DECOMPOSITIONS INTO CODIMENSION-TWO MANIFOLDS

BY

R. J. DAVERMAN AND J. J. WALSH

ABSTRACT. Let \( M \) denote an orientable \((n + 2)\)-manifold and let \( G \) denote an upper semicontinuous decomposition of \( M \) into continua having the shape of closed, orientable \( n \)-manifolds. The main result establishes that the decomposition space \( M/G \) is a 2-manifold.

1. Introduction. Let \( M \) denote an orientable \((n + 2)\)-manifold and let \( G \) denote an upper semicontinuous decomposition of \( M \) into continua having the shape of closed, orientable \( n \)-manifolds. The main result here establishes that the decomposition space \( M/G \) is a 2-manifold.

This theorem lifts what is known about decompositions into codimension-two submanifolds to nearly the same level as what is known about decompositions into codimension-one submanifolds. Liem [L] proved that if \( G \) is a decomposition of an \((n + 1)\)-manifold \( N \) into compacta having the shape of \( S^n \), then \( N/G \) is a 1-manifold, and for \( n > 4 \) the decomposition map \( p: N \rightarrow N/G \) can be approximated by a locally trivial bundle map. Daverman [D] showed that if \( G \) is a decomposition of \( N \) into continua having the shape of arbitrary closed \( n \)-manifolds, then \( N/G \) is a 1-manifold (possibly with boundary if various orientability conditions are not all met).

In an earlier paper [DW1] we derived the conclusion that \( M/G \) is a 2-manifold in case \( M \) is an \((n + 2)\)-manifold and each \( g \) in \( G \) has the shape of \( S^n \). Then the natural map \( p: M \rightarrow M/G \) must be particularly nice, being an approximate fibration when \( n > 1 \) (the same is "almost" true when \( n = 1 \)). The present work evolved from techniques similar to those of [DW1], which were inspired for the most part by methods of Coram and Duvall [CD1, CD2]. However, the strong conclusion that \( p: M \rightarrow M/G \) is an approximate fibration cannot be attained in the more general situation at hand.

In the fourth section we determine the structure of a certain sheaf \( \mathcal{H}^n \) on \( M/G \) induced by \( p \). The properties of this sheaf can be used to recover the conclusions in [DW1] just mentioned, in the special situation where the elements of \( G \) have the shape of a \( n \)-sphere.

Received by the editors April 5, 1984 and, in revised form, July 31, 1984.

1980 Mathematics Subject Classification. Primary 57N15, 57N05; Secondary 55P55, 54B15.

Key words and phrases. Codimension-two submanifold, winding function, upper semicontinuous decomposition, 2-manifold.

\(^1\)Research of the first author was supported in part by NSF Grant MCS 81-20741; research of the second author was supported in part by NSF Grant MCS 82-00913 and Alfred P. Sloan Foundation Grant.

\(^{\dagger}\)1985 American Mathematical Society

0002-9947/85 $1.00 + $.25 per page
The manifold features present in such decomposition spaces $M/G$ do not persist through successively larger codimensions. If $G$ is a decomposition of a manifold $M$ into codimension-three submanifolds, $M/G$ need not be a manifold nor need it even be a generalized manifold. It seems that the fundamental question to address is whether $M/G$, if finite dimensional, is an ANR.

It should be pointed out that, in our terminology, a manifold is separable, metric and boundaryless. We will speak of a manifold with boundary as the need to permit a boundary occurs.

The symbol $\approx$ will be used to mean “is homeomorphic to” and the symbol $\cong$ will be used to mean “is isomorphic to.” Unless expressly stated otherwise, homology and cohomology groups are determined with integral ($\mathbb{Z}$) coefficients.

2. The winding function and its continuity set. Throughout the remainder of the paper we shall employ the following notation: $M$ for an orientable $(n + 2)$-manifold, $n \geq 1$, $G$ for a usc (the standard abbreviation of “upper semicontinuous”) decomposition of $M$ into compact connected elements each having the shape of a closed orientable $n$-manifold, $B$ for the decomposition space $M/G$, and $p$ for the decomposition map $M \to B$.

We begin with a variation on the Coram-Duvall definition of local winding number function. Temporarily fix $b_0 \in B$. Since $g_0 = p^{-1}b_0$ is movable, there exist connected neighborhoods $U$ and $U_0$ of $b_0$ in $B$ such that $b_0 \in U_0 \subset U$ and (the inclusion induced)

$$\phi_0: \hat{H}_n(p^{-1}b_0) \to H_n(p^{-1}U)$$

is an isomorphism onto the image of

$$\phi: H_n(p^{-1}U_0) \to H_n(p^{-1}U).$$

For any $b \in U_0$, the image of

$$\phi_b: H_n(p^{-1}b) \to H_n(p^{-1}U)$$

is contained in $\text{Im} \phi$. Hence,

$$\phi_0 \circ \phi_b: \hat{H}_n(p^{-1}b) \to \hat{H}_n(p^{-1}b_0)$$

is a well-defined homomorphism between copies of $\mathbb{Z}$, meaning that it amounts to multiplication by some integer $q_b \geq 0$. Define the winding function $\alpha: U_0 \to \mathbb{Z}$ by $\alpha(b) = q_b$.

Let $K$ denote the set of all points $b_0$ in $B$ such that every neighborhood $W$ of $b_0$ (with $W \subset U_0$) contains some point $w$ for which $\alpha(w) = 0$. As in previous analyses in [CD1, CD2, DW1], the set $K$ is a troublesome spot. Clearly, $K$ is closed in $B$.

**Lemma 2.1.** $K$ is nowhere dense in $B$.

**Proof.** See [CD2, Lemma 3.1].

Define $B_+ = B - K$. In the rest of this section we outline an argument showing that the complement in $B_+$ of a certain countable closed subset is a 2-manifold. Although operating under less limiting hypotheses, we are able to extract enough from arguments in [CD1 and DW1, §3] to satisfy our needs.
We shall continue to work locally throughout most of this section. Assume that $b_0 \in B$ mentioned near the beginning lies in $B_+$ and that the neighborhood $U_0$ mentioned there satisfies $U_0 \subset B_+$ and

$$\text{Im}\{ \mathcal{H}_1(p^{-1}b_0) \rightarrow H_1(p^{-1}U) \} = \text{Im}\{ H_1(p^{-1}U_0) \rightarrow H_1(p^{-1}U) \}$$

as well. The three facts stated below are direct analogues of Lemmas 1–3 in [CD1].

1. If $b \in U_0$, then there is a neighborhood $V$ of $b$ in $U_0$ such that to each $b' \in V$ there corresponds a positive integer $k$ such that $a(b') = ka(b)$.

2. $a$ is lower semicontinuous.

3. The continuity set $C = \{ b \in U_0; a$ is continuous at $b \}$ is open and dense in $U_0$; its complement $D = U_0 - C$ can be written as $D_1 \cup D_2$ where $D_2$ is countable, $D_1$ is closed, and each neighborhood of $d \in D_1$ contains uncountably many points of $D_1$.

(Here $D_2 = \{ d \in D; some$ neighborhood of $d$ in $D$ is countable$\}$ and $D_1 = D - D_2$.)

**Lemma 2.2.** Let $A$ be an arc in $C$. Then, using either $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$-coefficients,

1. $\mathcal{H}^{n+1}(p^{-1}A) \equiv 0$;
2. for $x \in A$, $i^*: \mathcal{H}^n(p^{-1}A) \rightarrow \mathcal{H}^n(p^{-1}x)$ is surjective; and
3. for $x, y \in A$, $\ker\{ \mathcal{H}^n(p^{-1}A) \rightarrow \mathcal{H}^n(p^{-1}x) \} = \ker\{ \mathcal{H}^n(p^{-1}A) \rightarrow \mathcal{H}^n(p^{-1}y) \}$.

**Proof.** We assume the coefficient group is $\mathbb{Z}$, the argument being the same for other coefficient groups. For each $x \in A$, there is a closed subarc $A_x$, a neighborhood of $x$ in $A$, for which there is a shape strong deformation retraction of $p^{-1}A_x$ to $p^{-1}x$, the deformation occurring in a neighborhood of $p^{-1}x$. Consequently, the diagram

$$
\begin{array}{ccc}
\mathcal{H}^n(p^{-1}A_x) & \xrightarrow{i^*} & \mathcal{H}^n(p^{-1}x) \\
\downarrow \cong & & \downarrow \cong \\
\mathcal{H}^n(p^{-1}y) & \xleftarrow{i^*} & \mathcal{H}^n(p^{-1}x)
\end{array}
$$

commutes for each $y \in A_x$, where $\psi$ is induced by the retraction and is an isomorphism as we are working in the continuity set. It follows easily that the arc $A_x$ satisfies conclusions (ii) and (iii).

If a closed subarc $A'$ of $A$ satisfies conclusion (ii), then obviously so does each closed subarc $A''$; moreover, inclusion induces a surjection $\mathcal{H}^n(p^{-1}A') \rightarrow \mathcal{H}^n(p^{-1}A'')$.

