LOWEST ORDER INVARIANTS
FOR REAL-ANALYTIC SURFACES IN C²

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ABSTRACT. Suppose M is a general real-analytic surface in complex euclidean
two-space with complex tangent space at a point p. Further suppose M is
tangent to order k at p. This paper determines a complete set of kth order
local holomorphic invariants for M at p.

1. Introduction. This paper considers the problem of how to locally distin-
guish one real surface in C² from another. The present study is motivated by recent
work of Moser and Webster [2], who in turn were inspired by an earlier study of
Bishop [1]. Before stating the main result of Moser and Webster, we must establish
some notation and recall some definitions.

We always let M denote a real-analytic real two-dimensional surface in C².
Because we are concerned only with local properties of M, we always assume the
origin O is in M. Moreover, M will also denote M ∩ U for any open ball U about
O. Likewise, C² will denote any (possibly small) open ball about O in C². The
description “near O” will be used only occasionally to remind the reader of this
abuse of notation; however, “near O” is always understood.

DEFINITION 1.1. Two surfaces M and M are (holomorphically) equivalent (near
O), M ≈ M, provided there exists an invertible holomorphic map Γ: C² → C²
(near O) so that Γ(O) = O and Γ(M) = M.

DEFINITION 1.2. A property P of a surface M is an (holomorphic) invariant
for M provided any surface which is equivalent to M also has property P.

The purpose of this study is to find some interesting invariants for a general
surface M. If M is CR, then it must be either complex and equivalent to C (near
O), or it must be totally real and equivalent to R². Thus the only surfaces of
interest to this study are those for which O is a CR singularity. That is, the surface
is totally real for points arbitrarily near O but is tangent to C at O. A point on a
surface in C² with a complex tangent space is called an exceptional point by Bishop.
Bishop, as well as Moser and Webster, notes that any surface with a CR singularity
at O can be assumed to have the form

(1.3) M = {(z, h(z, \overline{z}))},

where h is a power series in z and \overline{z} with complex coefficients and centered at O;
moreover, h vanishes at O to order at least 2. We adopt the notation h ∈ O². If
h ∈ O³, Bishop calls the origin exceptionally exceptional.

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Under the assumption that \( O \) is not exceptionally exceptional, Bishop showed that \( h \) in the form (1.3) can be written in the form

\[
h(z, \bar{z}) = \begin{cases} 
\gamma z^2 + z\bar{z} + \gamma \bar{z}^2 + \lambda(z, \bar{z}) \\
or \\
\bar{z}^2 + z^2 + \lambda(z, \bar{z}) ,
\end{cases}
\]

where \( \lambda \in \mathcal{O}^3 \). Moreover, \( \gamma \) can be assumed to be a nonnegative real number and, as such, is a holomorphic invariant of \( M \). The main result of Moser and Webster is the following:

**Theorem [Moser and Webster].** Let \( M \) be in the form of (1.3) with \( h \) in the form of (1.4). Further suppose \( 0 < \gamma < 1/2 \) (the elliptic case). Then there exists a holomorphic coordinate system \( (z_1, z_2) \) in which \( M \) has the form

\[
(1.5) \quad \{(z_1, z_1\bar{z}_1 + \Gamma(x_2)) (z_1^2 + \bar{z}_1^2)\},
\]

where \( \Gamma(x_2) = \gamma + \delta x_2^2; \delta = \pm 1 \) for s a nonnegative integer, or \( \delta = 0 \) for \( s = \infty \). Moreover, the set \( \{\gamma, \delta, s\} \) is a complete set of invariants for \( M \). That is, \( \{\gamma, \delta, s\} \) completely characterizes \( M \) up to equivalence.

Bishop’s paper considers questions about the basic structure of \( M \): Does \( M \) bound an analytic subvariety? How can we describe the hull of holomorphy of \( M \)? The approach is to first construct a form of \( M \) which yields invariants for \( M \), then answer the questions by considering properties of the invariants. Therefore, a normal form (with a complete set of invariants) is most desirable. For a surface \( M \) with an elliptic not exceptionally exceptional point at \( O \), the normal form of Moser-Webster (1.5) could not be more elegant. However, the problem of finding a normal form for more general types of surfaces is left open by their results and techniques. In addition, notice that (1.5) is implicit (dependent on \( z_2 \)), whereas the original form of \( M \) (1.3) is explicit as a graph. We might ask if it is possible to find a normal form for \( M \) which retains the property of being a graph. It is this question which provides the motivation for our study.

To begin a study of this question it is natural to seek an invariant normal form for the lowest order terms in the power series \( h \) in (1.3). Given \( h \) in (1.3) with \( h \in \mathcal{O}^k \) for \( k \geq 2 \), we seek an analogue of Bishop’s form (1.4). Deriving such an analogue is the point of this paper, which culminates in the proof of our main result.

