,p_k \in \pi(S)$, and let $G = G_{p_1} \times \cdots \times G_{p_k}$ be a finite nilpotent group. Then each smooth homomorphism $\phi\colon \Gamma(S) \to G$ determines k smooth homomorphisms

$$\psi_{p_i}\colon \Gamma(S_{p_i}) \to G_{p_i} \qquad (i = 1,\ldots,k)$$

such that if $\gamma \in \Gamma(S)$ and $g_i = \psi_{p_i} \circ l_{p_i}(\gamma) \in G_{p_i}$, then $\phi(\gamma) = g_1,\ldots,g_k \in G$. Thus to find all the covering maps $\Phi\colon \Gamma(S) \to G$ one can find all smooth homomorphisms $\psi_i\colon \Gamma(S_{p_i}) \to G_{p_i}$, where G_{p_i} is a p_i-Sylow subgroup of G.

1.6 *The p-Frattini series of a group.*

1.6.1. Let G be a finitely generated group, and let p be a prime number. Define

$$G^p = \left\langle a^p,\, bcb^{-1}c^{-1} \middle| a,\, b,\, c \in G \right\rangle.$$

G^p is characteristic in G and is called the *p-Frattini* subgroup of G [**15**]. The factor group G/G^p is an elementary abelian p-group. Suppose G has the presentation $\langle g_i | R(g_i)\rangle$. Then the presentation for G/G^p is obtained from G by adding the extra relators a_i^p and $[a_k, a_l]$; $i, k, l = 1,\dots,m$.

The p-Frattini series of G is defined by:

$$G = G_0^p \supseteq G_1^p \supseteq \cdots \supseteq G_k^p \supseteq \cdots,$$

where

$$G_{k+1}^p = (G_k^p)^p, \qquad k = 0,1,2,\dots.$$

Then G_i^p is also characteristic in G and G/G_i^p is a finite p-group for all $i = 1,2,\dots$. Next we consider the p-Frattini series of $\Gamma(S_p)$, where

$$S_p = (g;\, p^{\alpha_1},\dots,p^{\alpha_r}).$$

1.6.2. Let $N = \max\{\alpha_1, \alpha_2,\dots,\alpha_r\}$, and let $\chi(S_p) \leqslant 0$.

THEOREM 1.6.1 [**14**]. *Let Γ have signature $S = (g;\, m_1,\dots,m_r)$. Then Γ contains a subgroup Γ_1 with signature*

$$S_1 = \left(g', n_{11}, n_{12},\dots,n_{1k_1}, n_{21}, n_{22},\dots,n_{2k_2},\dots,n_{r_1}, n_{r_2},\dots,n_{rk_r}\right)$$

such that $[\Gamma : \Gamma_1] = N$ if and only if there exists a finite permutation group G transitive on N points and a homomorphism $\Phi\colon \Gamma \to G$ onto G with the properties:

(i) *The permutation $\Phi(x_i)$ has precisely k_i cycles of lengths*

$$\frac{m_i}{n_{i1}}, \frac{m_i}{n_{i2}},\dots, \frac{m_i}{n_{ik_i}}.$$

(ii) $N = [\Gamma : \Gamma_1] = \chi(\Gamma_1)/\chi(\Gamma)$.

The following lemma is by A. M. Macbeath [**11**].

LEMMA 1.6.1. *If $r > 2$, then the maximum period of the group $(\Gamma(S_p))^p$ is p^{N-1}.*

LEMMA 1.6.2. *If $r = 1$ and $\chi(S_p) \leqslant 0$, then the number of periods of $(\Gamma(S_p))^p$ is greater than or equal to* 4.

PROOF. In this case $S_p = (g;\, p^N)$ and

$$\Gamma(S_p) = \left\langle x_1, a_1, b_1,\dots,a_g, b_g \,\middle|\, x_1^{m_1} = x_1 \prod_{j=1}^{g} [a_j, b_j] = 1 \right\rangle.$$

Thus $x_1 = (\prod_{j=1}^g a_j b_j a_j^{-1} b_j^{-1})^{-1} \in G' \subset (\Gamma(S_p))^p$. And $\chi(S_p) = 1 + p^{-N} - 2g$ and so we must have $g \geqslant 1$. In [**11**, Lemma 6.4], it is proved that the number of periods is $\geqslant p^{2g}$, which gives the result.