To see why this is so, express $A' = E_1 \cup A'' \cup E_2$ as the union of closed subarcs (one of $E_1$ or $E_2$ being empty, should $A''$ contain an endpoint of $A'$), where $E_1 \cap A''$ and $E_2 \cap A''$ are endpoints of $A''$. Given an element $z \in \mathcal{H}^n(p^{-1}A'')$, use conclusion (ii) to find $z_j \in \mathcal{H}^n(p^{-1}E_j)$ for $j \in \{1, 2\}$ such that

$$\text{Im}\{ z_j \rightarrow \mathcal{H}^n(p^{-1}(A'' \cap E_j)) \} = \text{Im}\{ z \rightarrow \mathcal{H}^n(p^{-1}(A'' \cap E_j)) \}.$$

From the Mayer-Vietoris sequence of $(p^{-1}(A'' \cup E_1), p^{-1}A'', p^{-1}E_1)$,

$$\mathcal{H}^n(p^{-1}(A'' \cup E_2)) \rightarrow \mathcal{H}^n(p^{-1}A'') \oplus \mathcal{H}^n(p^{-1}E_1) \xrightarrow{\Delta} \mathcal{H}^n(p^{-1}(A'' \cap E_1)),$$
we see that $\Delta(z \oplus z_1) = 0$, which gives $z'' \in \tilde{H}^n(\mathcal{D})$, whose restrictions (= inclusion-induced images) in $\tilde{H}^n(\mathcal{D})$ and $\tilde{H}^n(\mathcal{D}_1)$ are $z$ and $z_1$, respectively. Similarly, starting from part of the Mayer-Vietoris sequence for

$$
p^{-1}(E_1 \cup A'' \cup E_2), p^{-1}(E_1 \cup A'''), p^{-1}E_2,
$$
we obtain $z' \in H^n(p^{-1}A')$ whose restriction to $H^n(p^{-1}A'')$ is $z$. It now follows easily that if, in addition, $\mathcal{A}$ satisfies conclusion (iii), then so does each closed subarc.

Let $A_1$ and $A_2$ denote closed subarcs of $\mathcal{A}$ such that $A_1 \cap A_2 \neq \emptyset$. By the preceding argument, if $A_1$ and $A_2$ satisfy conclusion (ii), so does their union $A_1 \cup A_2$. If both $A_1$ and $A_2$ satisfy conclusion (iii), we can verify that the same is true of $\mathcal{A}$ by supposing $y \in A_1 \cap A_2$ and $x \in A_1$, say, and considering the commutative diagram:

$$
\begin{array}{ccc}
\tilde{H}^n(p^{-1}A) & \leftarrow & \tilde{H}^n(p^{-1}A_1) \\
\downarrow & & \downarrow \\
\tilde{H}^n(p^{-1}x) & \leftarrow & \tilde{H}^n(p^{-1}y)
\end{array}
$$

Let $A'$ be a maximal subarc of $\mathcal{A}$ such that conclusions (ii) and (iii) are valid for each proper closed subarc of $A'$. Each endpoint of $A'$ is contained in $\mathcal{A}$, else the arc $A_x$, where $x$ is an endpoint of $A'$, together with the conclusions of the previous paragraph would lead to a contradiction of the maximality of $A'$. For the same reason, $A' = \mathcal{A}$.

Finally, conclusion (i) is a consequence of conclusion (ii) since repeated applications of the Mayer-Vietoris sequence reveal

$$
\tilde{H}^{n+1}(p^{-1}A) \cong \tilde{H}^{n+1}(p^{-1}A_1) \oplus \cdots \oplus \tilde{H}^{n+1}(p^{-1}A_k)
$$

for any “partition” $A_1, \ldots, A_k$ of $\mathcal{A}$ into subarcs. The nontriviality of $\tilde{H}^{n+1}(p^{-1}A)$ would ultimately detect the nontriviality of $\tilde{H}^{n+1}(p^{-1}x)$ for some $x \in \mathcal{A}$, but this is not possible.

**Lemma 2.3.** Neither a point nor an arc in $C$ separates $C$, but every simple closed curve there does separate $C$.

**Proof.** Let $\mathcal{A}$ denote an arc in $C$ and $V$ the component of $C$ containing $\mathcal{A}$. From the exact sequence of the pair $(p^{-1}V, p^{-1}V - p^{-1}A)$, we obtain

$$
H_1(p^{-1}V, p^{-1}V - p^{-1}A) \to \tilde{H}_0(p^{-1}V - p^{-1}A) \to \tilde{H}_0(p^{-1}A) \cong 0
$$

and, by duality [S, p. 296] and Lemma 2.2,

$$
\tilde{H}_1(p^{-1}V, p^{-1}V - p^{-1}A) \cong \tilde{H}^{n+1}(p^{-1}A) \cong 0.
$$

Thus $p^{-1}V - p^{-1}A$ and $V - A$ are connected.

Similarly, no point separates $C$.

Now, let $J$ denote a simple closed curve in some component $V$ of $C$. Express $J$ as the union of two arcs $A_1$ and $A_2$ whose intersection is the boundary $\{a_1, a_2\}$ of each. From the Mayer-Vietoris sequence for $p^{-1}J = p^{-1}A_1 \cup p^{-1}A_2$, we analyse with $\mathbb{Z}/2\mathbb{Z}$-coefficients

$$
\cdots \to \tilde{H}^n(p^{-1}A_1) \oplus \tilde{H}^n(p^{-1}A_2) \xrightarrow{\mathcal{H}} \tilde{H}^n(p^{-1}\{a_1, a_2\}) \to \tilde{H}^{n+1}(p^{-1}J) \to 0
$$

$$
\cong \downarrow \phi
$$

$$
\tilde{H}^n(p^{-1}a_1) \oplus \tilde{H}^n(p^{-1}a_2) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}
$$
(where the zero at the right comes from Lemma 2.2). Write $\phi \circ \beta = \delta_1 \oplus \delta_2$, where, for $i = 1, 2, \delta_i; \hat{H}^n(p^{-1}A_i) \oplus \hat{H}^n(p^{-1}A_2) \to \hat{H}^n(p^{-1}A_i)$. The conclusion in part (iii) of Lemma 2.2 reveals that $\delta_1(z, 0) = 0$ if and only if $\delta_2(z, 0) = 0$ and, likewise, that $\delta_1(0, z) = 0$ if and only if $\delta_2(0, z) = 0$. It follows that the image of $\phi \beta$ is contained in the diagonal subgroup $\{(0, 0), (1, 1)\}$ and, in light of part (ii) of Lemma 2.2, is equal to this subgroup. Consequently,

\[
\hat{H}^{n+1}(p^{-1}J; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}.
\]

Continuing throughout the remainder of the proof to use $\mathbb{Z}/2\mathbb{Z}$-coefficients, we establish that $p^{-1}J$ separates $p^{-1}V$ by showing that the homomorphism $\gamma$ in the sequence below is trivial:

\[
\begin{array}{cccccccccc}
H_1(p^{-1}V) & \to & H_1(p^{-1}V, p^{-1}V - p^{-1}J) & \to & \hat{H}_0(p^{-1}V - p^{-1}J) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \equiv \\
H_1(p^{-1}U_0 - p^{-1}J) & \to & H_1(p^{-1}U_0) & \to & H_1(p^{-1}U_0, p^{-1}U_0 - p^{-1}J) & \to & \hat{H}^{n+1}(p^{-1}J) & \cong & \mathbb{Z}/2\mathbb{Z} \\
\end{array}
\]

A simple chase through the diagram

\[
\begin{array}{cccccccccc}
H_1(p^{-1}V - p^{-1}J) & \to & H_1(p^{-1}V) & \to & H_1(p^{-1}V, p^{-1}V - p^{-1}J) \\
\downarrow & & \downarrow & & \downarrow & & \equiv \\
H_1(p^{-1}U_0 - p^{-1}J) & \to & H_1(p^{-1}U_0) & \to & H_1(p^{-1}U_0, p^{-1}U_0 - p^{-1}J) \\
\downarrow j & & \downarrow j & & \downarrow j & & \equiv \\
H_1(p^{-1}U - p^{-1}J) & \to & H_1(p^{-1}U) & \to & H_1(p^{-1}U, p^{-1}U - p^{-1}J) \\
\end{array}
\]

establishes the triviality of $\gamma$ provided $\text{Im } \eta \supset \text{Im } j$. One can detect the last containment by choosing $c_0 \in U_0 - J$ so close to $b_0$ that $c_0$ lies in a neighborhood $W$ of $b_0$ for which there is a shape deformation retraction in $p^{-1}U_0$, say $r: p^{-1}W \to p^{-1}b_0$. Since the restriction of $r$ induces an isomorphism $\hat{H}_n(p^{-1}c_0) \to \hat{H}_n(p^{-1}b_0)$ with $\mathbb{Z}$ and, hence, with $\mathbb{Z}/2\mathbb{Z}$-coefficients (as we are working in the continuity set), $r$ induces a surjection $\hat{H}_1(p^{-1}c_0) \to \hat{H}_1(p^{-1}b_0)$. It follows that

\[
\text{Im} \{ \hat{H}_1(p^{-1}c_0) \to H_1(p^{-1}U) \} = \text{Im} \{ \hat{H}_1(p^{-1}b_0) \to H_1(p^{-1}U) \}.
\]

We stipulated earlier (for $\mathbb{Z}$ and, hence, $\mathbb{Z}/2\mathbb{Z}$-coefficients) that

\[
\text{Im} \{ \hat{H}_1(p^{-1}b_0) \to H_1(p^{-1}U) \} = \text{Im} \{ H_1(p^{-1}U_0) \to H_1(p^{-1}U) \},
\]

and, as

\[
\text{Im } \eta \supset \text{Im} \{ \hat{H}_1(p^{-1}c_0) \to H_1(p^{-1}U) \},
\]

we conclude that $\text{Im } \eta \supset \text{Im } j$.