**Theorem 1.6.** Suppose \( M \) is a real-analytic two-dimensional surface in \( \mathbb{C}^2 \) with exceptional point at \( O \). If \( M \neq \mathcal{C} \), then, near \( O \), \( M \) can be written in the form

\[
(1.7) \quad M = \{(z, P_k(z, \bar{z}) + z\mathcal{O}^k)\}
\]

for a homogeneous polynomial \( P_k \) of the form \( z^{k-\alpha_0}z^{\alpha_0} \) for integers \( k \) and \( \alpha_0 \) satisfying \( k \geq 2 \) and \( 1 \leq \alpha_0 \leq k \), or \( z^{k-\alpha_0}z^{\alpha_0} + \gamma z^{k-\alpha_1}z^{\alpha_1} \) for integers \( k \), \( \alpha_0 \), and \( \alpha_1 \) satisfying \( k \geq 2 \), \( 1 \leq \alpha_0 \leq \alpha_1 \leq k \), and real number \( \gamma > 0 \), or

\[
z^{k-\alpha_0}z^{\alpha_0} + \gamma z^{k-\alpha_1}z^{\alpha_1} + \sum_{j=2}^{\tau} a_{k-\alpha_j}z^{k-\alpha_j}z^{\alpha_j}
\]

for integers \( k, \tau, \alpha_0, \ldots, \alpha_\tau \) satisfying \( k \geq 2 \), \( 2 \leq \tau \leq k-1 \), \( 1 \leq \alpha_0 < \alpha_1 < \cdots < \alpha_\tau \leq k \), a real number \( \gamma > 0 \), and nonzero complex numbers \( a_{k-\alpha_j} \) (\( 2 \leq j \leq \tau \))
satisfying $0 \leq \operatorname{Arg} a_k - \alpha_j - \alpha_j < 2\pi/k_j$, where the integers $k_j$ are defined as follows. For each $j = 2, 3, \ldots, \tau$ let $\rho_j$ be the integer defined by
\[ 0 \leq \rho_j < \alpha_j - \alpha_0 \quad \text{and} \quad (\alpha_j - \alpha_0) = m_j(\alpha_1 - \alpha_0) + \rho_j \]
for some integer $m_j$. Let $l_1 \equiv \alpha_1 - \alpha_0$. For given $l_1, \ldots, l_j$ let $l_j+1 \equiv \gcd(l_j, \rho_j+1)$ or $l_{j+1} = l_j$ if $\rho_{j+1} = 0$. For each $j = 2, \ldots, \tau$ let $k_j \equiv l_{j-1}/l_j$. Moreover, $P_k$ in (1.7) is a local holomorphic invariant for $M$.

Notice that if $k = 2$ in (1.7) (i.e., $O$ is not exceptionally exceptional) then the form of $P_2$ is either $z\bar{z}$, $z^2$, or $z\bar{z} + \gamma z^2$ for some $\gamma > 0$. Thus it is easy to see, applying Proposition 2.2, that (1.7) is simply another version of Bishop’s form (1.4) in the case $k = 2$. Of course, the interesting aspect of Theorem 1.6 is that it produces an invariant normal form for the lowest order term for any real-analytic surface; i.e., for any value of $k$. For example, if $M$ is tangent to $C$ to order exactly 3 (it follows from the following discussion that the order of tangency to $C$ is an invariant of $M$ (Proposition 2.9)) then $M$ has the form $M = \{(z, P_3(z, z) + \bar{z}O^3)\}$, with $P_3$ in the invariant normal form $z^2\bar{z}, z^2\bar{z}, z^3, z^2\bar{z} + \gamma z^2, z^2\bar{z} + \gamma z^2, z\bar{z}^2 + \gamma z^3$ for some real $\gamma > 0$ or $z^2\bar{z} + \gamma z\bar{z}^2 + c\bar{z}^3$ for some real $\gamma > 0$ and complex $c \neq 0$. In the latter case $k_2 = 1$ and the condition $0 \leq \operatorname{Arg} c < 2\pi$ is vacuous. The presence of the complex parameter “$c$” is somewhat surprising in the form of $P_3$. Notice that if $M = \{(z, z^2\bar{z} + \gamma z\bar{z}^2 + c\bar{z}^3 + O^4)\}$ and $\tilde{M} = \{(z, z^2\bar{z} + \gamma z\bar{z}^2 + c\bar{z}^3 + O^4)\}$ with $\gamma > 0$ and $c \neq \tilde{c}$, then $M \neq \tilde{M}$. Indeed, Theorem 1.6 yields an interesting collection of holomorphically distinct surfaces parametrized over $\mathbb{R}^+ \times (\mathbb{C} \setminus \{O\})$.

We now begin the proof of Theorem 1.6, leaving other observations and examples until §4.