We now give a presentation for the quotient group $\Gamma(S_p)/(\Gamma(S_p))^p = \Gamma/\Gamma^p$, say, in terms of the generators $x_1' = x_1\Gamma^p$, $a_i' = a_i\Gamma^p$, $b_j' = b_j\Gamma^p$, where $i, j = 1,\dots,g$. We have relators

$$x_1'^{p^N},\ x_1' \prod_{j=1}^{g} [a_j', b_j'] \tag{1}$$

from the original presentation of Γ_1 together with the relators

(2) $\quad x_1'^p, a_i'^p, b_j'^p, [x_1', a_i'], [x_1', b_j'], [a_i', a_j'], [b_i', b_j'], [a_i', b_j'], [b_i', a_j']$.

Since Γ^p contains all commutators, the second relator in (1) can be reduced to $x_1' = 1$, so the relators $x_1'^p, x_1'^{p^N}$ in (1) and (2) can be omitted, and we have for Γ/Γ^p the elementary abelian group of rank $2g$ generated by a_i', b_i'.

Thus the order of Γ/Γ^p must be $p^{2g} \geqslant 2^2 = 4$. We now apply Theorem 1.6.1 to $\Gamma = \Gamma(S_p)$.

If we let Φ: $\Gamma \to \Gamma/\Gamma^p$ be the natural homomorphism, the group Γ/Γ^p can be realized as a permutation subgroup of the group $S_{p^{2g}}$ transitive on p^{2g} points. Now Φ maps x_1 onto the identity element of Γ/Γ^p, i.e., $\Phi(x_1) = (1)(2)\cdots(p^{2g})$, a permutation with p^{2g} cycles all of length one. Thus

$$n_{ij} = \frac{m_i}{\text{length of the cycle}} = \frac{p^N}{1} = p^N, \qquad i = 1,\ j = 1,\ldots,p^{2g}.$$

Therefore

$$n_{11} = n_{12} = \cdots = n_{1p^{2g}} = p^N$$

and $S_1 = (g', p^N, p^N, \ldots, p^N)$, so the number of periods of $\Gamma^p(S_p)$ is $p^{2g} \geqslant 4$.

Using the Riemann-Hurwitz identity,

$$N = p^{2g} = \frac{2 - 2g' + p^{2g}(p^{-N} - 1)}{2 - 2g + p^{-N} - 1},$$

or $p^{2g}(2 - 2g) = 2 - 2g'$, $g' = (g-1)p^{2g} + 1$. Next, combining Lemmas 1.6.1 and 1.6.2, we have

$$\text{maximum period of } \Gamma_2^p < \text{maximum period of } \Gamma.$$

Thus we can conclude the following result:

THEOREM 1.6.2. *If S_p is a p-local signature with $\chi(S_p) \leqslant 0$, then Γ_k^p is torsion-free if k is sufficiently large.*

Since the natural homomorphism ϕ: $\Gamma \to \Gamma/\Gamma_k^p$ is smooth if and only if Γ_k^p is torsion-free, we can deduce the following

COROLLARY 1.6.1. *If S_p is a p-local signature of nonpositive Euler characteristic, then $\Gamma(S_p)$ covers infinitely many Riemann surface automorphism groups which are finite p-groups.*

1.7 *Relationship between the lower central series and localization.*

1.7.1. Let $S = (g; m_1, \ldots, m_r)$; l_p: $\Gamma(S) \to \Gamma(S_p)$ is the p-localization homomorphism

$$\Pi(S) = \left\{ p_1, \ldots, p_k : p_i \Big| \prod_{i=1}^{r} m_i,\ i = 1, \ldots, k \right\}.$$

Let $\Gamma_f(S)$ be the characteristic subgroup of Γ generated by the set of all elements of finite order in Γ. If $y \in \Gamma$ has finite order, then $y = t^{-1}x_i^{n_i}t$ for some periodic generator x_i in Γ. Therefore we have

$$\Gamma_f = \text{Normal closure } \{x_1, \ldots, x_r\},$$

and the following

LEMMA 1.7.1. *For all prime numbers* p, $\ker l_p \subseteq \Gamma_f(S)$, *and equality holds if and only if* $p \notin \Pi(S)$.