ADDENDUM. Having established that $p^{-1}J$ separates, we can conclude that $\hat{H}^{n+1}(p^{-1}J; \mathbb{Z}) \cong \mathbb{Z}$. The only point in the preceding argument that made essential use of $\mathbb{Z}/2\mathbb{Z}$-coefficients arose in the conclusion that the image of $\beta$ equaled a diagonal subgroup. For $\mathbb{Z}$-coefficients, we can draw the analogous conclusion provided we know $\hat{H}^{n+1}(p^{-1}J; \mathbb{Z}) \neq 0$, which holds because $p^{-1}J$ separates $V$ (and since the homomorphism $\gamma$ analysed in the proof is trivial for $\mathbb{Z}$-coefficients as well).
Theorem 2.4. The space $C$ is a 2-manifold.

Proof. This is a consequence of Lemma 2.3 and [Wi, p. 95].

Lemma 2.5. The set $D$ is countable.

Proof. By fact (3), $D = D_1 \cup D_2$ where $D_1$ is closed in $D$ and dense-in-itself, and where $D_2$ is countable. We shall prove that $D_1 = \emptyset$.

Suppose $D_1 \neq \emptyset$. Since $\alpha|D_1$ is lower semicontinuous, the set where $\alpha|D_1$ is continuous forms a dense open subset of $D_1$. Thus, we can find $b \in D_1$ and some neighborhood $W$ of $b$ in $U_0$ such that $\alpha|D_1 \cap W$ is constant. Accordingly, as in fact (1), $W$ can be restricted so that for all $w \in W$ there exists an integer $k(w) > 0$ such that $\alpha(w) = k(w) \cdot \alpha(b)$. Since $b$ is a discontinuity of $\alpha$, there is some $w \in W - D_1$ for which $k(w) > 1$. This yields $k(w') \geq k(w) > 1$ for all $w' \in$ some neighborhood of $w$, so we can assume $w \in W - D$.

Using the countability of $D_2$, we build an arc $A$ in $W$ with endpoints $c, d$ in $D_1$ and with $A - \{c, d\} \subset W - D$ such that, for $x \in A - \{c, d\}$,

$$\alpha(x) = k \alpha(c) = k \alpha(d) \quad \text{(some } k > 1).$$

It suffices to compute that $\tilde{H}^{n+1}(p^{-1}A) \cong \mathbb{Z} / k \mathbb{Z}$, for one can verify that this is impossible by inspecting the sequence

$$H_1(p^{-1}W) \to H_1(p^{-1}W, p^{-1}W - p^{-1}A) \to \tilde{H}_0(p^{-1}W - p^{-1}A) \to 0$$

while recalling the argument in Lemma 2.3 that $\gamma$ is the trivial homomorphism.

Express $A$ as the union of two subarcs $A_1, A_2$ having a common endpoint $e$. It suffices to determine that, for $i = 1, 2$, $\tilde{H}^{n+1}(p^{-1}A_i) \equiv 0$ and that

$$\operatorname{Im}\{\tilde{H}^n(p^{-1}A_i) \to \tilde{H}^n(p^{-1}e)\} = k \mathbb{Z}.$$

Then the tail of a Mayer-Vietoris sequence

$$\tilde{H}^n(p^{-1}A_1) \oplus \tilde{H}^n(p^{-1}A_2) \to \tilde{H}^n(p^{-1}e) \to \tilde{H}^{n+1}(p^{-1}A) \to 0$$

detects that $\tilde{H}^{n+1}(p^{-1}A) \cong \mathbb{Z} / k \mathbb{Z}$.

Sublemma 2.6. Let $W$ and $D$ be as above and let $A$ be an arc with one endpoint $c \in D$, with $A \setminus \{c\} \subset W$, and with $\alpha(x) = k \alpha(c)$ for $x \in A - \{c\}$ (some $k > 1$). Then

(i) $\tilde{H}^{n+1}(p^{-1}A) = 0$, and

(ii) $\operatorname{Im}\{\tilde{H}^n(p^{-1}A) \to \tilde{H}^n(p^{-1}e)\} = k \mathbb{Z}$, where $e$ is the endpoint of $A - \{c\}$.

Proof. Conclusion (i) follows from conclusion (ii) of Lemma 2.2, just as conclusion (i) of Lemma 2.2 followed. (The key observation is that, expressing $A = A_1 \cup A_2$ as the union of subarcs having just an endpoint in common,

$$\tilde{H}^n(p^{-1}A_1) \oplus \tilde{H}^n(p^{-1}A_2) \to \tilde{H}^n(p^{-1}(A_1 \cap A_2))$$

is surjective, since one of $A_1$ and $A_2$ is contained in the continuity set.)
Take an arbitrary neighborhood $N$ of $p^{-1}A$. Let $A_1$ be a proper subarc of $A$ containing $c$ for which there exists a shape strong deformation of $p^{-1}A_1$ to $p^{-1}c$, the deformation taking place in $N$. Let $d$ denote the other endpoint of $A_1$ and $A' = \text{Cl}(A - A_1)$. It follows that

$$\text{Im}\{\tilde{H}^n(p^{-1}A_1) \to \tilde{H}^n(p^{-1}d)\} = k\mathbb{Z}$$

because $a(d) = k\alpha(c)$. From the cohomology ladder

$$\begin{array}{ccc}
\tilde{H}^n(p^{-1}A) & \to & \tilde{H}^n(p^{-1}A') \\
\downarrow & & \downarrow \\
\tilde{H}^n(p^{-1}A_1) & \to & \tilde{H}^n(p^{-1}d)
\end{array} \quad \begin{array}{ccc}
\tilde{H}^{n+1}(p^{-1}A, p^{-1}A') & \to & 0 \\
\downarrow & & \downarrow \\
\tilde{H}^{n+1}(p^{-1}A_1, p^{-1}d) & \to & 0
\end{array}$$

we see that $\text{Im}\{\tilde{H}^n(p^{-1}A) \to \tilde{H}^n(p^{-1}d)\} = k\mathbb{Z}$. The desired conclusion that

$$\text{Im}\{\tilde{H}^n(p^{-1}A) \to \tilde{H}^n(p^{-1}e)\} = k\mathbb{Z}$$

follows from the following diagram and the conclusion of Lemma 2.2 that the surjections $i^*_e$ and $i^*_d$ have equal kernels:

$$\begin{array}{ccc}
\tilde{H}^n(p^{-1}A) & \leftarrow & \tilde{H}^n(p^{-1}e) \\
\downarrow & & \downarrow \\
\tilde{H}^n(p^{-1}A') & \rightarrow & \tilde{H}^n(p^{-1}d)
\end{array}$$

This completes the proof that $D$ is countable.

Now it is appropriate to let the point $b_0$, identified shortly after the statement of Lemma 2.1, take on different values in $B_+ = B - K$. For each $b \in B_+$ we have a neighborhood $U_b \subset B_+$, a closed and countable subset $D_b$ of $U_b$, a dense open subset $C_b = U_b - D_b$ of $U_b$, and a winding function $\alpha_b: U_b \to \mathbb{Z}$ with the properties described at the outset of this section. In case $c \in U_b(1) \cap D_b(2)$, then $c \in C_b(1)$ iff $c \in C_b(2)$, since continuity of $\alpha_b(i)$ at $c$ depends only on the behavior of point inverses near $p^{-1}c$. Select a countable subcollection $\{U_b(i)\}$ of $\{U_b\}$ covering $B_+$ and define $C_+$ as $\bigcup_i C_b(i)$. From Theorem 2.4 and Lemma 2.5 we obtain

**Theorem 2.7.** The set $C_+$ is open and dense in $B$ and $D_+ = B_+ - C_+$ is a countable, relatively closed subset of $B_+$. Furthermore, $C_+$ is a 2-manifold.