2. Preliminaries to the proof. Suppose $M$ is a surface of the form
\[ M = \{(z, h(z, \bar{z}))\} \]
for some $h \in \mathcal{O}^k$, $k \geq 2$. Further suppose $\Gamma : \mathbb{C}^2 \to \mathbb{C}^2$ and $f : \mathbb{C} \to \mathbb{C}$ are holomorphic coordinate changes which preserve the origin. Let

\[ \tilde{M} \equiv \{\Gamma(f(z), h(f(z), \bar{f}(\bar{z}))\} \]

where $\bar{f}$ denotes the power series obtained by conjugating the coefficients of the power series $f$. It follows that $M \approx \tilde{M}$. The idea is to choose $\Gamma$ and $f$ so that $\tilde{M}$ is of the form (1.7). By way of illustration we obtain

**Proposition 2.2.** Any surface of the form (2.1) can be written in the form
\[ M = \{(z, h(z, \bar{z}))\} \]
for some $h \in \mathcal{O}^k \cap \mathcal{O}(\bar{z})$, $k \geq 2$. Here we mean $h \in \mathcal{O}^k$ and $h$ is divisible by $\bar{z}$; we will also use the notation $h \in \mathcal{O}^{k-1}$ to mean the same thing.

**Proof.** Simply choose $\Gamma : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $\Gamma(z_1, z_2) \equiv (z_1, z_2 - h(z_1, 0))$. □

Showing that we can obtain holomorphic invariants for $M$ by using $\Gamma$ and $f$ as above is complicated by the fact that existence of such $\Gamma$ and $f$ is not a necessary condition for $M$ to be equivalent to $\tilde{M}$. This difficulty will be overcome by use of
Lemma 2.4. Let \( M = \{(z, h(z, \bar{z}))\} \) and \( \tilde{M} = \{(z, \tilde{h}(z, \bar{z}))\} \) be two surfaces in the form (2.3). If \( M \approx \tilde{M} \) then there exists a holomorphic coordinate change \( \Gamma: \mathbb{C}^2 \to \mathbb{C}^2 \) and real-analytic \( f: \mathbb{C} \to \mathbb{C} \) of the form
\[
(2.5) \quad f(z, \bar{z}) = bz + O^2, \quad b \neq 0,
\]
such that
\[
(2.6) \quad \Gamma(z, h(z, \bar{z})) = (f(z, \bar{z}), \tilde{h}(f(z, \bar{z}), \bar{f}(z, \bar{z}))).
\]

Proof. Suppose \( \tau: \mathbb{C}^2 \to \mathbb{C}^2 \) is a holomorphic coordinate change with \( \tau(0) = 0 \) and \( \Gamma(M) = \tilde{M} \). Let \( \Gamma = (\Gamma_1, \Gamma_2) \) for \( \Gamma_1, \Gamma_2: \mathbb{C}^2 \to \mathbb{C} \). Define a function \( f: \mathbb{C} \to \mathbb{C} \) by
\[
(2.7) \quad f(z, \bar{z}) = \Gamma(z, h(z, \bar{z})).
\]
It follows that \( f \) is real-analytic and invertible. Suppose \( \Gamma_1(z_1, z_2) = bz_1 + O(z_1^2) \cup O(z_2) \). Then, since \( h \in O^k \),
\[
(2.8) \quad f(z, \bar{z}) = bz + O(z^2) + O^k.
\]
Since \( f \) is invertible we have \( b \neq 0 \), and since \( k \geq 2 \) we have \( f \) in the form (2.5). Equation (2.6) follows because \( \Gamma(M) = \tilde{M} \).

The importance of Lemma 2.4 lies in the fact that any \( \bar{z} \) dependence in \( f \) (2.5) can occur only in the higher-order terms. It is this fact which eventually will allow us to conclude that our form (1.7) yields holomorphic invariants for \( M \).

We now apply Lemma 2.4 to obtain a well-known invariant of \( M \), the order of tangency to \( \mathbb{C} \).

Proposition 2.9. Suppose \( M = \{(z, h(z, \bar{z}))\} \) is in the form (2.3). Then \( k \leq \text{Ord}_0 h \) is a holomorphic invariant of \( M \).

Proof. Suppose \( \tilde{M} = \{(z, \tilde{h}(z, \bar{z}))\} \) is also in the form (2.3) and \( M \approx \tilde{M} \). Let \( P_k \), respectively \( \tilde{P}_k \), denote the leading nonvanishing homogeneous polynomial in the expansion of \( h \), respectively \( \tilde{h} \), into a sum of homogeneous polynomials of increasing degree. Thus
\[
h(z, \bar{z}) = P_k(z, \bar{z}) + O^{k+1} \quad \text{and} \quad \tilde{h}(z, \bar{z}) = \tilde{P}_k(z, \bar{z}) + O^{k+1}.
\]
Let \( \Gamma \) and \( f \) be the mappings guaranteed by Lemma 2.4. Again letting \( \Gamma = (\Gamma_1, \Gamma_2) \), it follows from (2.5) and (2.6) that
\[
(2.10) \quad \Gamma_2(z, P_k(z, \bar{z}) + zO^k) = \tilde{P}_k(bz, \bar{bz}) + O^{k+1}.
\]
Suppose \( \Gamma_2 \) has the form
\[
(2.11) \quad \Gamma_2(z_1, z_2) = \sum_{j=1}^{k} a_j z_1^j + cz_2 + O(z_1^{k+1}) \cup O(z_2^2).
\]
By interchanging \( k \) and \( \tilde{k} \), if necessary, we may assume without loss of generality that \( k \leq \tilde{k} \). It then follows from (2.10) and (2.11) that
\[
(2.12) \quad \sum_{j=1}^{k} a_j z^j + cP_k(z, \bar{z}) = \tilde{P}_k(bz, \bar{bz}) + O^{k+1}.
\]
Since \( \bar{z} \) divides both \( P_k(z, \bar{z}) \) and \( \tilde{P}_k(bz, \bar{b} \bar{z}) \), (2.12) implies \( a_j = 0 \) for \( 1 \leq j \leq k \); hence
\[
(2.13) \quad cP_k(z, \bar{z}) = \tilde{P}_k(bz, \bar{b} \bar{z}) + \mathcal{O}^{k+1}.
\]
But, \( \Gamma \cong (\Gamma_1, \Gamma_2) \) invertible with \( a_1 = 0 \) in (2.11) implies \( c \neq 0 \) in (2.11). Proposition 2.9 follows immediately from (2.13).