PROOF. By definition, $l_p\colon \Gamma(S) \to \Gamma(S_p)$ is called the p-localization homomorphism, and we have

$$l_p\left(x_i^{p^{\alpha_i}}\right) = (x_i')^{p^{\alpha_i}} = 1.$$

Therefore $\Gamma(S_p)$ is obtained from $\Gamma(S)$ by adjoining all the relators $x_1^{p^{\alpha_1}},\dots,x_r^{p^{\alpha_r}}$. Thus $\ker l_p =$ normal closure $\{x_1^{p^{\alpha_1}},\dots,x_r^{p^{\alpha_r}}\} \subset \Gamma_f(S)$.

Next suppose $p \notin \Pi(S)$, then S_p has no periods, i.e. $\Gamma(S_p)$ is a surface group. Hence we must have $\alpha_1 = \cdots = \alpha_r = 0$ which implies

$$\ker l_p = \text{normal closure}\{x_1,\dots,x_r\} = \Gamma_f(S).$$

This result states that if $q \notin \Pi(S)$, then $\ker l_p \subseteq \ker l_q$ for all prime numbers p.

1.7.2. *The lower central series for* $\Gamma(S)$. The normal series

$$\Gamma = \gamma_1(\Gamma) \triangleright \cdots \triangleright \gamma_i(\Gamma) \triangleright \cdots,$$

where $\gamma_{k+1}(\Gamma) = [\Gamma, \gamma_k(\Gamma)]$, is the lower central series of $\Gamma(S)$.

Let also $\gamma_\infty(\Gamma) = \bigcap_{k=1}^{\infty} \gamma_k(\Gamma)$ where $\gamma_\infty(\Gamma)$ is called the "nilpotent residual" of Γ.

Then $\gamma_\infty(\Gamma)$ satisfies the following identities:

(a) $\gamma_\infty(\Gamma) = \{\alpha \in \Gamma\colon \phi(\alpha) = 1$ for all homomorphisms $\phi\colon \Gamma \xrightarrow{\text{onto}} G$ with nilpotent $G\}$.

(b) $\gamma_\infty(\Gamma) = \bigcap_{p \in \Pi(S)} \ker l_p$.

PROOF OF (a). First if $y \notin \gamma_\infty(\Gamma)$, then $y \notin \gamma_K(\Gamma)$ for any K, i.e. $y\gamma_K(\Gamma) \neq \gamma_K(\Gamma)$.

Now let $\phi\colon \Gamma \to \Gamma/\gamma_K(\Gamma)$ be the canonical homomorphism; then $\phi(y) = y\gamma_K(\Gamma)$ and so $\phi(y) \neq 1$. Conversely, if $\phi\colon \Gamma \to G$ (G a nilpotent group of class K) is a homomorphism from Γ onto G such that $\phi(y) \neq 1$ for some $y \in \Gamma$, then $y \notin \ker \phi$, and we have

$$1 = \gamma_{K+1}(G) = \gamma_{K+1}(\phi(\Gamma)) = \phi(\gamma_{K+1}(\Gamma)).$$

Thus $\gamma_{K+1}(\Gamma) \subset \ker \phi$, that is $\gamma_\infty(\Gamma) \subset \ker \phi$. Therefore $y \notin \gamma_\infty(\Gamma)$.

PROOF OF (b). By Lemma 1.7.1 if $q \notin \Pi(S)$, then $\ker l_p \subset \ker l_q$; thus we need to show only $\gamma_\infty(\Gamma(S)) = \bigcap_p \ker l_p$ where p is any prime. Next to show this we let $x \notin \gamma_\infty(\Gamma(S))$, thus there is a nilpotent group G_1 not necessarily finite, and a homomorphism $\phi\colon \Gamma(S) \to G_1$ such that $\phi(x) \neq 1$. But $\Gamma(S)$ is finitely generated, thus $\phi(\Gamma) = G_1$ is also finitely generated. Therefore by a theorem of (Gruenberg) there is a second homomorphism $\psi\colon \phi(\Gamma) \to G_2$ where G_2 is a finite nilpotent group, such that $\psi(\phi(x)) \neq 1$. Let $G_2 = \prod_p \otimes G_p$; then for at least one projection $\omega\colon G_2 \to G_p$ for some prime p, $\omega(\psi(\phi(x))) \neq 1$. Letting $\omega \circ \psi \circ \phi = \delta$, we find $\delta\colon \Gamma(S) \to G_p$ is a homomorphism of $\Gamma(S)$ onto a finite p-group such that $x \notin \ker \delta$.