**3. The structure of the decomposition space.** At this point we launch an investigation of the subset $K = B - B_+$, with an eye toward ultimately concluding it is locally finite. The crucial step is detecting that $K$ is totally disconnected, for we have

**Lemma 3.1.** If $K$ is totally disconnected, then $B$ is a 2-manifold.

**Sketch of the Proof.** The argument in Lemma 2.3 shows each $y_0 \in K \cup D_+$ to have a connected neighborhood $U_0$ in $B$ such that every simple closed curve in $U_0 \cap C_+$ separates $U_0 \cap C_+$. Thus, as argued in [DW1, Lemma 4.1], $U_0 \cap C_+$ is homeomorphic to the complement in $S^2$ of some closed 0-dimensional subset. The
points of \((K \cup D_+) \cap U_0\) can be considered to be (some of the) ends of \(U_0\) and the homeomorphism of \(U_0 \cap C_+\) into \(S^2\) can be extended, in an end-preserving way, to one from \(U_0\) onto an open subset of \(S^2\).

Suppose now that \(K_1 \subset K\) is a closed subset such that \(K - K_1\) is totally disconnected (equivalently, \(0\)-dimensional, as \(K - K_1\) is locally compact). Using [DH, Theorem 2.10], we name a point \(b_0 \in K_1\), a neighborhood \(U_0\) of \(b_0\) as above, and then another neighborhood \(W\) of \(b_0\) in \(U_0\) for which there exists a shape retraction \(r: p^{-1}W \to p^{-1}b_0\) such that the restriction of \(r\) to \(p^{-1}b\) provides a shape equivalence from \(p^{-1}b\) to \(p^{-1}b_0\) for each \(b \in K_1 \cap W\).

**Proposition 3.2.** There is a nonempty relatively open subset \(\emptyset\) of \(K_1 \cap W\) that is totally disconnected (equivalently, in this setting, \(0\)-dimensional).

Before embarking on the proof, which occupies most of the remainder of this section, we derive the important

**Corollary 3.3.** \(K\) is totally disconnected.

**Proof.** Specify \(K_1\) by requiring \(K - K_1\) to be a maximal relatively open subset of \(K\) that is \(0\)-dimensional. The separability of \(K\) and the Sum Theorem for \(0\)-dimensional sets [HW, p. 18] ensure that such a subset exists. Then \(K_1 = \emptyset\), for otherwise a nonempty, \(0\)-dimensional, relatively open subset \(\emptyset\) of \(K_1\) would exist, and \((K - K_1) \cup \emptyset\) would be larger than \(K - K_1\). It would be obvious that \((K - K_1) \cup \emptyset\) is relatively open in \(K\) and the Sum Theorem of [HW] again would reveal that \((K - K_1) \cup \emptyset\) is \(0\)-dimensional. Thus, \(K\) itself is \(0\)-dimensional.

**Lemma 3.4.** There is a connected neighborhood \(W_1 \subset W\) of \(b_0\) and a component \(V\) of \(W_1 - K_1\) such that every arc \(A\) with \(\partial A \subset K_1\) and \(A - \partial A \subset V \cap C_+\) separates \(B\) and one component of \(B - A\) is contained in \(U_0\).

**Proof.** Choose \(W_1 \subset W\) so that \(H_{n+1}(p^{-1}W_1) \to H_{n+1}(p^{-1}U_0)\) is trivial. Since \(b_0 \in K\), there exists \(w \in W_1\) such that \(H_n(p^{-1}w) \to H_n(p^{-1}W_1)\) is trivial. Because \(\alpha[K_1 \cap W\) is nonzero, \(w \not\in K_1\). It follows from the \(0\)-dimensionality of \([(K - K_1) \cup D_+\] \cap W_1\) (and from the fact that \(H_n(p^{-1}w) \to H_n(p^{-1}W_1)\) is trivial for all \(w\) in some neighborhood of \(w\)) that we can choose \(w \in W_1 \cap C_+\).

Let \(V\) denote the component of \(W_1 - K_1\) containing \(w\). Observe that \(V \cap K \subset K - K_1\) is a relatively closed \(0\)-dimensional subset of \(V\) and, hence, \(V \cap C_+ = V - K\) is a dense, connected, open subset of \(V\).

Let \(A\) be an arc with \(\partial A \subset K_1\) and \(A - \partial A \subset V \cap C_+\). Split \(A\) into subarcs \(A_1, A_2\) having just an endpoint \(e\) in common. Then \(e \in V \cap C_+\). An argument like that in Lemma 2.5 verifies that \(\tilde{H}^{n+1}(p^{-1}A; \mathbb{Z}) \cong \mathbb{Z}\). The diagram

\[
\begin{array}{c}
H_1(p^{-1}W_1) \xrightarrow{\gamma} H_1(p^{-1}W_1, p^{-1}W_1 - p^{-1}A) \to \tilde{H}_0(p^{-1}W_1 - p^{-1}A) \to 0 \\
\downarrow \\
\tilde{H}^{n+1}(p^{-1}A) \cong \mathbb{Z}
\end{array}
\]
indicates that \( p^{-1}A \) separates \( p^{-1}W_1 \) (recall the previous argument in the proof of Lemma 2.3 that \( \gamma \) is trivial). Similarly, \( p^{-1}A \) separates \( p^{-1}U_0 \). Hence, \( A \) separates both \( W_1 \) and \( U_0 \).

Our initial specification that \( H_{n+1}(p^{-1}W_1) \to H_{n+1}(p^{-1}U_0) \) be trivial starts the chase through the diagram

\[
\begin{array}{ccc}
H_{n+1}(p^{-1}U_0) & \cong & H^1(\text{Cl } p^{-1}U_0, \text{Fr } p^{-1}U_0) \\
\uparrow j & & \uparrow j \\
H^1(\text{Cl } p^{-1}U_0, \text{Cl } p^{-1}U_0 - p^{-1}W_1) & \cong & H^1(\text{Cl } p^{-1}W_1, \text{Fr } p^{-1}W_1)
\end{array}
\]

leading to the detection that \( j \) is trivial. (The isomorphisms on the left arise from Lefshetz Duality [S, p. 297] while that on the right is an excision.) Since \( j \) is trivial, the diagram

\[
\begin{array}{ccc}
0 & \to & H^0(\text{Fr } p^{-1}U_0) \\
\uparrow i & & \uparrow j \\
0 & \to & H^0(\text{Cl } p^{-1}U_0 - p^{-1}W_1) \\
\uparrow i & & \uparrow j \\
0 & \to & H^1(\text{Cl } p^{-1}U_0, \text{Cl } p^{-1}U_0 - p^{-1}W_1)
\end{array}
\]

can be used to detect that \( i \) is trivial and, consequently, \( \text{Fr } p^{-1}U_0 \) is contained in a single component of \( (\text{Cl } p^{-1}U_0) - p^{-1}W_1 \). Thus, \( \text{Fr } p^{-1}U_0 \) is contained in a single component of \( (\text{Cl } p^{-1}U_0) - p^{-1}A \). This shows that one component of \( (\text{Cl } p^{-1}U_0) - p^{-1}A \) is also a component of \( M - p^{-1}A \), as well as that one component of \( B - A \) is a subset of \( U_0 \). This completes the proof of Lemma 3.4.

It remains to analyze the set \( W_1 \cap \text{Fr } V \) with the aim of determining that it is totally disconnected. Thereby we shall discover that \( V = W_1 \) and that \( W_1 \cap K_1 \) is a nonempty \((b_0 \in W_1 \cap K_1)\), totally disconnected subset of \( K_1 \).

**Lemma 3.5.** \( W_1 \cap \text{Fr } V \) contains no arc.

**Proof.** Suppose that \( A \subset W_1 \cap \text{Fr } V \) is an arc. Recall that the choice of \( W \) and \( b_0 \) was based on the stipulation that the shape retraction \( r: p^{-1}W \to p^{-1}b_0 \) restricts to a shape equivalence from \( p^{-1}b \) to \( p^{-1}b_0 \), for each \( b \in K_1 \cap W \). It follows that, for each subarc \( A' \) of \( A \) and \( x \in A' \), the inclusion \( p^{-1}x \to p^{-1}A' \) is a shape equivalence. Consequently, we can choose connected open sets \( P \) and \( N \), with \( P \subset N \subset W_1 \), such that

\[
N \cap A = A - \partial A = P \cap A,
\]

the inclusion induces an injection of \( \pi_1(p^{-1}(A - \partial A)) \) to \( \pi_1(p^{-1}N) \), and

\[
\text{Im}\{ \pi_1(p^{-1}(A - \partial A)) \to \pi_1(p^{-1}N) \} = \text{Im}\{ \pi_1(p^{-1}P) \to \pi_1(p^{-1}N) \}.
\]

**Claim.** The set \( p^{-1}(A - \partial A) \) separates \( p^{-1}P \) into precisely two connected pieces. Furthermore, if \( A' \) is a subarc of \( A - \partial A \) and if \( P' \subset N' \subset P \) are connected open sets chosen so that \( N' \cap A' = A' - \partial A' = P' \cap A' \), inclusion induces an injection of \( \pi_1(p^{-1}(A' - \partial A')) \) to \( \pi_1(p^{-1}N') \), and

\[
\text{Im}\{ \pi_1(p^{-1}(A' - \partial A')) \to \pi_1(p^{-1}N') \} = \text{Im}\{ \pi_1(p^{-1}P') \to \pi_1(p^{-1}N) \}.
\]
then distinct components of \( p^{-1}P' - p^{-1}(A' - \partial A') \) are contained in distinct components of \( p^{-1}P - p^{-1}(A - \partial A) \).