As indicated above, Proposition 2.9 is a well-known result. We have included this particular proof because it yields the following useful

**Lemma 2.14.** Suppose \( M \) and \( \hat{M} \) are two surfaces in the form (2.3) with \( M = \{(z, P_k(z, \bar{z}) + \mathcal{O}^{k+1})\} \) and \( \hat{M} = \{(z, \tilde{P}_k(z, \bar{z}) + \mathcal{O}^{k+1})\} \) for nonvanishing homogeneous polynomials \( P \) and \( \tilde{P} \) of degree \( k \). If \( M \approx \hat{M} \) then there are nonzero complex numbers \( b \) and \( c \) such that
\[
(2.15) \quad cP_k(z, \bar{z}) = \tilde{P}_k(bz, \bar{b} \bar{z}).
\]

**Proof.** Equation (2.15) follows immediately from (2.13).

Lemma 2.14 is the key to finding an invariant normal form for the leading homogeneous term. Indeed, because of Lemma 2.14 we need to consider only those coordinate changes of the forms \( f(z) = bz \) and \( \Gamma(z_1, z_2) = (z_1, cz_2) \) for nonzero complex numbers \( b \) and \( c \).

3. The proof of Theorem 1.6. Suppose \( M \) is a surface of the form (2.3) with \( h(z, \bar{z}) = P_k(z, \bar{z}) + \mathcal{O}^{k+1} \). Let \( P_k \) have the explicit description
\[
(3.1) \quad P_k(z, \bar{z}) = \sum_{\alpha=1}^{k} a_{k-\alpha, \alpha} z^{k-\alpha} \bar{z}^{\alpha}.
\]
Note that \( \alpha \geq 1 \) because \( \bar{z} \) divides \( P_k \). Let \( \alpha_0 = \min \{ \alpha : a_{k-\alpha, \alpha} \neq 0 \} \). It follows from Lemma 2.14 that \( \alpha_0 \) is a holomorphic invariant for \( M \). By choosing \( \Gamma(z_1, z_2) = (z_1, (1/a_{k-\alpha_0, \alpha_0})z_2) \), we may assume (3.1) has the form
\[
(3.2) \quad P_k(z, \bar{z}) = z^{k-\alpha_0} \bar{z}^{\alpha_0} + \sum_{\alpha > \alpha_0} a_{k-\alpha, \alpha} z^{k-\alpha} \bar{z}^{\alpha}.
\]
If \( a_{k-\alpha, \alpha} = 0 \) for all \( \alpha_0 < \alpha \leq k \), it follows from Lemma 2.14 that \( P_k(z, \bar{z}) = z^{k-\alpha_0} \bar{z}^{\alpha_0} \) is the desired invariant normal form. Otherwise let
\[
\alpha_1 = \min \{ \alpha > \alpha_0 : a_{k-\alpha, \alpha} \neq 0 \}.
\]
It follows, again by Lemma 2.14, that \( \alpha_1 \) is an invariant for \( M \). The idea is to find an invariant normal form for the coefficient \( a_{k-\alpha_1, \alpha_1} \) while preserving the form (3.2), namely, preserving \( a_{k-\alpha_0, \alpha_0} = 1 \). To do this we no longer have complete freedom to choose \( c \) and \( b \); in particular, Lemma 2.14 implies that \( c \) and \( b \) must be chosen to satisfy the condition
\[
(3.3) \quad c = b^{k-\alpha_0} b^{\alpha_0}.
\]
If we let \( f(z) = bz \) for an arbitrary nonzero \( b \) in \( \mathbb{C} \) and let \( \Gamma(z_1, z_2) = (z_1, z_2/c) \) for \( c \) defined by (3.3), we may assume (3.2) has the form
\[
(3.4) \quad P_k(z, \bar{z}) = z^{k-\alpha_0} \bar{z}^{\alpha_0} + \left[ \frac{b}{b} \right]^{\alpha_1-\alpha_0} a_{k-\alpha_1, \alpha_1} z^{k-\alpha_1} \bar{z}^{\alpha_1}
\]
\[
+ \sum_{\alpha > \alpha_1} \left[ \frac{b}{b} \right]^{\alpha-\alpha_0} a_{k-\alpha, \alpha} z^{k-\alpha} \bar{z}^{\alpha}.
\]
Since \( a_{k-\alpha_1,\alpha_1} \neq 0 \) and \( b \) in (3.4) can be any nonzero complex number, we may choose \( b \) so that \( \frac{[b/b]^1}{[b/b]^1} = \gamma \) is a positive real number. Thus we may assume (3.4) has the form