By Theorem 1.5.1, there exists a unique homomorphism $\delta_p\colon \Gamma(S_p) \to G_p$ such that $\delta = \delta_p \circ l_p$ where l_p is the p-local homomorphism. Therefore $x \notin \ker(\delta_p \circ l_p)$, i.e., $x \notin \ker l_p$ for this prime p which implies $x \notin \bigcap_p \ker l_p$. Conversely let $x \notin \ker l_q$ for some prime q (i.e. $l_q(x) \neq 1$). Since $l_q(x) \in \Gamma(S_q)$, and $\Gamma(S_q)$ is a residually finite

q-group, there exists a homomorphism ψ_q: $\Gamma(S_q) \to G_q$, where G_q is a finite q-group such that $\psi_q(l_q(x)) \neq 1$. Letting $\phi = \psi_q \circ l_q$, then ϕ is a homomorphism of $\Gamma(S)$ onto G_q such that $\phi(x) \neq 1$, which implies $x \notin \gamma_\infty(\Gamma(S))$, and this proves (b).

1.8 *Covering groups of nilpotent Riemann surface automorphism groups.* In the previous subsections of this paper we have dealt with problems of obtaining information about the relationship between nilpotent groups of automorphisms and the family of p-local signatures of a given signature. In this subsection we want to characterize precisely those signatures $S = (g; m_1, \ldots, m_r)$ for which the group $\Gamma(S)$ actually can cover at least one nilpotent automorphism group of some Riemann surface. If $\Gamma(S)$ is a finite group having positive $\chi(S)$ Euler characteristic, then $\Gamma(S)$ can only cover itself. Thus we shall assume $\chi(S) \leqslant 0$.

DEFINITION 1.8.1. We call a signature S *nilpotent-admissible* if every p-local signature S_p of S is nondegenerate.

We require the following two important theorems by A. M. Macbeath [**11**].

THEOREM 1.8.1. The following are equivalent:

(i) *S is a nilpotent-admissible signature.*

(ii) *$\Gamma(S)$ can cover at least one nilpotent group of automorphisms of a Riemann surface.*

(iii) *The intersection $\gamma_\infty(\Gamma(S))$ of the lower central series of $\Gamma(S)$ is torsion-free.*

The next theorem relates the number of nilpotent automorphism groups covered by a *nilpotent-admissible* signature to the nature of the Euler-characteristic of its p-local signature.

THEOREM 1.8.2. *Let S be a nilpotent-admissible signature; then one of the following holds:*

(i) *If $\chi(S_p) > 0$ for every prime $p \in \Pi(S)$, then there is only one nilpotent Riemann surface automorphism group G covered by $\Gamma(S)$. Moreover, the lower central series of $\Gamma(S)$ in this case becomes constant after a finite number of steps, and all the terms of the series have finite index, only the constant one being torsion-free.*

(ii) *If $\chi(S_p) \leqslant 0$ for at least one $p \in \Pi(S)$, then there are infinitely many nilpotent Riemann surface automorphism groups covered by $\Gamma(S)$. In this case, on the other hand, all the terms in the lower central series of $\Gamma(S)$ are distinct.*

EXAMPLE. The only nilpotent Riemann surface automorphism group G covered by $\Gamma(S)$ when $S = (0; 2, 2g+1, 2(2g+1))$ is the cyclic group $Z_{2(2g+1)}$, which was discovered by A. Wiman [**16**] and W. J. Harvey [**4**] to be the largest cyclic group of automorphisms of a Riemann surface of genus $g \geqslant 2$.

Finally in the next theorem we consider all finitely generated cocompact Fuchsian groups having nilpotent-admissible signatures. Using the fact that every Fuchsian group has a fundamental region of positive hyperbolic area, we will find the minimum value of this area.