Before proving this, we use it to arrive at the contradiction that establishes Lemma 3.5. Transferring the conclusions via the map \( p \), we have that \( P - A = Z_1 \cup Z_2 \), where \( Z_1 \) and \( Z_2 \) are distinct components. An easy consequence of the “furthermore” in the Claim is that every point of \( A - \partial A \) is arcwise accessible from each of \( Z_1 \) and \( Z_2 \). As one of \( Z_1 \) or \( Z_2 \) meets \( V \), say \( Z_1 \cap V \neq \emptyset \), there is a 4-od \( T \) (i.e., \( T \) is the cone on a four-point set) with \( T \subseteq (Z_1 \cap C_+) \cup (A - \partial A) \), with the cone point in \( V \) and with \( T \cap (A - \partial A) \) equal to the four endpoints of \( T \). Let \( T' \) denote the subset of \( T \) that is the “sub-4-od” whose endpoints lie in \( \text{Fr} \, V \) but is otherwise contained in \( V \cap C_+ \). Name the endpoints of \( T \) to be \( a_1, a_2, a_3, a_4 \) and the corresponding endpoints of \( T' \) to be \( a'_1, a'_2, a'_3, a'_4 \). Let \( \langle x, y \rangle \) denote the subarc of \( T \) connecting any given pair of points \( x, y \in T \). Lemma 3.4 implies that for some pair, say \( a'_1, a'_2, p^{-1}(\langle a'_1, a'_2 \rangle) \) separates \( p^{-1}a'_1 \) from \( p^{-1}a'_2 \) and, therefore, that \( \langle a'_1, a'_2 \rangle \) separates \( a'_3 \) from \( a'_4 \). This is an impossibility, since \( \langle a'_1, a'_2 \rangle \cup Z_2 \cup \langle a'_3, a'_4 \rangle \) is a connected set missing \( \langle a'_1, a'_2 \rangle \).

Proof of the Claim. Since \( p^{-1}x \) and \( p^{-1}y \) are “connected” by a shape equivalence for \( x, y \in A \), an argument like the one given for Lemma 2.2 shows that inclusion induces an isomorphism \( \tilde{H}^n(\langle A' - \partial A' \rangle) \rightarrow \tilde{H}^n(\langle p^{-1}A' \rangle, p^{-1}(A' - \partial A')) = 0 \) for each subarc \( A' \) of \( A \) and each \( a \in A' \). The exact cohomology sequence for the pair \( (p^{-1}A', p^{-1}\partial A') \) and properties of cohomology with compact supports [S, p. 321] give

\[
\tilde{H}^{n+1}(\langle p^{-1}(A' - \partial A') \rangle) \equiv \tilde{H}^{n+1}(\langle p^{-1}A', p^{-1}\partial A' \rangle) = 0,
\]

and a comparison of two such sequences reveals that inclusion induces an isomorphism

\[
H_c^{n+1}(\langle p^{-1}(A' - \partial A') \rangle) \rightarrow H_c^{n+1}(\langle p^{-1}(A'' - \partial A'') \rangle)
\]

for subarcs \( A'' \subset A' \) of \( A \). The diagram

\[
\begin{array}{ccc}
H_1(\langle p^{-1}N \rangle) & \xrightarrow{\beta} & H_1(\langle p^{-1}N, p^{-1}(N - A) \rangle) \\
\uparrow \phi & & \uparrow \cong \\
H_1(\langle p^{-1}P \rangle) & \xrightarrow{\gamma} & H_1(\langle p^{-1}P, p^{-1}(P - A) \rangle) \rightarrow \tilde{H}_0(\langle p^{-1}(P - A) \rangle) \rightarrow 0 \\
\uparrow \cong & & \\
H_c^{n+1}(\langle p^{-1}(A - \partial A) \rangle) \equiv Z
\end{array}
\]

can be used to establish the Claim, provided \( \gamma \) is the trivial homomorphism. [The “furthermore” follows by comparing the diagrams for \( A \subset P \subset N \) and \( A' \subset P' \subset N' \) and by using the inclusion-induced isomorphism

\[
H_c^{n+1}(\langle p^{-1}(A - \partial A) \rangle) \rightarrow H_c^{n+1}(\langle p^{-1}(A' - \partial A') \rangle).
\]

The triviality of \( \gamma \) is immediate if \( \pi_1(\langle p^{-1}a \rangle) = 0 \) for some (hence, any) \( a \in A \), as the homomorphism \( \phi \) is then trivial. An outline of the argument that \( \gamma \) is trivial in
general will follow; this will be done by showing \( \beta(2\phi(e)) = 0 \) for each \( e \in H_1(p^{-1}P) \) and, hence, as \( \beta \) is a homomorphism to \( \mathbb{Z}, \beta\phi \) is trivial. Of course, it suffices to show that \( 2\phi(e) \) is an element of

\[
\text{Im}\{ H_1(p^{-1}(N - A)) \to H_1(p^{-1}N) \}.
\]

In fact, we show that in the diagram

\[
\begin{array}{ccc}
\pi_1(p^{-1}(N - A), x_0) & \delta & \pi_1(p^{-1}N, x_0) \\
\downarrow & & \downarrow \\
\pi_1(p^{-1}P, x_0)
\end{array}
\]

\(2n(e) \in \text{Im} \delta \) for each \( e \in \pi_1(p^{-1}P, x_0) \), where the basepoint \( x_0 \) lies in \( p^{-1}(P - A) \).

Let \( f: \tilde{N} \to p^{-1}N \) denote the universal covering space. Choose \( y_0 \in f^{-1}(x_0) \). Let \( \tilde{P}_0 \) denote the component of \( f^{-1}(P) \) containing \( y_0 \) and let

\[
\tilde{A}_0 = \tilde{P}_0 \cap f^{-1}p^{-1}(A - \partial A).
\]

It follows from the \( \pi_1 \) conditions stipulated prior to the statement of the Claim that \( \tilde{A}_0 \) is the only component of \( f^{-1}p^{-1}(A - \partial A) \) in \( \tilde{P}_0 \) and that \( \pi_1(\tilde{A}_0) \) is trivial. A computation outlined below establishes \( H^{n+1}_c(\tilde{A}) \equiv \mathbb{Z} \), and then the simply-connected case of this result applies to reveal \( \tilde{P}_0 - \tilde{A}_0 = C_1 \cup C_2 \), where \( C_1 \) and \( C_2 \) are distinct components. For an arbitrary representative \( e: (I, \{0,1\}) \to (p^{-1}P, x_0) \) of an element (also called) \( e \) of \( \pi_1(p^{-1}P, x_0) \), if there is a lift \( \tilde{e}: I \to \tilde{P}_0 \) of \( e \) with \( \tilde{e}(0) \) and \( \tilde{e}(1) \) contained in the same \( C_i \), then the loop obtained by connecting \( \tilde{e}(0) \) and \( \tilde{e}(1) \) via a path in that \( C_i \) is necessarily null homotopic in \( \tilde{N} \) and, consequently, \( e \) is homotopic rel \( (0,1) \) in \( p^{-1}N \) to \( e': (I, \{0,1\}) \to (f(C_i), x_0) \). As \( f(C_i) \cap p^{-1}A = \emptyset \), the element \( \eta(e) = \eta(e') \) is in \( \text{Im} \delta \). If no lift \( \tilde{e} \) to \( \tilde{P}_0 \) has \( \tilde{e}(0) \) and \( \tilde{e}(1) \) contained in the same \( C_i \), then each lift of \( 2e \) to \( \tilde{P}_0 \) must have this property. Thus, in either case, \( \eta(2e) \in \text{Im} \delta \).