\[
(3.5) \quad P_k(z, \bar{z}) = z^{k-a_0} + \gamma z^{k-\alpha_1} + \sum_{\alpha > \alpha_1} a_{k-\alpha,\alpha} z^{k-\alpha} \gamma
\]

for \( 1 \leq \alpha_0 < \alpha_1 \leq k \) and some real number \( \gamma > 0 \). Moreover, \( \gamma \) in (3.5) is an invariant for \( M \). To see this suppose \( \tilde{M} \) is of the form (2.1) with \( \tilde{h}(z, \bar{z}) = \tilde{P}_k(z, \bar{z}) + O^{k+1} \) and

\[
\tilde{P}_k(z, \bar{z}) = z^{k-\alpha_0} + \gamma^2 z^{k-\alpha_1} + \sum_{\alpha > \alpha_1} \tilde{a}_{k-\alpha,\alpha} z^{k-\alpha} \gamma
\]

for some positive real number \( \tilde{\gamma} \). (Recall that we already know \( k, \alpha_0, \) and \( \alpha_1 \) are invariants of \( M \).) If \( M \simeq \tilde{M} \) then applying Lemma 2.14 and condition (3.3) implies the existence of a nonzero complex number \( b \) such that

\[
(3.6) \quad \gamma = \frac{\tilde{h}/b}^{\alpha_1-a_0} \tilde{\gamma} \quad \text{and} \quad a_{k-\alpha,\alpha} = \left[\frac{\tilde{h}/b}^{\alpha_1-a_0} \tilde{a}_{k-\alpha,\alpha}\right] \quad \text{for all} \quad \alpha_0 < \alpha \leq k.
\]

Not only does (3.6) yield that \( \gamma = \tilde{\gamma} \) (i.e., \( \gamma \) in form (3.5) is an invariant of \( M \)), it also yields that \( |a_{k-\alpha,\alpha}| \) in form (3.5) is an invariant for \( M \) for each \( \alpha_0 < \alpha \leq k \). In addition, it follows from (3.6) that in order to preserve the normalization (3.5), we are constrained to require a condition on our future choice of \( b \), namely,

\[
(3.7) \quad \left[\frac{\tilde{h}/b}^{\alpha_1-a_0} = 1.
\]

If \( \alpha_1 = k \) or \( a_{k-\alpha,\alpha} = 0 \) for \( \alpha_0 < \alpha \leq k \), then we have obtained the invariant normal form

\[
P_k(z, \bar{z}) = z^{k-a_0} + \gamma z^{k-\alpha_1} \quad \text{for} \quad 1 \leq \alpha_0 < \alpha_1 \leq k
\]

and positive real \( \gamma \). Otherwise, let \( \alpha_2, \ldots, \alpha_r \) denote the integers satisfying \( \alpha_1 < \alpha_2 < \cdots < \alpha_r \leq k \) for which (3.5) has the form

\[
(3.8) \quad P_k(z, \bar{z}) = z^{k-a_0} + \gamma z^{k-\alpha_1} + \sum_{j=2}^{r} a_{k-\alpha_j,\alpha_j} z^{k-\alpha_j} \gamma_j
\]

for \( \gamma > 0 \) and nonzero \( a_{k-\alpha_j,\alpha_j}, 2 \leq j \leq r \). Choosing any complex number \( b \) satisfying (3.7), defining \( c \) by (3.3), and applying the coordinate changes \( f(z) = bz \) and \( \Gamma(z_1, z_2) = \left(z_1, z_2/c\right) \) transforms (3.8) to

\[
(3.11) \quad P_k(z, \bar{z}) = z^{k-a_0} + \gamma z^{k-\alpha_1} + \sum_{j=2}^{r} \left[\frac{b}{c}\right]^{\alpha_j} a_{k-\alpha_j,\alpha_j} z^{k-\alpha_j} \gamma_j.
\]

We see that our freedom to choose \( b \) so as to normalize the coefficient of \( z^{k-\alpha_2} \alpha_2 \) in (3.9) is completely determined by the set

\[
\mathcal{G}(M; 2) = \left\{ \frac{[\tilde{h}/b]^{\alpha_2-a_0}}{b \in \mathbb{C} \text{ and } [\tilde{h}/b]^{\alpha_1-a_0} = 1} \right\}.
\]