THEOREM 1.8.3. *Let Γ be a finitely generated cocompact Fuchsian group with a nilpotent-admissible signature $S = (g; m_1, \ldots, m_r)$, then $\mu(F_\Gamma) \geqslant \pi/4$, and equality occurs only when Γ is the $(2, 4, 8)$ triangle group (i.e. the group of signature $(0; 2, 4, 8)$).*

PROOF. Write $\mu(F_\Gamma) = \mu$. If Γ has the above signature, then by (1.1.3)

$$\mu = 2\pi\left[2g - 2 + \sum_{j=1}^{r}\left(1 - \frac{1}{m_j}\right)\right], \qquad 2 \leqslant m_1 \leqslant \cdots \leqslant m_r < \infty.$$

(Of course r may be zero, in which case the sum by definition is zero.)

The proof is made by considering three cases.

Case 1. $g \geqslant 2$.

$$\mu > 2\pi\left[2 + \sum_{j=1}^{r}\left(1 - \frac{1}{m_j}\right)\right] \geqslant 4\pi.$$

Case 2. $g = 1$.

$$\mu = 2\pi \sum_{j=1}^{r}\left(1 - \frac{1}{m_j}\right) \geqslant 2\pi\frac{r}{2} \geqslant \pi.$$

Case 3. $g = 0$.

$$\mu = 2\pi\left[-2 + \sum_{j=1}^{r}\left(1 - \frac{1}{m_j}\right)\right] \geqslant 2\pi\left(-2 + \frac{r}{2}\right).$$

(i) $r \geqslant 5$. $\mu \geqslant \pi$.

(ii) $r = 4$. If all $m_j = 2$, $\mu = 0$ and Γ is not Fuchsian. Hence assume $m_1, m_2, m_3 \geqslant 2$, $m_4 \geqslant 3$; then $\mu \geqslant 2\pi(-2 + 3/2 + 2/3) = \pi/3$.

(iii) $r = 2$. $\mu < 0$ and Γ cannot be a Fuchsian group.

Therefore the only case left to be considered is $g = 0$, $r = 3$, i.e. the *triangle groups*. Then $\mu = 2\pi[1 - 1/m_1 - 1/m_2 - 1/m_3]$, $2 \leqslant m_1 \leqslant m_2 \leqslant m_3 \leqslant \infty$ and $\mu > 0$ rules out $m_j = 2, j = 1, 2, 3$, as well as $m_1 = m_2 = 2$.

Subcase 1. $m_j \geqslant 3, j = 1, 2, 3$, which can be divided into four parts.

(i) $m_1 = 3$, $m_2 \geqslant 4$, $m_3 \geqslant 4$. $\mu \geqslant \pi/3$.

(ii) $m_1 = m_2 = 3$, $m_3 \geqslant 4$. $\mu = 2\pi(1/3 - 1/m_3)$. If $\mu < \pi/4$, then $m_3 = 4$. Hence $S = (0; 3, 3, 4)$ and the 2-local signature $(0; 4)$ is degenerate.

(iii) $m_1 = m_2 = m_3 = 3$. Then $\mu = 0$.

(iv) $m_j \geqslant 4$ for all $j = 1, 2, 3$. $\mu > \pi/2$.

Subcase 2. $m_1 = 2$, $m_2 \geqslant 3$, $m_3 \geqslant 3$. $\mu = 2\pi(1/2 - 1/m_1 - 1/m_2)$.

(a) $m_2 \geqslant 6$, $m_3 \geqslant 6$. Then $\mu > \pi/3$.

(b) $3 \leqslant m_2 < 6$, $m_3 \geqslant m_2$. There are three possibilities for this case.

(i) $S = (0; 2, 3, m)$, $m \geqslant 7$. $\mu = 2\pi(1/6 - 1/m)$. Now $\mu < \pi/4$ only if $m \leqslant 23$, or $7 \leqslant m \leqslant 23$. But among these 17 integers all those divisible by a prime $p \neq 2, 3$ must be dropped out, because then the p-local signature S_p would be degenerate. Thus $m = 8, 9, 12, 16, 18$. Moreover, if $2^\alpha | m$ $(3^\alpha | m)$ for some $\alpha \geqslant 2$, then the 2-local (3-local) signature is degenerate.