Finally, to finish off the Claim, we describe the computation of \( H^{n+1}_c(\tilde{A}_0) \). Set \( \tilde{E} \) equal to that component of \( f^{-1}p^{-1}E \) contained in \( \tilde{A}_0 \), where \( E \) is a closed subarc of \( A \). Given any point \( a_0 \in E \), we specify a shape equivalence \( s \) from \( p^{-1}a_0 \) to a closed orientable \( n \)-manifold \( g_0 \) and a shape deformation retraction \( r': p^{-1}E \to p^{-1}a_0 \). Denote by \( \tilde{g}_0 \) the universal cover of \( g_0 \) and by \( \tilde{s}: \tilde{E} \cap f^{-1}p^{-1}a_0 \to \tilde{g}_0 \) and \( \tilde{r}': \tilde{E} \to \tilde{E} \cap f^{-1}p^{-1}a_0 \) lifts of \( s \) and \( r' \), respectively. Computing with \( \tilde{s} \) and \( \tilde{r}' \) establishes that \( H^n_c(E \cap f^{-1}p^{-1}a_0) \equiv \mathbb{Z} \), that \( \tilde{H}_c^{n+1}(\tilde{E}) = 0 \), and that inclusion induces an isomorphism

\[
\tilde{H}_c^n(\tilde{E}) \to \tilde{H}_c^n(\tilde{E} \cap f^{-1}p^{-1}a_0).
\]

The exact cohomology sequence (with compact supports) of the pair \( (\tilde{E}, \tilde{E} \cap f^{-1}p^{-1}E) \) displays that

\[
\tilde{H}_c^{n+1}(\tilde{E}, \tilde{E} \cap f^{-1}p^{-1}\partial E) \equiv \tilde{H}_c^{n+1}(\tilde{E} - f^{-1}p^{-1}\partial E) \equiv \mathbb{Z}.
\]

For a pair of closed subarcs \( E \subset F \) of \( A \), a comparison of the computation just described for each reveals that inclusion induces an isomorphism

\[
\tilde{H}_c^{n+1}(\tilde{E} - f^{-1}p^{-1}\partial E) \equiv \tilde{H}_c^{n+1}(\tilde{F} - f^{-1}p^{-1}\partial F).
\]
(Specifically, compare the pair \((\hat{F}, \hat{F} \cap f^{-1}p^{-1}\partial F)\) and the pair \((\tilde{F}, \tilde{F} \cap f^{-1}p^{-1}\text{Cl}(F – E))\) to determine that
\[
\hat{H}^{n+1}_c(\hat{F}, \hat{F} \cap f^{-1}p^{-1}\partial F) \equiv \hat{H}^{n+1}_c(\tilde{F}, \tilde{F} \cap f^{-1}p^{-1}\text{Cl}(F – E))
\]
and, then, use the excision axiom to deduce that the latter group is isomorphic to
\[
\hat{H}^{n+1}_c(\tilde{F} – f^{-1}p^{-1}\partial E).
\]
Finally, \(\hat{H}^{n+1}_c(\tilde{A}_0)\) is the direct limit of the groups \(\hat{H}^{n+1}_c(\tilde{E} – f^{-1}p^{-1}\partial E)\) where \(E\) ranges over closed subarcs of \(A\) and, hence, \(\hat{H}^{n+1}_c(\tilde{A}_0) \equiv \mathbb{Z}\).

Proof of Proposition 3.2. Since \(W_1 \cap \text{Fr} V\) contains no arc, no compact connected subset of it, other than a singleton, can be locally connected. Consequently, we can show \(W_1 \cap \text{Fr} V\) to be totally disconnected by proving it contains no sequence \(Z_1, Z_2, \ldots\) of pairwise disjoint, compact connected subsets, each with at least two points, that converges in the Hausdorff metric to another compact connected subset \(Z\), also with at least two points. The possibility that such a sequence might exist is eliminated by considering the following two cases.

First, \(Z\) contains four points \(a_1, a_2, a_3, a_4\) which are arcwise accessible from \(V\). Then there exists a 4-od \(T\) in \((V \cap C_+) \cup Z\), with \(T \cap Z = \{a_1, a_2, a_3, a_4\}\). Lemma 3.4 implies that for some pair, say \(\{a_1, a_2\}\), the subarc \(\langle a_1, a_2 \rangle\) of \(T\) separates \(a_3\) from \(a_4\) in \(V\). This is not possible because the connected sets \(Z_1, Z_2, \ldots\) miss \(T\) but have both \(a_3\) and \(a_4\) as limit points.

Second, the set of points in \(Z\) that are arcwise accessible from \(V\) is not dense in \(Z\). In this case, there exists a 4-od \(T\) in \((V \cap C_+) \cup (W_1 \cap \text{Fr} V)\), having its endpoints \(a_1, \ldots, a_4\) in \(Z\) and containing a "sub-4-od" \(T'\), which, in turn, has its endpoints \(a'_1, \ldots, a'_4\) in \(W_1 \cap \text{Fr} V\), disjoint from \(\{a_1, \ldots, a_4\}\), and satisfying \(T' - \{a_1, \ldots, a_4\} \subset V \cap C_+\). Lemma 3.4 again implies that for some pair, say \(\{a'_1, a'_2\}\), the subarc \(\langle a'_1, a'_2 \rangle\) of \(T'\) separates \(a'_3\) from \(a'_4\). This is not possible because \(\langle a'_3, a'_4 \rangle \cup Z\) is a connected subset of \(B\) missing \(\langle a'_1, a'_2 \rangle\) but containing both \(a'_3\) and \(a'_4\).

Lemma 3.1 and Corollary 3.3 combine to form our main result.

Theorem 3.6. If \(M\) is an orientable \((n + 2)\)-manifold and \(G\) is a usc decomposition of \(M\) into continua having the shape of closed, orientable \(n\)-manifolds, then \(B = M/G\) is a 2-manifold (without boundary).

Orientability of the elements of \(G\) is a fundamental hypothesis in Theorem 3.6; orientability of the source manifold \(M\) is less crucial, as the next result demonstrates.

Theorem 3.7. If \(G\) is a usc decomposition of an \((n + 2)\)-manifold \(M\) into continua having the shape of closed, orientable \(n\)-manifolds, then \(B = M/G\) is a 2-manifold possibly with boundary.

Proof. Assume \(M\) to be nonorientable. Consider the orientable double covering \(\theta: \tilde{M} \to M\) and usc decomposition \(\tilde{G}\) into the components of \(\theta^{-1}(g), g \in G\). The
nontrivial covering homeomorphism $h: \tilde{M} \to \tilde{M}$, which respects elements of $\tilde{G}$, induces an involution $u$ on $\tilde{M}/\tilde{G}$, and the diagram

\[
\begin{array}{ccc}
\tilde{M} & \xrightarrow{h} & M \\
\downarrow & & \downarrow \\
\tilde{M}/\tilde{G} & \xrightarrow{T} & M/G
\end{array}
\]

induces a map $T: \tilde{M}/\tilde{G} \to M/G$ behaving just like the map of $\tilde{M}/\tilde{G}$ to the orbit space of $u$. It is well known [Wh] that the orbit space of every involution on a 2-manifold is a 2-manifold with boundary (the boundary may be empty).

Before closing down our analysis of the troublesome spots $D$ and $K$, the discontinuity and degeneracy subsets of $B$, we provide one last analysis.

**Proposition 3.8.** Assume $M$ (orientable) satisfies $H_1(M) = 0$ and $K \neq \emptyset$. Then $K \cup D$ consists of exactly one point.

**Proof.** Find an arc $A$ in $B$ with $\partial A \subset K \cup D$, $\partial A \cap K \neq \emptyset$, and $A - \partial A \subset C_+$. If $\partial A \subset K$, then as before $\tilde{H}^{n+1}(p^{-1}A; \mathbb{Z}) \equiv \mathbb{Z}$; if $d \in \partial A \cap D$, then

$$\tilde{H}^{n+1}(p^{-1}A; \mathbb{Z}/k\mathbb{Z}) \equiv \mathbb{Z}/k\mathbb{Z},$$

where $\alpha_d(x) = k$ for all $x \neq d$ sufficiently close to $d$. As a result, with the appropriate coefficients the diagram

$$0 = H_1(M) \to H_1(M, M - p^{-1}A) \to \tilde{H}_0(M - p^{-1}A) \to \tilde{H}_0(M) = 0$$

reveals $\tilde{H}_0(M - p^{-1}A)$ to be nontrivial, implying that the arc $A$ separates the 2-manifold $B$, an impossibility.

As applications of Theorem 3.6, we mention some structural limitations upon the manifolds $M$ admitting decompositions into codimension-two submanifolds.

**Proposition 3.9.** Suppose $M$ is a connected $(n + 2)$-manifold with at least two ends and with $H_1(M; \mathbb{Z}/2\mathbb{Z}) = 0$. Then $M$ admits no usc decomposition $\mathcal{G}$ into continua having the shape of closed, orientable $n$-manifolds.

**Proof.** If it did, the (noncompact) decomposition space $B$ would satisfy $H_1(B; \mathbb{Z}/2\mathbb{Z}) = 0$ [B], indicating that $B$ is the plane. Here the key property is that $B$ has one end. The desired contradiction stems from the fact that proper, monotone maps preserve the number of ends.

**Proposition 3.10.** Suppose $M$ is a noncompact, connected $(n + 2)$-manifold such that for each compact subset $C$ of $M$ there exists another compact set $C' \supset C$ such that

$$H_1(M - C'; \mathbb{Z}/2\mathbb{Z}) \to H_1(M - C; \mathbb{Z}/2\mathbb{Z})$$

is trivial. Then $M$ admits no usc decomposition $\mathcal{G}$ into continua having the shape of closed, orientable $n$-manifolds.
Proof. Suppose otherwise. Then \( B = M/G \) is a noncompact 2-manifold for which 
\( H_1(B; \mathbb{Z}/2\mathbb{Z}) \) is trivial at infinity, by [B] and the hypothesis. This is impossible: every noncompact 2-manifold contains loops arbitrarily close to infinity that fail to be null homologous in the complement of a given compactum.

Our chief interest was in determining whether \( E^{n+2} \) admits a decomposition into codimension-two submanifolds, which now is settled.