Let \( \rho_2 \) be the integer defined by \( 0 \leq \rho_2 < \alpha_1 - \alpha_0 \) and \( (\alpha_2 - \alpha_0) = m_2(\alpha_1 - \alpha_0) + \rho_2 \) for some nonnegative integer \( m_2 \). It follows that

\[
\mathcal{G}(M; 2) = \left\{ \frac{b}{c} \right\}^{\rho_2}.
\]
the group of $k_2$ roots of unity, where $k_2 \equiv (\alpha_1 - \alpha_0)/\gcd(\alpha_1 - \alpha_0, \rho_2)$. We are using the convention $\gcd(l,0) \equiv l$ for all $l > 0$. Thus we may choose $b$ (and the appropriate coordinate changes) so that

$$0 \leq \Arg\left[\frac{\bar{b}}{b}\right]^{\alpha_2 - \alpha_0}a_{k - \alpha_2, \alpha_2} < 2\pi/k_2.$$  

That is, the form of $P_k(z, \bar{z})$ in (3.8) can be assumed to satisfy the additional property

$$0 \leq \Arg a_{k - \alpha_2, \alpha_2} < 2\pi/k_2. \tag{3.10}$$

It follows from (3.6) that $a_{k - \alpha_2, \alpha_2}$, normalized to satisfy (3.10), is an invariant of $M$. Furthermore, to preserve the normalization (3.5) (with (3.8)) requires yet another constraint on our future choice of $b$, namely,

$$[\bar{b}/b]^{\alpha_2 - \alpha_0} = 1. \tag{3.11}$$

We proceed in this way to find an invariant normal form for the coefficient of $z^{k - \alpha_2}z^{\alpha_2}$, then the coefficient of $z^{k - \alpha_3}z^{\alpha_3}$, and so on, until we have obtained the invariant normal form (1.7).

4. Conclusion. We now present some interesting consequences of Theorem 1.6 and the above proof. To begin, we observe

COROLLARY 4.1. Let $M$ be any surface of the form

$$M = \left\{ \left( z, z^{k - \alpha_0}z^{\alpha_0} + \sum_{\alpha = \alpha_0 + 1}^{k} a_{k - \alpha, \alpha}z^{k - \alpha}z^{\alpha} + \bar{z}O^k \right) \right\}.$$  
The numbers $|a_{k - \alpha, \alpha}|$, $\alpha = \alpha_0 + 1, \ldots, k$, are invariants for $M$.

PROOF. Corollary 4.1 follows from the above discussion since $\alpha_0, \ldots, \alpha_\tau$ are invariants for $M$ which pick out the nonzero coefficients, and for any $b \in \mathbb{C}\{O\}$ and any integer $\rho$ we always have $|([\bar{b}/b])^\rho| = 1$. □

COROLLARY 4.2. Suppose $M$ is any surface of the form $\{(z, P_k(z, \bar{z}) + \bar{z}O^k)\}$ with $P_k(z, \bar{z})$ in the invariant normal form of Theorem 1.6. There are exactly $K \equiv \prod_{j=2}^{\tau} k_j$ surfaces of the form

$$\left\{ \left( z, z^{k - \alpha_0}z^{\alpha_0} + \gamma z^{k - \alpha_1}z^{\alpha_1} + \sum_{\alpha = \alpha_1 + 1}^{k} \tilde{a}_{k - \alpha, \alpha}z^{k - \alpha}z^{\alpha} + \bar{z}O^k \right) \right\}$$

which have the same $k$th order invariants as $M$. (\(\gamma\) is the positive real number fixed by Theorem 1.6.)

PROOF. As before, all $\tilde{a}_{k - \alpha, \alpha}$ must be 0 except for $\tilde{a}_{k - \alpha_j, \alpha_j}$, $2 \leq j \leq \tau$. Corollary 4.2 follows by counting from back to front. That is, fix $\tilde{a}_{k - \alpha_j, \alpha_j} = a_{k - \alpha_j, \alpha_j}$ for all $j = 2, \ldots, \tau - 1$. Clearly, there are $k_\tau$ distinct possible choices for $\tilde{a}_{k - \alpha, \alpha_\tau}$. Similarly, there are $k_{\tau - 1}$ choices for $\tilde{a}_{k - \alpha_{\tau - 1}, \alpha_{\tau - 1}}$, etc. □

To emphasize the explicit nature of the proof of Theorem 1.6 we prove the following proposition.
PROPOSITION 4.3. Let $M$ be any surface as in Corollary 4.2. The set of all surfaces of the form \( \{(z, \tilde{P}_k(z, \bar{z}) + \mathcal{O}^{k+1})\} \) with the same $k$th order invariants as $M$ is

\[
\left\{ \left( z, z^{k-\alpha_0}z^\alpha_0 + \gamma e^{i\theta}z^{k-\alpha_1}z^\alpha_1 \right. \right.
\]
\[
\left. \left. + \sum_{j=2}^\tau \left\{ \exp \left( \frac{\mu_j}{k_j} 2\pi i \right) \exp \left( \frac{\alpha_j - \alpha_0}{\alpha_1 - \alpha_0} \theta_i \right) \times \exp \left( \sum_{l=2}^j \frac{\mu_{l-1}}{k_{l-1} \rho_{l-1}} \rho_j 2\pi i \right) \right\}\right\} \times \exp \left( \sum_{j=2}^\tau \frac{\mu_{l-1}}{k_{l-1} \rho_{l-1}} \rho_j 2\pi i \right) \times a_k^{-\alpha_j, \alpha_j, z^{k-\alpha_j}z^\alpha_j + \mathcal{O}^{k+1}} \right\}
\]
\[
0 \leq \theta < 2\pi; \text{ for all } j = 2, \ldots, \tau, \ 0 \leq \mu_j < k_j \}
\]

(For notational convenience we have employed the convention $k_1 = \rho_1 = 1$ and $\mu_1 = 0$. All other terms are from Theorem 1.6.)