(ii) $S = (0; 2, 5, m)$, $m \geqslant 5$. $\mu = 2\pi(3/10 - 1/m)$. Again $\mu < \pi/4$ only for $m \leqslant 5$. Thus the only possibility is $m = 5$. But if $S = (0; 2, 5, 5)$, then the 2-local signature is $(0; 2)$ and is degenerate.

(iii) $S = (0; 2, 4, m)$, $m \geqslant 4$. In this final case $\mu = 2\pi[1/4 - 1/m]$, and $\mu > 0$ implies $m \geqslant 5$. And $\mu < \pi/4$ only when $m \leqslant 8$. Hence $m = 5, 6, 7, 8$ are the only

possible numbers for the last period. Therefore we have:

(i) $S = (0; 2, 4, 5)$, which has the 5-local signature $(0; 5)$ degenerate.

(ii) $S = (0; 2, 4, 6)$, which has the 3-local signature $(0; 3)$ degenerate.

(iii) $S = (0; 2, 4, 7)$, which has the 2-local $(0; 2, 4)$ and 7-local $(0; 7)$ signatures, both degenerate.

Thus a bound for a nilpotent-admissible signature occurs when S has the exact form $(0; 2, 4, 8)$, which is in its own 2-local form, and for that group $\mu(F_\Gamma) = \pi/4$. This completes the proof.

This leads immediately to the first main result. Define Γ_0 to be the group of signature $(0; 2, 4, 8)$, a notation we shall use from now on.

THEOREM 1.8.4. *Let G be a finite nilpotent group acting on some compact Riemann surface X of genus $g \geqslant 2$. Then G has order $|G| \leqslant 16(g-1)$. Equality occurs if and only if $X = H^2/\Gamma$, where Γ is a proper normal subgroup of finite index in Γ_0.*

PROOF. Let $\tilde{X}$ be the universal covering space of X, then by subsection 1.4 there is a group $\tilde{G}$ which covers G. In that case there is a smooth homomorphism ϕ of the covering group $\tilde{G}$ onto G, such that the kernel $\pi_1(X)$ of ϕ is the fundamental group of the surface X and is the group with signature $(0; g)$. Here $\tilde{X}$ is the complex upper-half plane H^2 and $\tilde{G}$ is a Fuchsian group. By 1.1.5,

$$|G| = \frac{\operatorname{Area}\left(H^2/\pi_1(X)\right)}{\operatorname{Area}\left(H^2/\Gamma(S)\right)}.$$

By the area formula 1.1.3, $\operatorname{Area}(H^2/\pi_1(x)) = 4\pi(g-1)$. By Theorem 1.8.3, $\operatorname{Area}(H^2/\Gamma) \geqslant \pi/4$, and equality occurs if and only if $\Gamma(S) = \Gamma_0$. The result now follows.

2.0 The structure of the $(2, 4, 8)$-triangle group.

2.1 In view of Theorem 1.8.4 of §1.0, which shows that $\Gamma_0 = (0; 2, 4, 8)$ is the unique *nilpotent-admissible* signature with Euler characteristic of minimum absolute value, it is natural for us to look closely at the properties of this group, and in particular its *nilpotent smooth quotient groups*.

The argument in Theorem 1.8.4 shows that if $|\operatorname{Aut}(X)| = 16(g-1)$, then $X = H^2/\Gamma$ where Γ is a proper normal subgroup of Γ_0. (Here Γ is defined to be the group of covering transformations of the universal covering $\phi\colon H^2 \to X$.)

We discovered that the restriction of nilpotency makes it possible for us to answer completely the question: "Which values of g are possible for a nilpotent automorphism group of order $16(g-1)$?" In fact, g is a possible value if and only if $g-1$ is a power of 2.

THEOREM 2.1.1. *Let S be a signature with genus 0, and G a nilpotent automorphism group covered by $\Gamma(S)$. Then all prime factors of $|G|$ are factors of periods of $\Gamma(S)$. In particular if S is p-local with genus zero, then every nilpotent automorphism group covered by $\Gamma(S)$ is a p-group.*

PROOF. Let S be the signature $(0; m_1, \ldots, m_r)$ and let $\Pi(S)$ denote, as before, the set of prime divisors of the periods of S. Let p be a prime number such that $p \notin \Pi(S)$. Then the p-localization signature S_p is $(0;\ )$, and so $\Gamma(S_p)$ represents the trivial group. But now by Theorem 1.5.2 the p-Sylow subgroup of G must be a factor group of $\Gamma(S_p)$, therefore trivial, i.e. p does not divide the order of G. This proves that the only prime factors of $|G|$ are precisely those which divide the periods of $\Gamma(S)$. In particular, if $S = S_p$ is a p-local signature for some prime p, then every period of $\Gamma(S)$ is a power of this prime p. Thus the only prime divisor of the order of G is p, which implies that G is a p-group.