Corollary 3.11. There is no usc decomposition of \( E^{n+2} \) into continua having the shape of closed, orientable \( n \)-manifolds.

Remark. The upper semicontinuity of the decompositions \( G \) has been fundamental to the discussion throughout this paper. Reinforcing this point, a remarkably explicit partition of \( E^3 \) into (round) circles is given in [Sz].

Finally, in the compact case, we have

Proposition 3.12. Suppose \( M^{n+2} \) is a closed, orientable \( n \)-manifold for which 
\( H_2(M; \mathbb{Z}/2\mathbb{Z}) = 0 \). Then \( M \) admits no usc decomposition \( G \) into continua having the shape of closed \( n \)-manifolds with \( \tilde{H}_1(g; \mathbb{Z}/2\mathbb{Z}) = 0 \) for all \( g \in G \).

Proof. If it did, \( B \) would be a closed 2-manifold, and \( H_2(B; \mathbb{Z}/2\mathbb{Z}) \neq 0 \). Under these hypotheses, the Vietoris-Begle Theorem [B] ensures that

\[
p_*: H_2(M; \mathbb{Z}/2\mathbb{Z}) \to H_2(B; \mathbb{Z}/2\mathbb{Z})
\]

is surjective, a contradiction.

4. Structure of the sheaf \( \mathcal{H}^n \). We shall denote by \( \mathcal{H}^n \) the presheaf on \( B \) determined by setting \( \mathcal{H}^n(U) = H^n(p^{-1}U; \mathbb{Z}) \) (and using inclusion induced homomorphisms) as well as the associated sheaf, whose stalk at a point \( x \in B \) is \( \tilde{H}^n(p^{-1}x; \mathbb{Z}) \). The path leading to the conclusion that \( B \) is a 2-manifold also brought us to a determination that \( \mathcal{H}^n \) is locally constant except at points of the sparse set \( K \cup D^+ \). A final refinement is

Theorem 4.1. Suppose \( M \) is an orientable \( (n+2) \)-manifold and \( G \) is a usc decomposition of \( M \) into continua having the shape of closed, orientable \( n \)-manifolds. Then the decomposition space \( B \) is a 2-manifold and \( B \) contains a locally finite (closed) subset \( F \) such that \( \mathcal{H}^n \) is locally constant at each point of \( B - F \). Furthermore, if \( \tilde{H}_1(g; \mathbb{Z}) \equiv 0 \) for each \( g \in G \), then \( F = \emptyset \).

Proof. Since it is already established that \( B \) is a 2-manifold and \( \mathcal{H}^n \) is locally constant over \( B \setminus K \cup D^+ \), we need to show that \( K \cup D^+ \) is locally finite and, under the additional constraints of the "furthermore", that it is empty.

The local finiteness of \( K \cup D^+ \) is detected by precluding the existence of three possible types of convergent sequences from \( K \cup D^+ \).

First, suppose there is a sequence \( \{x_i\} \subset K \) with \( x_i \to x \in K \). It had been previously observed that fact (1) detects that the local winding function at any point of \( K \) is zero at some point of the continuity set \( C_+ \). As we currently know that \( U \cap C_+ \) is connected whenever \( U \) is an open connected set, each such local winding function is zero at every point of \( C_+ \) in its domain. Thus, the argument in Lemma...
3.4 would show that an arc \( A \) connecting \( x, \) and \( x_{i+1}, \) where \( i \) is chosen so that \( A \) is contained in an appropriately small neighborhood of \( x \) and \( A - \partial A \subset C_+, \) would separate a connected neighborhood of \( x. \) This clearly is not possible as \( B \) is a 2-manifold (without boundary).

Second, suppose that some point \( x \in D_+ \) is a limit point of \( D_+ - x. \) Let \( \alpha \) denote the local winding function at \( x \) defined on a connected neighborhood \( U \) of \( x \) and, for \( d \in D_+ \cap U, \) let \( \alpha_d \) denote the local winding function of \( d \) defined on a connected neighborhood \( V_d \subset U \) of \( d. \) Since \( U \cap C_+ \) is connected, \( \alpha \) is constant on \( U \cap C_+ \), say \( \alpha(b) = r. \) Similarly, \( \alpha_d \) is constant on \( V_d \cap C_+, \) say \( \alpha_d(b) = r_d. \) According to fact (1), \( r = k \cdot r_d \) for \( b \in V_d \cap C_+ \) and, consequently, the integers \( r_d \) are bounded (by \( r). \) Of course, \( r_d > 1 \) as \( d \in D_+. \) Thus, we can specify a sequence \( \{ x_i \} \subset D_+ - x \) with \( x_i \to x \) and with the local winding functions at the \( x_i \)'s taking on the single value \( q > 1 \) at points in the continuity set \( C_+. \) An argument as in Lemma 3.4 using \( \mathbb{Z}/q\mathbb{Z} \)-coefficients would lead to the contradiction that an arc \( A, \) chosen as in the preceding paragraph, separates a connected neighborhood of \( x. \)

Third, suppose that there is a sequence \( \{ x_i \} \subset D_+ \) with \( x_i \to x \in K. \) As before, we assume that the \( x_i \)'s are contained in a neighborhood \( U \) of \( x \) on which the local winding function \( \alpha \) at \( x \) is defined. Specify connected neighborhoods \( V_i \subset U \) of \( x_i \) on which the local winding function \( \alpha_i \) at \( x_i \) is defined and set \( r_i \) equal to the constant value of \( \alpha_i \) on \( V_i \cap C_+. \) In contrast with the previous situation, we cannot conclude that the \( r_i \)'s are bounded, as \( \alpha(b) = 0 \) for \( b \in U_0 \cap C_+. \) Consider an arc \( A \) connecting \( x \) and \( x_1 \) where \( A \) is chosen in a small neighborhood of \( x \) containing \( x_i \) and where \( A - \partial A \subset C. \) An argument as in Lemma 3.4 using \( \mathbb{Z}/r_i\mathbb{Z} \)-coefficients produces the contradiction that \( A \) separates a connected neighborhood of \( x \) provided the homomorphism \( H_1(p^{-1}W_1) \to H_1(p^{-1}W_1, p^{-1}W_1 - p^{-1}A) \) is trivial (\( W_1 \) being chosen near the start of the proof at Lemma 3.4). In the preceding two paragraphs, the fact that \( x \notin A \) facilitates the computation. In the current situation, we must insist that, after choices of neighborhoods \( W_1 \subset U_0 \subset U \) of \( x \) are made, \( i \) is chosen with \( p^{-1}A \to p^{-1}x \) so close to \( p^{-1}x \) that

\[
\text{Im}\{ \tilde{H}_1(p^{-1}x) \to H_1(p^{-1}U) \} = \text{Im}\{ H_1(p^{-1}(U_0 - A)) \to H_1(p^{-1}U) \}.
\]

This poses no difficulties since generators for \( \text{Im}\{ \tilde{H}_1(p^{-1}x) \to H_1(p^{-1}U) \} \) can be found in \( H_i(p^{-1}(U_0 - x)). \)

We now assume that \( \tilde{H}_1(p^{-1}x; \mathbb{Z}) = 0 \) for each \( x \in B. \) Suppose, contrary to the Claim of the "furthermore", that \( b \in K \cup D_+. \) Specify an open 2-cell \( \Delta \subset B \) with \( \Delta \cap (K \cup D_+) = \{ b \} \) so that there is a shape strong deformation retraction \( r: p^{-1}\Delta \to p^{-1}b, \) the deformation occurring in a neighborhood of \( p^{-1}b. \) Set \( \Delta^* = \Delta - \{ b \} \) and choose any \( c \in \Delta^*. \) Consider the diagram

\[
\begin{array}{ccc}
H^n(p^{-1}\Delta) & \xrightarrow{\alpha} & H^n(p^{-1}\Delta^*) \\
\downarrow j^* & & \downarrow i^* \\
\tilde{H}^n(p^{-1}b) & \xrightarrow{\phi} & \tilde{H}^n(p^{-1}c)
\end{array}
\]

where the top row is part of the exact sequence of the pair \( (p^{-1}\Delta, p^{-1}\Delta^*), \) \( \phi \) is induced by the restriction of \( r, \) and the remaining homomorphisms are induced by
inclusions. In particular, \( \phi j^* = i^*\alpha \). The existence of \( r \) assures that \( j^* \) is into. The assumption that \( \check{H}_1(p^{-1}b) = 0 \) yields, by duality [S, p. 296] that \( H^{n+1}(p^{-1}\Delta, p^{-1}\Delta^*) \equiv 0 \). Consequently, the contradiction that \( \phi \) is onto (and hence an isomorphism) will be reached once \( i^* \) is shown to be onto.