PROOF. For any fixed $\theta$ in (4.4) there are $\prod_{j=2}^\tau k_j$ surfaces in the set (4.4). Thus by Corollary 4.2 it suffices to show that each surface in (4.4) reduces to the form of $M$. To this end, fix $\theta$ and choose $b$ such that $(\bar{b}/b)^{\alpha_1 - \alpha_0} = e^{-i\theta}$. Observe that for each $j = 2, \ldots, \tau$,

\[
\left( \frac{\bar{b}}{b} \right)^{\alpha_j - \alpha_0} = \exp \left( \frac{-\alpha_j - \alpha_0}{\alpha_1 - \alpha_0} \theta_i \right).
\]

The surface in (4.4) for this $\theta$ reduces to

\[
\left\{ \left( z, z^{k-\alpha_0}z^\alpha_0 + \gamma z^{k-\alpha_1}z^\alpha_1 \right. \right.
\]
\[
\left. \left. + \exp \left( \frac{\mu_2}{k_2} 2\pi i \right) a_k^{-\alpha_2, \alpha_2, z^{k-\alpha_2}z^\alpha_2} \right\} \times \exp \left( \sum_{j=3}^\tau \frac{\mu_{l-1}}{k_{l-1} \rho_{l-1}} \rho_j 2\pi i \right) \times a_k^{-\alpha_j, \alpha_j, z^{k-\alpha_j}z^\alpha_j + \mathcal{O}^{k+1}} \right\}
\]

(4.5)

Choose $b$ such that $(\bar{b}/b)^{\alpha_1 - \alpha_2} = 1$ and recall the definitions of $\rho_2, \ldots, \rho_\tau$ to see
that (4.5) reduces to
\[
\left\{ z, z^{k-\alpha_0}z^\alpha_0 + \gamma z^{k-\alpha_1}z^\alpha_1 \\
+ \exp\left( \frac{\mu_2 \cdot 2\pi i}{k_2} \right) \left( \frac{b}{\bar{b}} \right)^{\rho_2} a_{k-\alpha_2,\alpha_2} z^{k-\alpha_2}z^\alpha_2 \\
+ \sum_{j=3}^{\tau} \left\{ \exp\left( \frac{\mu_j \cdot 2\pi i}{k_j} \right) \exp\left( \sum_{l=3}^{j} \frac{\mu_l-1}{k_l-1} \rho_j 2\pi i \right) \left( \frac{b}{\bar{b}} \right)^{\rho_j} \right\} \right. \\
\times a_{k-\alpha_j,\alpha_j} z^{k-\alpha_j}z^\alpha_j + \mathcal{O}^k + 1 \left. \right\}.
\]
(4.6)

Recalling the definition of \( k_2 \) and the fact that \( 0 < \rho_2 < k_2 \), we can choose \( b \) such that
\[
\left( \frac{b}{\bar{b}} \right)^{\rho_2} = \exp\left( -\frac{\mu_2 \cdot 2\pi i}{k_2} \right).
\]
Observe that for each \( j = 3, \ldots, \tau \),
\[
\left( \frac{b}{\bar{b}} \right)^{\rho_j} = \exp\left( -\frac{\mu_j \cdot 2\pi i}{k_j \rho_j} \right).
\]
It follows that (4.6) reduces to
\[
\left\{ z, \ldots, a_{k-\alpha_2,\alpha_2} z^{k-\alpha_2}z^\alpha_2 + \exp\left( \frac{\mu_3 \cdot 2\pi i}{k_3} \right) a_{k-\alpha_3,\alpha_3} z^{k-\alpha_3}z^\alpha_3 \\
+ \sum_{j=4}^{\tau} \left\{ \exp\left( \frac{\mu_j \cdot 2\pi i}{k_j} \right) \exp\left( \sum_{l=4}^{j} \frac{\mu_l-1}{k_l-1} \rho_j 2\pi i \right) \right\} \right. \\
\times a_{k-\alpha_j,\alpha_j} z^{k-\alpha_j}z^\alpha_j \left. \right\}.
\]
(4.7)

We can now choose \( b \) so that
\[
\left( \frac{b}{\bar{b}} \right)^{\alpha_1-\alpha_0} = \left( \frac{b}{\bar{b}} \right)^{\rho_2} = 1 \quad \text{and} \quad \left( \frac{b}{\bar{b}} \right)^{\rho_3} = \exp\left( -\frac{\mu_3 \cdot 2\pi i}{k_3} \right).
\]
Observe that
\[
\left( \frac{b}{\bar{b}} \right)^{\rho_j} = \exp\left( -\frac{\mu_3 \cdot 2\pi i}{k_3 \rho_3} \right) \quad \text{for all } j = 4, \ldots, \tau.
\]
The argument proceeds and the proof of Proposition 4.3 is clear. \( \square \)