COROLLARY 2.1.1. *Every nilpotent automorphism group covered by Γ_0 is a 2-group. Thus if a surface of genus g admits a nilpotent automorphism group G of order $16(g-1)$, then $g-1$ must be a power of 2.*

Note. There are many *nonnilpotent* groups covered by Γ_0. (See, for instance, Macbeath [**12**].) It is only because we restrict ourselves to nilpotent groups that we obtain such complete results arithmetically. We shall prove, conversely, later, that if $g-1$ is a power of 2, then there is always at least one nilpotent automorphism group covered by Γ_0, so that the values of $\underline{g}$ such that some surface of genus $\underline{g}$ admits a nilpotent automorphism group are completely characterized. But first let us consider the case when $n = 4$, i.e. G is a 2-group of order 16, and $g = 2$. We ask: "Does there exist a compact Riemann surface of genus 2 and a nilpotent automorphism group of order 16 covered by Γ_0?" There are precisely nine types of nonabelian groups of order 16 and five types of abelian ones; see Burnside [**1**]. Among these there is only one $(2,4,8)$-group given by $G = \langle a, b \mid a^2 = b^8 = 1, aba = b^3 \rangle$. It can be seen easily that ab is of order 4. Since $ab = b^3a^{-1} = b^3a$, $(ab)^2 = b^3a^2b = b^4$. Hence $(ab)^4 = b^8 = a^2 = 1$. Therefore, the only $(2,4,8)$-group of order 16 is the group $G_1 = \langle a, b \mid a^2 = (ab)^4 = b^8 = 1, aba = b^3 \rangle$. Now let Γ_0 be generated by P and Q where $P^2 = Q^8 = (PQ)^4 = 1$. Let $\Theta\colon \Gamma \to G_1$ be a homomorphism defined by $\Theta(P) = a$, $\Theta(Q) = b$, $\Theta(PQ) = \Theta(p)\Theta(Q) = ab$. Hence Θ is smooth because every element of finite order belong to $\ker(\Theta)$ must be conjugate to some power of P or Q or PQ. Therefore, $\ker(\Theta)$ is a Fuchsian surface group of genus 2, and G_1 is a *smooth quotient group* for Γ_0. We denote this kernel by N_1 and use it as the first step in an induction argument to prove the following existence theorem.

THEOREM (2.1.2) (EXISTENCE). *For any integer $n \geqslant 4$, there exists a nilpotent 2-group G of order 2^n acting on a compact Riemann surface X of genus $g = 2^{n-4} + 1$.*

In the proof of Theorem 2.1.2 we need the following elementary but technical lemma.

LEMMA 2.1.1. *Let G be a finite p-group and let $\{1\} \neq N \lhd G$. Then there exists a series of subgroups $N = N_1 \supset \cdots \supset N_s = \{1\}$ each normal in G with $[N_i : N_{i+1}] = p$.*

PROOF. Since N is normal in G, it is a union of G-conjugacy classes, each of which contains p^m elements for some m. Partitioning N into its G-conjugacy classes we have: $|N| = |N \cap Z(G)| + \sum_{i=1}^{k} \alpha_i$, ($Z(G)$ is the center of G), where the α_i are the