The universal coefficient theorem [S, p. 243] yields the short exact sequence

\[
0 \rightarrow \text{Ext}(H_n(p^{-1}\Delta, p^{-1}\Delta^*), \mathbb{Z}) \rightarrow H^{n+1}(p^{-1}\Delta, p^{-1}\Delta^*) \rightarrow \text{Hom}(H_{n+1}(p^{-1}\Delta, p^{-1}\Delta^*), \mathbb{Z}) \rightarrow 0
\]

and duality [S, p. 296] produces

\[
H_n(p^{-1}\Delta, p^{-1}\Delta^*) \equiv \check{H}_2(p^{-1}b) \quad \text{and} \quad H_{n+1}(p^{-1}\Delta, p^{-1}\Delta^*) \equiv \check{H}_1(p^{-1}b).
\]

The same exact sequence applied to \( \check{H}_1(p^{-1}b) \) reveals that \( \check{H}_1(p^{-1}b) = 0 \) and, then, to \( \check{H}_2(p^{-1}b) \) reveals that \( \check{H}_2(p^{-1}b) \) is free. It follows that \( H^{n+1}(p^{-1}\Delta, p^{-1}\Delta^*) \equiv 0 \).

We begin the computation of \( i^* \)'s surjectiveness by expressing \( \Delta - b = \Delta^* \) as the union of two open disks \( R_1 \) and \( R_2 \) whose intersection is a pair of open disks \( V_1 \) and \( V_2 \) with \( c \in V_1 \). Further, we specify a point \( d \in V_2 \) and arcs \( A_1 \subset R_1 \) and \( A_2 \subset R_2 \) connecting \( e \) and \( d \). The strategy is to decompose \( i^* \) as

\[
\check{H}^n(p^{-1}\Delta^*) \rightarrow \check{H}^n(p^{-1}(A_1 \cup A_2)) \rightarrow \check{H}^n(p^{-1}c),
\]

and then to determine that both inclusion induced homomorphisms \( \delta \) and \( \beta \) are surjective.

The Mayer-Vietoris sequence

\[
\check{H}^n(p^{-1}(A_1 \cap A_2)) \rightarrow \check{H}^{n+1}(p^{-1}(A_1 \cup A_2)) \rightarrow 0
\]

contains adequate information for discovering that \( \beta \) is surjective. The computation of \( \check{H}^{n+1}(p^{-1}(A_1 \cup A_2)) \) is essentially done in Lemma 2.3 while the zero at the right is a consequence of Lemma 2.2. A further consequence of the latter is that, for \( i = 1, 2 \), there is an element \( a_i \in \check{H}^n(p^{-1}A_i) \) that restricts to a generator in both \( \check{H}^n(p^{-1}c) \) and \( \check{H}^n(p^{-1}d) \). Hence, essentially either \( \gamma(a_1, a_2) = (0, 0) \) or \( \gamma(a_1, a_2) = (2, 0) \). The latter would force \( \check{H}^{n+1}(p^{-1}(A_1 \cup A_2)) \) to be a torsion group, which it is not, while the former readily shows that \( \check{H}^n(p^{-1}(A_1 \cup A_1)) \rightarrow \check{H}^n(p^{-1}A_1) \) is surjective, and, hence, \( \beta \) is surjective.

Detecting the surjectivity of

\[
\check{H}^n(p^{-1}\Delta^*) \rightarrow \check{H}^n(p^{-1}(A_1 \cup A_2))
\]

is accomplished by comparing the spectral sequences of \( p|: p^{-1}\Delta^* \rightarrow \Delta^* \) and \( p|: p^{-1}(A_1 \cup A_2) \rightarrow A_1 \cup A_2 \). (The sequence is derived in [G] and is described adequately for our purpose in [DW2].) We only need to extract the commuting diagram

\[
\begin{array}{ccc}
\check{H}^n(p^{-1}\Delta^*; \mathbb{Z}) & \rightarrow & H^0(\Delta^*; \mathcal{F}^n) \\
\downarrow \delta & & \downarrow \\
\check{H}^n(p^{-1}A_1 \cup A_2; \mathbb{Z}) & \rightarrow & H^0(A_1 \cup A_2; \mathcal{F}^n)
\end{array}
\]

\( \rightarrow 0 \)
where the right-hand vertical homomorphism is inclusion induced. Furthermore, the homomorphism is an isomorphism since the early computation that
\[ \tilde{H}^{n+1}(p^{-1}(A_1 \cup A_2); \mathbb{Z}) \cong \mathbb{Z} \]
detects that the sheaf \( \mathcal{H}^n \) is constant on \( \Delta^* \). The surjectivity of \( \delta \) follows once we determine that the homomorphism \( \eta \) is an isomorphism. From the spectral sequence of
\[ p|: p^{-1}(A_1 \cup A_2) \to A_1 \cup A_2 \]
we determine that kernel \( \eta \equiv H^1(A_1 \cup A_2; \mathcal{H}^{n-1}) \) but \( \mathcal{H}^{n-1} \) is the zero-sheaf as each stalk \( \tilde{H}^{n-1}(p^{-1}x; \mathbb{Z}) \) is dually isomorphic to \( \tilde{H}_1(p^{-1}x; \mathbb{Z}) \equiv 0 \) (by assumption).

5. Examples. As a summary, we want to compare what is known about usc decompositions \( G \) of orientable \((n + k)\)-manifolds \( M \) into connected, orientable \( n \)-manifolds in the two low-codimensional cases where \( k = 1 \) and \( k = 2 \). In either case \( B = M/G \) is now known to be a boundaryless \( k \)-manifold.

Sometimes the decomposition map \( p: M \to B \) can be employed to study the structure of \( M \). This investigative tool is quite effective when \( p \) is an approximate fibration. Unfortunately, that is not always the prevailing situation. Example 3 of [D] provides a decomposition \( G \) of an \((n + 1)\)-manifold \( M \) into homology \( n \)-spheres, some simply connected and others not; the decomposition map cannot be an approximate fibration because the decomposition elements have different shapes (different homotopy types). There is a similar decomposition of the \((n + 2)\)-manifold \( M \times E^1 \) into homology \( n \)-spheres.

When \( k = 1 \) the structure of \( M \) is exposed, to some extent, by one unifying feature: the elements of \( G \) are all pairwise homologically equivalent [D, Corollary 6.3]. Consequently, the shape (neighborhood) retractions \( V \to g \) induce homology isomorphisms \( H_j(g') \to H_j(g) \) for all \( g' \) in \( V \) sufficiently close to \( g \). The first fact may appear to stem from the property that the continuity set \( C_+ \) of \( B \) coincides with \( B \) itself. However, when \( k = 2 \) the elements of \( G \) need not all be homologically equivalent, not even if \( C_+ = B \).

Example. A usc decomposition \( G \) of a connected, orientable \((n + 2)\)-manifold \( M \) \((n \geq 2) \) into orientable \( n \)-manifolds such that (1) in the resulting decomposition space \( B \), each point belongs to the continuity set \( C_+ \), and (2) \( G \) contains a pair of homologically inequivalent elements.

The crux of the matter is manifested when \( n = 2 \), so focus on that case. Start with a knot \( K \) in \( S^3 \), like the trefoil, whose closed complement fibers over \( S^1 \) with fiber \( T \), where \( T \) is a disk having a positive number of handles. Attach a \((4\text{-dimensional})\) 2-handle \( H \) to \( B^4 \) along \( U(K) \), a tubular neighborhood of \( K \) in \( S^3 = \partial B^4 \), via the framing determined by the longitude \( \partial T \) of the knot space \( \text{Cl}(S^3 - U(K)) \). The interior \( M \) of the resulting 4-manifold with boundary represents the desired manifold.

The exceptional element \( g_0 \) of the decomposition \( G \) yet to be specified is the 2-sphere in \( M \) formed by the cone over \( K \) from the center of \( B^4 \) and the core of the
attached 2-handle, \( h \). Topologically the other, more standard elements of \( G \) all are the union \( \hat{T} \) of \( T \) and a 2-cell capping off \( \partial T \). Each lies in \( M \) as a copy of \( \hat{T} \) in

\[
r \cdot (S^3 - U(K)) \subset B^4 \quad (\text{where } 0 < r < 1)
\]

concentric with one of the given fibers of \( S^3 - U(K) \), a copy of \( B^2 \) in

\[
B^2 \times r \cdot \partial D^2 \subset B^2 \times D^2 = h,
\]

and an annulus connecting the boundaries of the first two in the cone over \( U(K) \) in \( B^4 \), as suggested by Figure 1.

![Figure 1](image-url)

Away from the exceptional element \( g_0 \), the decomposition map behaves like the projection of \( \hat{T} \times (E^2 - 0) \) to \( E^2 - 0 \). One can see that the winding function \( \alpha_h \) is locally constant at \( b = p(g_0) \) by observing the existence of a retraction \( r_0: M \to g_0 \) such that \( r_0(M \cap B^4) \subset g_0 \cap B^4 \) and \( r_0|h \cap M \) acts like the projection of \( B^2 \times \text{Int} \, D^2 \) to its core \( B^2 \times 0 \subset g_0 \). For geometric reasons, \( r_0|g': g' \to g_0 \) is a degree-one map, for each \( g' \in G \). Obviously, for each \( g \neq g_0 \), \( H_1(g') = H_1(\hat{T}) \neq 0 \).

**References**


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TENNESSEE, KNOXVILLE, TENNESSEE 37996-1300