Perhaps the most interesting feature of the invariant normal form of \( P_k \) in Theorem 1.6 is the presence of conditions on the principal arguments of the invariants \( a_{k-\alpha_j,\alpha_j} \) for \( 2 \leq j \leq \tau \). These conditions are determined by various groups of roots of unity. These groups, in turn, are determined by the invariants \( \alpha_0, \alpha_1, \ldots, \alpha_\tau \) and, therefore, are themselves invariants of the surface. We conclude this study with a brief look at these invariant groups.
Suppose $P_k(z, \bar{z})$ is in the invariant normal form of Theorem 1.6. Let $\alpha_0, \alpha_1, \ldots, \alpha_r$ be the invariants given in Theorem 1.6. The groups in question are described as follows: For notational consistency let $\mathcal{G}(M; O)$ denote $\mathbb{C}\{0\}$ and $\mathcal{G}(M; 1)$ denote the unit circle. For each $2 \leq j \leq r$ let

$$\mathcal{G}(M; j) = \{ra' : ra' = 1 \text{ for } 1 \leq l \leq j - 1\}.$$ 

The numbers $k_2, \ldots, k_r$ in Theorem 1.6 are the orders of $\mathcal{G}(M; 2), \ldots, \mathcal{G}(M; r)$, respectively; indeed, for each $2 \leq j \leq r$ it follows that $\mathcal{G}(M; j)$ is the group of $k_j$ roots of unity. Notice that $\mathcal{G}(M; j)$ is the group containing all the numbers which can be used to reduce (by multiplication) the coefficient of $z^{k_2}a_2z^{k_3}a_3 \cdots$ to the invariant form of Theorem 1.6. The groups $\mathcal{G}(M; O)$ and $\mathcal{G}(M; 1)$ are the same for all surfaces (with trivial exception being those for which $P_k(z, \bar{z}) = z^{k_0}a_0z^{k_1}a_1$ and yield all possible information if $r < k$. Notice this includes the case $k = 2$, since the construction requires $r \leq k - 1$. Hence, the invariant groups are interesting only in the case $k \geq 3$ and $r \geq 2$, that is, in the case $P_k(z, \bar{z})$ has at least three nonvanishing terms.

In the case $k = 3$ and $r = 2$ we see that $\mathcal{G}(M; 2) = \{id\}$, thus accounting for the fact that the coefficient of $z^3$ is invariant without any condition on its argument. Indeed, if $\alpha_1 - \alpha_0 = 1$ then $\mathcal{G}(M; j) = \{id\}$ for all $2 \leq j \leq r$, and we obtain no restrictive condition on any of the remaining nonzero invariant coefficients of $P_k(z, \bar{z})$. Hence, the first occurrence of a restrictive condition on the argument of an invariant coefficient is for $k = 4$ and $r = 2$. Thus, we consider the following example.

Suppose $M$ is tangent to $\mathbb{C}$ to order exactly 4. Then $M$ has the form $M = \{(z, P_4(z, \bar{z}) + z^{4a})\}$ with $P_4$ in the invariant form $z^4-a_0z^{a_0}$ for some $1 \leq a_0 \leq 4$; or $z^4-a_0z^{a_0} + \gamma z^{a_1-z^{a_1}}$ for some $1 \leq a_0 < a_1 \leq 4$ and $\gamma > 0$; or $z^3z + \gamma z^{a_2} + a_1 z^{a_3} + a_4 z^4$ for some $\gamma > 0$ and complex $a_3, a_4$ not both equal to 0; or $z^2z^2 + \gamma z^{a_3} + c z^4$ for some $\gamma > 0$ and complex $c \neq 0$; or $z^3z + \gamma z^{a_3} + c z^4$ for some $\gamma > 0$ and nonzero $c$ satisfying $0 \leq \text{Arg } c < \pi$.

For a given $k \geq 3$ and $r \geq 2$, suppose we wish to give an example for which the invariant form of $P_k(z, \bar{z})$ places the most stringent condition possible on the argument of some coefficient of $P_k(z, \bar{z})$. It is evident that we must choose the example so that $\mathcal{G}(M; 2)$ has greatest possible order. This is accomplished by choosing $a_0 = 1$, $a_1 = k - 1$, and $a_2 = k$, thus producing the invariant form

$$P_k(z, \bar{z}) = z^{k-1}z + \gamma z^{k-1}z + c z^k$$

for some $\gamma > 0$ and nonzero complex number $c$ satisfying $0 \leq \text{Arg } c < 2\pi/(k - 2)$.

We close with the observation that we can apply elementary number-theoretic arguments to the invariants $\alpha_0, \alpha_1, \ldots, \alpha_r$ and produce many observations concerning the restrictive conditions on the invariant coefficients of $P_k(z, \bar{z})$. Although some of these observations are amusing, we have overcome the temptation to present them here.

REFERENCES


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