sizes of the distinct conjugacy classes of noncentral elements in N. Suppose $a_i \in N$ is not in $Z(G)$; then $C(a_i)$, the centralizer of a_i, is a proper subgroup of G, and so $\alpha_i = [G : C(a_i)]$ is a power of p. Thus p divides each α_i and therefore also $|N \cap Z(G)|$. Hence $N \cap Z(G)$ is nontrivial, and has order a power of p. We now prove the assertion of the lemma by induction on the order of N. Assume that the lemma in question is true for all subgroups N^* of any p-group G^* where $|G^*| < |G|$. Let N_{s-1} be a subgroup of $N \cap Z(G)$ of order p. Let ρ be the natural homomorphism ρ: $G \to G/N_{s-1} = G^*$. Now $|G^*| < |G|$, so by the induction hypothesis applied to $N^* = N/N_{s-1}$, there exists a series $N^* = N_1^* \supset \cdots \supset N_{s-1}^* = \{1\}$ with N_i^* normal in $G^* = G/N_{s-1}$. Letting $N_i = \rho^{-1}(N_i^*)$ for $i = 1, \ldots, s-1$ we obtain the desired series for N, G. This enables us to prove that, for every $n \geqslant 4$, there exists a surface X of genus $g = 2^{n-4} + 1$, and a nilpotent 2-group (covered by Γ_0) of automorphisms of X.

PROOF OF THEOREM 2.1.2. Let $\Gamma_0 = \Gamma(S)$, and let G_1 be the unique $(2, 4, 8)$-group of order 16, i.e., the group generated by a and b satisfying the relators $a^2 = b^8 = 1$, $aba = b^3$. Let $N_1 = \ker \Theta$, where as before Θ is the smooth homomorphism Θ: $\Gamma_0 \to G_1$ with the smooth quotient group G_1. We have shown that N_1 is a surface subgroup of Γ_0 with genus $g = 2$. Since $\chi(S) = -1/8$ is negative and S is a 2-local signature, we can use Corollary 1.6.1 to deduce that Γ_0 contains normal subgroups N_1 and N_2 with $\Gamma_0 \triangleright N_1 \triangleright N_2$ such that

(i) genus$(N_1) = 2$,

(ii) genus$(N_2) \geqslant 2^{n-4} + 1$,

(iii) Γ_0/N_2 is a finite 2-group.

Now let G be the finite 2-group Γ_0/N_2, and ϕ the natural map ϕ: $\Gamma_0 \to \Gamma_0/N_2$. Let $\phi(N_1) = N = N_1/N_2$, thus $N \triangleleft G$. By Lemma 2.1.1, there is a 2-group N_α with $N \supset N_\alpha$, $N_\alpha \triangleleft G$ and $[N : N_\alpha] = 2^{n-4}$. Let $N_3 = \phi^{-1}(N_\alpha)$. Then N_3 is normal in $\Gamma_0 = \phi^{-1}(G)$ and $N_1 \supset N_3$; thus N_3 must also be a surface group. Now by the Riemann-Hurwitz relation N_3 has genus $2^{n-4} + 1$, and by standard theory Γ_0/N_3 is a group of automorphisms of the compact Riemann surface H^2/N_3.

The observation that all nilpotent groups of maximum order turn out to be 2-groups suggests the problem (obviously closely related to the general nilpotent problem in view of the localization techniques used in our attack) of determining for each odd prime p, the "p-group" analogue of Hurwitz' bound. It turns out, as often happens in questions of this nature, that $p = 3$ is exceptional and harder to deal with, whereas all primes $p \geqslant 5$ can be dealt with at once.

The following results will be shown in a later paper, using similar techniques.

(i) *If G is a 3-group, then* $|G| \leqslant 9(g-1)$. *If* $g - 1 = 3^n$, $n \geqslant 4$, *then there is a surface X of genus g with* $9(g-1)$ *automorphisms. There is no automorphism group of order* 9 *acting on genus* 2, *there is no automorphism group of order* 27 *acting on genus* 4, *and there is no automorphism group of order* 81 *acting on genus* 10.

(ii) *If G is a p-group for any prime* $p \geqslant 5$, *then*

$$|G| \leqslant \frac{2p}{p-3}(g-1).$$

Conversely, if

$$g - 1 = \frac{p-3}{2} p^n, \qquad n \geqslant 0,$$

then there is a surface of genus g with an automorphism group of order p^{n+1},

The bounds (i), (ii), correspond to specific Fuchsian groups, given by the signatures $(0; 3, 3, 9)$ for $p = 3$ and $(0; p, p, p)$ for $p \geqslant 5$, which cover the two types of automorphism groups. I have also made a study of the lower central series of each of these groups, by computing the terms to the point where a torsion-free subgroup is reached.

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